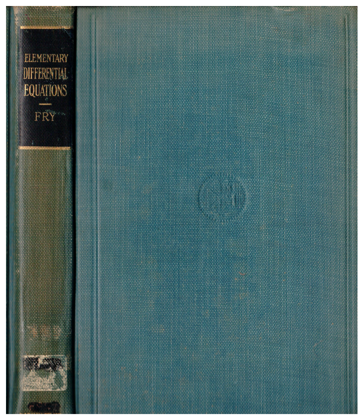
A Solution Manual For

Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)



Nasser M. Abbasi February 5, 2025 Compiled on February 5, 2025 at 3:26pm

Contents

CHAPTER 1

Lookup tables for all problems in current book

1.1 Chapter 1. section 5. Problems at page 19

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE
18531	2	$x^2y'' - \frac{x^2{y'}^2}{2y} + 4xy' + 4y = 0$
18532	3	$y' + cy = a$
18533	4	$y'' + \frac{y'}{x} + k^2y = 0$
18534	5	$\cos(x)\, y' + \sin(x)\, y'' + ny\sin(x) = 0$
18535	6	$y' = \frac{\sqrt{1-y^2}\, \arcsin(y)}{x}$
18536	16 (a)	$v'' = \left(\frac{1}{v} + {v'}^4\right)^{1/3}$
18537	16 (b)	$v' + u^2v = \sin(u)$
18538	17 (a)	$\sqrt{y' + y} = (y'' + 2x)^{1/4}$
18539	18	$v' + \frac{2v}{u} = 3$

1.2 Chapter IV. Methods of solution: First order equations. section 24. Problems at page 62

Table 1.2: Lookup table for all problems in current section

ID	problem	ODE
18540	4 (a)	$\sin(x)\cos(y)^2 + \cos(x)^2\, y' = 0$
18541	4 (b)	$y' + \sqrt{\frac{1-y^2}{-x^2+1}} = 0$
18542	4 (c)	$y - xy' = b(1 + x^2y')$
18543	5	$x' = k(A - nx)\,(M - mx)$
18544	6	$y' = 1 + \frac{1}{x} - \frac{1}{y^2+2} - \frac{1}{x(y^2+2)}$

1.3 Chapter IV. Methods of solution: First order equations. section 29. Problems at page 81

Table 1.3: Lookup table for all problems in current section

ID	problem	ODE
18545	1	$y^2 = x(y-x)\,y'$
18546	2	$2x^2y + y^3 - x^3y' = 0$
18547	3	$2ax + by + (2cy + bx + e)\,y' = g$
18548	4	$\sec(x)^2 \tan(y)\,y' + \sec(y)^2 \tan(x) = 0$
18549	5	$x + y'y = my$
18550	6	$\frac{2x}{y^3} + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right) y' = 0$
18551	8	$\left(T + \frac{1}{\sqrt{t^2-T^2}}\right) T' = \frac{T}{t\sqrt{t^2-T^2}} - t$

1.4 Chapter IV. Methods of solution: First order equations. section 31. Problems at page 85

Table 1.4: Lookup table for all problems in current section

ID	problem	ODE
18552	1	$y' + xy = x$
18553	2	$y' + \frac{y}{x} = \sin(x)$
18554	3	$y' + \frac{y}{x} = \frac{\sin(x)}{y^3}$
18555	4	$p' = \frac{p + a\,t^3 - 2pt^2}{t(-t^2+1)}$
18556	5	$(T\ln(t) - 1)\,T = tT'$
18557	6	$y' + y\cos(x) = \frac{\sin(2x)}{2}$

Continued on next page

Table 1.4 Lookup table
Continued from previous page

ID	problem	ODE
18558	7	$y - \cos(x) y' = y^2 \cos(x) (-\sin(x) + 1)$

1.5 Chapter IV. Methods of solution: First order equations. section 32. Problems at page 89

Table 1.5: Lookup table for all problems in current section

ID	problem	ODE
18559	2	$xy'^2 - y + 2y' = 0$
18560	3	$2y'^3 + y'^2 - y = 0$
18561	4	$y' = e^{z-y'}$
18562	5	$\sqrt{t^2 + T} = T'$
18563	7	$(x^2 - 1) y'^2 = 1$
18564	8	$y' = (x + y)^2$

1.6 Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91

Table 1.6: Lookup table for all problems in current section

ID	problem	ODE
18565	1	$\theta'' = -p^2\theta$
18566	2 (eq 39)	$\sec(\theta)^2 = \frac{ms'}{k}$
18567	3 (eq 41)	$y'' = \frac{m\sqrt{y'^2+1}}{k}$
18568	4 (eq 50)	$\phi'' = \frac{4\pi nc}{\sqrt{v_0^2 + \frac{2e(\phi - V_0)}{m}}}$

Continued on next page

Table 1.6 Lookup table
Continued from previous page

ID	problem	ODE
18569	8 (eq 68)	$y' = x(ay^2 + b)$
18570	8 (eq 69)	$n' = (n^2 + 1)\,x$
18571	9 (a)	$v' + \frac{2v}{u} = 3v$
18572	9 (b)	$\sqrt{-u^2+1}\,v' = 2u\sqrt{1-v^2}$
18573	9 (c)	$\sqrt{1+v'} = \frac{e^u}{2}$
18574	9 (d)	$\frac{y'}{x} = y\sin(x^2-1) - \frac{2y}{\sqrt{x}}$
18575	9 (e)	$y' = 1 + \frac{2y}{x-y}$
18576	10 (a)	$v' + 2vu = 2u$
18577	10 (b)	$1 + v^2 + (u^2+1)\,vv' = 0$
18578	10 (c)	$u\ln(u)\,v' + \sin(v)^2 = 1$

1.7 Chapter V. Singular solutions. section 36. Problems at page 99

Table 1.7: Lookup table for all problems in current section

ID	problem	ODE
18579	1 (eq 98)	$4yy'^3 - 2x^2y'^2 + 4xyy' + x^3 = 16y^2$

1.8 Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163

Table 1.8: Lookup table for all problems in current section

ID	problem	ODE
18580	1 (eq 100)	$\theta'' - p^2\theta = 0$
18581	2	$y'' + y = 0$
18582	3	$y'' + 12y = 7y'$
18583	4	$r'' - a^2 r = 0$
18584	5	$y'''' - a^4 y = 0$
18585	6	$v'' - 6v' + 13v = \mathrm{e}^{-2u}$
18586	7	$y'' + 4y' - y = \sin(t)$
18587	8	$y'' + 3y = \sin(x) + \frac{\sin(3x)}{3}$
18588	10	$5x' + x = \sin(3t)$
18589	11	$x'''' - 6x''' + 11x'' - 6x' = \mathrm{e}^{-3t}$
18590	14	$x^4 y'''' + x^3 y''' - 20x^2 y'' + 20xy' = 17x^6$
18591	15	$t^4 x'''' - 2t^3 x''' - 20t^2 x'' + 12x't + 16x = \cos(3\ln(t))$

1.9 Chapter VII. Linear equations of order higher than the first. section 63. Problems at page 196

Table 1.9: Lookup table for all problems in current section

ID	problem	ODE
18592	1	$y''' - y'' - y' + y = 0$
18593	2	$y'''' - 3y''' + 3y'' - y' = \mathrm{e}^{2x}$
18594	3	$y''' - y'' + y' - y = \cos(x)$
18595	8	$x^2 y'' + 3xy' + y = \frac{1}{x}$

CHAPTER 2

Book Solved Problems

2.1 Chapter 1. section 5. Problems at page 19

2.1.1 Problem 2

Internal problem ID [18531]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter 1. section 5. Problems at page 19
Problem number : 2
Date solved : `Tuesday, January 28, 2025 at 08:28:41 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _reducible, _mu_xy]]`

Solve

$$x^2y'' - \frac{x^2{y'}^2}{2y} + 4xy' + 4y = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)+12*(x^2*(diff(diff(y
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- 2nd order ODE linearizable_by_differentiation successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 20

```
dsolve(x^2*diff(diff(y(x),x),x)-1/2*x^2/y(x)*diff(y(x),x)^2+4*x*diff(y(x),x)+4*y(x) =
```

$$y(x) = \frac{\left(c_2 x + \frac{c_1}{2}\right)^2}{c_2 x^4}$$

Mathematica DSolve solution

Solving time : 0.245 (sec)
Leaf size : 19

```
DSolve[{x^2*D[y[x],{x,2}]-x^2/(2*y[x])*D[y[x],x]^2+4*x*D[y[x],x]+4*y[x]==0,{}},y[x],x,Includ
```

$$y(x) \to \frac{c_2(x + 2c_1)^2}{x^4}$$

2.1.2 Problem 3

Internal problem ID [18532]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter 1. section 5. Problems at page 19
Problem number : 3
Date solved : `Tuesday, January 28, 2025 at 11:54:24 AM`
CAS classification : `[_quadrature]`

Solve

$$y' + cy = a$$

Solved as first order autonomous ode

Time used: 0.164 (sec)

Integrating gives

$$\int \frac{1}{-cy + a} dy = dx$$
$$-\frac{\ln(-cy + a)}{c} = x + c_1$$

Singular solutions are found by solving

$$-cy + a = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{a}{c}$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

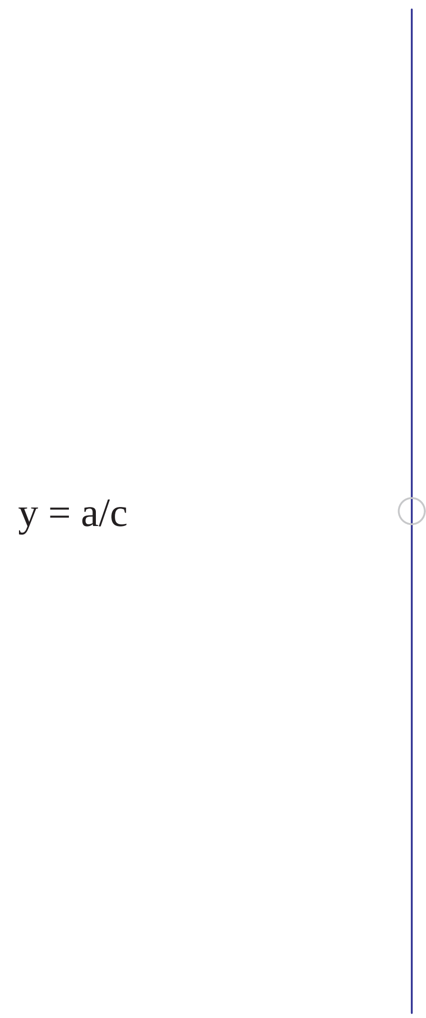


Figure 2.1: Phase line diagram

Solving for y gives

$$y = \frac{a}{c}$$

$$y = -\frac{e^{-c_1 c - x c} - a}{c}$$

Summary of solutions found

$$y = \frac{a}{c}$$

$$y = -\frac{e^{-c_1 c - x c} - a}{c}$$

Solved as first order Exact ode

Time used: 0.109 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (-cy + a)\,\mathrm{d}x\\ (cy - a)\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= cy - a\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(cy - a) \\ &= c\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((c) - (0)) \\ &= c\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int c \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{xc} \\ &= \mathrm{e}^{xc}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{xc}(cy - a) \\ &= -(-cy + a)\,\mathrm{e}^{xc}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{xc}(1) \\ &= \mathrm{e}^{xc}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-(-cy+a)\,\mathrm{e}^{xc}\right) + \left(\mathrm{e}^{xc}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y \\ \int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \mathrm{e}^{xc}\,\mathrm{d}y \\ \phi &= \mathrm{e}^{xc}y + f(x)\end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = c\,\mathrm{e}^{xc}y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -(-cy+a)\,\mathrm{e}^{xc}$. Therefore equation (4) becomes

$$-(-cy+a)\,\mathrm{e}^{xc} = c\,\mathrm{e}^{xc}y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\mathrm{e}^{xc}a$$

Integrating the above w.r.t x gives

$$\int f'(x)\,dx = \int (-e^{xc}a)\,dx$$
$$f(x) = -\frac{e^{xc}a}{c} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = e^{xc}y - \frac{e^{xc}a}{c} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = e^{xc}y - \frac{e^{xc}a}{c}$$

Solving for y gives

$$y = \frac{(e^{xc}a + c_1c)\,e^{-xc}}{c}$$

Summary of solutions found

$$y = \frac{(e^{xc}a + c_1c)\,e^{-xc}}{c}$$

Solved using Lie symmetry for first order ode

Time used: 0.279 (sec)

Writing the ode as

$$y' = -cy + a$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (-cy + a)(b_3 - a_2) - (-cy + a)^2 a_3 + c(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-c^2y^2a_3 + 2acya_3 - a^2a_3 + cxb_2 + cya_2 - aa_2 + ab_3 + cb_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-c^2y^2a_3 + 2acya_3 - a^2a_3 + cxb_2 + cya_2 - aa_2 + ab_3 + cb_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-c^2a_3v_2^2 + 2aca_3v_2 - a^2a_3 + ca_2v_2 + cb_2v_1 - aa_2 + ab_3 + cb_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$cb_2v_1 - c^2a_3v_2^2 + (2aca_3 + ca_2)\,v_2 - a^2a_3 - aa_2 + ab_3 + cb_1 + b_2 = 0 \qquad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} cb_2 &= 0 \\ -c^2a_3 &= 0 \\ 2aca_3 + ca_2 &= 0 \\ -a^2a_3 - aa_2 + ab_3 + cb_1 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= -\frac{ab_3}{c} \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= -\frac{-cy + a}{c} \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{-cy+a}{c}} dy
\end{aligned}
$$

Which results in

$$
S = \ln(-cy + a)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -cy + a
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 0 \\
S_y &= -\frac{c}{-cy+a}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -c \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -c
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -c\, dR$$
$$S(R) = -cR + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(-cy + a) = -xc + c_2$$

Which gives

$$y = -\frac{e^{-xc+c_2} - a}{c}$$

Summary of solutions found

$$y = -\frac{e^{-xc+c_2} - a}{c}$$

Maple step by step solution

Let's solve
$y' + cy = a$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = -cy + a$
- Separate variables
  $\frac{y'}{-cy+a} = 1$
- Integrate both sides with respect to x
  $\int \frac{y'}{-cy+a}dx = \int 1dx + C1$
- Evaluate integral
  $-\frac{\ln(-cy+a)}{c} = x + C1$
- Solve for y
  $y = -\frac{e^{-C1c-xc}-a}{c}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 18

```
dsolve(diff(y(x),x)+c*y(x) = a,y(x),singsol=all)
```

$$y(x) = \frac{e^{-cx} c_1 c + a}{c}$$

Mathematica DSolve solution

Solving time : 0.044 (sec)
Leaf size : 29

```
DSolve[{D[y[x],x]+c*y[x]==a,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{a}{c} + c_1 e^{-cx}$$
$$y(x) \to \frac{a}{c}$$

2.1.3 Problem 4

Internal problem ID [18533]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter 1. section 5. Problems at page 19
Problem number : 4
Date solved : `Tuesday, January 28, 2025 at 11:54:25 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \frac{y'}{x} + k^2 y = 0$$

Solved as second order Bessel ode

Time used: 0.056 (sec)

Writing the ode as

$$x^2 y'' + x y' + k^2 x^2 y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= k\\ n &= 0\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}(0, kx) + c_2 \operatorname{BesselY}(0, kx)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \operatorname{BesselJ}(0, kx) + c_2 \operatorname{BesselY}(0, kx)$$

Solved as second order ode adjoint method

Time used: 0.618 (sec)

In normal form the ode

$$y'' + \frac{y'}{x} + k^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= k^2\\ r(x) &= 0\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi'' - (\xi\, p)' + \xi q &= 0\\ \xi'' - \left(\frac{\xi(x)}{x}\right)' + \left(k^2 \xi(x)\right) &= 0\\ \frac{\xi(x)\, k^2 x^2 + \xi''(x)\, x^2 - \xi'(x)\, x + \xi(x)}{x^2} &= 0\end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' x^2 - \xi' x + \left(k^2 x^2 + 1\right)\xi = 0 \tag{1}$$

Bessel ode has the form

$$\xi'' x^2 + \xi' x + \left(-n^2 + x^2\right)\xi = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$\xi'' x^2 + (1 - 2\alpha)\, x\xi' + \left(\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)\xi = 0 \tag{3}$$

With the standard solution

$$\xi = x^{\alpha}(c_3 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_4 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 1\\ \beta &= k\\ n &= 0\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$\xi = c_3 x \operatorname{BesselJ}(0, kx) + c_4 x \operatorname{BesselY}(0, kx)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' + y\left(\frac{1}{x} - \frac{c_3 \operatorname{BesselJ}(0, kx) - c_3 x \operatorname{BesselJ}(1, kx)\, k + c_4 \operatorname{BesselY}(0, kx) - c_4 x \operatorname{BesselY}(1, kx)\, k}{c_3 x \operatorname{BesselJ}(0, kx) + c_4 x \operatorname{BesselY}(0, kx)}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{k(\mathrm{BesselJ}\,(1, kx)\, c_3 + \mathrm{BesselY}\,(1, kx)\, c_4)}{c_3 \,\mathrm{BesselJ}\,(0, kx) + c_4 \,\mathrm{BesselY}\,(0, kx)}$$
$$p(x) = 0$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= \mathrm{e}^{\int \frac{k(\mathrm{BesselJ}(1,kx)c_3+\mathrm{BesselY}(1,kx)c_4)}{c_3 \,\mathrm{BesselJ}(0,kx)+c_4 \,\mathrm{BesselY}(0,kx)} dx} \\
&= \frac{1}{c_3 \,\mathrm{BesselJ}\,(0, kx) + c_4 \,\mathrm{BesselY}\,(0, kx)}
\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x} \mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{y}{c_3 \,\mathrm{BesselJ}\,(0, kx) + c_4 \,\mathrm{BesselY}\,(0, kx)} \right) = 0$$

Integrating gives

$$\begin{aligned}
\frac{y}{c_3 \,\mathrm{BesselJ}\,(0, kx) + c_4 \,\mathrm{BesselY}\,(0, kx)} &= \int 0 \, dx + c_5 \\
&= c_5
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_3 \,\mathrm{BesselJ}(0,kx)+c_4 \,\mathrm{BesselY}(0,kx)}$ gives the final solution

$$y = (c_3 \,\mathrm{BesselJ}\,(0, kx) + c_4 \,\mathrm{BesselY}\,(0, kx))\, c_5$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = (c_3 \,\mathrm{BesselJ}\,(0, kx) + c_4 \,\mathrm{BesselY}\,(0, kx))\, c_5$$

The constants can be merged to give

$$y = c_3 \,\mathrm{BesselJ}\,(0, kx) + c_4 \,\mathrm{BesselY}\,(0, kx)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_3 \,\mathrm{BesselJ}\,(0, kx) + c_4 \,\mathrm{BesselY}\,(0, kx)$$

Maple step by step solution

Let's solve

$y'' + \frac{y'}{x} + k^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = k^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $k^2 y x + y'' x + y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- > k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ◦ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

  ◦ Shift index using $k- > k + 1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$
- Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + k^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1(1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + k^2 a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 + k^2 a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2 a_k}{(k+2)^2}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2 a_k}{(k+2)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{k^2 a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
```

```
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 19

```
dsolve(diff(diff(y(x),x),x)+1/x*diff(y(x),x)+k^2*y(x) = 0,y(x),singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}(0, kx) + c_2 \operatorname{BesselY}(0, kx)$$

Mathematica DSolve solution

Solving time : 0.018 (sec)
Leaf size : 22

```
DSolve[{D[y[x],{x,2}]+1/x*D[y[x],x]+k^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \operatorname{BesselJ}(0, kx) + c_2 \operatorname{BesselY}(0, kx)$$

2.1.4 Problem 5

Internal problem ID [18534]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter 1. section 5. Problems at page 19
Problem number : 5
Date solved : `Tuesday, January 28, 2025 at 08:28:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\cos(x)\, y' + \sin(x)\, y'' + ny \sin(x) = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
```

```
      <- Legendre successful
   <- special function solution successful
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      (-n*t^2+n)*u(t)+(2*t^3-2*t)*diff(u(t),t)+(t^4-2*t^2+1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

Maple dsolve solution

Solving time : `0.048` (sec)
Leaf size : 37

```
dsolve(cos(x)*diff(y(x),x)+sin(x)*diff(diff(y(x),x),x)+n*y(x)*sin(x) = 0,y(x),singsol=
```

$$y(x) = c_1 \operatorname{LegendreP}\left(\frac{\sqrt{4n+1}}{2} - \frac{1}{2}, \cos(x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{\sqrt{4n+1}}{2} - \frac{1}{2}, \cos(x)\right)$$

Mathematica DSolve solution

Solving time : `0.112` (sec)
Leaf size : 48

```
DSolve[{D[Sin[x]*D[y[x],x],x]+n*y[x]*Sin[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \operatorname{LegendreP}\left(\frac{1}{2}\left(\sqrt{4n+1} - 1\right), \cos(x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{1}{2}\left(\sqrt{4n+1} - 1\right), \cos(x)\right)$$

2.1.5 Problem 6

Internal problem ID [18535]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter 1. section 5. Problems at page 19
Problem number : 6
Date solved : `Tuesday, January 28, 2025 at 11:54:28 AM`
CAS classification : `[_separable]`

Solve

$$y' = \frac{\sqrt{1-y^2}\,\arcsin(y)}{x}$$

Solved as first order separable ode

Time used: 2.078 (sec)

The ode

$$y' = \frac{\sqrt{1-y^2}\,\arcsin(y)}{x} \tag{2.1}$$

is separable as it can be written as

$$\begin{aligned} y' &= \frac{\sqrt{1-y^2}\,\arcsin(y)}{x} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(y) &= \sqrt{-y^2+1}\,\arcsin(y) \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$
$$\int \frac{1}{\sqrt{-y^2+1}\,\arcsin(y)}\,dy = \int \frac{1}{x}\,dx$$

$$\ln(\arcsin(y)) = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or

$$\sqrt{-y^2+1}\,\arcsin(y) = 0$$

for y gives

$$y = -1$$
$$y = 0$$
$$y = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(\arcsin(y)) = \ln(x) + c_1$$
$$y = -1$$
$$y = 0$$
$$y = 1$$

Summary of solutions found

$$\ln(\arcsin(y)) = \ln(x) + c_1$$
$$y = -1$$
$$y = 0$$
$$y = 1$$

Solved as first order Exact ode

Time used: 0.573 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\frac{\sqrt{-y^2+1}\,\arcsin(y)}{x}\right)\mathrm{d}x$$

$$\left(-\frac{\sqrt{-y^2+1}\,\arcsin(y)}{x}\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{\sqrt{-y^2+1}\,\arcsin(y)}{x}\\N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sqrt{-y^2+1}\,\arcsin(y)}{x}\right)\\&= \frac{-1+\frac{\arcsin(y)y}{\sqrt{-y^2+1}}}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\&= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\&= 1\left(\left(\frac{\arcsin(y)\,y}{x\sqrt{-y^2+1}} - \frac{1}{x}\right) - (0)\right)\\&= \frac{-1+\frac{\arcsin(y)y}{\sqrt{-y^2+1}}}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\&= -\frac{x}{\sqrt{-y^2+1}\,\arcsin(y)}\left((0) - \left(\frac{\arcsin(y)\,y}{x\sqrt{-y^2+1}} - \frac{1}{x}\right)\right)\\&= \frac{-\arcsin(y)\,y + \sqrt{-y^2+1}}{\arcsin(y)\,(y^2-1)}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y}\\ &= e^{\int \frac{-\arcsin(y)y+\sqrt{-y^2+1}}{\arcsin(y)\left(y^2-1\right)}\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(\arcsin(y))-\frac{\ln(y-1)}{2}-\frac{\ln(y+1)}{2}}\\ &= \frac{1}{\arcsin(y)\sqrt{y-1}\,\sqrt{y+1}}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\arcsin(y)\sqrt{y-1}\,\sqrt{y+1}}\left(-\frac{\sqrt{-y^2+1}\,\arcsin(y)}{x}\right)\\ &= -\frac{\sqrt{-y^2+1}}{x\sqrt{y-1}\,\sqrt{y+1}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{\arcsin(y)\sqrt{y-1}\,\sqrt{y+1}}(1)\\ &= \frac{1}{\arcsin(y)\sqrt{y-1}\,\sqrt{y+1}}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\left(-\frac{\sqrt{-y^2+1}}{x\sqrt{y-1}\,\sqrt{y+1}}\right)+\left(\frac{1}{\arcsin(y)\sqrt{y-1}\,\sqrt{y+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \overline{N} \, \mathrm{d}y \\
\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \frac{1}{\arcsin(y) \sqrt{y-1} \sqrt{y+1}} \, \mathrm{d}y \\
\phi &= \int \frac{1}{\arcsin(y) \sqrt{y-1} \sqrt{y+1}} dy + f(x) \qquad (3)
\end{aligned}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \qquad (4)$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -\frac{\sqrt{-y^2+1}}{x\sqrt{y-1}\sqrt{y+1}}$. Therefore equation (4) becomes

$$-\frac{\sqrt{-y^2+1}}{x\sqrt{y-1}\sqrt{y+1}} = 0 + f'(x) \qquad (5)$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{\sqrt{-y^2+1}}{x\sqrt{y-1}\sqrt{y+1}}$$

Integrating the above w.r.t x gives

$$\begin{aligned}
\int f'(x) \, \mathrm{d}x &= \int \left(-\frac{\sqrt{-y^2+1}}{x\sqrt{y-1}\sqrt{y+1}} \right) \mathrm{d}x \\
f(x) &= -\frac{\sqrt{-y^2+1} \, \ln(x)}{\sqrt{y-1}\sqrt{y+1}} + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \int \frac{1}{\arcsin(y) \sqrt{y-1} \sqrt{y+1}} dy - \frac{\sqrt{-y^2+1} \, \ln(x)}{\sqrt{y-1}\sqrt{y+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \int \frac{1}{\arcsin(y)\sqrt{y-1}\,\sqrt{y+1}}dy - \frac{\sqrt{-y^2+1}\,\ln(x)}{\sqrt{y-1}\,\sqrt{y+1}}$$

Summary of solutions found

$$\int^y \frac{1}{\arcsin(\_a)\sqrt{\_a-1}\,\sqrt{\_a+1}}d\_a - \frac{\sqrt{1-y^2}\,\ln(x)}{\sqrt{y-1}\,\sqrt{y+1}} = c_1$$

Solved as first order isobaric ode

Time used: 2.315 (sec)

Solving for y' gives

$$y' = \frac{\sqrt{1-y^2}\,\arcsin(y)}{x} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{\sqrt{1-y^2}\,\arcsin(y)}{x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 0$$

Since the ode is isobaric of order $m = 0$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= u \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u'(x) = \frac{\sqrt{1-u(x)^2}\,\arcsin(u(x))}{x}$$

The ode

$$u'(x) = \frac{\sqrt{1-u(x)^2}\,\arcsin(u(x))}{x} \tag{2.2}$$

is separable as it can be written as

$$
\begin{aligned}
u'(x) &= \frac{\sqrt{1-u(x)^2}\,\arcsin(u(x))}{x} \\
&= f(x)g(u)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
g(u) &= \sqrt{-u^2+1}\,\arcsin(u)
\end{aligned}
$$

Integrating gives

$$
\int \frac{1}{g(u)}\,du = \int f(x)\,dx
$$

$$
\int \frac{1}{\sqrt{-u^2+1}\,\arcsin(u)}\,du = \int \frac{1}{x}\,dx
$$

$$
\ln(\arcsin(u(x))) = \ln(x) + c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$
\sqrt{-u^2+1}\,\arcsin(u) = 0
$$

for $u(x)$ gives

$$
\begin{aligned}
u(x) &= -1 \\
u(x) &= 0 \\
u(x) &= 1
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln(\arcsin(u(x))) &= \ln(x) + c_1 \\
u(x) &= -1 \\
u(x) &= 0 \\
u(x) &= 1
\end{aligned}
$$

Converting $\ln(\arcsin(u(x))) = \ln(x) + c_1$ back to y gives

$$\ln(\arcsin(y)) = \ln(x) + c_1$$

Converting $u(x) = -1$ back to y gives

$$y = -1$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = 1$ back to y gives

$$y = 1$$

<u>Summary of solutions found</u>

$$\ln(\arcsin(y)) = \ln(x) + c_1$$
$$y = -1$$
$$y = 0$$
$$y = 1$$

Maple step by step solution

Let's solve

$y' = \frac{\sqrt{1-y^2}\,\arcsin(y)}{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $y' = \frac{\sqrt{1-y^2}\,\arcsin(y)}{x}$

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}\,\arcsin(y)} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}\,\arcsin(y)}dx = \int \frac{1}{x}dx + C1$

- Evaluate integral

  $\ln(\arcsin(y)) = \ln(x) + C1$

- Solve for y

  $y = \sin\left(x\,\mathrm{e}^{C1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 8

```
dsolve(diff(y(x),x) = 1/x*(1-y(x)^2)^(1/2)*arcsin(y(x)),y(x),singsol=all)
```

$$y(x) = \sin(c_1 x)$$

Mathematica DSolve solution

Solving time : 0.333 (sec)
Leaf size : 27

```
DSolve[{D[y[x],x]==1/x*Sqrt[1-y[x]^2]*ArcSin[y[x]],{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to \sin\left(e^{c_1} x\right)$$
$$y(x) \to -1$$
$$y(x) \to 0$$
$$y(x) \to 1$$

2.1.6 Problem 16 (a)

Internal problem ID [18536]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter 1. section 5. Problems at page 19
Problem number : 16 (a)
Date solved : `Tuesday, January 28, 2025 at 11:54:36 AM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$v'' = \left(\frac{1}{v} + v'^4\right)^{1/3}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, _b(_a)*(diff(_b(_a), _a))-((_b(_a)^4*_a+1)/_a)^(1/3)
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   differential order: 1; looking for linear symmetries
   trying exact
```

```
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
   --- Trying Lie symmetry methods, 1st order ---
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4
   `, `-> Computing symmetries using: way = 5
   trying symmetry patterns for 1st order ODEs
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
   -> trying a symmetry pattern of the form [F(x),G(x)]
   -> trying a symmetry pattern of the form [F(y),G(y)]
   -> trying a symmetry pattern of the form [F(x)+G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)+G(y)]
   -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
   -> trying a symmetry pattern of conformal type
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(x,y)
-> trying 2nd order, the S-function method
   -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, the S-function method
   -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
-> trying 2nd order, dynamical_symmetries, only a reduction of order through one integ
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
`, `-> Computing symmetries using: way = formal
            *** Sublevel 2 ***
            Methods for first order ODEs:
            --- Trying classification methods ---
            trying a quadrature
            trying 1st order linear
```

```
            <- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.082 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(diff(v(u),u),u) = (1/v(u)+diff(v(u),u)^4)^(1/3),v(u),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[v[u],{u,2}]==(1/v[u]+D[v[u],u]^4)^(1/3),{}},v[u],u,IncludeSingularSolutions->True]
```

Not solved

2.1.7 Problem 16 (b)

Internal problem ID [18537]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter 1. section 5. Problems at page 19
Problem number : 16 (b)
Date solved : `Tuesday, January 28, 2025 at 11:54:40 AM`
CAS classification : `[_linear]`

Solve

$$v' + u^2 v = \sin(u)$$

Solved as first order linear ode

Time used: 0.374 (sec)

In canonical form a linear first order is

$$v' + q(u)v = p(u)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(u) &= u^2 \\ p(u) &= \sin(u) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,du} \\ &= \mathrm{e}^{\int u^2 du} \\ &= \mathrm{e}^{\frac{u^3}{3}} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mu v) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mu v) = (\mu)(\sin(u))$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(v\,\mathrm{e}^{\frac{u^3}{3}}\right) = \left(\mathrm{e}^{\frac{u^3}{3}}\right)(\sin(u))$$

$$\mathrm{d}\left(v\,\mathrm{e}^{\frac{u^3}{3}}\right) = \left(\sin(u)\,\mathrm{e}^{\frac{u^3}{3}}\right)\,\mathrm{d}u$$

Integrating gives

$$\begin{aligned} v\,\mathrm{e}^{\frac{u^3}{3}} &= \int \sin(u)\,\mathrm{e}^{\frac{u^3}{3}}\,du \\ &= \int \sin(u)\,\mathrm{e}^{\frac{u^3}{3}}\,du + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{u^3}{3}}$ gives the final solution

$$v = \mathrm{e}^{-\frac{u^3}{3}}\left(\int \sin(u)\,\mathrm{e}^{\frac{u^3}{3}}\,du + c_1\right)$$

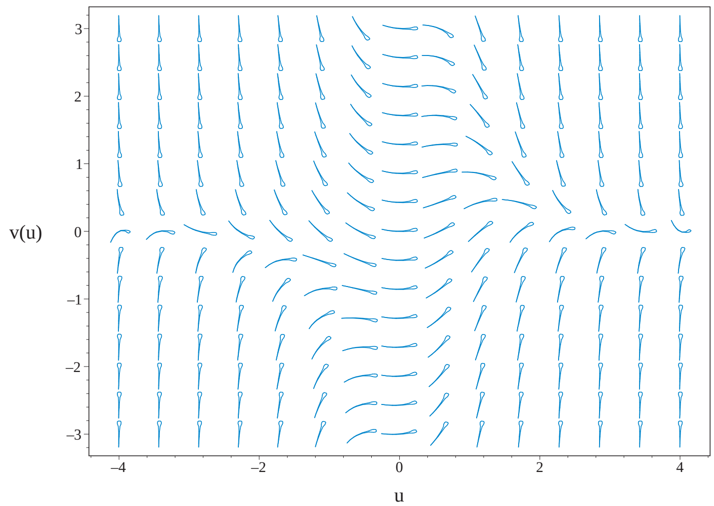


Figure 2.2: Slope field plot
$v' + u^2 v = \sin(u)$

<u>Summary of solutions found</u>

$$v = \mathrm{e}^{-\frac{u^3}{3}}\left(\int \sin(u)\,\mathrm{e}^{\frac{u^3}{3}}\,du + c_1\right)$$

Solved as first order Exact ode

Time used: 0.127 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(u,v)\,\mathrm{d}u+N(u,v)\,\mathrm{d}v=0 \tag{1A}$$

Therefore

$$\mathrm{d}v=\left(-u^2v+\sin\left(u\right)\right)\mathrm{d}u$$
$$\left(u^2v-\sin\left(u\right)\right)\mathrm{d}u+\mathrm{d}v=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(u,v)=u^2v-\sin\left(u\right)$$
$$N(u,v)=1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}\left(u^2 v - \sin(u)\right)\\ &= u^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial u} &= \frac{\partial}{\partial u}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial u}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u}\right)\\ &= 1\left(\left(u^2\right) - (0)\right)\\ &= u^2\end{aligned}$$

Since A does not depend on v, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}u}\\ &= e^{\int u^2 \, \mathrm{d}u}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{u^3}{3}}\\ &= \mathrm{e}^{\frac{u^3}{3}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{\frac{u^3}{3}}\left(u^2 v - \sin(u)\right)\\ &= \left(u^2 v - \sin(u)\right)\mathrm{e}^{\frac{u^3}{3}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{\frac{u^3}{3}}(1) \\ &= \mathrm{e}^{\frac{u^3}{3}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}v}{\mathrm{d}u} &= 0 \\ \left(\left(u^2 v - \sin(u)\right)\mathrm{e}^{\frac{u^3}{3}}\right) + \left(\mathrm{e}^{\frac{u^3}{3}}\right)\frac{\mathrm{d}v}{\mathrm{d}u} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(u, v)$

$$\frac{\partial \phi}{\partial u} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial v} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. v gives

$$\int \frac{\partial \phi}{\partial v}\,\mathrm{d}v = \int \overline{N}\,\mathrm{d}v$$

$$\int \frac{\partial \phi}{\partial v}\,\mathrm{d}v = \int \mathrm{e}^{\frac{u^3}{3}}\,\mathrm{d}v$$

$$\phi = v\,\mathrm{e}^{\frac{u^3}{3}} + f(u) \tag{3}$$

Where $f(u)$ is used for the constant of integration since ϕ is a function of both u and v. Taking derivative of equation (3) w.r.t u gives

$$\frac{\partial \phi}{\partial u} = v\,u^2 \mathrm{e}^{\frac{u^3}{3}} + f'(u) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial u} = \left(u^2 v - \sin(u)\right)\mathrm{e}^{\frac{u^3}{3}}$. Therefore equation (4) becomes

$$\left(u^2 v - \sin(u)\right)\mathrm{e}^{\frac{u^3}{3}} = v\,u^2 \mathrm{e}^{\frac{u^3}{3}} + f'(u) \tag{5}$$

Solving equation (5) for $f'(u)$ gives

$$f'(u) = -\sin(u)\,\mathrm{e}^{\frac{u^3}{3}}$$

Integrating the above w.r.t u gives

$$\int f'(u)\,\mathrm{d}u = \int \left(-\sin(u)\,\mathrm{e}^{\frac{u^3}{3}}\right)\mathrm{d}u$$

$$f(u) = \int_0^u -\sin(\tau)\,\mathrm{e}^{\frac{\tau^3}{3}}\,d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(u)$ into equation (3) gives ϕ

$$\phi = v\,\mathrm{e}^{\frac{u^3}{3}} + \int_0^u -\sin(\tau)\,\mathrm{e}^{\frac{\tau^3}{3}}\,d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = v\,\mathrm{e}^{\frac{u^3}{3}} + \int_0^u -\sin(\tau)\,\mathrm{e}^{\frac{\tau^3}{3}}\,d\tau$$

Solving for v gives

$$v = -\left(\int_0^u -\sin(\tau)\,\mathrm{e}^{\frac{\tau^3}{3}}\,d\tau - c_1\right)\mathrm{e}^{-\frac{u^3}{3}}$$

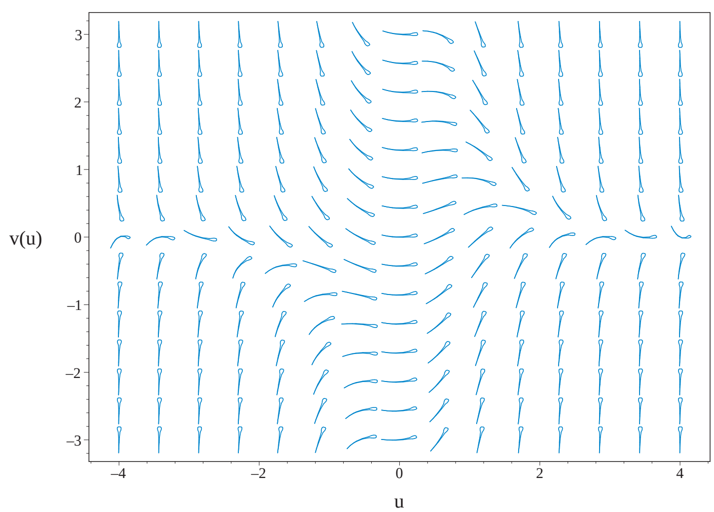


Figure 2.3: Slope field plot
$v' + u^2 v = \sin(u)$

<u>Summary of solutions found</u>

$$v = -\left(\int_0^u -\sin(\tau)\,\mathrm{e}^{\frac{\tau^3}{3}}\,d\tau - c_1\right)\mathrm{e}^{-\frac{u^3}{3}}$$

Maple step by step solution

Let's solve

$v' + u^2 v = \sin(u)$

- Highest derivative means the order of the ODE is 1

  v'

- Solve for the highest derivative

  $v' = -u^2 v + \sin(u)$

- Group terms with v on the lhs of the ODE and the rest on the rhs of the ODE

  $v' + u^2 v = \sin(u)$

- The ODE is linear; multiply by an integrating factor $\mu(u)$

  $\mu(u)\,(v' + u^2 v) = \mu(u)\sin(u)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{du}(v\mu(u))$

  $\mu(u)\,(v' + u^2 v) = v'\mu(u) + v\mu'(u)$

- Isolate $\mu'(u)$

  $\mu'(u) = \mu(u)\,u^2$

- Solve to find the integrating factor

  $\mu(u) = \mathrm{e}^{\frac{u^3}{3}}$

- Integrate both sides with respect to u

  $\int \left(\frac{d}{du}(v\mu(u))\right) du = \int \mu(u)\sin(u)\,du + C1$

- Evaluate the integral on the lhs

  $v\mu(u) = \int \mu(u)\sin(u)\,du + C1$

- Solve for v

  $v = \frac{\int \mu(u)\sin(u)du + C1}{\mu(u)}$

- Substitute $\mu(u) = \mathrm{e}^{\frac{u^3}{3}}$

  $v = \frac{\int \sin(u)\mathrm{e}^{\frac{u^3}{3}}du + C1}{\mathrm{e}^{\frac{u^3}{3}}}$

- Simplify

  $v = \mathrm{e}^{-\frac{u^3}{3}}\left(\int \sin(u)\,\mathrm{e}^{\frac{u^3}{3}}du + C1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 24

```
dsolve(diff(v(u),u)+u^2*v(u) = sin(u),v(u),singsol=all)
```

$$v = \left(\int \sin\left(u\right) \mathrm{e}^{\frac{u^{3}}{3}}d u +c_{1} \right) \mathrm{e}^{-\frac{u^{3}}{3}}$$

Mathematica DSolve solution

Solving time : 0.058 (sec)
Leaf size : 39

```
DSolve[{D[v[u],u]+u^2*v[u]==Sin[u],{}},v[u],u,IncludeSingularSolutions->True]
```

$$v(u)\to e^{-\frac{u^3}{3}}\left(\int_1^u e^{\frac{K[1]^3}{3}}\sin(K[1])dK[1]+c_1\right)$$

2.1.8 Problem 17 (a)

Internal problem ID [18538]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter 1. section 5. Problems at page 19
Problem number : 17 (a)
Date solved : `Tuesday, January 28, 2025 at 11:54:42 AM`
CAS classification : `[NONE]`

Solve

$$\sqrt{y' + y} = (y'' + 2x)^{1/4}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one inte
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
```

```
trying 2nd order, integrating factor of the form mu(y,y)
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
differential order: 2; looking for linear symmetries
-> trying 2nd order, the S-function method
   -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, the S-function method
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods
      -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods
      -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods
      -> trying 2nd order, No Point Symmetries Class V
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
-> trying 2nd order, dynamical_symmetries, only a reduction of order through one integ
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, dynamical_symmetries, only a reduction of order through one in
solving 2nd order ODE of high degree, Lie methods
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`
```

Maple dsolve solution

Solving time : 0.069 (sec)
Leaf size : maple_leaf_size

```
dsolve((diff(y(x),x)+y(x))^(1/2) = (diff(diff(y(x),x),x)+2*x)^(1/4),y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{Sqrt[D[y[x],x]+y[x]]== (D[y[x],{x,2}]+2*x)^(1/4),{}},y[x],x,IncludeSingularSolutions
```

Not solved

2.1.9 Problem 18

Internal problem ID [18539]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter 1. section 5. Problems at page 19
Problem number : 18
Date solved : `Tuesday, January 28, 2025 at 11:54:42 AM`
CAS classification : `[_linear]`

Solve

$$v' + \frac{2v}{u} = 3$$

Solved as first order linear ode

Time used: 0.039 (sec)

In canonical form a linear first order is

$$v' + q(u)v = p(u)$$

Comparing the above to the given ode shows that

$$q(u) = \frac{2}{u}$$
$$p(u) = 3$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,du} \\ &= \mathrm{e}^{\int \frac{2}{u}du} \\ &= u^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}(\mu v) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}u}(\mu v) &= (\mu)(3) \\ \frac{\mathrm{d}}{\mathrm{d}u}\left(v\,u^2\right) &= \left(u^2\right)(3) \\ \mathrm{d}\left(v\,u^2\right) &= \left(3u^2\right)\,\mathrm{d}u\end{aligned}$$

Integrating gives

$$\begin{aligned}v\,u^2 &= \int 3u^2\,du \\ &= u^3 + c_1\end{aligned}$$

Dividing throughout by the integrating factor u^2 gives the final solution

$$v = \frac{u^3 + c_1}{u^2}$$

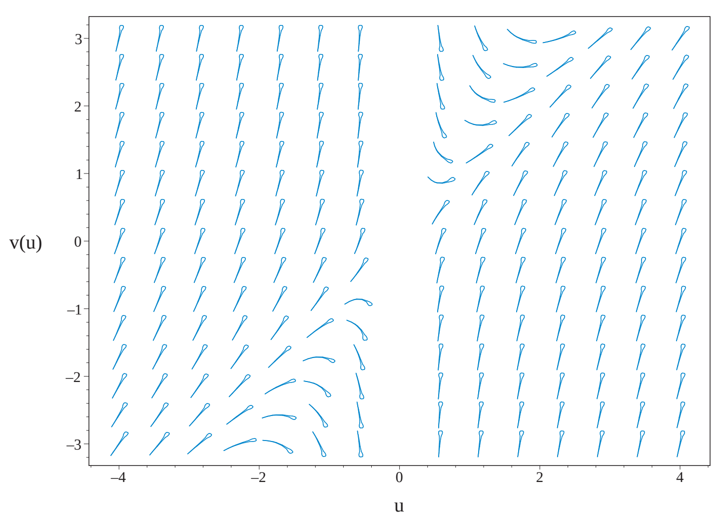


Figure 2.4: Slope field plot
$v' + \frac{2v}{u} = 3$

Summary of solutions found

$$v = \frac{u^3 + c_1}{u^2}$$

Solved as first order homogeneous class A ode

Time used: 0.237 (sec)

In canonical form, the ODE is

$$\begin{aligned} v' &= F(u, v) \\ &= -\frac{-3u + 2v}{u} \end{aligned} \tag{1}$$

An ode of the form $v' = \frac{M(u,v)}{N(u,v)}$ is called homogeneous if the functions $M(u, v)$ and $N(u, v)$ are both homogeneous functions and of the same order. Recall that a function $f(u, v)$ is homogeneous of order n if

$$f(t^n u, t^n v) = t^n f(u, v)$$

In this case, it can be seen that both $M = 3u - 2v$ and $N = u$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{v}{u}$, or $v = uu$. Hence

$$\frac{dv}{du} = \frac{du}{du}u + u$$

Applying the transformation $v = uu$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{du}u + u &= 3 - 2u \\ \frac{du}{du} &= \frac{3 - 3u(u)}{u} \end{aligned}$$

Or

$$u'(u) - \frac{3 - 3u(u)}{u} = 0$$

Or

$$u'(u)\, u + 3u(u) - 3 = 0$$

Which is now solved as separable in $u(u)$.

The ode

$$u'(u) = -\frac{3(u(u) - 1)}{u} \tag{2.3}$$

is separable as it can be written as

$$\begin{aligned} u'(u) &= -\frac{3(u(u)-1)}{u} \\ &= f(u)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(u) &= \frac{1}{u} \\ g(u) &= -3u+3 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(u)\, du \\ \int \frac{1}{-3u+3}\, du &= \int \frac{1}{u}\, du \end{aligned}$$

$$-\frac{\ln(u(u)-1)}{3} = \ln(u) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$-3u+3=0$$

for $u(u)$ gives

$$u(u) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} -\frac{\ln(u(u)-1)}{3} &= \ln(u) + c_1 \\ u(u) &= 1 \end{aligned}$$

Solving for $u(u)$ gives

$$\begin{aligned} u(u) &= 1 \\ u(u) &= \frac{u^3 + \mathrm{e}^{-3c_1}}{u^3} \end{aligned}$$

Converting $u(u) = 1$ back to v gives

$$v = u$$

Converting $u(u) = \frac{u^3+e^{-3c_1}}{u^3}$ back to v gives

$$v = \frac{u^3 + e^{-3c_1}}{u^2}$$

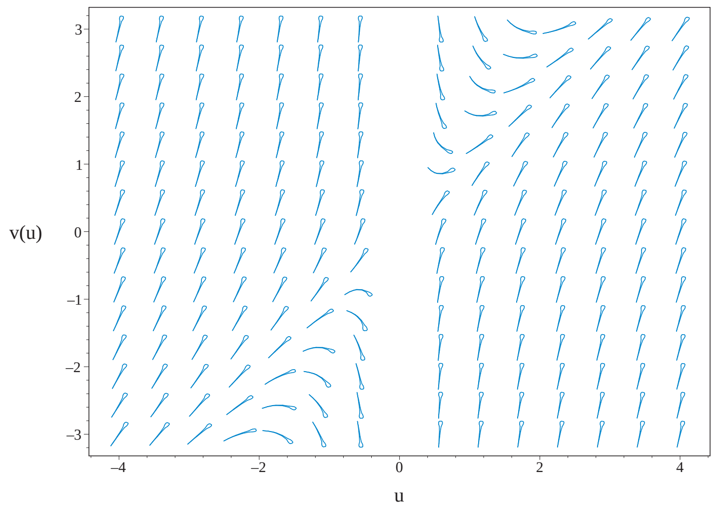


Figure 2.5: Slope field plot
$v' + \frac{2v}{u} = 3$

Summary of solutions found

$$v = u$$

$$v = \frac{u^3 + e^{-3c_1}}{u^2}$$

Solved as first order homogeneous class D2 ode

Time used: 0.117 (sec)

Applying change of variables $v = u(u)\, u$, then the ode becomes

$$u'(u)\, u + 3u(u) = 3$$

Which is now solved The ode

$$u'(u) = -\frac{3(u(u) - 1)}{u} \tag{2.4}$$

is separable as it can be written as

$$\begin{aligned} u'(u) &= -\frac{3(u(u)-1)}{u} \\ &= f(u)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(u) &= \frac{1}{u} \\ g(u) &= -3u+3 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(u)\, du \\ \int \frac{1}{-3u+3}\, du &= \int \frac{1}{u}\, du \end{aligned}$$

$$-\frac{\ln(u(u)-1)}{3} = \ln(u) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-3u + 3 = 0$$

for $u(u)$ gives

$$u(u) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} -\frac{\ln(u(u)-1)}{3} &= \ln(u) + c_1 \\ u(u) &= 1 \end{aligned}$$

Solving for $u(u)$ gives

$$\begin{aligned} u(u) &= 1 \\ u(u) &= \frac{u^3 + e^{-3c_1}}{u^3} \end{aligned}$$

Converting $u(u) = 1$ back to v gives

$$v = u$$

Converting $u(u) = \frac{u^3+e^{-3c_1}}{u^3}$ back to v gives

$$v = \frac{u^3 + e^{-3c_1}}{u^2}$$

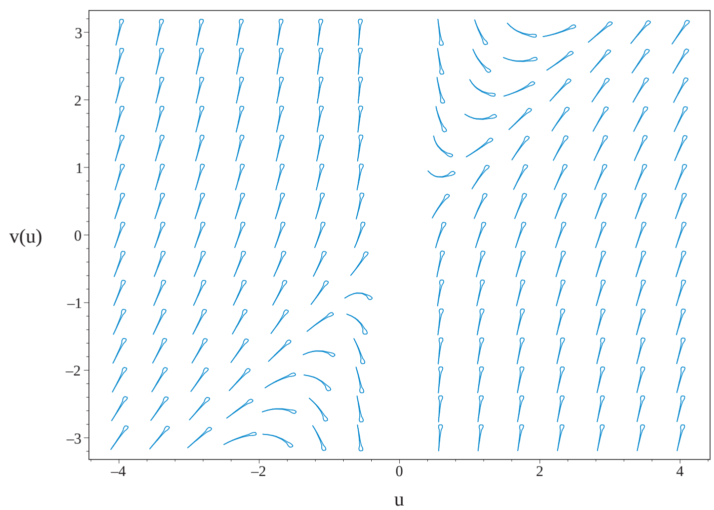


Figure 2.6: Slope field plot
$v' + \frac{2v}{u} = 3$

Summary of solutions found

$$v = u$$

$$v = \frac{u^3 + e^{-3c_1}}{u^2}$$

Solved as first order homogeneous class Maple C ode

Time used: 0.235 (sec)

Let $Y = v - y_0$ and $X = u - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{-3x_0 - 3X + 2Y(X) + 2y_0}{x_0 + X}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$

$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{-3X + 2Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{-3X + 2Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 3X - 2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= 3 - 2u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{3 - 3u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{3 - 3u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 3u(X) - 3 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{3(u(X) - 1)}{X} \tag{2.5}$$

is separable as it can be written as

$$\begin{aligned} \frac{d}{dX}u(X) &= -\frac{3(u(X) - 1)}{X} \\ &= f(X)g(u) \end{aligned}$$

Where

$$f(X) = \frac{1}{X}$$
$$g(u) = -3u + 3$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(X)\, dX$$
$$\int \frac{1}{-3u+3}\, du = \int \frac{1}{X}\, dX$$

$$-\frac{\ln (u(X) - 1)}{3} = \ln (X) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-3u + 3 = 0$$

for $u(X)$ gives

$$u(X) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln (u(X) - 1)}{3} = \ln (X) + c_1$$
$$u(X) = 1$$

Solving for $u(X)$ gives

$$u(X) = 1$$
$$u(X) = \frac{X^3 + e^{-3c_1}}{X^3}$$

Converting $u(X) = 1$ back to $Y(X)$ gives

$$Y(X) = X$$

Converting $u(X) = \frac{X^3+e^{-3c_1}}{X^3}$ back to $Y(X)$ gives

$$Y(X) = \frac{X^3 + e^{-3c_1}}{X^2}$$

Using the solution for $Y(X)$

$$Y(X) = X \tag{A}$$

And replacing back terms in the above solution using

$$Y = v + y_0$$
$$X = u + x_0$$

Or

$$Y = v$$
$$X = u$$

Then the solution in v becomes using EQ (A)

$$v = u$$

Using the solution for $Y(X)$

$$Y(X) = \frac{X^3 + e^{-3c_1}}{X^2} \tag{A}$$

And replacing back terms in the above solution using

$$Y = v + y_0$$
$$X = u + x_0$$

Or

$$Y = v$$
$$X = u$$

Then the solution in v becomes using EQ (A)

$$v = \frac{u^3 + e^{-3c_1}}{u^2}$$

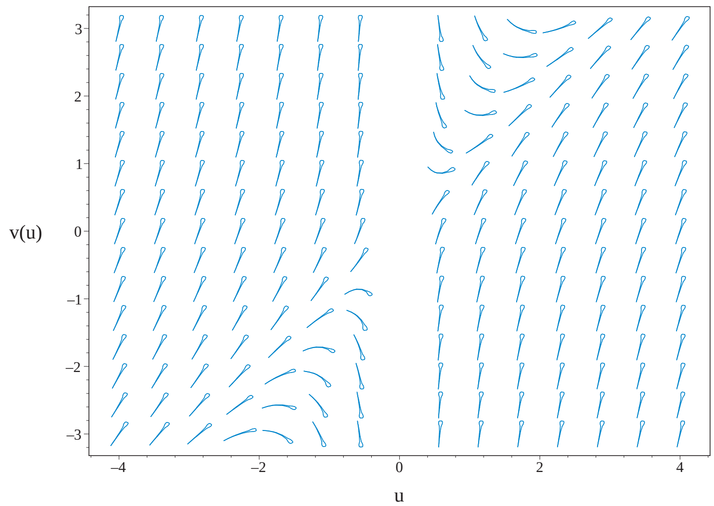


Figure 2.7: Slope field plot
$v' + \frac{2v}{u} = 3$

Solved as first order Exact ode

Time used: 0.148 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(u,v)\,\mathrm{d}u+N(u,v)\,\mathrm{d}v=0 \tag{1A}$$

Therefore

$$\mathrm{d}v=\left(3-\frac{2v}{u}\right)\mathrm{d}u$$

$$\left(\frac{2v}{u}-3\right)\mathrm{d}u+\mathrm{d}v=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(u,v)&=\frac{2v}{u}-3\\N(u,v)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v}=\frac{\partial N}{\partial u}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial v}&=\frac{\partial}{\partial v}\left(\frac{2v}{u}-3\right)\\&=\frac{2}{u}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial u}&=\frac{\partial}{\partial u}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial v}\neq\frac{\partial N}{\partial u}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}\right)\\&=1\left(\left(\frac{2}{u}\right)-(0)\right)\\&=\frac{2}{u}\end{aligned}$$

Since A does not depend on v, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}u}\\ &= e^{\int \frac{2}{u}\,\mathrm{d}u}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2\ln(u)}\\ &= u^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= u^2\left(\frac{2v}{u} - 3\right)\\ &= -3u^2 + 2uv\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= u^2(1)\\ &= u^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}v}{\mathrm{d}u} &= 0\\ \left(-3u^2 + 2uv\right) + \left(u^2\right)\frac{\mathrm{d}v}{\mathrm{d}u} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(u, v)$

$$\frac{\partial \phi}{\partial u} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial v} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. v gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial v}\,\mathrm{d}v &= \int \overline{N}\,\mathrm{d}v\\ \int \frac{\partial \phi}{\partial v}\,\mathrm{d}v &= \int u^2\,\mathrm{d}v\end{aligned}$$

$$\phi = v\,u^2 + f(u) \tag{3}$$

Where $f(u)$ is used for the constant of integration since ϕ is a function of both u and v. Taking derivative of equation (3) w.r.t u gives

$$\frac{\partial \phi}{\partial u} = 2uv + f'(u) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial u} = -3u^2 + 2uv$. Therefore equation (4) becomes

$$-3u^2 + 2uv = 2uv + f'(u) \tag{5}$$

Solving equation (5) for $f'(u)$ gives

$$f'(u) = -3u^2$$

Integrating the above w.r.t u gives

$$\int f'(u)\,\mathrm{d}u = \int \left(-3u^2\right) \mathrm{d}u$$
$$f(u) = -u^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(u)$ into equation (3) gives ϕ

$$\phi = -u^3 + v\,u^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -u^3 + v\,u^2$$

Solving for v gives

$$v = \frac{u^3 + c_1}{u^2}$$

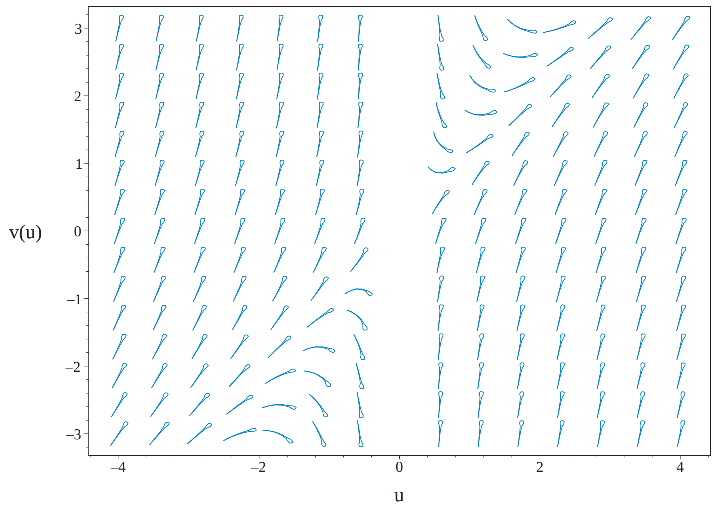


Figure 2.8: Slope field plot
$v' + \frac{2v}{u} = 3$

Summary of solutions found

$$v = \frac{u^3 + c_1}{u^2}$$

Solved as first order isobaric ode

Time used: 0.101 (sec)

Solving for v' gives

$$v' = -\frac{-3u + 2v}{u} \tag{1}$$

Each of the above ode's is now solved An ode $v' = f(u, v)$ is isobaric if

$$f(tu, t^m v) = t^{m-1} f(u, v) \tag{1}$$

Where here

$$f(u, v) = -\frac{-3u + 2v}{u} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} v &= u u^m \\ &= u u \end{aligned}$$

Converts the ODE to a separable in $u(u)$. Performing this substitution gives

$$u(u) + uu'(u) = -\frac{-3u + 2uu(u)}{u}$$

The ode

$$u'(u) = -\frac{3(u(u) - 1)}{u} \tag{2.6}$$

is separable as it can be written as

$$\begin{aligned} u'(u) &= -\frac{3(u(u) - 1)}{u} \\ &= f(u)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(u) &= \frac{1}{u} \\ g(u) &= -3u + 3 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(u)\, du \\ \int \frac{1}{-3u + 3}\, du &= \int \frac{1}{u}\, du \end{aligned}$$

$$-\frac{\ln(u(u) - 1)}{3} = \ln(u) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-3u + 3 = 0$$

for $u(u)$ gives

$$u(u) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln(u(u)-1)}{3} = \ln(u) + c_1$$
$$u(u) = 1$$

Solving for $u(u)$ gives

$$u(u) = 1$$
$$u(u) = \frac{u^3 + e^{-3c_1}}{u^3}$$

Converting $u(u) = 1$ back to v gives

$$\frac{v}{u} = 1$$

Converting $u(u) = \frac{u^3+e^{-3c_1}}{u^3}$ back to v gives

$$\frac{v}{u} = \frac{u^3 + e^{-3c_1}}{u^3}$$

Solving for v gives

$$v = u$$
$$v = \frac{u^3 + e^{-3c_1}}{u^2}$$

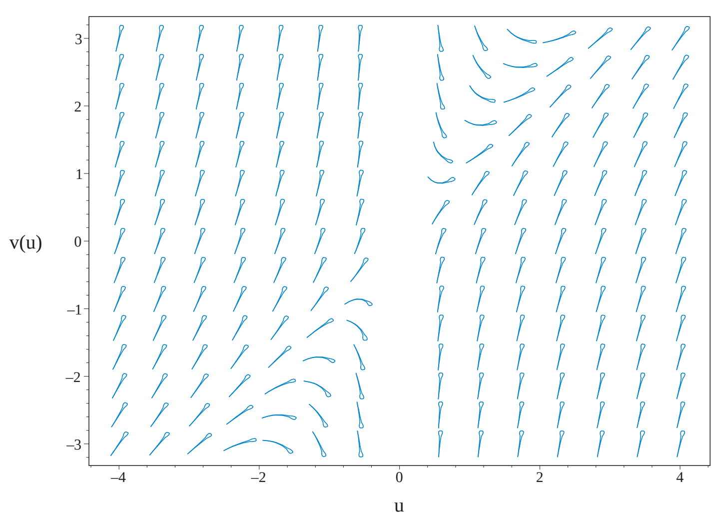


Figure 2.9: Slope field plot
$v' + \frac{2v}{u} = 3$

Summary of solutions found

$$v = u$$
$$v = \frac{u^3 + e^{-3c_1}}{u^2}$$

Solved using Lie symmetry for first order ode

Time used: 0.477 (sec)

Writing the ode as

$$v' = -\frac{-3u + 2v}{u}$$
$$v' = \omega(u, v)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_u + \omega(\eta_v - \xi_u) - \omega^2 \xi_v - \omega_u \xi - \omega_v \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ua_2 + va_3 + a_1 \tag{1E}$$
$$\eta = ub_2 + vb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(-3u + 2v)(b_3 - a_2)}{u} - \frac{(-3u + 2v)^2 a_3}{u^2} - \left(\frac{3}{u} + \frac{-3u + 2v}{u^2}\right)(ua_2 + va_3 + a_1) + \frac{2ub_2 + 2vb_3 + 2b_1}{u} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3u^2 a_2 + 9u^2 a_3 - 3b_2 u^2 - 3u^2 b_3 - 12uva_3 + 6v^2 a_3 - 2ub_1 + 2va_1}{u^2} = 0$$

Setting the numerator to zero gives

$$-3u^2a_2 - 9u^2a_3 + 3b_2u^2 + 3u^2b_3 + 12uva_3 - 6v^2a_3 + 2ub_1 - 2va_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{u, v\}$ in them.

$$\{u, v\}$$

The following substitution is now made to be able to collect on all terms with $\{u, v\}$ in them

$$\{u = v_1, v = v_2\}$$

The above PDE (6E) now becomes

$$-3a_2v_1^2 - 9a_3v_1^2 + 12a_3v_1v_2 - 6a_3v_2^2 + 3b_2v_1^2 + 3b_3v_1^2 - 2a_1v_2 + 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-3a_2 - 9a_3 + 3b_2 + 3b_3)\, v_1^2 + 12a_3v_1v_2 + 2b_1v_1 - 6a_3v_2^2 - 2a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ -6a_3 &= 0 \\ 12a_3 &= 0 \\ 2b_1 &= 0 \\ -3a_2 - 9a_3 + 3b_2 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_2 + b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = u$$
$$\eta = u$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(u,v)\,\xi \\ &= u - \left(-\frac{-3u+2v}{u}\right)(u) \\ &= -2u+2v \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(u,v) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{du}{\xi} = \frac{dv}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v}\right)S(u,v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = u$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy \\ &= \int \frac{1}{-2u+2v}dy\end{aligned}$$

Which results in

$$S = \frac{\ln(-u+v)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_u + \omega(u,v)S_v}{R_u + \omega(u,v)R_v} \tag{2}$$

Where in the above R_u, R_v, S_u, S_v are all partial derivatives and $\omega(u,v)$ is the right hand side of the original ode given by

$$\omega(u,v) = -\frac{-3u+2v}{u}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_u &= 1 \\ R_v &= 0 \\ S_u &= \frac{1}{2u-2v} \\ S_v &= -\frac{1}{2u-2v} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{u} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for u, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int -\frac{1}{R}\,dR \\ S(R) &= -\ln(R) + c_2 \end{aligned}$$

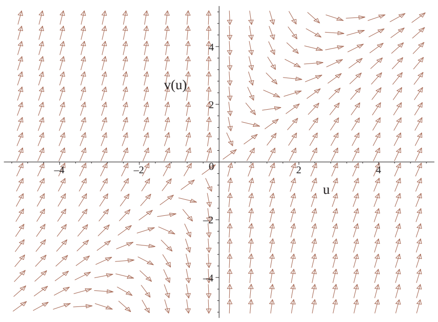
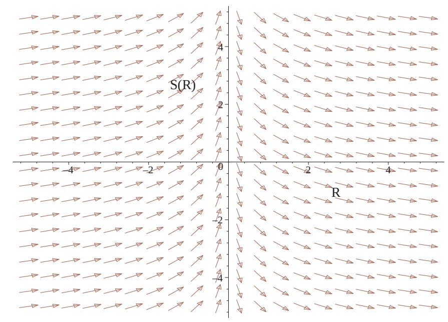
To complete the solution, we just need to transform the above back to u, v coordinates. This results in

$$\frac{\ln(-u+v)}{2} = -\ln(u) + c_2$$

Which gives

$$v = \frac{u^3 + e^{2c_2}}{u^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in u, v coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dv}{du} = -\frac{-3u+2v}{u}$ 	$R = u$ $S = \frac{\ln(-u+v)}{2}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

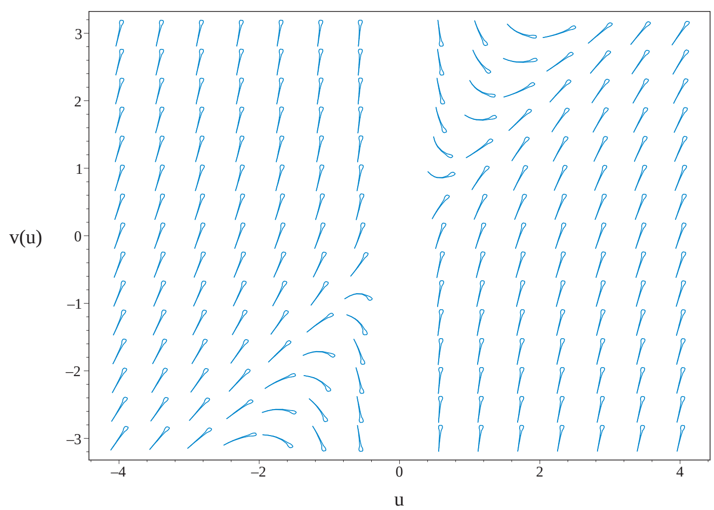


Figure 2.10: Slope field plot
$v' + \frac{2v}{u} = 3$

<u>Summary of solutions found</u>

$$v = \frac{u^3 + e^{2c_2}}{u^2}$$

Maple step by step solution

Let's solve

$v' + \frac{2v}{u} = 3$

- Highest derivative means the order of the ODE is 1

  v'

- Solve for the highest derivative

  $v' = 3 - \frac{2v}{u}$

- Group terms with v on the lhs of the ODE and the rest on the rhs of the ODE

  $v' + \frac{2v}{u} = 3$

- The ODE is linear; multiply by an integrating factor $\mu(u)$

  $\mu(u)\left(v' + \frac{2v}{u}\right) = 3\mu(u)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{du}(v\mu(u))$

  $\mu(u)\left(v' + \frac{2v}{u}\right) = v'\mu(u) + v\mu'(u)$

- Isolate $\mu'(u)$

  $\mu'(u) = \frac{2\mu(u)}{u}$

- Solve to find the integrating factor

  $\mu(u) = u^2$

- Integrate both sides with respect to u

  $\int \left(\frac{d}{du}(v\mu(u))\right) du = \int 3\mu(u)\, du + C1$

- Evaluate the integral on the lhs

  $v\mu(u) = \int 3\mu(u)\, du + C1$

- Solve for v

  $v = \frac{\int 3\mu(u)du + C1}{\mu(u)}$

- Substitute $\mu(u) = u^2$

  $v = \frac{\int 3u^2 du + C1}{u^2}$

- Evaluate the integrals on the rhs

  $v = \frac{u^3 + C1}{u^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.000 (sec)
Leaf size : 11

```
dsolve(diff(v(u),u)+2*v(u)/u = 3,v(u),singsol=all)
```

$$v = u + \frac{c_1}{u^2}$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 13

```
DSolve[{D[v[u],u]+2*v[u]/u==3,{}},v[u],u,IncludeSingularSolutions->True]
```

$$v(u) \to u + \frac{c_1}{u^2}$$

2.2 Chapter IV. Methods of solution: First order equations. section 24. Problems at page 62

2.2.1 Problem 4 (a)

Internal problem ID [18540]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 24. Problems at page 62
Problem number : 4 (a)
Date solved : `Tuesday, January 28, 2025 at 11:54:45 AM`
CAS classification : `[_separable]`

Solve

$$\sin(x)\cos(y)^2 + \cos(x)^2 y' = 0$$

Solved as first order separable ode

Time used: 0.207 (sec)

The ode

$$y' = -\frac{\sin(x)\cos(y)^2}{\cos(x)^2} \tag{2.7}$$

is separable as it can be written as

$$\begin{aligned} y' &= -\frac{\sin(x)\cos(y)^2}{\cos(x)^2} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{\sin(x)}{\cos(x)^2} \\ g(y) &= \cos(y)^2 \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$
$$\int \frac{1}{\cos(y)^2}\,dy = \int -\frac{\sin(x)}{\cos(x)^2}\,dx$$

$$\tan(y) = -\sec(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or

$$\cos(y)^2 = 0$$

for y gives

$$y = \frac{\pi}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\tan(y) = -\sec(x) + c_1$$
$$y = \frac{\pi}{2}$$

Solving for y gives

$$y = \frac{\pi}{2}$$
$$y = \arctan(-\sec(x) + c_1)$$

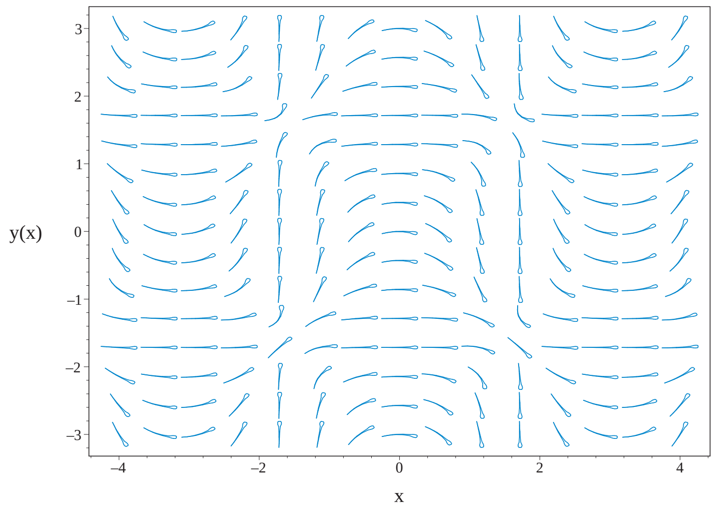


Figure 2.11: Slope field plot
$\sin(x)\cos(y)^2+\cos(x)^2 y'=0$

Summary of solutions found

$$y=\frac{\pi}{2}$$
$$y=\arctan(-\sec(x)+c_1)$$

Maple step by step solution

Let's solve
$\sin(x)\cos(y)^2+\cos(x)^2 y'=0$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y'=-\frac{\sin(x)\cos(y)^2}{\cos(x)^2}$
- Separate variables
  $\frac{y'}{\cos(y)^2}=-\frac{\sin(x)}{\cos(x)^2}$
- Integrate both sides with respect to x
  $\int \frac{y'}{\cos(y)^2}dx=\int -\frac{\sin(x)}{\cos(x)^2}dx+C1$
- Evaluate integral
  $\tan(y)=-\frac{1}{\cos(x)}+C1$
- Solve for y

$$y = \arctan\left(\frac{C1\cos(x)-1}{\cos(x)}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 11

```
dsolve(sin(x)*cos(y(x))^2+cos(x)^2*diff(y(x),x) = 0,y(x),singsol=all)
```

$$y(x) = -\arctan(\sec(x) + c_1)$$

Mathematica DSolve solution

Solving time : 1.507 (sec)
Leaf size : 31

```
DSolve[{Sin[x]*Cos[y[x]]^2+ Cos[x]^2*D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \arctan(-\sec(x) + c_1)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

2.2.2 Problem 4 (b)

Internal problem ID [18541]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 24. Problems at page 62
Problem number : 4 (b)
Date solved : `Tuesday, January 28, 2025 at 11:54:49 AM`
CAS classification : `[[_1st_order, '_with_symmetry_[F(x),G(x)*y+H(x)]']]`

Solve

$$y' + \sqrt{\frac{1 - y^2}{-x^2 + 1}} = 0$$

Solved as first order Exact ode

Time used: 22.789 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(-\sqrt{\frac{-y^2+1}{-x^2+1}}\right)\mathrm{d}x$$

$$\left(\sqrt{\frac{-y^2+1}{-x^2+1}}\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\sqrt{\frac{-y^2+1}{-x^2+1}}\\N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(\sqrt{\frac{-y^2+1}{-x^2+1}}\right)\\&=\frac{y}{\sqrt{\frac{y^2-1}{x^2-1}}\,(x^2-1)}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{y}{\sqrt{\frac{-y^2+1}{-x^2+1}}\,(-x^2+1)}\right) - (0)\right) \\
&= \frac{y}{\sqrt{\frac{y^2-1}{x^2-1}}\,(x^2-1)}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{\sqrt{\frac{y^2-1}{x^2-1}}}\left((0) - \left(-\frac{y}{\sqrt{\frac{-y^2+1}{-x^2+1}}\,(-x^2+1)}\right)\right) \\
&= -\frac{y}{y^2-1}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{y}{y^2-1}\,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln(y-1)}{2} - \frac{\ln(y+1)}{2}} \\
&= \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}\left(\sqrt{\frac{-y^2+1}{-x^2+1}}\right) \\
&= \frac{\sqrt{\frac{y^2-1}{x^2-1}}}{\sqrt{y-1}\,\sqrt{y+1}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}(1) \\
&= \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\sqrt{\frac{y^2-1}{x^2-1}}}{\sqrt{y-1}\,\sqrt{y+1}}\right) + \left(\frac{1}{\sqrt{y-1}\,\sqrt{y+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (2) w.r.t. y gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y \\
\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}\,\mathrm{d}y \\
\phi &= \frac{\sqrt{y^2-1}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}} + f(x)
\end{aligned} \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \frac{\sqrt{\frac{y^2-1}{x^2-1}}}{\sqrt{y-1}\,\sqrt{y+1}}$. Therefore equation (4) becomes

$$
\frac{\sqrt{\frac{y^2-1}{x^2-1}}}{\sqrt{y-1}\,\sqrt{y+1}} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$f'(x)=\frac{\sqrt{\frac{y^2-1}{x^2-1}}}{\sqrt{y-1}\,\sqrt{y+1}}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x=\int\left(\frac{\sqrt{\frac{y^2-1}{x^2-1}}}{\sqrt{y-1}\,\sqrt{y+1}}\right)\mathrm{d}x$$

$$f(x)=\frac{\sqrt{\frac{y^2-1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi=\frac{\sqrt{y^2-1}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}}+\frac{\sqrt{\frac{y^2-1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1=\frac{\sqrt{y^2-1}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}}+\frac{\sqrt{\frac{y^2-1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}}$$

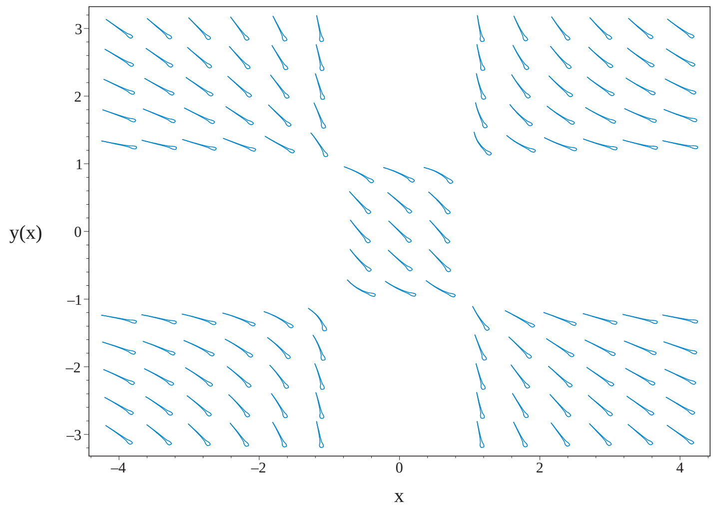


Figure 2.12: Slope field plot
$y' + \sqrt{\frac{1-y^2}{-x^2+1}} = 0$

Summary of solutions found

$$\frac{\sqrt{-1+y^2}\, \ln\left(y+\sqrt{-1+y^2}\right)}{\sqrt{y-1}\,\sqrt{y+1}} + \frac{\sqrt{\frac{-1+y^2}{x^2-1}}\,\sqrt{x^2-1}\, \ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}} = c_1$$

Maple step by step solution

Let's solve

$y' + \sqrt{\frac{1-y^2}{-x^2+1}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $y' = -\sqrt{\frac{1-y^2}{-x^2+1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 84

```
dsolve(diff(y(x),x)+((1-y(x)^2)/(-x^2+1))^(1/2) = 0,y(x),singsol=all)
```

$$\frac{\sqrt{\frac{-1+y(x)^2}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{y(x)-1}\,\sqrt{y(x)+1}} + \frac{\sqrt{-1+y(x)^2}\,\ln\left(y(x)+\sqrt{-1+y(x)^2}\right)}{\sqrt{y(x)-1}\,\sqrt{y(x)+1}} + c_1 = 0$$

Mathematica DSolve solution

Solving time : 0.367 (sec)
Leaf size : 39

```
DSolve[{D[y[x],x]+Sqrt[ (1-y[x]^2)/(1-x^2)]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\cosh\left(2\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x-1}{x+1}}}\right) - c_1\right)$$
$$y(x) \to -1$$
$$y(x) \to 1$$

2.2.3 Problem 4 (c)

Internal problem ID [18542]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 24. Problems at page 62
Problem number : 4 (c)
Date solved : `Tuesday, January 28, 2025 at 11:55:15 AM`
CAS classification : `[_separable]`

Solve

$$y - xy' = b(1 + x^2 y')$$

Solved as first order linear ode

Time used: 0.059 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{1}{x(bx+1)}$$
$$p(x) = -\frac{b}{x(bx+1)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\frac{1}{x(bx+1)}dx} \\ &= \frac{bx+1}{x}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(-\frac{b}{x\,(bx+1)}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y(bx+1)}{x}\right) = \left(\frac{bx+1}{x}\right)\left(-\frac{b}{x\,(bx+1)}\right)$$
$$\mathrm{d}\left(\frac{y(bx+1)}{x}\right) = \left(-\frac{b}{x^2}\right)\,\mathrm{d}x$$

Integrating gives

$$\begin{aligned}\frac{y(bx+1)}{x} &= \int -\frac{b}{x^2}\,dx \\ &= \frac{b}{x} + c_1\end{aligned}$$

Dividing throughout by the integrating factor $\frac{bx+1}{x}$ gives the final solution

$$y = \frac{c_1 x + b}{bx+1}$$

Summary of solutions found

$$y = \frac{c_1 x + b}{bx+1}$$

Solved as first order separable ode

Time used: 0.150 (sec)

The ode

$$y' = \frac{y-b}{x\,(bx+1)} \tag{2.8}$$

is separable as it can be written as

$$\begin{aligned}y' &= \frac{y-b}{x\,(bx+1)} \\ &= f(x)g(y)\end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x\,(bx+1)} \\
g(y) &= y-b
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(y)}\,dy &= \int f(x)\,dx \\
\int \frac{1}{y-b}\,dy &= \int \frac{1}{x\,(bx+1)}\,dx
\end{aligned}
$$

$$
\ln\left(-y+b\right) = \ln\left(\frac{x}{bx+1}\right) + c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y)=0$ or

$$
y-b=0
$$

for y gives

$$
y=b
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(-y+b\right) &= \ln\left(\frac{x}{bx+1}\right) + c_1 \\
y &= b
\end{aligned}
$$

Solving for y gives

$$
\begin{aligned}
y &= b \\
y &= -\frac{-b^2x+\mathrm{e}^{c_1}x-b}{bx+1}
\end{aligned}
$$

<u>Summary of solutions found</u>

$$
\begin{aligned}
y &= b \\
y &= -\frac{-b^2x+\mathrm{e}^{c_1}x-b}{bx+1}
\end{aligned}
$$

Solved as first order Exact ode

Time used: 0.167 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-b\,x^2-x\right)\mathrm{d}y&=(-y+b)\,\mathrm{d}x\\ (y-b)\,\mathrm{d}x+\left(-b\,x^2-x\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y-b\\ N(x,y)&=-b\,x^2-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - b) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-b\,x^2 - x\right) \\ &= -2bx - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{x\,(bx+1)}((1) - (-2bx - 1)) \\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}(y - b) \\ &= \frac{y - b}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^2}(-b\,x^2 - x) \\ &= \frac{-bx-1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{y-b}{x^2}\right) + \left(\frac{-bx-1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y \\ \int \frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int \frac{-bx-1}{x}\,\mathrm{d}y \\ \phi &= -\frac{y(bx+1)}{x} + f(x) \qquad (3)\end{aligned}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{y(bx+1)}{x^2} - \frac{yb}{x} + f'(x) \qquad (4) \\ &= \frac{y}{x^2} + f'(x)\end{aligned}$$

But equation (1) says that $\frac{\partial\phi}{\partial x} = \frac{y-b}{x^2}$. Therefore equation (4) becomes

$$\frac{y-b}{x^2} = \frac{y}{x^2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{b}{x^2}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{b}{x^2}\right) \mathrm{d}x$$
$$f(x) = \frac{b}{x} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{y(bx+1)}{x} + \frac{b}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{y(bx+1)}{x} + \frac{b}{x}$$

Solving for y gives

$$y = -\frac{c_1 x - b}{bx + 1}$$

Summary of solutions found

$$y = -\frac{c_1 x - b}{bx + 1}$$

Solved using Lie symmetry for first order ode

Time used: 0.390 (sec)

Writing the ode as

$$y' = \frac{y - b}{x\,(bx + 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(y-b)(b_3 - a_2)}{x(bx+1)} - \frac{(y-b)^2 a_3}{x^2(bx+1)^2} - \left(-\frac{y-b}{x^2(bx+1)} - \frac{(y-b)b}{x(bx+1)^2}\right)(xa_2 + ya_3 + a_1) - \frac{xb_2 + yb_3 + b_1}{x(bx+1)} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{b^2x^4b_2 - b^2x^2a_2 - b^2x^2b_3 - 2b^2xya_3 + b\,x^3b_2 + b\,x^2ya_2 + 2bx\,y^2a_3 - 2b^2xa_1 - b\,x^2b_1 + 2bxya_1 - b^2a_3 - bx}{x^2(bx+1)^2} = 0$$

Setting the numerator to zero gives

$$b^2x^4b_2 - b^2x^2a_2 - b^2x^2b_3 - 2b^2xya_3 + b\,x^3b_2 + b\,x^2ya_2 + 2bx\,y^2a_3 - 2b^2xa_1 - b\,x^2b_1 + 2bxya_1 - b^2a_3 - bxb_3 + bya_3 - ba_1 - xb_1 + ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
b^2b_2v_1^4 - b^2a_2v_1^2 - 2b^2a_3v_1v_2 - b^2b_3v_1^2 + ba_2v_1^2v_2 + 2ba_3v_1v_2^2 + bb_2v_1^3 - 2b^2a_1v_1 \\
+ 2ba_1v_1v_2 - bb_1v_1^2 - b^2a_3 + ba_3v_2 - bb_3v_1 - ba_1 + a_1v_2 - b_1v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
b^2b_2v_1^4 + bb_2v_1^3 + ba_2v_1^2v_2 + \left(-b^2a_2 - b^2b_3 - bb_1\right)v_1^2 + 2ba_3v_1v_2^2 \\
+ \left(-2b^2a_3 + 2ba_1\right)v_1v_2 + \left(-2b^2a_1 - bb_3 - b_1\right)v_1 + (ba_3 + a_1)v_2 - b^2a_3 - ba_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
ba_2 &= 0 \\
bb_2 &= 0 \\
b^2b_2 &= 0 \\
2ba_3 &= 0 \\
ba_3 + a_1 &= 0 \\
-2b^2a_3 + 2ba_1 &= 0 \\
-b^2a_3 - ba_1 &= 0 \\
-2b^2a_1 - bb_3 - b_1 &= 0 \\
-b^2a_2 - b^2b_3 - bb_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= -bb_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 0$$
$$\eta = y - b$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y - b} dy \end{aligned}$$

Which results in

$$S = \ln (y - b)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y - b}{x (bx + 1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 0 \\
S_y &= \frac{1}{y-b}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x(bx+1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(Rb+1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int \frac{1}{R(Rb+1)}\, dR \\
S(R) &= \ln(R) - \ln(Rb+1) + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y-b) = \ln(x) - \ln(bx+1) + c_2$$

Which gives

$$y = \frac{b^2x + x\,\mathrm{e}^{c_2} + b}{bx+1}$$

Summary of solutions found

$$y = \frac{b^2x + x\,\mathrm{e}^{c_2} + b}{bx+1}$$

Maple step by step solution

Let's solve

$y - y'x = b(1 + y'x^2)$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $y' = \frac{-y+b}{-b\,x^2-x}$

- Separate variables

  $\frac{y'}{-y+b} = \frac{1}{-b\,x^2-x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y+b}dx = \int \frac{1}{-b\,x^2-x}dx + C1$

- Evaluate integral

  $-\ln(-y+b) = \ln(bx+1) - \ln(x) + C1$

- Solve for y

  $y = \frac{e^{C1}b^2x+e^{C1}b-x}{e^{C1}(bx+1)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 17

```
dsolve(y(x)-x*diff(y(x),x) = b*(1+x^2*diff(y(x),x)),y(x),singsol=all)
```

$$y(x) = \frac{c_1x + b}{bx + 1}$$

Mathematica DSolve solution

Solving time : 0.043 (sec)
Leaf size : 24

```
DSolve[{y[x]-x*D[y[x],x]==b*(1+x^2*D[y[x],x]),{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{b + c_1 x}{bx + 1}$$
$$y(x) \to b$$

2.2.4 Problem 5

Internal problem ID [18543]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 24. Problems at page 62
Problem number : 5
Date solved : `Tuesday, January 28, 2025 at 11:55:16 AM`
CAS classification : `[_quadrature]`

Solve

$$x' = k(A - nx)(M - mx)$$

Solved as first order autonomous ode

Time used: 0.647 (sec)

Integrating gives

$$\int \frac{1}{k(-nx + A)(-mx + M)} dx = dt$$

$$\frac{\ln(-nx + A) - \ln(-mx + M)}{k(Am - Mn)} = t + c_1$$

Singular solutions are found by solving

$$k(-nx + A)(-mx + M) = 0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x = \frac{A}{n}$$

$$x = \frac{M}{m}$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

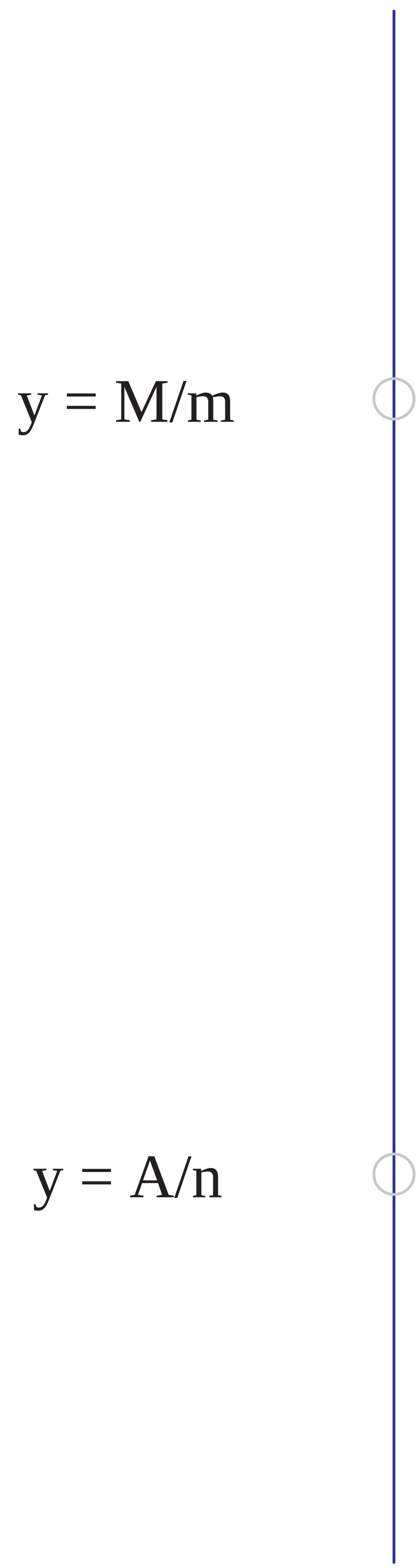


Figure 2.13: Phase line diagram

Solving for x gives

$$x = \frac{A}{n}$$

$$x = \frac{M}{m}$$

$$x = \frac{A\,\mathrm{e}^{-Ac_1km-Akmt+Mc_1kn+Mknt} - M}{\mathrm{e}^{-Ac_1km-Akmt+Mc_1kn+Mknt}n - m}$$

Summary of solutions found

$$x = \frac{A}{n}$$

$$x = \frac{M}{m}$$

$$x = \frac{A\,\mathrm{e}^{-Ac_1km-Akmt+Mc_1kn+Mknt} - M}{\mathrm{e}^{-Ac_1km-Akmt+Mc_1kn+Mknt}n - m}$$

Solved as first order Exact ode

Time used: 0.437 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}x &= (k(-nx + A)(-mx + M))\, \mathrm{d}t \\ (-k(-nx + A)(-mx + M))\, \mathrm{d}t + \mathrm{d}x &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, x) &= -k(-nx + A)(-mx + M) \\ N(t, x) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-k(-nx + A)(-mx + M)) \\ &= ((-2nx + A)m + Mn)k \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\ &= 1((kn(-mx + M) + k(-nx + A)m) - (0)) \\ &= ((-2nx + A)m + Mn)k \end{aligned}$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right) \\
&= -\frac{1}{k\left(-nx+A\right)\left(-mx+M\right)}\left(\left(0\right) - \left(kn(-mx+M) + k(-nx+A)\, m\right)\right) \\
&= \frac{\left(-2nx+A\right)m + Mn}{\left(-nx+A\right)\left(-mx+M\right)}
\end{aligned}
$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}x} \\
&= e^{\int \frac{(-2nx+A)m+Mn}{(-nx+A)(-mx+M)} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln((-nx+A)(-mx+M))} \\
&= \frac{1}{\left(-nx+A\right)\left(-mx+M\right)}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\left(-nx+A\right)\left(-mx+M\right)}\left(-k(-nx+A)\left(-mx+M\right)\right) \\
&= -k
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\left(-nx+A\right)\left(-mx+M\right)}(1) \\
&= \frac{1}{\left(-nx+A\right)\left(-mx+M\right)}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(-k\right) + \left(\frac{1}{\left(-nx+A\right)\left(-mx+M\right)}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -k\,\mathrm{d}t$$
$$\phi = -kt + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{(-nx+A)(-mx+M)}$. Therefore equation (4) becomes

$$\frac{1}{(-nx+A)(-mx+M)} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \frac{1}{(-nx+A)(-mx+M)}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(\frac{1}{(-nx+A)(-mx+M)}\right)\mathrm{d}x$$
$$f(x) = \frac{\ln(-nx+A)}{Am-Mn} - \frac{\ln(-mx+M)}{Am-Mn} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -kt + \frac{\ln(-nx+A)}{Am-Mn} - \frac{\ln(-mx+M)}{Am-Mn} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -kt + \frac{\ln(-nx + A)}{Am - Mn} - \frac{\ln(-mx + M)}{Am - Mn}$$

Solving for x gives

$$x = \frac{A\,e^{-Akmt+Mknt-c_1mA+c_1nM} - M}{e^{-Akmt+Mknt-c_1mA+c_1nM}n - m}$$

Summary of solutions found

$$x = \frac{A\,e^{-Akmt+Mknt-c_1mA+c_1nM} - M}{e^{-Akmt+Mknt-c_1mA+c_1nM}n - m}$$

Solved using Lie symmetry for first order ode

Time used: 1.179 (sec)

Writing the ode as

$$\begin{aligned} x' &= k(-nx + A)(-mx + M) \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$

$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + k(-nx + A)(-mx + M)(b_3 - a_2) - k^2(-nx + A)^2(-mx + M)^2 a_3 \\ &\quad - (-kn(-mx + M) - k(-nx + A)m)(tb_2 + xb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-k^2m^2n^2x^4a_3+2A\,k^2m^2n\,x^3a_3+2M\,k^2m\,n^2x^3a_3-A^2k^2m^2x^2a_3 \\ -4AM\,k^2mn\,x^2a_3-M^2k^2n^2x^2a_3+2A^2M\,k^2mxa_3+2A\,M^2k^2nxa_3 \\ -A^2M^2k^2a_3-2kmntxb_2-kmn\,x^2a_2-kmn\,x^2b_3+Akmtb_2+Akmxa_2 \\ +Mkntb_2+Mknxa_2-2kmnxb_1-AMka_2+AMkb_3+Akmb_1+Mknb_1+b_2=0$$

Setting the numerator to zero gives

$$-k^2m^2n^2x^4a_3+2A\,k^2m^2n\,x^3a_3+2M\,k^2m\,n^2x^3a_3-A^2k^2m^2x^2a_3 \\ -4AM\,k^2mn\,x^2a_3-M^2k^2n^2x^2a_3+2A^2M\,k^2mxa_3 \\ +2A\,M^2k^2nxa_3-A^2M^2k^2a_3-2kmntxb_2-kmn\,x^2a_2 \\ -kmn\,x^2b_3+Akmtb_2+Akmxa_2+Mkntb_2+Mknxa_2 \\ -2kmnxb_1-AMka_2+AMkb_3+Akmb_1+Mknb_1+b_2=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t,x\}$ in them.

$$\{t,x\}$$

The following substitution is now made to be able to collect on all terms with $\{t,x\}$ in them

$$\{t=v_1,x=v_2\}$$

The above PDE (6E) now becomes

$$-k^2m^2n^2a_3v_2^4+2A\,k^2m^2na_3v_2^3+2M\,k^2m\,n^2a_3v_2^3-A^2k^2m^2a_3v_2^2 \\ -4AM\,k^2mna_3v_2^2-M^2k^2n^2a_3v_2^2+2A^2M\,k^2ma_3v_2 \\ +2A\,M^2k^2na_3v_2-A^2M^2k^2a_3-kmna_2v_2^2-2kmnb_2v_1v_2 \\ -kmnb_3v_2^2+Akma_2v_2+Akmb_2v_1+Mkna_2v_2+Mknb_2v_1 \\ -2kmnb_1v_2-AMka_2+AMkb_3+Akmb_1+Mknb_1+b_2=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2\}$$

Equation (7E) now becomes

$$-2kmnb_2v_1v_2+(Akmb_2+Mknb_2)\,v_1 \\ -k^2m^2n^2a_3v_2^4+\left(2A\,k^2m^2na_3+2M\,k^2m\,n^2a_3\right)v_2^3 \\ +\left(-A^2k^2m^2a_3-4AM\,k^2mna_3-M^2k^2n^2a_3-kmna_2-kmnb_3\right)v_2^2 \\ +\left(2A^2M\,k^2ma_3+2A\,M^2k^2na_3+Akma_2+Mkna_2-2kmnb_1\right)v_2 \\ -A^2M^2k^2a_3-AMka_2+AMkb_3+Akmb_1+Mknb_1+b_2=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2kmnb_2 &= 0\\
-k^2m^2n^2a_3 &= 0\\
2A\,k^2m^2na_3 + 2M\,k^2m\,n^2a_3 &= 0\\
Akmb_2 + Mknb_2 &= 0\\
2A^2M\,k^2ma_3 + 2A\,M^2k^2na_3 + Akma_2 + Mkna_2 - 2kmnb_1 &= 0\\
-A^2k^2m^2a_3 - 4AM\,k^2mna_3 - M^2k^2n^2a_3 - kmna_2 - kmnb_3 &= 0\\
-A^2M^2k^2a_3 - AMka_2 + AMkb_3 + Akmb_1 + Mknb_1 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1\\
a_2 &= 0\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1\\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t,x)\,\xi\\
&= 0 - (k(-nx + A)\,(-mx + M))\,(1)\\
&= -x^2kmn + Axkm + Mxkn - AMk\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial x}\right)S(t,x)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=t$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{-x^2kmn+Axkm+Mxkn-AMk}dy\end{aligned}$$

Which results in

$$S=-\frac{\ln(-nx+A)}{k\,(Am-Mn)}+\frac{\ln(-mx+M)}{k\,(Am-Mn)}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_t+\omega(t,x)S_x}{R_t+\omega(t,x)R_x}\tag{2}$$

Where in the above R_t,R_x,S_t,S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x)=k(-nx+A)\,(-mx+M)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t&=1\\R_x&=0\\S_t&=0\\S_x&=-\frac{1}{k\,(-nx+A)\,(-mx+M)}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=-1\tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t,x in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=-1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -1\, dR$$
$$S(R) = -R + c_2$$

To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$\frac{-\ln(A - nx) + \ln(M - mx)}{k(Am - Mn)} = -t + c_2$$

Which gives

$$x = \frac{A\,\mathrm{e}^{Ac_2km - Akmt - Mc_2kn + Mknt} - M}{\mathrm{e}^{Ac_2km - Akmt - Mc_2kn + Mknt}n - m}$$

Summary of solutions found

$$x = \frac{A\,\mathrm{e}^{Ac_2km - Akmt - Mc_2kn + Mknt} - M}{\mathrm{e}^{Ac_2km - Akmt - Mc_2kn + Mknt}n - m}$$

Maple step by step solution

Let's solve
$x' = k(A - nx)(M - mx)$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = k(A - nx)(M - mx)$
- Separate variables
  $\frac{x'}{(A-nx)(M-mx)} = k$
- Integrate both sides with respect to t
  $\int \frac{x'}{(A-nx)(M-mx)}dt = \int kdt + C1$
- Evaluate integral
  $-\frac{\ln(M-mx)}{Am-Mn} + \frac{\ln(A-nx)}{Am-Mn} = tk + C1$

- Solve for x

  $x = \frac{A\,\mathrm{e}^{-Akmt+Mknt-AC1m+C1Mn}-M}{\mathrm{e}^{-Akmt+Mknt-AC1m+C1Mn}n-m}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 47

```
dsolve(diff(x(t),t) = k*(A-n*x(t))*(M-m*x(t)),x(t),singsol=all)
```

$$x = \frac{-A\,\mathrm{e}^{-k(c_1+t)(Am-Mn)}+M}{-\mathrm{e}^{-k(c_1+t)(Am-Mn)}n+m}$$

Mathematica DSolve solution

Solving time : `2.839` (sec)
Leaf size : 82

```
DSolve[{D[x[t],t]==k*(A-n*x[t])*(M-m*x[t]),{}},x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{Ae^{Mn(kt+c_1)} - Me^{Am(kt+c_1)}}{ne^{Mn(kt+c_1)} - me^{Am(kt+c_1)}}$$
$$x(t) \to \frac{M}{m}$$
$$x(t) \to \frac{A}{n}$$

2.2.5 Problem 6

Internal problem ID [18544]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 24. Problems at page 62
Problem number : 6
Date solved : `Tuesday, January 28, 2025 at 11:55:19 AM`
CAS classification : `[_separable]`

Solve

$$y' = 1 + \frac{1}{x} - \frac{1}{y^2+2} - \frac{1}{x\,(y^2+2)}$$

Solved as first order separable ode

Time used: 0.176 (sec)

The ode

$$y' = \frac{y^2x + y^2 + x + 1}{x\,(y^2+2)} \tag{2.9}$$

is separable as it can be written as

$$\begin{aligned} y' &= \frac{y^2x + y^2 + x + 1}{x\,(y^2+2)} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{x+1}{x} \\ g(y) &= \frac{y^2+1}{y^2+2} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$
$$\int \frac{y^2+2}{y^2+1}\,dy = \int \frac{x+1}{x}\,dx$$

$$y + \arctan(y) = x + \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or

$$\frac{y^2+1}{y^2+2} = 0$$

for y gives

$$y = -i$$
$$y = i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$y + \arctan(y) = x + \ln(x) + c_1$$
$$y = -i$$
$$y = i$$

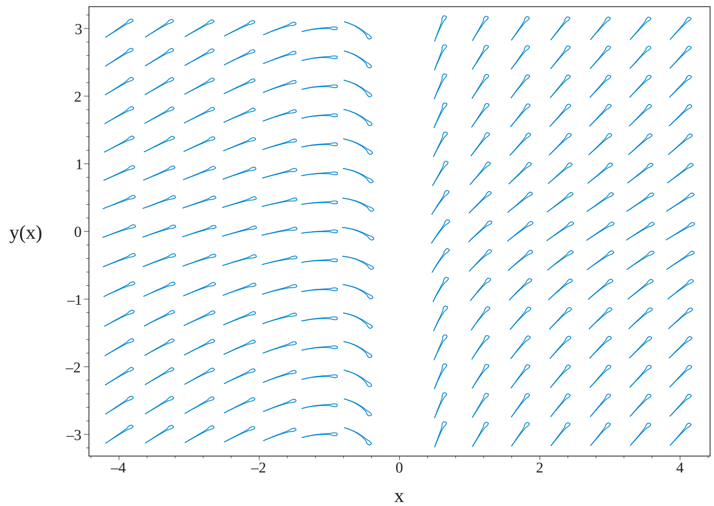


Figure 2.14: Slope field plot
$y' = 1 + \frac{1}{x} - \frac{1}{y^2+2} - \frac{1}{x(y^2+2)}$

Summary of solutions found

$$y + \arctan(y) = x + \ln(x) + c_1$$
$$y = -i$$
$$y = i$$

Solved as first order Exact ode

Time used: 0.162 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(1+\frac{1}{x}-\frac{1}{y^2+2}-\frac{1}{x\,(y^2+2)}\right)\mathrm{d}x$$

$$\left(-1-\frac{1}{x}+\frac{1}{y^2+2}+\frac{1}{x\,(y^2+2)}\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-1-\frac{1}{x}+\frac{1}{y^2+2}+\frac{1}{x\,(y^2+2)}\\N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-1-\frac{1}{x}+\frac{1}{y^2+2}+\frac{1}{x\,(y^2+2)}\right)\\&=-\frac{2y(x+1)}{x\,(y^2+2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{2y}{(y^2+2)^2} - \frac{2y}{x\,(y^2+2)^2}\right) - (0)\right) \\
&= -\frac{2y(x+1)}{x\,(y^2+2)^2}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{x(y^2+2)}{(y^2+1)\,(x+1)}\left((0) - \left(-\frac{2y}{(y^2+2)^2} - \frac{2y}{x\,(y^2+2)^2}\right)\right) \\
&= -\frac{2y}{(y^2+2)\,(y^2+1)}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{2y}{(y^2+2)\,(y^2+1)}\,\mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\ln(y^2+1)+\ln(y^2+2)} \\
&= \frac{y^2+2}{y^2+1}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{y^2+2}{y^2+1}\left(-1 - \frac{1}{x} + \frac{1}{y^2+2} + \frac{1}{x\,(y^2+2)}\right) \\
&= \frac{-x-1}{x}
\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{y^2+2}{y^2+1}(1)\\ &= \frac{y^2+2}{y^2+1}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-x-1}{x}\right) + \left(\frac{y^2+2}{y^2+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-x-1}{x}\,\mathrm{d}x\\ \phi &= -x - \ln(x) + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2+2}{y^2+1}$. Therefore equation (4) becomes

$$\frac{y^2+2}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y^2 + 2}{y^2 + 1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y^2 + 2}{y^2 + 1}\right) \mathrm{d}y$$
$$f(y) = y + \arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \ln(x) + y + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -x - \ln(x) + y + \arctan(y)$$

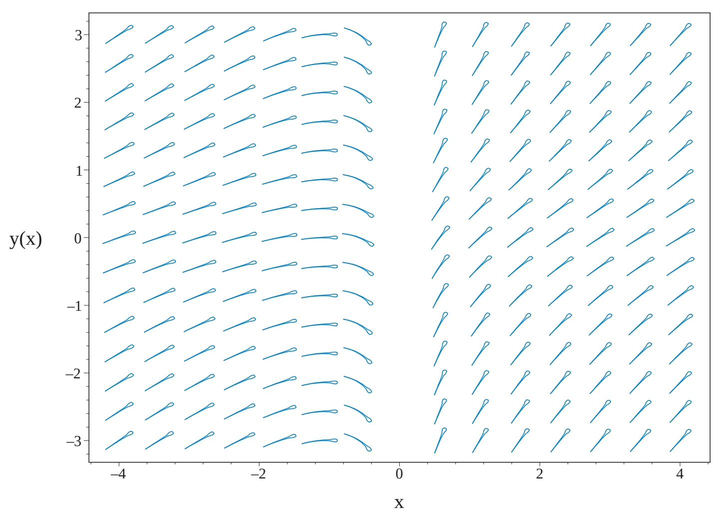


Figure 2.15: Slope field plot
$y' = 1 + \frac{1}{x} - \frac{1}{y^2+2} - \frac{1}{x(y^2+2)}$

Summary of solutions found

$$-x - \ln(x) + y + \arctan(y) = c_1$$

Maple step by step solution

Let's solve
$y' = 1 + \frac{1}{x} - \frac{1}{y^2+2} - \frac{1}{x(y^2+2)}$

- Highest derivative means the order of the ODE is 1
y'
- Solve for the highest derivative
$y' = 1 + \frac{1}{x} - \frac{1}{y^2+2} - \frac{1}{x(y^2+2)}$
- Separate variables
$\frac{y'(y^2+2)}{y^2+1} = \frac{x+1}{x}$
- Integrate both sides with respect to x
$\int \frac{y'(y^2+2)}{y^2+1} dx = \int \frac{x+1}{x} dx + C1$
- Evaluate integral
$y + \arctan(y) = x + \ln(x) + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 18

```
dsolve(diff(y(x),x) = 1+1/x-1/(y(x)^2+2)-1/x/(y(x)^2+2),y(x),singsol=all)
```

$$y(x) = \tan\left(\mathrm{RootOf}\left(\ln(x) + x - \tan(\_Z) - \_Z + c_1\right)\right)$$

Mathematica DSolve solution

Solving time : 0.304 (sec)
Leaf size : 19

```
DSolve[{D[y[x],x]==1+1/x-1/(y[x]^2+2)-1/(x*(y[x]^2+2)),{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \text{InverseFunction}[\arctan(\#1) + \#1\&][x + \log(x) + c_1]$$

2.3 Chapter IV. Methods of solution: First order equations. section 29. Problems at page 81

2.3.1 Problem 1

Internal problem ID [18545]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 29. Problems at page 81
Problem number : 1
Date solved : `Tuesday, January 28, 2025 at 11:55:21 AM`
CAS classification :
`[[_homogeneous, ‘class A‘], _rational, [_Abel, ‘2nd type‘, ‘class B‘]]`

Solve

$$y^2 = x(y - x)\, y'$$

Solved as first order homogeneous class A ode

Time used: 0.279 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2}{x\,(y-x)} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = -y^2$ and $N = x(-y + x)$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dx}x + u &= \frac{u^2}{u-1} \\ \frac{du}{dx} &= \frac{\frac{u(x)^2}{u(x)-1} - u(x)}{x} \end{aligned}$$

Or

$$u'(x) - \frac{\frac{u(x)^2}{u(x)-1} - u(x)}{x} = 0$$

Or

$$u'(x)\, xu(x) - u'(x)\, x - u(x) = 0$$

Or

$$x(u(x) - 1)\, u'(x) - u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode

$$u'(x) = \frac{u(x)}{x\,(u\,(x) - 1)} \tag{2.10}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)}{x\,(u\,(x) - 1)} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= \frac{u}{u-1} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{u-1}{u}\, du &= \int \frac{1}{x}\, dx \end{aligned}$$

$$u(x) + \ln\left(\frac{1}{u(x)}\right) = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u}{u-1} = 0$$

for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$u(x) + \ln\left(\frac{1}{u(x)}\right) = \ln(x) + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$ back to y gives

$$y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

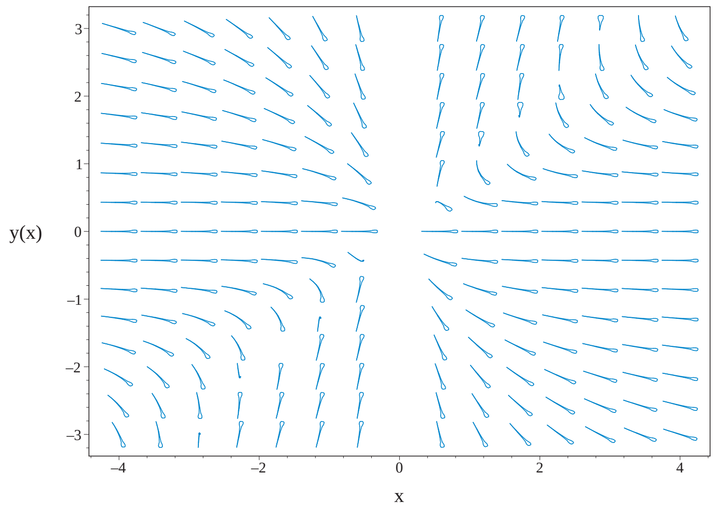


Figure 2.16: Slope field plot
$y^2 = x(y - x)\, y'$

Summary of solutions found

$$
\begin{aligned}
y &= 0 \\
y &= -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)
\end{aligned}
$$

Solved as first order homogeneous class D2 ode

Time used: 0.158 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$
u(x)^2 x^2 = x(u(x)\, x - x)\,(u'(x)\, x + u(x))
$$

Which is now solved The ode

$$
u'(x) = \frac{u(x)}{(u(x) - 1)\, x} \tag{2.11}
$$

is separable as it can be written as

$$
\begin{aligned}
u'(x) &= \frac{u(x)}{(u(x) - 1)\, x} \\
&= f(x)g(u)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
g(u) &= \frac{u}{u-1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)} \, du &= \int f(x) \, dx \\
\int \frac{u-1}{u} \, du &= \int \frac{1}{x} \, dx
\end{aligned}
$$

$$
u(x) + \ln\left(\frac{1}{u(x)}\right) = \ln(x) + c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$
\frac{u}{u-1} = 0
$$

for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
u(x) + \ln\left(\frac{1}{u(x)}\right) &= \ln(x) + c_1 \\
u(x) &= 0
\end{aligned}
$$

Solving for $u(x)$ gives

$$
\begin{aligned}
u(x) &= 0 \\
u(x) &= -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)
\end{aligned}
$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$ back to y gives

$$y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

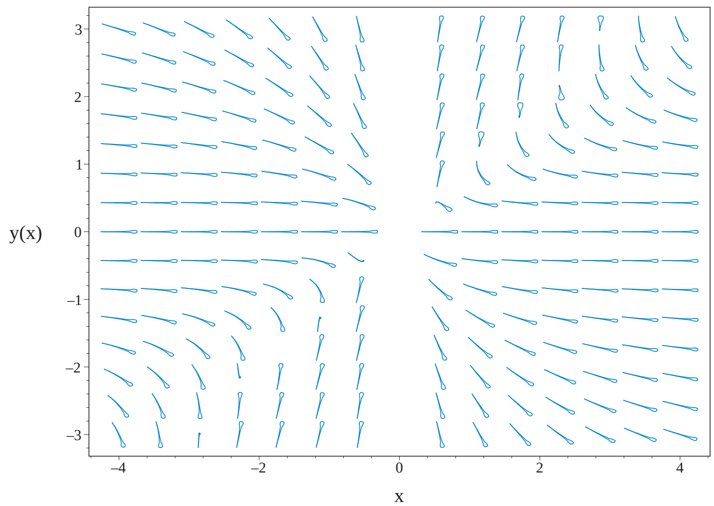


Figure 2.17: Slope field plot
$y^2 = x(y - x)\, y'$

<u>Summary of solutions found</u>

$$y = 0$$

$$y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

Solved as first order homogeneous class Maple C ode

Time used: 0.364 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{(Y(X) + y_0)^2}{(x_0 + X)\,(Y\,(X) + y_0 - x_0 - X)}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$
\begin{aligned}
x_0 &= 0 \\
y_0 &= 0
\end{aligned}
$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$
\frac{d}{dX}Y(X) = \frac{Y(X)^2}{-X^2 + XY(X)}
$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{Y^2}{X(Y - X)} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -Y^2$ and $N = X(X - Y)$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u^2}{u - 1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)^2}{u(X)-1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{u(X)^2}{u(X)-1} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right) Xu(X) - \left(\frac{d}{dX}u(X)\right) X - u(X) = 0
$$

Or

$$X(u(X)-1)\left(\frac{d}{dX}u(X)\right)-u(X)=0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X)=\frac{u(X)}{X\left(u\left(X\right)-1\right)} \tag{2.12}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X)&=\frac{u(X)}{X\left(u\left(X\right)-1\right)}\\&=f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X)&=\frac{1}{X}\\g(u)&=\frac{u}{u-1}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int\frac{1}{g(u)}\,du&=\int f(X)\,dX\\\int\frac{u-1}{u}\,du&=\int\frac{1}{X}\,dX\end{aligned}$$

$$u(X)+\ln\left(\frac{1}{u\left(X\right)}\right)=\ln\left(X\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$\frac{u}{u-1}=0$$

for $u(X)$ gives

$$u(X)=0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$u(X) + \ln\left(\frac{1}{u(X)}\right) = \ln(X) + c_1$$
$$u(X) = 0$$

Solving for $u(X)$ gives

$$u(X) = 0$$
$$u(X) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{X}\right)$$

Converting $u(X) = 0$ back to $Y(X)$ gives

$$Y(X) = 0$$

Converting $u(X) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{X}\right)$ back to $Y(X)$ gives

$$Y(X) = -X \operatorname{LambertW}\left(-\frac{e^{-c_1}}{X}\right)$$

Using the solution for $Y(X)$

$$Y(X) = 0 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = 0$$

Using the solution for $Y(X)$

$$Y(X) = -X \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{X}\right) \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = -x \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)$$

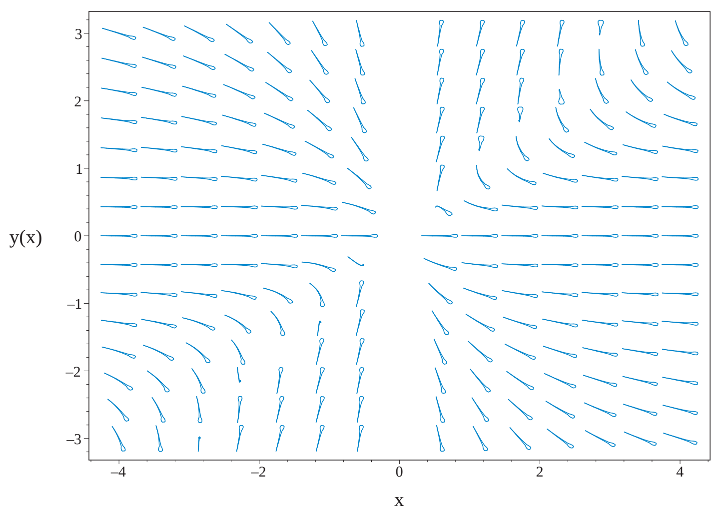


Figure 2.18: Slope field plot
$y^2 = x(y - x)\, y'$

Solved as first order Exact ode

Time used: 0.211 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x(y-x))\,\mathrm{d}y&=\left(-y^2\right)\mathrm{d}x\\ \left(y^2\right)\mathrm{d}x+(-x(y-x))\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^2\\ N(x,y)&=-x(y-x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x(y-x))\\ &= -y+2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = y^2$ and $N = -x(y-x)$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{y}{x^2}$$

$$N = -\frac{y-x}{xy}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{y-x}{xy}\right)\mathrm{d}y &= \left(-\frac{y}{x^2}\right)\mathrm{d}x\\ \left(\frac{y}{x^2}\right)\mathrm{d}x + \left(-\frac{y-x}{xy}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \frac{y}{x^2}\\ N(x, y) &= -\frac{y-x}{xy}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{x^2}\right)\\ &= \frac{1}{x^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y-x}{xy}\right)\\ &= \frac{1}{x^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\frac{y}{x^2}\,\mathrm{d}x$$

$$\phi=-\frac{y}{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=-\frac{1}{x}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=-\frac{y-x}{xy}$. Therefore equation (4) becomes

$$-\frac{y-x}{xy}=-\frac{1}{x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int\left(\frac{1}{y}\right)\mathrm{d}y$$

$$f(y)=\ln(y)+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y}{x} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{y}{x} + \ln(y)$$

Solving for y gives

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)+c_1}$$

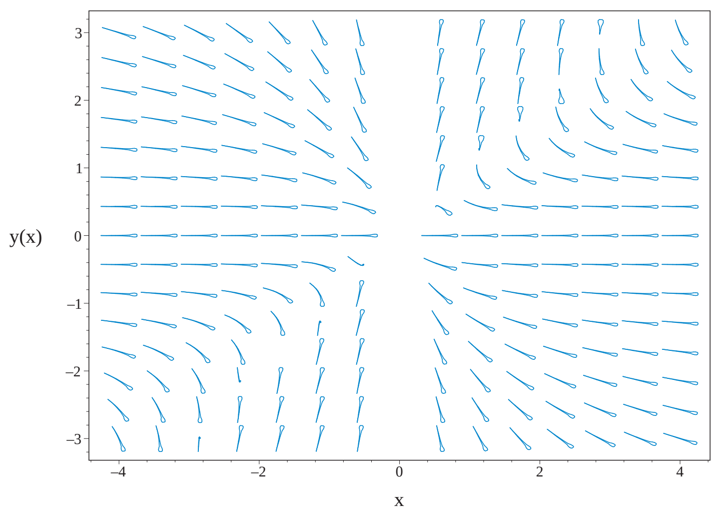


Figure 2.19: Slope field plot
$y^2 = x(y-x)\,y'$

Summary of solutions found

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)+c_1}$$

Solved as first order isobaric ode

Time used: 0.115 (sec)

Solving for y' gives

$$y' = \frac{y^2}{x(y-x)} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = \frac{y^2}{x(y-x)} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = \frac{xu(x)^2}{xu(x) - x}$$

The ode

$$u'(x) = \frac{u(x)}{(u(x)-1)x} \tag{2.13}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)}{(u(x)-1)x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= \frac{u}{u-1} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$
$$\int \frac{u-1}{u}\, du = \int \frac{1}{x}\, dx$$

$$u(x) + \ln\left(\frac{1}{u(x)}\right) = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u}{u-1} = 0$$

for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$u(x) + \ln\left(\frac{1}{u(x)}\right) = \ln(x) + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

Converting $u(x) = 0$ back to y gives

$$\frac{y}{x} = 0$$

Converting $u(x) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$ back to y gives

$$\frac{y}{x} = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

Solving for y gives

$$y = 0$$

$$y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

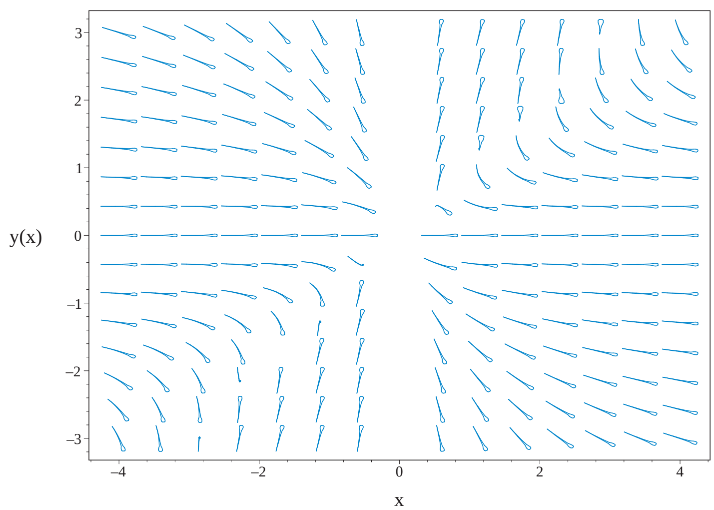


Figure 2.20: Slope field plot
$y^2 = x(y - x)\, y'$

Summary of solutions found

$$y = 0$$

$$y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

Solved using Lie symmetry for first order ode

Time used: 0.474 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y^2}{x(y-x)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 + \frac{y^2(b_3 - a_2)}{x(y-x)} - \frac{y^4 a_3}{x^2 (y-x)^2} - \left(-\frac{y^2}{x^2(y-x)} + \frac{y^2}{x(y-x)^2}\right)(x a_2 + y a_3 + a_1) \\
- \left(\frac{2y}{x(y-x)} - \frac{y^2}{x(y-x)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{x^4 b_2 - x^2 y^2 a_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 - x\,y^2 b_1 + y^3 a_1}{x^2(-y+x)^2} = 0
$$

Setting the numerator to zero gives

$$
x^4 b_2 - x^2 y^2 a_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 - x\,y^2 b_1 + y^3 a_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 v_2^2 - 2a_3 v_1 v_2^3 + b_2 v_1^4 + b_3 v_1^2 v_2^2 - 2a_1 v_1 v_2^2 + a_1 v_2^3 + 2b_1 v_1^2 v_2 - b_1 v_1 v_2^2 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^4 + (b_3 - a_2) v_1^2 v_2^2 + 2b_1 v_1^2 v_2 - 2a_3 v_1 v_2^3 + (-2a_1 - b_1) v_1 v_2^2 + a_1 v_2^3 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_2 &= 0 \\ -2a_3 &= 0 \\ 2b_1 &= 0 \\ -2a_1 - b_1 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y^2}{x\,(y-x)}\right)(x) \\
&= \frac{yx}{-y+x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{yx}{-y+x}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{y}{x} + \ln(y)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2}{x (y - x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x^2} \\ S_y &= \frac{-y + x}{yx} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 0 \, dR + c_2 \\ S(R) &= c_2 \end{aligned}$$

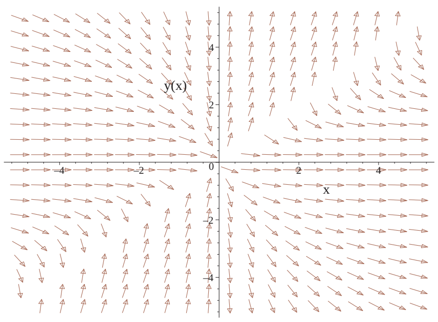
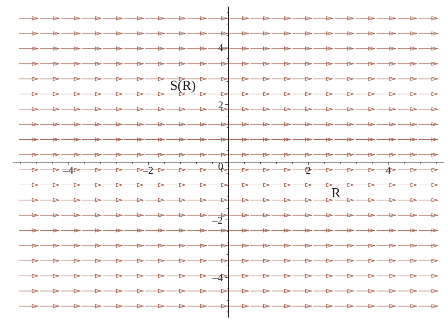
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln (y) x - y}{x} = c_2$$

Which gives

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{c_2}}{x}\right) + c_2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x(y-x)}$ 	$R = x$ $S = \frac{\ln(y)\,x - y}{x}$	$\frac{dS}{dR} = 0$ 

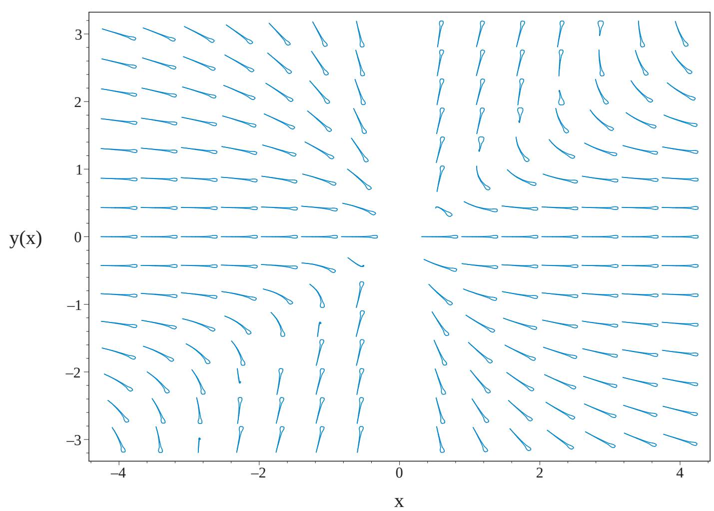


Figure 2.21: Slope field plot
$y^2 = x(y - x)\,y'$

Summary of solutions found

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{c_2}}{x}\right)+c_2}$$

Solved as first order ode of type dAlembert

Time used: 15.526 (sec)

Let $p = y'$ the ode becomes

$$y^2 = x(y - x)\, p$$

Solving for y from the above results in

$$y = \left(\frac{p}{2} + \frac{\sqrt{p^2 - 4p}}{2}\right) x \tag{1}$$

$$y = \left(\frac{p}{2} - \frac{\sqrt{p^2 - 4p}}{2}\right) x \tag{2}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved.

Solving ode 1A

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p}{2} + \frac{\sqrt{p\,(p - 4)}}{2}$$
$$g = 0$$

Hence (2) becomes

$$\frac{p}{2} - \frac{\sqrt{p\,(p - 4)}}{2} = \left(\frac{x}{2} + \frac{xp}{2\sqrt{p^2 - 4p}} - \frac{x}{\sqrt{p^2 - 4p}}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$\frac{p}{2} - \frac{\sqrt{p\,(p - 4)}}{2} = 0$$

Solving the above for p results in

$$p_1 = 0$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{\frac{p(x)}{2} - \frac{\sqrt{p(x)(p(x)-4)}}{2}}{\frac{x}{2} + \frac{xp(x)}{2\sqrt{p(x)^2-4p(x)}} - \frac{x}{\sqrt{p(x)^2-4p(x)}}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = \frac{\left(p(x) - \sqrt{p(x)\left(p(x)-4\right)}\right)\sqrt{p(x)\left(p(x)-4\right)}}{x\left(\sqrt{p(x)\left(p(x)-4\right)} + p(x) - 2\right)} \tag{2.14}$$

is separable as it can be written as

$$\begin{aligned}
p'(x) &= \frac{\left(p(x) - \sqrt{p(x)\left(p(x)-4\right)}\right)\sqrt{p(x)\left(p(x)-4\right)}}{x\left(\sqrt{p(x)\left(p(x)-4\right)} + p(x) - 2\right)} \\
&= f(x)g(p)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= \frac{1}{x} \\
g(p) &= \frac{\left(p - \sqrt{p\left(p-4\right)}\right)\sqrt{p\left(p-4\right)}}{\sqrt{p\left(p-4\right)} + p - 2}
\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(x)\,dx$$

$$\int \frac{\sqrt{p\left(p-4\right)} + p - 2}{\left(p - \sqrt{p\left(p-4\right)}\right)\sqrt{p\left(p-4\right)}}\,dp = \int \frac{1}{x}\,dx$$

$$\ln\left(\frac{1}{\sqrt{\sqrt{p(x)(p(x)-4)}+p(x)-2}\,\sqrt{p(x)}}\right)+\frac{\sqrt{p(x)(p(x)-4)}}{2}+\frac{p(x)}{2}=\ln(x)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or

$$\frac{\left(p-\sqrt{p(p-4)}\right)\sqrt{p(p-4)}}{\sqrt{p(p-4)}+p-2}=0$$

for $p(x)$ gives

$$p(x)=0$$
$$p(x)=4$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(\frac{1}{\sqrt{\sqrt{p(x)(p(x)-4)}+p(x)-2}\,\sqrt{p(x)}}\right)+\frac{\sqrt{p(x)(p(x)-4)}}{2}+\frac{p(x)}{2}=\ln(x)+c_1$$
$$p(x)=0$$
$$p(x)=4$$

Solving for $p(x)$ gives

$$p(x)=0$$
$$p(x)=4$$
$$p(x)=-\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2}{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1}$$
$$p(x)=-\frac{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2}{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1}$$

Substituing the above solution for p in (2A) gives

$$y = \frac{x\left(\sqrt{\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2\left(\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+2\right)^2}{\left(\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1\right)^2}}\ \operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right) - \operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2 + \cdots}{2\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+2}$$

$$y = \frac{x\left(\sqrt{\frac{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2\left(\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+2\right)^2}{\left(\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1\right)^2}}\ \operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right) - \operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2 \cdots}{2\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+2}$$

Solving ode 2A

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{p}{2} - \frac{\sqrt{p\,(p-4)}}{2} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$\frac{p}{2} + \frac{\sqrt{p\,(p-4)}}{2} = \left(\frac{x}{2} - \frac{xp}{2\sqrt{p^2-4p}} + \frac{x}{\sqrt{p^2-4p}}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$\frac{p}{2} + \frac{\sqrt{p\,(p-4)}}{2} = 0$$

Solving the above for p results in

$$p_1 = 0$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{\frac{p(x)}{2} + \frac{\sqrt{p(x)(p(x)-4)}}{2}}{\frac{x}{2} - \frac{xp(x)}{2\sqrt{p(x)^2-4p(x)}} + \frac{x}{\sqrt{p(x)^2-4p(x)}}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = -\frac{\left(\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)} + p(x)\right)\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)}}{x\left(-\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)} + p\left(x\right) - 2\right)} \tag{2.15}$$

is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{\left(\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)} + p(x)\right)\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)}}{x\left(-\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)} + p\left(x\right) - 2\right)} \\ &= f(x)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(p) &= \frac{\left(p + \sqrt{p\left(p-4\right)}\right)\sqrt{p\left(p-4\right)}}{-\sqrt{p\left(p-4\right)} + p - 2} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\, dp = \int f(x)\, dx$$

$$\int \frac{-\sqrt{p\left(p-4\right)} + p - 2}{\left(p + \sqrt{p\left(p-4\right)}\right)\sqrt{p\left(p-4\right)}}\, dp = \int -\frac{1}{x}\, dx$$

$$\ln\left(\frac{\sqrt{p\left(x\right)}}{\sqrt{\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)} + p\left(x\right) - 2}}\right) + \frac{\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)}}{2} - \frac{p(x)}{2} = \ln\left(\frac{1}{x}\right) + c_2$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$\frac{\left(p+\sqrt{p\left(p-4\right)}\right)\sqrt{p\left(p-4\right)}}{-\sqrt{p\left(p-4\right)}+p-2}=0$$

for $p(x)$ gives

$$\begin{aligned} p(x) &= 0 \\ p(x) &= 4 \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln\left(\frac{\sqrt{p\left(x\right)}}{\sqrt{\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)}+p\left(x\right)-2}}\right)+\frac{\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)}}{2}-\frac{p(x)}{2} &= \ln\left(\frac{1}{x}\right)+c_2 \\ p(x) &= 0 \\ p(x) &= 4 \end{aligned}$$

Substituing the above solution for p in (2A) gives

$$y = x\left(\frac{\left(\operatorname{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}}+4\_Zx^2+4x^2\right)+2\right)^2}{4\operatorname{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}}+4\_Zx^2+4x^2\right)}-\frac{\sqrt{2}\sqrt{\frac{\left(\operatorname{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_}{}}\right.\right.}{}}}{}\right.$$

The solution

$$y = 2x$$

was found not to satisfy the ode or the IC. Hence it is removed.

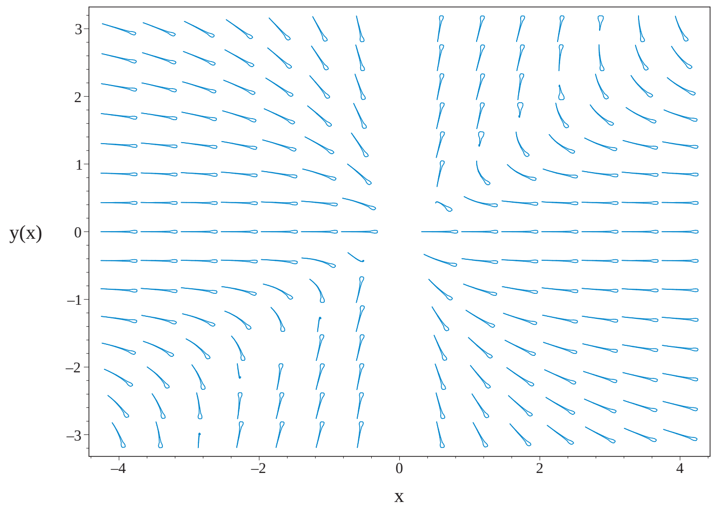


Figure 2.22: Slope field plot
$y^2 = x(y - x)\, y'$

Summary of solutions found

$y = 0$

$$y = x\left(\frac{\left(\operatorname{RootOf}\left(\_Z^2x^2 - 2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}} + 4\_Zx^2 + 4x^2\right) + 2\right)^2}{4\operatorname{RootOf}\left(\_Z^2x^2 - 2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}} + 4\_Zx^2 + 4x^2\right)}\right.$$

$$-\frac{\sqrt{2}\sqrt{\frac{\left(\operatorname{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}}+4\_Zx^2+4x^2\right)+2\right)^2\left(\frac{\left(\operatorname{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}}+4\_Zx^2+4x^2\right)\cdots\right.}{2\operatorname{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}}+4\_Zx^2+4x^2\cdots\right.}\cdots\right.}{\operatorname{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}}+4\_Zx^2+4x^2\right)}\cdots}}{4}$$

$$y = \frac{x\left(\sqrt{\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2\left(\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+2\right)^2}{\left(\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+1\right)^2}}\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right) - \operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2 + \sqrt{\cdots}\right.}{2\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right) + 2}$$

$$y = \frac{x\left(\sqrt{\frac{\operatorname{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2\left(\operatorname{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+2\right)^2}{\left(\operatorname{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+1\right)^2}}\operatorname{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right) - \operatorname{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2 + \sqrt{\frac{\operatorname{Lambert}\cdots}{\cdots}}\right.}{2\operatorname{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right) + 2}$$

Maple step by step solution

Let's solve

$y^2 = x(y-x)\, y'$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $y' = \frac{y^2}{x(y-x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.016` (sec)
Leaf size : 17

```
dsolve(y(x)^2 = x*(-x+y(x))*diff(y(x),x),y(x),singsol=all)
```

$$y(x) = -x \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)$$

Mathematica DSolve solution

Solving time : 2.01 (sec)
Leaf size : 25

```
DSolve[{y[x]^2==x*(y[x]-x)*D[y[x],x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -xW\left(-\frac{e^{-c_1}}{x}\right)$$
$$y(x) \to 0$$

2.3.2 Problem 2

Internal problem ID [18546]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 29. Problems at page 81
Problem number : 2
Date solved : `Tuesday, January 28, 2025 at 11:55:39 AM`
CAS classification : `[[_homogeneous, 'class A'], _rational, _Bernoulli]`

Solve

$$2x^2y + y^3 - x^3y' = 0$$

Solved as first order homogeneous class A ode

Time used: 0.467 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(2x^2+y^2)}{x^3} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y(2x^2 + y^2)$ and $N = x^3$ are both homogeneous and of the same order $n = 3$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x}x + u &= u^3 + 2u \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{u(x)^3 + u(x)}{x} \end{aligned}$$

Or

$$u'(x) - \frac{u(x)^3 + u(x)}{x} = 0$$

Or

$$-u(x)^3 + u'(x)\,x - u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode

$$u'(x) = \frac{u(x)\left(u(x)^2 + 1\right)}{x} \tag{2.16}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)\left(u(x)^2 + 1\right)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= u\left(u^2 + 1\right) \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u\,(u^2 + 1)}\,du &= \int \frac{1}{x}\,dx \end{aligned}$$

$$\ln\left(\frac{u(x)}{\sqrt{u\left(x\right)^{2}+1}}\right)=\ln\left(x\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$u\left(u^2+1\right)=0$$

for $u(x)$ gives

$$\begin{aligned}u(x)&=0\\u(x)&=-i\\u(x)&=i\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(\frac{u(x)}{\sqrt{u\left(x\right)^{2}+1}}\right)=\ln\left(x\right)+c_1$$
$$u(x)=0$$
$$u(x)=-i$$
$$u(x)=i$$

Solving for $u(x)$ gives

$$\begin{aligned}u(x)&=0\\u(x)&=-i\\u(x)&=i\\u(x)&=\frac{\mathrm{e}^{c_1}x}{\sqrt{1-x^2\mathrm{e}^{2c_1}}}\\u(x)&=-\frac{\mathrm{e}^{c_1}x}{\sqrt{1-x^2\mathrm{e}^{2c_1}}}\end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = -i$ back to y gives

$$y = -ix$$

Converting $u(x) = i$ back to y gives

$$y = ix$$

Converting $u(x) = \frac{e^{c_1}x}{\sqrt{1-x^2 e^{2c_1}}}$ back to y gives

$$y = \frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

Converting $u(x) = -\frac{e^{c_1}x}{\sqrt{1-x^2 e^{2c_1}}}$ back to y gives

$$y = -\frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

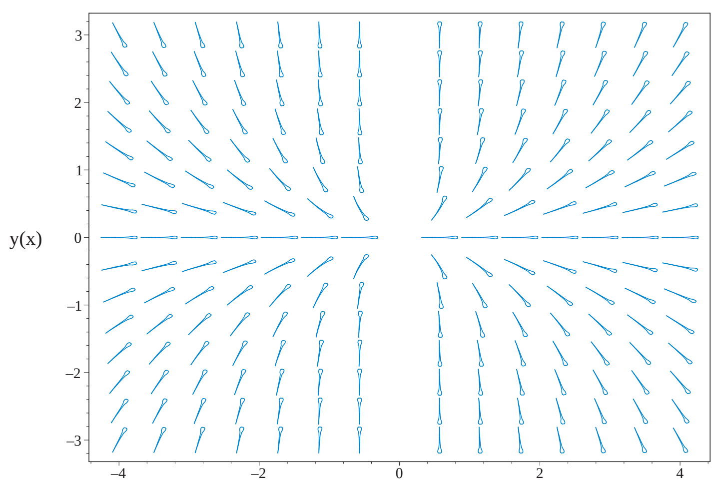


Figure 2.23: Slope field plot
$2x^2y + y^3 - x^3y' = 0$

Summary of solutions found

$$y = 0$$
$$y = -ix$$
$$y = ix$$
$$y = \frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$
$$y = -\frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

Solved as first order homogeneous class D2 ode

Time used: 0.142 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$2x^3 u(x) + u(x)^3 x^3 - x^3(u'(x)\,x + u(x)) = 0$$

Which is now solved The ode

$$u'(x) = \frac{u(x)\left(u(x)^2 + 1\right)}{x} \tag{2.17}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)\left(u(x)^2 + 1\right)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= u\left(u^2 + 1\right) \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{1}{u\,(u^2 + 1)}\,du = \int \frac{1}{x}\,dx$$

$$\ln\left(\frac{u(x)}{\sqrt{u\left(x\right)^{2}+1}}\right)=\ln\left(x\right)+c_{1}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$u\left(u^{2}+1\right)=0$$

for $u(x)$ gives

$$\begin{aligned}u(x)&=0\\u(x)&=-i\\u(x)&=i\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(\frac{u(x)}{\sqrt{u\left(x\right)^{2}+1}}\right)=\ln\left(x\right)+c_{1}$$
$$u(x)=0$$
$$u(x)=-i$$
$$u(x)=i$$

Solving for $u(x)$ gives

$$u(x)=0$$
$$u(x)=-i$$
$$u(x)=i$$
$$u(x)=\frac{\mathrm{e}^{c_{1}}x}{\sqrt{1-x^{2}\mathrm{e}^{2c_{1}}}}$$
$$u(x)=-\frac{\mathrm{e}^{c_{1}}x}{\sqrt{1-x^{2}\mathrm{e}^{2c_{1}}}}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = -i$ back to y gives

$$y = -ix$$

Converting $u(x) = i$ back to y gives

$$y = ix$$

Converting $u(x) = \frac{e^{c_1} x}{\sqrt{1-x^2 e^{2c_1}}}$ back to y gives

$$y = \frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

Converting $u(x) = -\frac{e^{c_1} x}{\sqrt{1-x^2 e^{2c_1}}}$ back to y gives

$$y = -\frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

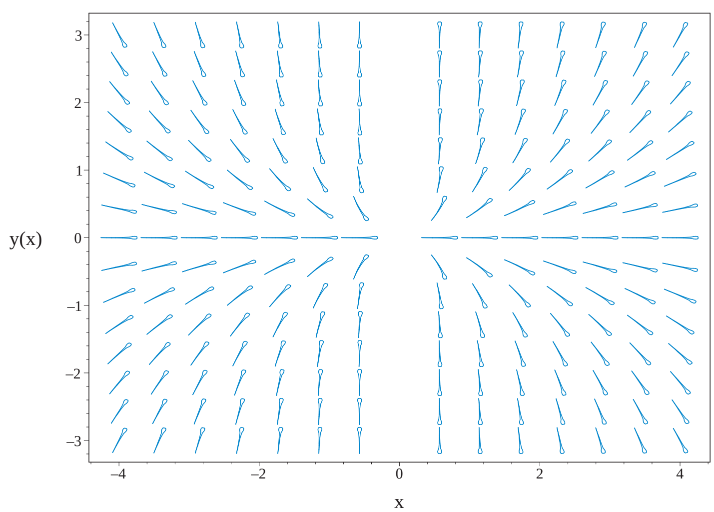


Figure 2.24: Slope field plot
$2x^2y + y^3 - x^3y' = 0$

Summary of solutions found

$$y = 0$$
$$y = -ix$$
$$y = ix$$
$$y = \frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$
$$y = -\frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

Solved as first order homogeneous class Maple C ode

Time used: 0.546 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{(Y(X) + y_0)\left((Y(X) + y_0)^2 + 2(x_0 + X)^2\right)}{(x_0 + X)^3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X)\,X^2 + Y(X)^3}{X^3}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{Y(2X^2 + Y^2)}{X^3} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = Y(2X^2 + Y^2)$ and $N = X^3$ are both homogeneous and of the same order $n = 3$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= u^3 + 2u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{u(X)^3 + u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{u(X)^3 + u(X)}{X} = 0$$

Or

$$-u(X)^3 + \left(\frac{d}{dX}u(X)\right)X - u(X) = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = \frac{u(X)\left(u(X)^2 + 1\right)}{X} \tag{2.18}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= \frac{u(X)\left(u(X)^2 + 1\right)}{X} \\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= \frac{1}{X} \\ g(u) &= u\left(u^2 + 1\right)\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(X)\,dX \\ \int \frac{1}{u\left(u^2 + 1\right)}\,du &= \int \frac{1}{X}\,dX\end{aligned}$$

$$\ln\left(\frac{u(X)}{\sqrt{u(X)^2+1}}\right) = \ln(X) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$u(u^2+1) = 0$$

for $u(X)$ gives

$$u(X) = 0$$
$$u(X) = -i$$
$$u(X) = i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(\frac{u(X)}{\sqrt{u(X)^2+1}}\right) = \ln(X) + c_1$$
$$u(X) = 0$$
$$u(X) = -i$$
$$u(X) = i$$

Solving for $u(X)$ gives

$$u(X) = 0$$
$$u(X) = -i$$
$$u(X) = i$$
$$u(X) = \frac{e^{c_1} X}{\sqrt{1 - X^2 e^{2c_1}}}$$
$$u(X) = -\frac{e^{c_1} X}{\sqrt{1 - X^2 e^{2c_1}}}$$

Converting $u(X) = 0$ back to $Y(X)$ gives

$$Y(X) = 0$$

Converting $u(X) = -i$ back to $Y(X)$ gives

$$Y(X) = -iX$$

Converting $u(X) = i$ back to $Y(X)$ gives

$$Y(X) = iX$$

Converting $u(X) = \frac{e^{c_1} X}{\sqrt{1-X^2 e^{2c_1}}}$ back to $Y(X)$ gives

$$Y(X) = \frac{X^2 e^{c_1}}{\sqrt{1 - X^2 e^{2c_1}}}$$

Converting $u(X) = -\frac{e^{c_1} X}{\sqrt{1-X^2 e^{2c_1}}}$ back to $Y(X)$ gives

$$Y(X) = -\frac{X^2 e^{c_1}}{\sqrt{1 - X^2 e^{2c_1}}}$$

Using the solution for $Y(X)$

$$Y(X) = 0 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = 0$$

Using the solution for $Y(X)$

$$Y(X) = -iX \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$y = -ix$$

Using the solution for $Y(X)$

$$Y(X) = iX \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$y = ix$$

Using the solution for $Y(X)$

$$Y(X) = \frac{X^2 e^{c_1}}{\sqrt{1 - X^2 e^{2c_1}}} \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$y = \frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

Using the solution for $Y(X)$

$$Y(X) = -\frac{X^2 e^{c_1}}{\sqrt{1 - X^2 e^{2c_1}}} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = -\frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

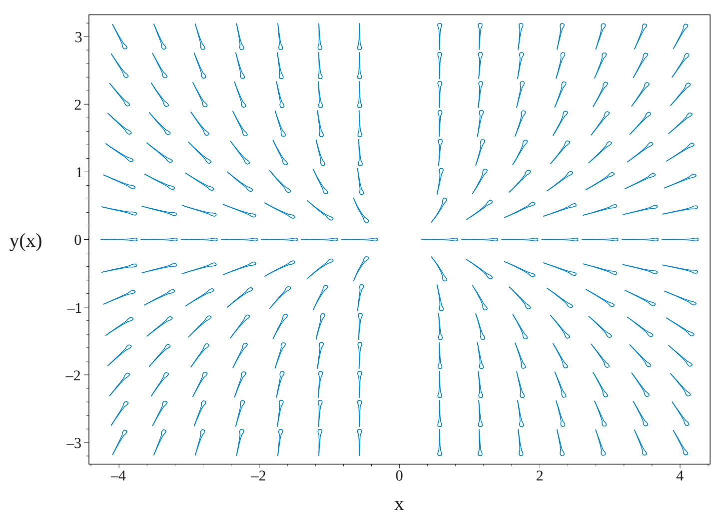


Figure 2.25: Slope field plot
$2x^2y + y^3 - x^3y' = 0$

Solved as first order Bernoulli ode

Time used: 0.179 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(2x^2 + y^2)}{x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \left(\frac{2}{x}\right) y + \left(\frac{1}{x^3}\right) y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= \frac{2}{x} \\ f_1 &= \frac{1}{x^3} \end{aligned}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{2}{x} \\ f_1(x) &= \frac{1}{x^3} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = \frac{2}{x\,y^2} + \frac{1}{x^3} \tag{4}$$

Let

$$
\begin{aligned}
v &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
v' = -\frac{2}{y^3}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{v'(x)}{2} &= \frac{2v(x)}{x} + \frac{1}{x^3} \\
v' &= -\frac{4v}{x} - \frac{2}{x^3}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $v(x)$ which is now solved.

In canonical form a linear first order is

$$
v'(x) + q(x)v(x) = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{4}{x} \\
p(x) &= -\frac{2}{x^3}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int \frac{4}{x}dx} \\
&= x^4
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= (\mu)\left(-\frac{2}{x^3}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(v\,x^4\right) &= \left(x^4\right)\left(-\frac{2}{x^3}\right) \\
\mathrm{d}\left(v\,x^4\right) &= (-2x)\;\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$\begin{aligned} v\,x^4 &= \int -2x\,dx \\ &= -x^2 + c_1 \end{aligned}$$

Dividing throughout by the integrating factor x^4 gives the final solution

$$v(x) = \frac{-x^2 + c_1}{x^4}$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$\frac{1}{y^2} = \frac{-x^2 + c_1}{x^4}$$

Solving for y gives

$$y = \frac{x^2}{\sqrt{-x^2 + c_1}}$$

$$y = -\frac{x^2}{\sqrt{-x^2 + c_1}}$$

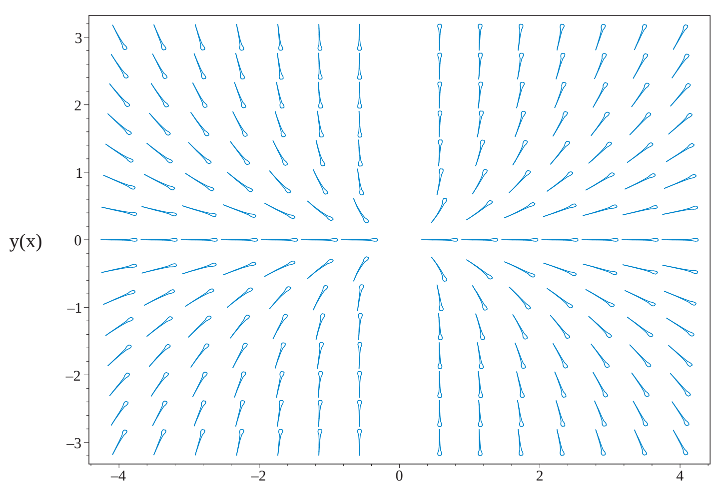


Figure 2.26: Slope field plot
$2x^2y + y^3 - x^3y' = 0$

Summary of solutions found

$$y = \frac{x^2}{\sqrt{-x^2 + c_1}}$$
$$y = -\frac{x^2}{\sqrt{-x^2 + c_1}}$$

Solved as first order isobaric ode

Time used: 0.234 (sec)

Solving for y' gives

$$y' = \frac{y(y^2 + 2x^2)}{x^3} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{y(y^2 + 2x^2)}{x^3} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = \frac{u(x)\left(x^2 u(x)^2 + 2x^2\right)}{x^2}$$

The ode

$$u'(x) = \frac{u(x)\left(u(x)^2 + 1\right)}{x} \tag{2.19}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)\left(u(x)^2 + 1\right)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$f(x)=\frac{1}{x}$$
$$g(u)=u(u^2+1)$$

Integrating gives

$$\int \frac{1}{g(u)}\,du=\int f(x)\,dx$$
$$\int \frac{1}{u\,(u^2+1)}\,du=\int \frac{1}{x}\,dx$$

$$\ln\left(\frac{u(x)}{\sqrt{u\,(x)^2+1}}\right)=\ln\,(x)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$u(u^2+1)=0$$

for $u(x)$ gives

$$u(x)=0$$
$$u(x)=-i$$
$$u(x)=i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(\frac{u(x)}{\sqrt{u\,(x)^2+1}}\right)=\ln\,(x)+c_1$$
$$u(x)=0$$
$$u(x)=-i$$
$$u(x)=i$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = -i$$
$$u(x) = i$$
$$u(x) = \frac{e^{c_1} x}{\sqrt{1 - x^2 e^{2c_1}}}$$
$$u(x) = -\frac{e^{c_1} x}{\sqrt{1 - x^2 e^{2c_1}}}$$

Converting $u(x) = 0$ back to y gives

$$\frac{y}{x} = 0$$

Converting $u(x) = -i$ back to y gives

$$\frac{y}{x} = -i$$

Converting $u(x) = i$ back to y gives

$$\frac{y}{x} = i$$

Converting $u(x) = \frac{e^{c_1} x}{\sqrt{1-x^2 e^{2c_1}}}$ back to y gives

$$\frac{y}{x} = \frac{e^{c_1} x}{\sqrt{1 - x^2 e^{2c_1}}}$$

Converting $u(x) = -\frac{e^{c_1} x}{\sqrt{1-x^2 e^{2c_1}}}$ back to y gives

$$\frac{y}{x} = -\frac{e^{c_1} x}{\sqrt{1 - x^2 e^{2c_1}}}$$

Solving for y gives

$$y = 0$$
$$y = -ix$$
$$y = ix$$
$$y = \frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$
$$y = -\frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

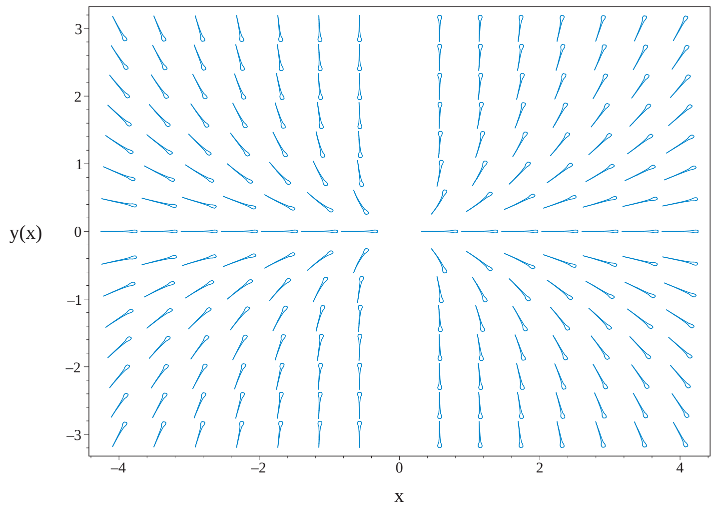


Figure 2.27: Slope field plot
$2x^2y + y^3 - x^3y' = 0$

Summary of solutions found

$$y = 0$$
$$y = -ix$$
$$y = ix$$
$$y = \frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$
$$y = -\frac{x^2 e^{c_1}}{\sqrt{1 - x^2 e^{2c_1}}}$$

Solved using Lie symmetry for first order ode

Time used: 0.923 (sec)

Writing the ode as

$$y' = \frac{y(2x^2 + y^2)}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{y(2x^2+y^2)(b_3-a_2)}{x^3} - \frac{y^2(2x^2+y^2)^2 a_3}{x^6} \\ &- \left(\frac{4y}{x^2} - \frac{3y(2x^2+y^2)}{x^4}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{2x^2+y^2}{x^3} + \frac{2y^2}{x^3}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{b_2x^6 + 2x^4y^2a_3 + 3x^4y^2b_2 - 2x^3y^3a_2 + 2x^3y^3b_3 + x^2y^4a_3 + y^6a_3 + 2x^5b_1 - 2x^4ya_1 + 3x^3y^2b_1 - 3x^2y^3a_1}{x^6} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-b_2x^6 - 2x^4y^2a_3 - 3x^4y^2b_2 + 2x^3y^3a_2 - 2x^3y^3b_3 - x^2y^4a_3 \\ &- y^6a_3 - 2x^5b_1 + 2x^4ya_1 - 3x^3y^2b_1 + 3x^2y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
2a_2v_1^3v_2^3 - 2a_3v_1^4v_2^2 - a_3v_1^2v_2^4 - a_3v_2^6 - b_2v_1^6 - 3b_2v_1^4v_2^2 \\
- 2b_3v_1^3v_2^3 + 2a_1v_1^4v_2 + 3a_1v_1^2v_2^3 - 2b_1v_1^5 - 3b_1v_1^3v_2^2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
-b_2v_1^6 - 2b_1v_1^5 + (-2a_3 - 3b_2)\, v_1^4v_2^2 + 2a_1v_1^4v_2 \\
+ (2a_2 - 2b_3)\, v_1^3v_2^3 - 3b_1v_1^3v_2^2 - a_3v_1^2v_2^4 + 3a_1v_1^2v_2^3 - a_3v_2^6 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 &= 0 \\
3a_1 &= 0 \\
-a_3 &= 0 \\
-3b_1 &= 0 \\
-2b_1 &= 0 \\
-b_2 &= 0 \\
2a_2 - 2b_3 &= 0 \\
-2a_3 - 3b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y(2x^2+y^2)}{x^3}\right)(x) \\
&= \frac{-x^2y - y^3}{x^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2y-y^3}{x^2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2+y^2)}{2} - \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2x^2+y^2)}{x^3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{x^2+y^2} \\
S_y &= -\frac{x^2}{y\,(x^2+y^2)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

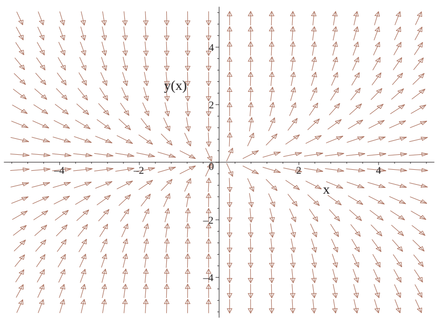
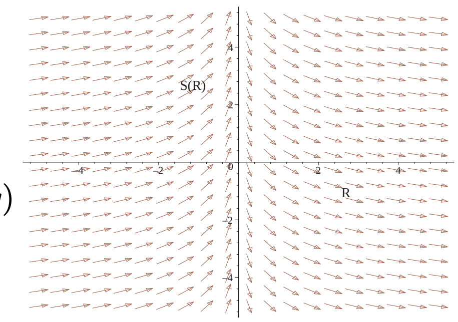
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -\frac{1}{R}\,dR \\
S(R) &= -\ln(R) + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
\frac{\ln(x^2+y^2)}{2} - \ln(y) = -\ln(x) + c_2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x^2+y^2)}{x^3}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \ln(y)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Solving for y gives

$$y = \frac{x^2}{\sqrt{e^{2c_2} - x^2}}$$

$$y = -\frac{x^2}{\sqrt{e^{2c_2} - x^2}}$$

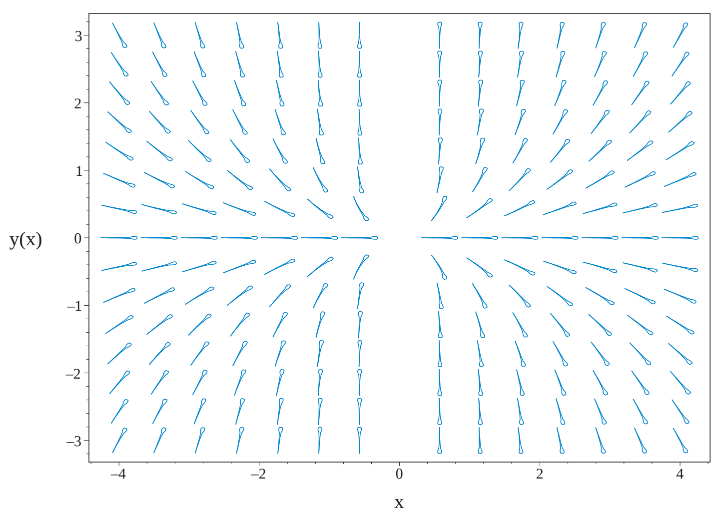


Figure 2.28: Slope field plot
$2x^2y + y^3 - x^3y' = 0$

Summary of solutions found

$$y = \frac{x^2}{\sqrt{e^{2c_2} - x^2}}$$

$$y = -\frac{x^2}{\sqrt{e^{2c_2} - x^2}}$$

Solved as first order ode of type dAlembert

Time used: 84.512 (sec)

Let $p = y'$ the ode becomes

$$-x^3 p + 2x^2 y + y^3 = 0$$

Solving for y from the above results in

$$y = \left(\frac{\left(108p + 12\sqrt{81p^2 + 96}\right)^{1/3}}{6} - \frac{4}{\left(108p + 12\sqrt{81p^2 + 96}\right)^{1/3}}\right) x \tag{1}$$

$$y = \left(-\frac{\left(108p + 12\sqrt{81p^2 + 96}\right)^{1/3}}{12} + \frac{2}{\left(108p + 12\sqrt{81p^2 + 96}\right)^{1/3}} + \frac{i\sqrt{3}\left(\frac{\left(108p+12\sqrt{81p^2+96}\right)^{1/3}}{6} + \frac{4}{\left(108p+12\sqrt{81p^2+96}\right)^{1/3}}\right)}{2}\right) x \tag{2}$$

$$y = \left(-\frac{\left(108p + 12\sqrt{81p^2 + 96}\right)^{1/3}}{12} + \frac{2}{\left(108p + 12\sqrt{81p^2 + 96}\right)^{1/3}} - \frac{i\sqrt{3}\left(\frac{\left(108p+12\sqrt{81p^2+96}\right)^{1/3}}{6} + \frac{4}{\left(108p+12\sqrt{81p^2+96}\right)^{1/3}}\right)}{2}\right) x \tag{3}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved.

Solving ode 1A

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{\left(108p + 12\sqrt{81p^2 + 96}\right)^{2/3} - 24}{6\left(108p + 12\sqrt{81p^2 + 96}\right)^{1/3}} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{\left(108p + 12\sqrt{81p^2 + 96}\right)^{2/3} - 24}{6\left(108p + 12\sqrt{81p^2 + 96}\right)^{1/3}} = \left(\frac{6x}{\left(108p + 12\sqrt{81p^2 + 96}\right)^{2/3}} + \frac{54xp}{\left(108p + 12\sqrt{81p^2 + 96}\right)^{2/3}\sqrt{81p^2 + 9} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{\left(108p + 12\sqrt{81p^2 + 96}\right)^{2/3} - 24}{6\left(108p + 12\sqrt{81p^2 + 96}\right)^{1/3}} = 0$$

Solving the above for p results in

$$p_1 = 0$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}-24}{6\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{1/3}}}{\frac{6x}{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}} + \frac{54xp(x)}{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}\sqrt{81p(x)^2+96}} + \frac{144x}{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{4/3}} + \frac{}{\left(108p(x)+12\right.}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = -\frac{\sqrt{81p(x)^2+96}\left(\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3} - 6p(x)\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{1/3} - 2\right.}{3\left(\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}+24\right)x} \tag{2.20}$$

is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{\sqrt{81p(x)^2+96}\left(\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3} - 6p(x)\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{1/\right.}}{3\left(\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}+24\right)x} \\ &= f(x)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{3x} \\ g(p) &= \frac{\sqrt{81p^2+96}\left(\left(108p+12\sqrt{81p^2+96}\right)^{2/3} - 6p\left(108p+12\sqrt{81p^2+96}\right)^{1/3} - 24\right)}{\left(108p+12\sqrt{81p^2+96}\right)^{2/3}+24} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(x)\,dx$$

$$\int \frac{\left(108p+12\sqrt{81p^2+96}\right)^{2/3}+24}{\sqrt{81p^2+96}\left(\left(108p+12\sqrt{81p^2+96}\right)^{2/3} - 6p\left(108p+12\sqrt{81p^2+96}\right)^{1/3} - 24\right)}\,dp = \int -\frac{1}{3x}\,dx$$

$$\int^{p(x)} \frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+24}{\sqrt{81\tau^2+96}\,\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}-6\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}-24\right)} d\tau = \ln\left(\frac{1}{x^{1/3}}\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or

$$\frac{\sqrt{81p^2+96}\left(\left(108p+12\sqrt{81p^2+96}\right)^{2/3}-6p\left(108p+12\sqrt{81p^2+96}\right)^{1/3}-24\right)}{\left(108p+12\sqrt{81p^2+96}\right)^{2/3}+24}=0$$

for $p(x)$ gives

$$p(x)=0$$
$$p(x)=-\frac{4i\sqrt{6}}{9}$$
$$p(x)=\frac{4i\sqrt{6}}{9}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{p(x)} \frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+24}{\sqrt{81\tau^2+96}\,\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}-6\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}-24\right)} d\tau = \ln\left(\frac{1}{x^{1/3}}\right)+c_1$$

$$p(x)=0$$

$$p(x)=-\frac{4i\sqrt{6}}{9}$$

$$p(x)=\frac{4i\sqrt{6}}{9}$$

Substituing the above solution for p in (2A) gives

$$y = \frac{x\left(\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z} -\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+24}{\sqrt{81\tau^2+96}\left(-\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+6\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}+24\right)}d\tau\right) - \ln(x) \cdots}{6\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z} -\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+24}{\sqrt{81\tau^2+96}\left(-\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+6\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}+24\right)}d\tau\right) - \ln(\cdots}$$

Solving ode 2A

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{\left(i\sqrt{3}-1\right)\left(108p+12\sqrt{81p^2+96}\right)^{2/3}+24i\sqrt{3}+24}{12\left(108p+12\sqrt{81p^2+96}\right)^{1/3}} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{\left(i\sqrt{3}-1\right)\left(108p+12\sqrt{81p^2+96}\right)^{2/3}+24i\sqrt{3}+24}{12\left(108p+12\sqrt{81p^2+96}\right)^{1/3}} = \left(\frac{3ix\sqrt{3}}{\left(108p+12\sqrt{81p^2+96}\right)^{2/3}} + \frac{27i\cdots}{\left(108p+12\sqrt{81p^2\cdots}\right.}\right. \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{\left(i\sqrt{3}-1\right)\left(108p+12\sqrt{81p^2+96}\right)^{2/3}+24i\sqrt{3}+24}{12\left(108p+12\sqrt{81p^2+96}\right)^{1/3}} = 0$$

Solving the above for p results in

$$p_1 = \frac{i\left(3i\sqrt{3}-3+3\sqrt{30+30i\sqrt{3}}\right)\sqrt{3}+21i\sqrt{3}+27-3\sqrt{30+30i\sqrt{3}}}{12\sqrt{3i\sqrt{3}-3+3\sqrt{30+30i\sqrt{3}}}}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = ix$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) \tag{3}$$

$$= \frac{}{\frac{3ix\sqrt{3}}{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}} + \frac{27ix\sqrt{3}\,p(x)}{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}\sqrt{81p(x)^2+96}} - \frac{3x}{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}} - \frac{}{\left(108p(x)+12\right.}}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = \tag{2.21}$$

$$-\frac{\sqrt{81p(x)^2+96}\left(\sqrt{3}\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}+24\sqrt{3}+12ip(x)\left(108p(x)+12\sqrt{81p(x)^2\right.\right.}{3\left(\sqrt{3}\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}-24\sqrt{3}+i\left(108p(x)+12\sqrt{\right.\right.}}$$

is separable as it can be written as

$$p'(x) = -\frac{\sqrt{81p(x)^2+96}\left(\sqrt{3}\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}+24\sqrt{3}+12ip(x)\left(108p(x)+12\sqrt{81\right.\right.}{3\left(\sqrt{3}\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}-24\sqrt{3}+i\left(108p(x)+\right.\right.}$$

$$= f(x)g(p)$$

Where

$$f(x) = -\frac{1}{3x}$$

$$g(p) = \frac{\sqrt{81p^2+96}\left(\left(108p+12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}+12ip\left(108p+12\sqrt{81p^2+96}\right)^{1/3}+i(108p+\right.}{\left(108p+12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}+i\left(108p+12\sqrt{81p^2+96}\right)^{2/3}+2}$$

Integrating gives

$$\int \frac{1}{g(p)}\, dp = \int f(x)\, dx$$

$$\int \frac{\left(108p + 12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3} - 24\sqrt{3} + i\left(108p + 12\sqrt{81p^2+96}\right)^{2/3} + 24i}{\sqrt{81p^2+96}\,\left(\left(108p + 12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3} + 24\sqrt{3} + 12ip\left(108p + 12\sqrt{81p^2+96}\right)^{1/3} + i\left(108p + 12\right.\right.}$$

$$\int^{p(x)} \frac{\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3} - 24\sqrt{3} + i\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{2/3} + 24}{\sqrt{81\tau^2+96}\,\left(\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3} + 24\sqrt{3} + 12i\tau\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{1/3} + i\left(108\tau +\right.\right.}$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$\frac{\sqrt{81p^2+96}\,\left(\left(108p + 12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3} + 24\sqrt{3} + 12ip\left(108p + 12\sqrt{81p^2+96}\right)^{1/3} + i\left(108p + 12\sqrt{81\right.\right.}{\left(108p + 12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3} - 24\sqrt{3} + i\left(108p + 12\sqrt{81p^2+96}\right)^{2/3} + 24i}$$

for $p(x)$ gives

$$p(x) = -\frac{4i\sqrt{6}}{9}$$
$$p(x) = \frac{4i\sqrt{6}}{9}$$
$$p(x) = \frac{i\left(3i\sqrt{3} - 3 + 3\sqrt{30 + 30i\sqrt{3}}\right)\sqrt{3} + 21i\sqrt{3} + 27 - 3\sqrt{30 + 30i\sqrt{3}}}{12\sqrt{3i\sqrt{3} - 3 + 3\sqrt{30 + 30i\sqrt{3}}}}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{p(x)} \frac{\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3} - 24\sqrt{3} + i\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{2/3} + 24}{\sqrt{81\tau^2+96}\,\left(\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3} + 24\sqrt{3} + 12i\tau\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{1/3} + i\left(108\tau +\right.\right.}$$

$$p(x) = -\frac{4i\sqrt{6}}{9}$$

$$p(x) = \frac{4i\sqrt{6}}{9}$$

$$p(x) = \frac{i\left(3i\sqrt{3}-3+3\sqrt{30+30i\sqrt{3}}\right)\sqrt{3}+21i\sqrt{3}+27-3\sqrt{30+30i\sqrt{3}}}{12\sqrt{3i\sqrt{3}-3+3\sqrt{30+30i\sqrt{3}}}}$$

Substituing the above solution for p in (2A) gives

$$y = \frac{x\left(\left(i\sqrt{3}-1\right)\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z}\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}+i\left(108\tau+12\sqrt{81\tau^2+96}\right.}{\sqrt{81\tau^2+96}\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}+12i\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}+i\right.}\right.\right.\right.\right.\right.}{12\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z}\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}+i\left(108\tau+12\sqrt{81\tau^2+\right.}{\sqrt{81\tau^2+96}\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}+12i\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/\right.}\right.\right.\right.}$$

Solving ode 3A

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f+(xf'+g')\frac{dp}{dx} \\ p-f &= (xf'+g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf+g$ to (1A) shows that

$$f = -\frac{\left(1+i\sqrt{3}\right)\left(108p+12\sqrt{81p^2+96}\right)^{2/3}+24i\sqrt{3}-24}{12\left(108p+12\sqrt{81p^2+96}\right)^{1/3}}$$

$$g = 0$$

Hence (2) becomes

$$p + \frac{\left(1+i\sqrt{3}\right)\left(108p+12\sqrt{81p^2+96}\right)^{2/3}+24i\sqrt{3}-24}{12\left(108p+12\sqrt{81p^2+96}\right)^{1/3}} = \left(-\frac{3ix\sqrt{3}}{\left(108p+12\sqrt{81p^2+96}\right)^{2/3}}-\frac{27\ldots}{\left(108p+12\sqrt{81p\ldots\right.}\right. \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{(1+i\sqrt{3})\left(108p + 12\sqrt{81p^2+96}\right)^{2/3} + 24i\sqrt{3} - 24}{12\left(108p + 12\sqrt{81p^2+96}\right)^{1/3}} = 0$$

Solving the above for p results in

$$p_1 = -\frac{i\left(-3i\sqrt{3} - 3 + 3\sqrt{30 - 30i\sqrt{3}}\right)\sqrt{3} + 21i\sqrt{3} - 27 + 3\sqrt{30 - 30i\sqrt{3}}}{12\sqrt{-3i\sqrt{3} - 3 + 3\sqrt{30 - 30i\sqrt{3}}}}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = -ix$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) \tag{3}$$

$$= \frac{}{-\frac{3ix\sqrt{3}}{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}} - \frac{27ix\sqrt{3}\,p(x)}{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{2/3}\sqrt{81p(x)^2+96}} + \frac{72ix\sqrt{3}}{\left(108p(x)+12\sqrt{81p(x)^2+96}\right)^{4/3}} + \frac{}{\left(108p(x)+\cdots\right.}}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = -\frac{\left(\sqrt{3}\left(108p(x) + 12\sqrt{81p\left(x\right)^2+96}\right)^{2/3} + 24\sqrt{3} - 12ip(x)\left(108p(x) + 12\sqrt{81p\left(x\right)^2+96}\right)^{1/3} - i\left(\cdots\right.\right.}{3x\left(\sqrt{3}\left(108p\left(x\right) + 12\sqrt{81p\left(x\right)^2+96}\right)^{2/3} - 24\sqrt{3} - i\left(108p\left(x\right) + 12\cdots\right.\right.} \tag{2.22}$$

is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{\left(\sqrt{3}\left(108p(x) + 12\sqrt{81p\left(x\right)^2+96}\right)^{2/3} + 24\sqrt{3} - 12ip(x)\left(108p(x) + 12\sqrt{81p\left(x\right)^2+96}\right)^{1/3}\cdots\right.}{3x\left(\sqrt{3}\left(108p\left(x\right) + 12\sqrt{81p\left(x\right)^2+96}\right)^{2/3} - 24\sqrt{3} - i\left(108p\left(x\right)\cdots\right.\right.} \\ &= f(x)g(p) \end{aligned}$$

Where

$$f(x) = -\frac{1}{3x}$$

$$g(p) = \frac{\left(\left(108p+12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}-12ip\left(108p+12\sqrt{81p^2+96}\right)^{1/3}-i\left(108p+12\sqrt{81p^2+\right.}{\left(108p+12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}-i\left(108p+12\sqrt{81p^2+96}\right)^{2/3}-2}$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(x)\,dx$$

$$\int \frac{\left(108p+12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}-i\left(108p+12\sqrt{81p^2+96}\right)^{2/3}-24i}{\left(\left(108p+12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}-12ip\left(108p+12\sqrt{81p^2+96}\right)^{1/3}-i\left(108p+12\sqrt{81p^2+96}\right)^{2}\right.}$$

$$\int^{p(x)} \frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}-i\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}-24}{\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}-12i\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}-i\left(108\tau+12\sqrt{81\tau^2+\right.}$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$\frac{\left(\left(108p+12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}-12ip\left(108p+12\sqrt{81p^2+96}\right)^{1/3}-i\left(108p+12\sqrt{81p^2+96}\right)^{2/3}-\right.}{\left(108p+12\sqrt{81p^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}-i\left(108p+12\sqrt{81p^2+96}\right)^{2/3}-24i}$$

for $p(x)$ gives

$$p(x) = -\frac{4i\sqrt{6}}{9}$$
$$p(x) = \frac{4i\sqrt{6}}{9}$$
$$p(x) = -\frac{i\left(-3i\sqrt{3}-3+3\sqrt{30-30i\sqrt{3}}\right)\sqrt{3}+21i\sqrt{3}-27+3\sqrt{30-30i\sqrt{3}}}{12\sqrt{-3i\sqrt{3}-3+3\sqrt{30-30i\sqrt{3}}}}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{p(x)} \frac{\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3} - 24\sqrt{3} - i\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{2/3} - 24 \cdots}{\left(\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3} + 24\sqrt{3} - 12i\tau\left(108\tau + 12\sqrt{81\tau^2+96}\right)^{1/3} - i\left(108\tau + 12\sqrt{81\tau^2+ \cdots\right.\right.}$$

$$p(x) = -\frac{4i\sqrt{6}}{9}$$

$$p(x) = \frac{4i\sqrt{6}}{9}$$

$$p(x) = -\frac{i\left(-3i\sqrt{3} - 3 + 3\sqrt{30 - 30i\sqrt{3}}\right)\sqrt{3} + 21i\sqrt{3} - 27 + 3\sqrt{30 - 30i\sqrt{3}}}{12\sqrt{-3i\sqrt{3} - 3 + 3\sqrt{30 - 30i\sqrt{3}}}}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{x\left(\left(1+i\sqrt{3}\right)\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z} \frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}-i\left(108\tau+12\sqrt{81\tau^2+ \cdots\right.\right.}{\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}-12i\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}-i\left(108\tau+1 \cdots\right.\right.} \cdots\right.\right.\right.\right.\right.}{12\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z} \frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}-i\left(108\tau+12\sqrt{81\tau \cdots\right.\right.}{\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}-12i\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}-i\left(108 \cdots\right.\right.} \cdots\right.\right.\right.\right.}$$

The solution

$$y = -\frac{2i\sqrt{6}\,x}{3}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{i\sqrt{6}\,x}{3}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = \frac{i\sqrt{6}\,x}{3}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{i\sqrt{3}\,\sqrt{2}\,x}{3}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = \frac{2i\sqrt{3}\,\sqrt{2}\,x}{3}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x\left(\left(1+i\sqrt{3}\right)\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z}\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}-i\left(108\tau+12\sqrt{81\tau^2+96}\right.}{\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}-12i\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}-i\left(108\tau+12\right.\right.}\right.\right.\right.\right.}{12\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z}\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}-i\left(108\tau+12\sqrt{81\tau^2+\right.}{\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}-12i\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}-i\left(108\tau+\right.\right.}\right.\right.\right.}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = \frac{x\left(\left(i\sqrt{3}-1\right)\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z}\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}+i\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/}}{\sqrt{81\tau^2+96}\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}+12i\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}+i\left(10\right.\right.}\right.\right.\right.\right.}{12\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z}\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}-24\sqrt{3}+i\left(108\tau+12\sqrt{81\tau^2+96}\right.}{\sqrt{81\tau^2+96}\left(\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}\sqrt{3}+24\sqrt{3}+12i\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}+\right.}\right.\right.\right.}$$

was found not to satisfy the ode or the IC. Hence it is removed.

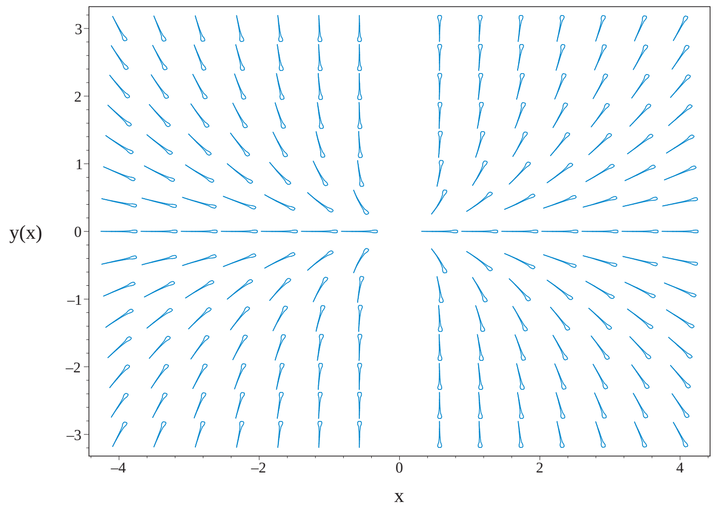


Figure 2.29: Slope field plot
$2x^2y + y^3 - x^3y' = 0$

Summary of solutions found

$$y = 0$$

$$y = -ix$$

$$y = ix$$

$$y = \frac{x\left(\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z} -\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+24}{\sqrt{81\tau^2+96}\left(-\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+6\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}+24\right)}d\tau\right) - \ln(x) + 3\right.\right.\right.}{6\left(108\,\mathrm{RootOf}\left(-3\left(\int^{-Z} -\frac{\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+24}{\sqrt{81\tau^2+96}\left(-\left(108\tau+12\sqrt{81\tau^2+96}\right)^{2/3}+6\tau\left(108\tau+12\sqrt{81\tau^2+96}\right)^{1/3}+24\right)}d\tau\right) - \ln(x)\right.\right.}$$

Maple step by step solution

Let's solve
$2x^2y + y^3 - x^3y' = 0$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = -\frac{-y^3-2x^2y}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 34

```
dsolve(2*x^2*y(x)+y(x)^3-x^3*diff(y(x),x) = 0,y(x),singsol=all)
```

$$y(x) = \frac{x^2}{\sqrt{-x^2+c_1}}$$
$$y(x) = -\frac{x^2}{\sqrt{-x^2+c_1}}$$

Mathematica DSolve solution

Solving time : 0.173 (sec)
Leaf size : 47

```
DSolve[{2*x^2*y[x]+y[x]^3-x^3*D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{x^2}{\sqrt{-x^2+c_1}}$$
$$y(x) \to \frac{x^2}{\sqrt{-x^2+c_1}}$$
$$y(x) \to 0$$

2.3.3 Problem 3

Internal problem ID [18547]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 29. Problems at page 81
Problem number : 3
Date solved : `Tuesday, January 28, 2025 at 11:57:08 AM`
CAS classification :
`[[_homogeneous, ‘class C‘], _exact, _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

Solve

$$2ax + by + (2cy + bx + e)\, y' = g$$

Solved as first order polynomial type ode

Time used: 0.962 (sec)

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_3}$$

Where $a_1 = -2a, b_1 = -b, c_1 = g, a_2 = b, b_2 = 2c, c_2 = e$. There are now two possible solution methods. The first case is when the two lines $a_1 x + b_1 y + c_1$,$a_2 x + b_2 y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1 x + b_1 y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$\begin{aligned} X &= x - x_0 \\ Y &= y - y_0 \end{aligned}$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$\begin{aligned} a_1x_0 + b_1y_0 + c_1 &= 0 \\ a_2x_0 + b_2y_0 + c_2 &= 0 \end{aligned}$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$\begin{aligned} -2ax_0 - by_0 + g &= 0 \\ bx_0 + 2cy_0 + e &= 0 \end{aligned}$$

Solving for x_0, y_0 from the above gives

$$\begin{aligned} x_0 &= \frac{be + 2cg}{4ac - b^2} \\ y_0 &= -\frac{2ae + bg}{4ac - b^2} \end{aligned}$$

Therefore the transformation becomes

$$\begin{aligned} X &= x - \left|\frac{be + 2cg}{4ac - b^2}\right| \\ Y &= y + \left|\frac{2ae + bg}{4ac - b^2}\right| \end{aligned}$$

Using this transformation in $2ax + by + (2cy + bx + e)\,y' = g$ result in

$$\frac{dY}{dX} = \frac{-2Xa - Yb}{Xb + 2Yc}$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{2Xa + Yb}{Xb + 2Yc} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -2Xa - Yb$ and $N = Xb + 2Yc$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-bu - 2a}{2cu + b} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-bu(X)-2a}{2cu(X)+b} - u(X)}{X}
\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-bu(X)-2a}{2cu(X)+b} - u(X)}{X} = 0$$

Or

$$2\left(\frac{d}{dX}u(X)\right)u(X)\,Xc + \left(\frac{d}{dX}u(X)\right)Xb + 2u(X)^2 c + 2bu(X) + 2a = 0$$

Or

$$X(2cu(X) + b)\left(\frac{d}{dX}u(X)\right) + 2u(X)^2 c + 2bu(X) + 2a = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{2\left(u(X)^2 c + bu(X) + a\right)}{X\left(2cu\left(X\right) + b\right)} \tag{2.23}$$

is separable as it can be written as

$$\begin{aligned}
\frac{d}{dX}u(X) &= -\frac{2\left(u(X)^2 c + bu(X) + a\right)}{X\left(2cu\left(X\right) + b\right)} \\
&= f(X)g(u)
\end{aligned}$$

Where

$$
\begin{aligned}
f(X) &= -\frac{2}{X} \\
g(u) &= \frac{u^2c + bu + a}{2cu + b}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(X)\,dX \\
\int \frac{2cu + b}{u^2c + bu + a}\,du &= \int -\frac{2}{X}\,dX
\end{aligned}
$$

$$
\ln\left(u(X)^2 c + bu(X) + a\right) = \ln\left(\frac{1}{X^2}\right) + c_2
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$
\frac{u^2c + bu + a}{2cu + b} = 0
$$

for $u(X)$ gives

$$
\begin{aligned}
u(X) &= \frac{-b + \sqrt{-4ac + b^2}}{2c} \\
u(X) &= -\frac{b + \sqrt{-4ac + b^2}}{2c}
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\ln\left(u(X)^2 c + bu(X) + a\right) = \ln\left(\frac{1}{X^2}\right) + c_2
$$

$$
u(X) = \frac{-b + \sqrt{-4ac + b^2}}{2c}
$$

$$
u(X) = -\frac{b + \sqrt{-4ac + b^2}}{2c}
$$

Solving for $u(X)$ gives

$$u(X) = \frac{-b+\sqrt{-4ac+b^2}}{2c}$$

$$u(X) = -\frac{b+\sqrt{-4ac+b^2}}{2c}$$

$$u(X) = \frac{-Xb+\sqrt{-4X^2ac+X^2b^2+4\,e^{c_2}c}}{2cX}$$

$$u(X) = -\frac{Xb+\sqrt{-4X^2ac+X^2b^2+4\,e^{c_2}c}}{2cX}$$

Converting $u(X) = \frac{-b+\sqrt{-4ac+b^2}}{2c}$ back to $Y(X)$ gives

$$Y(X) = \frac{X\left(-b+\sqrt{-4ac+b^2}\right)}{2c}$$

Converting $u(X) = -\frac{b+\sqrt{-4ac+b^2}}{2c}$ back to $Y(X)$ gives

$$Y(X) = -\frac{X\left(b+\sqrt{-4ac+b^2}\right)}{2c}$$

Converting $u(X) = \frac{-Xb+\sqrt{-4X^2ac+X^2b^2+4\,e^{c_2}c}}{2cX}$ back to $Y(X)$ gives

$$Y(X) = \frac{-Xb+\sqrt{-4X^2ac+X^2b^2+4\,e^{c_2}c}}{2c}$$

Converting $u(X) = -\frac{Xb+\sqrt{-4X^2ac+X^2b^2+4\,e^{c_2}c}}{2cX}$ back to $Y(X)$ gives

$$Y(X) = -\frac{Xb+\sqrt{-4X^2ac+X^2b^2+4\,e^{c_2}c}}{2c}$$

The solution is

$$Y(X) = \frac{-Xb+\sqrt{-4X^2ac+X^2b^2+4\,e^{c_2}c}}{2c}$$

Replacing $Y = y - y_0, X = x - x_0$ gives

$$y + \frac{2ae+bg}{4ac-b^2} = \frac{-\left(x-\frac{be+2cg}{4ac-b^2}\right)b+\sqrt{-4\left(x-\frac{be+2cg}{4ac-b^2}\right)^2ac+\left(x-\frac{be+2cg}{4ac-b^2}\right)^2b^2+4\,e^{c_2}c}}{2c}$$

Or

$$y = \frac{-\left(x-\frac{be+2cg}{4ac-b^2}\right)b+\sqrt{-4\left(x-\frac{be+2cg}{4ac-b^2}\right)^2ac+\left(x-\frac{be+2cg}{4ac-b^2}\right)^2b^2+4\,e^{c_2}c}}{2c} - \frac{2ae+bg}{4ac-b^2}$$

Which simplifies to

$$y=\frac{-bx-e+\sqrt{\frac{(16c^2a-4c\,b^2)\mathrm{e}^{c_2}-16\left(\left(ax-\frac{g}{2}\right)c-\frac{b(bx+e)}{4}\right)^2}{4ac-b^2}}}{2c}$$

The solution is

$$Y(X)=-\frac{Xb+\sqrt{-4X^2ac+X^2b^2+4\,\mathrm{e}^{c_2}c}}{2c}$$

Replacing $Y=y-y_0, X=x-x_0$ gives

$$y+\frac{2ae+bg}{4ac-b^2}=-\frac{\left(x-\frac{be+2cg}{4ac-b^2}\right)b+\sqrt{-4\left(x-\frac{be+2cg}{4ac-b^2}\right)^2ac+\left(x-\frac{be+2cg}{4ac-b^2}\right)^2b^2+4\,\mathrm{e}^{c_2}c}}{2c}$$

Or

$$y=-\frac{\left(x-\frac{be+2cg}{4ac-b^2}\right)b+\sqrt{-4\left(x-\frac{be+2cg}{4ac-b^2}\right)^2ac+\left(x-\frac{be+2cg}{4ac-b^2}\right)^2b^2+4\,\mathrm{e}^{c_2}c}}{2c}-\frac{2ae+bg}{4ac-b^2}$$

Which simplifies to

$$y=\frac{-bx-e-2\sqrt{\frac{(4c^2a-c\,b^2)\mathrm{e}^{c_2}-4\left(\left(ax-\frac{g}{2}\right)c-\frac{b(bx+e)}{4}\right)^2}{4ac-b^2}}}{2c}$$

The solution is

$$Y(X)=\frac{X\left(-b+\sqrt{-4ac+b^2}\right)}{2c}$$

Replacing $Y=y-y_0, X=x-x_0$ gives

$$y+\frac{2ae+bg}{4ac-b^2}=\frac{\left(x-\frac{be+2cg}{4ac-b^2}\right)\left(-b+\sqrt{-4ac+b^2}\right)}{2c}$$

Or

$$y=\frac{\left(x-\frac{be+2cg}{4ac-b^2}\right)\left(-b+\sqrt{-4ac+b^2}\right)}{2c}-\frac{2ae+bg}{4ac-b^2}$$

The solution is

$$Y(X)=-\frac{X\left(b+\sqrt{-4ac+b^2}\right)}{2c}$$

Replacing $Y=y-y_0, X=x-x_0$ gives

$$y+\frac{2ae+bg}{4ac-b^2}=-\frac{\left(x-\frac{be+2cg}{4ac-b^2}\right)\left(b+\sqrt{-4ac+b^2}\right)}{2c}$$

Or

$$y=-\frac{\left(x-\frac{be+2cg}{4ac-b^2}\right)\left(b+\sqrt{-4ac+b^2}\right)}{2c}-\frac{2ae+bg}{4ac-b^2}$$

Summary of solutions found

$$y = \frac{-bx - e - 2\sqrt{\frac{(4c^2a - c\,b^2)\mathrm{e}^{c_2} - 4\left(\left(ax - \frac{g}{2}\right)c - \frac{b(bx+e)}{4}\right)^2}{4ac - b^2}}}{2c}$$

$$y = \frac{-bx - e + \sqrt{\frac{(16c^2a - 4c\,b^2)\mathrm{e}^{c_2} - 16\left(\left(ax - \frac{g}{2}\right)c - \frac{b(bx+e)}{4}\right)^2}{4ac - b^2}}}{2c}$$

$$y = \frac{\left(x - \frac{be+2cg}{4ac-b^2}\right)\left(-b + \sqrt{-4ac + b^2}\right)}{2c} - \frac{2ae + bg}{4ac - b^2}$$

$$y = -\frac{\left(x - \frac{be+2cg}{4ac-b^2}\right)\left(b + \sqrt{-4ac + b^2}\right)}{2c} - \frac{2ae + bg}{4ac - b^2}$$

Solved as first order homogeneous class Maple C ode

Time used: 0.613 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{b(Y(X) + y_0) + 2a(x_0 + X) - g}{2c\,(Y\,(X) + y_0) + b\,(x_0 + X) + e}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = \frac{be + 2cg}{4ac - b^2}$$
$$y_0 = \frac{-2ae - bg}{4ac - b^2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2aX + bY(X) + \frac{2a(be+2cg)}{4ac-b^2} + \frac{b(-2ae-bg)}{4ac-b^2} - g}{bX + 2cY\,(X) + \frac{b(be+2cg)}{4ac-b^2} + \frac{2c(-2ae-bg)}{4ac-b^2} + e}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{2aX + bY}{bX + 2cY} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -2aX - bY$ and $N = bX + 2cY$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dX}X + u &= \frac{-bu - 2a}{2cu + b}\\ \frac{du}{dX} &= \frac{\frac{-bu(X)-2a}{2cu(X)+b} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-bu(X)-2a}{2cu(X)+b} - u(X)}{X} = 0$$

Or

$$2\left(\frac{d}{dX}u(X)\right)u(X)\,Xc + \left(\frac{d}{dX}u(X)\right)Xb + 2u(X)^2\,c + 2bu(X) + 2a = 0$$

Or

$$X(2cu(X) + b)\left(\frac{d}{dX}u(X)\right) + 2u(X)^2\,c + 2bu(X) + 2a = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{2(u(X)^2\,c + bu(X) + a)}{X\,(2cu\,(X) + b)} \tag{2.24}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{2(u(X)^2\,c + bu(X) + a)}{X\,(2cu\,(X) + b)}\\ &= f(X)g(u)\end{aligned}$$

Where

$$f(X) = -\frac{2}{X}$$
$$g(u) = \frac{u^2c + bu + a}{2cu + b}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(X)\,dX$$
$$\int \frac{2cu + b}{u^2c + bu + a}\,du = \int -\frac{2}{X}\,dX$$

$$\ln\left(u(X)^2 c + bu(X) + a\right) = \ln\left(\frac{1}{X^2}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2c + bu + a}{2cu + b} = 0$$

for $u(X)$ gives

$$u(X) = \frac{-b + \sqrt{-4ac + b^2}}{2c}$$
$$u(X) = -\frac{b + \sqrt{-4ac + b^2}}{2c}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(u(X)^2 c + bu(X) + a\right) = \ln\left(\frac{1}{X^2}\right) + c_1$$
$$u(X) = \frac{-b + \sqrt{-4ac + b^2}}{2c}$$
$$u(X) = -\frac{b + \sqrt{-4ac + b^2}}{2c}$$

Solving for $u(X)$ gives

$$u(X) = \frac{-b + \sqrt{-4ac + b^2}}{2c}$$
$$u(X) = -\frac{b + \sqrt{-4ac + b^2}}{2c}$$
$$u(X) = \frac{-bX + \sqrt{-4X^2ac + b^2X^2 + 4\,\mathrm{e}^{c_1}c}}{2cX}$$
$$u(X) = -\frac{bX + \sqrt{-4X^2ac + b^2X^2 + 4\,\mathrm{e}^{c_1}c}}{2cX}$$

Converting $u(X) = \frac{-b+\sqrt{-4ac+b^2}}{2c}$ back to $Y(X)$ gives

$$Y(X) = \frac{X\left(-b + \sqrt{-4ac + b^2}\right)}{2c}$$

Converting $u(X) = -\frac{b+\sqrt{-4ac+b^2}}{2c}$ back to $Y(X)$ gives

$$Y(X) = -\frac{X\left(b + \sqrt{-4ac + b^2}\right)}{2c}$$

Converting $u(X) = \frac{-bX+\sqrt{-4X^2ac+b^2X^2+4\,\mathrm{e}^{c_1}c}}{2cX}$ back to $Y(X)$ gives

$$Y(X) = \frac{-bX + \sqrt{-4X^2ac + b^2X^2 + 4\,\mathrm{e}^{c_1}c}}{2c}$$

Converting $u(X) = -\frac{bX+\sqrt{-4X^2ac+b^2X^2+4\,\mathrm{e}^{c_1}c}}{2cX}$ back to $Y(X)$ gives

$$Y(X) = -\frac{bX + \sqrt{-4X^2ac + b^2X^2 + 4\,\mathrm{e}^{c_1}c}}{2c}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{-bX + \sqrt{-4X^2ac + b^2X^2 + 4\,\mathrm{e}^{c_1}c}}{2c} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$
\begin{aligned}
Y &= y + \frac{-2ae - bg}{4ac - b^2} \\
X &= x + \frac{be + 2cg}{4ac - b^2}
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y - \frac{-2ae - bg}{4ac - b^2} = \frac{-\left(x - \frac{be+2cg}{4ac-b^2}\right) b + \sqrt{-4\left(x - \frac{be+2cg}{4ac-b^2}\right)^2 ac + \left(x - \frac{be+2cg}{4ac-b^2}\right)^2 b^2 + 4\,\mathrm{e}^{c_1} c}}{2c}
$$

Using the solution for $Y(X)$

$$
Y(X) = -\frac{bX + \sqrt{-4X^2ac + b^2X^2 + 4\,\mathrm{e}^{c_1} c}}{2c} \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y + \frac{-2ae - bg}{4ac - b^2} \\
X &= x + \frac{be + 2cg}{4ac - b^2}
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y - \frac{-2ae - bg}{4ac - b^2} = -\frac{\left(x - \frac{be+2cg}{4ac-b^2}\right) b + \sqrt{-4\left(x - \frac{be+2cg}{4ac-b^2}\right)^2 ac + \left(x - \frac{be+2cg}{4ac-b^2}\right)^2 b^2 + 4\,\mathrm{e}^{c_1} c}}{2c}
$$

Using the solution for $Y(X)$

$$
Y(X) = \frac{X\left(-b + \sqrt{-4ac + b^2}\right)}{2c} \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$Y = y + \frac{-2ae - bg}{4ac - b^2}$$
$$X = x + \frac{be + 2cg}{4ac - b^2}$$

Then the solution in y becomes using EQ (A)

$$y - \frac{-2ae - bg}{4ac - b^2} = \frac{\left(x - \frac{be+2cg}{4ac-b^2}\right)\left(-b + \sqrt{-4ac + b^2}\right)}{2c}$$

Using the solution for $Y(X)$

$$Y(X) = -\frac{X\left(b + \sqrt{-4ac + b^2}\right)}{2c} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + \frac{-2ae - bg}{4ac - b^2}$$
$$X = x + \frac{be + 2cg}{4ac - b^2}$$

Then the solution in y becomes using EQ (A)

$$y - \frac{-2ae - bg}{4ac - b^2} = -\frac{\left(x - \frac{be+2cg}{4ac-b^2}\right)\left(b + \sqrt{-4ac + b^2}\right)}{2c}$$

Solving for y gives

$$y = \frac{-bx + \sqrt{-4\left(x - \frac{be+2cg}{4ac-b^2}\right)^2 ac + \left(x - \frac{be+2cg}{4ac-b^2}\right)^2 b^2 + 4\,\mathrm{e}^{c_1} c} - e}{2c}$$

$$y = -\frac{bx + \sqrt{-4\left(x - \frac{be+2cg}{4ac-b^2}\right)^2 ac + \left(x - \frac{be+2cg}{4ac-b^2}\right)^2 b^2 + 4\,\mathrm{e}^{c_1} c} + e}{2c}$$

$$y = \frac{4\sqrt{-4ac + b^2}\,acx - \sqrt{-4ac + b^2}\,b^2 x - 4bcxa + b^3 x - \sqrt{-4ac + b^2}\,be - 2\sqrt{-4ac + b^2}\,cg - 4ace + b^2 e}{2c\left(4ac - b^2\right)}$$

$$y = -\frac{4\sqrt{-4ac + b^2}\,acx - \sqrt{-4ac + b^2}\,b^2 x + 4bcxa - b^3 x - \sqrt{-4ac + b^2}\,be - 2\sqrt{-4ac + b^2}\,cg + 4ace - b^2 e}{2c\left(4ac - b^2\right)}$$

Solved as first order Exact ode

Time used: 0.309 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(bx+2cy+e)\,\mathrm{d}y&=(-2ax-by+g)\,\mathrm{d}x\\ (2ax+by-g)\,\mathrm{d}x+(bx+2cy+e)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2ax+by-g\\ N(x,y)&=bx+2cy+e\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2ax + by - g) \\ &= b\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(bx + 2cy + e) \\ &= b\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 2ax + by - g \, \mathrm{d}x\end{aligned}$$

$$\phi = x(ax + by - g) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = bx + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = bx + 2cy + e$. Therefore equation (4) becomes

$$bx + 2cy + e = bx + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2cy + e$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (2cy + e)\,\mathrm{d}y$$
$$f(y) = y^2c + ey + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(ax + by - g) + y^2c + ey + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x(ax + by - g) + y^2c + ey$$

Solving for y gives

$$y = \frac{-bx - e + \sqrt{-4ac\,x^2 + b^2x^2 + 2bex + 4cgx + 4c_1c + e^2}}{2c}$$
$$y = -\frac{bx + \sqrt{-4ac\,x^2 + b^2x^2 + 2bex + 4cgx + 4c_1c + e^2} + e}{2c}$$

Summary of solutions found

$$y = \frac{-bx - e + \sqrt{-4ac\,x^2 + b^2x^2 + 2bex + 4cgx + 4c_1c + e^2}}{2c}$$
$$y = -\frac{bx + \sqrt{-4ac\,x^2 + b^2x^2 + 2bex + 4cgx + 4c_1c + e^2} + e}{2c}$$

Solved using Lie symmetry for first order ode

Time used: 0.621 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2ax + by - g}{bx + 2cy + e} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(2ax + by - g)(b_3 - a_2)}{bx + 2cy + e} - \frac{(2ax + by - g)^2 a_3}{(bx + 2cy + e)^2} \\
& - \left(-\frac{2a}{bx + 2cy + e} + \frac{(2ax + by - g) b}{(bx + 2cy + e)^2}\right)(x a_2 + y a_3 + a_1) \\
& - \left(-\frac{b}{bx + 2cy + e} + \frac{2(2ax + by - g) c}{(bx + 2cy + e)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{4a^2x^2a_3 - 2ab\,x^2a_2 + 2ab\,x^2b_3 + 4abxya_3 + 4ac\,x^2b_2 - 8acxya_2 + 8acxyb_3 - 4ac\,y^2a_3 - 2b^2x^2b_2 + 2b^2y^2}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-4a^2x^2a_3+2ab\,x^2a_2-2ab\,x^2b_3-4abxya_3-4ac\,x^2b_2+8acxya_2-8acxyb_3 \\
&+4ac\,y^2a_3+2b^2x^2b_2-2b^2y^2a_3+4bcxyb_2+2bc\,y^2a_2-2bc\,y^2b_3+4c^2y^2b_2 \\
&-4acxb_1+4acya_1+4aexa_2-2aexb_3+2aeya_3+4agxa_3+b^2xb_1 \\
&-b^2ya_1+3bexb_2+beya_2+bgxb_3+3bgya_3+4ceyb_2+2cgxb_2-2cgya_2 \\
&+4cgyb_3+2aea_1+beb_1+bga_1+2cgb_1+e^2b_2-ega_2+egb_3-g^2a_3=0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-4a^2a_3v_1^2+2aba_2v_1^2-4aba_3v_1v_2-2abb_3v_1^2+8aca_2v_1v_2+4aca_3v_2^2 \\
&-4acb_2v_1^2-8acb_3v_1v_2-2b^2a_3v_2^2+2b^2b_2v_1^2+2bca_2v_2^2+4bcb_2v_1v_2-2bcb_3v_2^2 \\
&+4c^2b_2v_2^2+4aca_1v_2-4acb_1v_1+4aea_2v_1+2aea_3v_2-2aeb_3v_1+4aga_3v_1 \\
&-b^2a_1v_2+b^2b_1v_1+bea_2v_2+3beb_2v_1+3bga_3v_2+bgb_3v_1+4ceb_2v_2-2cga_2v_2 \\
&+2cgb_2v_1+4cgb_3v_2+2aea_1+beb_1+bga_1+2cgb_1+e^2b_2-ega_2+egb_3-g^2a_3 \\
&=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-4a^2a_3+2aba_2-2abb_3-4acb_2+2b^2b_2\right)v_1^2 \\
&+\left(-4aba_3+8aca_2-8acb_3+4bcb_2\right)v_1v_2 \\
&+\left(-4acb_1+4aea_2-2aeb_3+4aga_3+b^2b_1+3beb_2+bgb_3+2cgb_2\right)v_1 \\
&+\left(4aca_3-2b^2a_3+2bca_2-2bcb_3+4c^2b_2\right)v_2^2 \\
&+\left(4aca_1+2aea_3-b^2a_1+bea_2+3bga_3+4ceb_2-2cga_2+4cgb_3\right)v_2 \\
&+2aea_1+beb_1+bga_1+2cgb_1+e^2b_2-ega_2+egb_3-g^2a_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-4aba_3 + 8aca_2 - 8acb_3 + 4bcb_2 &= 0 \\
4aca_3 - 2b^2a_3 + 2bca_2 - 2bcb_3 + 4c^2b_2 &= 0 \\
-4a^2a_3 + 2aba_2 - 2abb_3 - 4acb_2 + 2b^2b_2 &= 0 \\
4aca_1 + 2aea_3 - b^2a_1 + bea_2 + 3bga_3 + 4ceb_2 - 2cga_2 + 4cgb_3 &= 0 \\
-4acb_1 + 4aea_2 - 2aeb_3 + 4aga_3 + b^2b_1 + 3beb_2 + bgb_3 + 2cgb_2 &= 0 \\
2aea_1 + beb_1 + bga_1 + 2cgb_1 + e^2b_2 - ega_2 + egb_3 - g^2a_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= \frac{2acea_3 - b^2ea_3 - bceb_3 - bcga_3 - 2c^2gb_3}{c\,(4ac - b^2)} \\
a_2 &= \frac{ba_3 + cb_3}{c} \\
a_3 &= a_3 \\
b_1 &= \frac{abea_3 + 2aceb_3 + 2acga_3 + bcgb_3}{c\,(4ac - b^2)} \\
b_2 &= -\frac{aa_3}{c} \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4acx - b^2x - be - 2cg}{4ac - b^2} \\
\eta &= \frac{4acy - b^2y + 2ae + bg}{4ac - b^2}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= \frac{4acy - b^2y + 2ae + bg}{4ac - b^2} - \left(-\frac{2ax + by - g}{bx + 2cy + e}\right)\left(\frac{4acx - b^2x - be - 2cg}{4ac - b^2}\right) \\
&= \frac{8a^2c\,x^2 - 2a\,b^2x^2 + 8abcxy + 8a\,c^2y^2 - 2b^3xy - 2b^2c\,y^2 + 8acey - 8acgx - 2b^2ey + 2b^2gx + 2a\,e^2 + 2}{4bcxa + 8a\,c^2y - b^3x - 2b^2cy + 4ace - b^2e} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{8a^2 c\, x^2 - 2a\, b^2 x^2 + 8abcxy + 8a\, c^2 y^2 - 2b^3 xy - 2b^2 c\, y^2 + 8acey - 8acgx - 2b^2 ey + 2b^2 gx + 2a\, e^2 + 2beg + 2c\, g^2}{4bcxa + 8a\, c^2 y - b^3 x - 2b^2 cy + 4ace - b^2 e}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln \left(4a^2 c\, x^2 - a\, b^2 x^2 + 4abcxy + 4a\, c^2 y^2 - b^3 xy - b^2 c\, y^2 + 4acey - 4acgx - b^2 ey + b^2 gx + a\, e^2 + beg\right.}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2ax + by - g}{bx + 2cy + e}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{(2ax + by - g)\,(4ac - b^2)}{8a^2 c\, x^2 + (-2b^2 x^2 + 8cxyb + 8c^2 y^2 + (8ey - 8gx)\, c + 2e^2)\, a - 2\,(by - g)\,(b^2 x + (cy + e)\, b + cg)} \\ S_y &= \frac{(bx + 2cy + e)\,(4ac - b^2)}{8a^2 c\, x^2 + (-2b^2 x^2 + 8cxyb + 8c^2 y^2 + (8ey - 8gx)\, c + 2e^2)\, a - 2\,(by - g)\,(b^2 x + (cy + e)\, b + cg)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0\, dR + c_2$$
$$S(R) = c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln\left(4a^2c\,x^2 + \left(-b^2x^2 + 4ybcx - 4cgx + 4\left(cy + \frac{e}{2}\right)^2\right)a - (by - g)\left(b^2x + (cy + e)\,b + cg\right)\right)}{2} = c_2$$

Summary of solutions found

$$\frac{\ln\left(4a^2c\,x^2 + \left(-b^2x^2 + 4ybcx - 4cgx + 4\left(cy + \frac{e}{2}\right)^2\right)a - (by - g)\left(b^2x + (cy + e)\,b + cg\right)\right)}{2} = c_2$$

Solved as first order ode of type dAlembert

Time used: 0.868 (sec)

Let $p = y'$ the ode becomes

$$2ax + by + (bx + 2cy + e)\,p = g$$

Solving for y from the above results in

$$y = -\frac{(bp + 2a)\,x}{2cp + b} - \frac{ep - g}{2cp + b} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-bp - 2a}{2cp + b} \\ g &= \frac{-ep + g}{2cp + b} \end{aligned}$$

Hence (2) becomes

$$p - \frac{-bp - 2a}{2cp + b} = \left(-\frac{xb}{2cp + b} + \frac{2xcbp}{(2cp + b)^2} + \frac{4xca}{(2cp + b)^2} - \frac{e}{2cp + b} + \frac{2cep}{(2cp + b)^2} - \frac{2cg}{(2cp + b)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-bp - 2a}{2cp + b} = 0$$

Solving the above for p results in

$$\begin{aligned} p_1 &= \frac{-b + \sqrt{-4ac + b^2}}{2c} \\ p_2 &= -\frac{b + \sqrt{-4ac + b^2}}{2c} \end{aligned}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$\begin{aligned} y &= \frac{-bx\sqrt{-4ac + b^2} - 4acx + b^2x - e\sqrt{-4ac + b^2} + be + 2cg}{2c\sqrt{-4ac + b^2}} \\ y &= \frac{-bx\sqrt{-4ac + b^2} + 4acx - b^2x - e\sqrt{-4ac + b^2} - be - 2cg}{2\sqrt{-4ac + b^2}\, c} \end{aligned}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-bp(x)-2a}{2cp(x)+b}}{-\frac{xb}{2cp(x)+b} + \frac{2xcbp(x)}{(2cp(x)+b)^2} + \frac{4xca}{(2cp(x)+b)^2} - \frac{e}{2cp(x)+b} + \frac{2cep(x)}{(2cp(x)+b)^2} - \frac{2cg}{(2cp(x)+b)^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = \frac{2(2cp(x)+b)\left(p(x)^2 c + bp(x) + a\right)}{4acx - b^2x - be - 2cg} \tag{2.25}$$

is separable as it can be written as

$$\begin{aligned} p'(x) &= \frac{2(2cp(x)+b)\left(p(x)^2 c + bp(x) + a\right)}{4acx - b^2x - be - 2cg} \\ &= f(x)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{2}{4acx - b^2x - be - 2cg} \\ g(p) &= (2cp+b)\left(c\,p^2 + bp + a\right) \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(x)\,dx$$

$$\int \frac{1}{(2cp+b)\,(c\,p^2+bp+a)}\,dp = \int \frac{2}{4acx - b^2x - be - 2cg}\,dx$$

$$\frac{\ln\left(\frac{(2cp(x)+b)^2}{p(x)^2c+bp(x)+a}\right)}{4ac-b^2} = \frac{2\ln\left((4ac-b^2)\,x - be - 2cg\right)}{4ac-b^2} + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$(2cp+b)\left(c\,p^2 + bp + a\right) = 0$$

for $p(x)$ gives

$$
\begin{aligned}
p(x) &= -\frac{b}{2c} \\
p(x) &= \frac{-b+\sqrt{-4ac+b^2}}{2c} \\
p(x) &= -\frac{b+\sqrt{-4ac+b^2}}{2c}
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\frac{\ln\left(\frac{(2cp(x)+b)^2}{p(x)^2c+bp(x)+a}\right)}{4ac-b^2} = \frac{2\ln\left(\left(4ac-b^2\right)x-be-2cg\right)}{4ac-b^2}+c_1
$$

$$
p(x) = -\frac{b}{2c}
$$

$$
p(x) = \frac{-b+\sqrt{-4ac+b^2}}{2c}
$$

$$
p(x) = -\frac{b+\sqrt{-4ac+b^2}}{2c}
$$

Substituing the above solution for p in (2A) gives

Expression too large to display

$$
\begin{aligned}
y &= \frac{\sqrt{-4ac+b^2}\left(bx\sqrt{-4ac+b^2}+4acx-b^2x+e\sqrt{-4ac+b^2}-be-2cg\right)}{2c\left(4ac-b^2\right)} \\
y &= -\frac{\sqrt{-4ac+b^2}\left(-bx\sqrt{-4ac+b^2}+4acx-b^2x-e\sqrt{-4ac+b^2}-be-2cg\right)}{2c\left(4ac-b^2\right)}
\end{aligned}
$$

Summary of solutions found

$$
y = \frac{-bx\sqrt{-4ac+b^2}-4acx+b^2x-e\sqrt{-4ac+b^2}+be+2cg}{2c\sqrt{-4ac+b^2}}
$$

$$
y = \frac{-bx\sqrt{-4ac+b^2}+4acx-b^2x-e\sqrt{-4ac+b^2}-be-2cg}{2\sqrt{-4ac+b^2}\,c}
$$

$$y = -\frac{\sqrt{-4ac+b^2}\left(-bx\sqrt{-4ac+b^2}+4acx-b^2x-e\sqrt{-4ac+b^2}-be-2cg\right)}{2c\left(4ac-b^2\right)}$$

$$y = \frac{\sqrt{-4ac+b^2}\left(bx\sqrt{-4ac+b^2}+4acx-b^2x+e\sqrt{-4ac+b^2}-be-2cg\right)}{2c\left(4ac-b^2\right)}$$

Expression too large to display

Maple step by step solution

Let's solve

$2ax+by+(2cy+bx+e)\,y'=g$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - ◦ Evaluate derivatives

    $b=b$
  - ◦ Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y)=C1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(2ax+by-g)\,dx+\_F1(y)$
- Evaluate integral

  $F(x,y)=a\,x^2+bxy-gx+\_F1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $bx+2cy+e=bx+\frac{d}{dy}\_F1(y)$
- Isolate for $\frac{d}{dy}\_F1(y)$

  $\frac{d}{dy}\_F1(y)=2cy+e$
- Solve for $\_F1(y)$

$$\_F1(y) = y^2c + ey$$

- Substitute $\_F1(y)$ into equation for $F(x,y)$

$$F(x,y) = a\,x^2 + bxy + y^2c + ey - gx$$

- Substitute $F(x,y)$ into the solution of the ODE

$$a\,x^2 + bxy + y^2c + ey - gx = C1$$

- Solve for y

$$\left\{y = \frac{-bx-e+\sqrt{-4ac\,x^2+b^2x^2+2bex+4cgx+4C1\,c+e^2}}{2c}, y = -\frac{bx+\sqrt{-4ac\,x^2+b^2x^2+2bex+4cgx+4C1\,c+e^2}+e}{2c}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.105` (sec)
Leaf size : 90

```
dsolve(2*a*x+b*y(x)+(2*c*y(x)+b*x+e)*diff(y(x),x) = g,y(x),singsol=all)
```

$$y(x) = \frac{-\sqrt{-64\left(ac-\frac{b^2}{4}\right)\left(\left(ax-\frac{g}{2}\right)c-\frac{b(bx+e)}{4}\right)^2 c_1^2+4c}+\left(-4abcx+b^3x-4ace+b^2e\right)c_1}{8\left(ac-\frac{b^2}{4}\right)cc_1}$$

Mathematica DSolve solution

Solving time : 17.046 (sec)
Leaf size : 132

```
DSolve[{(2*a*x+b*y[x])+(2*c*y[x]+b*x+e)*D[y[x],x]==g,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to -\frac{\frac{\sqrt{\frac{4cx(g-ax)+b^2x^2+2bex+4c^2c_1+e^2}{c}}}{\sqrt{\frac{1}{c}}}+bx+e}{2c}$$

$$y(x) \to -\frac{-\frac{\sqrt{\frac{4cx(g-ax)+b^2x^2+2bex+4c^2c_1+e^2}{c}}}{\sqrt{\frac{1}{c}}}+bx+e}{2c}$$

2.3.4 Problem 4

Internal problem ID [18548]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 29. Problems at page 81
Problem number : 4
Date solved : `Tuesday, January 28, 2025 at 11:57:12 AM`
CAS classification : `[_separable]`

Solve

$$\sec(x)^2 \tan(y) y' + \sec(y)^2 \tan(x) = 0$$

Solved as first order separable ode

Time used: 0.231 (sec)

The ode

$$y' = -\frac{\sec(y)^2 \tan(x)}{\sec(x)^2 \tan(y)} \tag{2.26}$$

is separable as it can be written as

$$\begin{aligned} y' &= -\frac{\sec(y)^2 \tan(x)}{\sec(x)^2 \tan(y)} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{\tan(x)}{\sec(x)^2} \\ g(y) &= \frac{\sec(y)^2}{\tan(y)} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

$$\int \frac{\tan(y)}{\sec(y)^2}\, dy = \int -\frac{\tan(x)}{\sec(x)^2}\, dx$$

$$-\frac{\cos(y)^2}{2} = \frac{\cos(x)^2}{2} + c_1$$

Solving for y gives

$$y = \pi - \arccos\left(\sqrt{-\cos(x)^2 - 2c_1}\right)$$

$$y = \arccos\left(\sqrt{-\cos(x)^2 - 2c_1}\right)$$

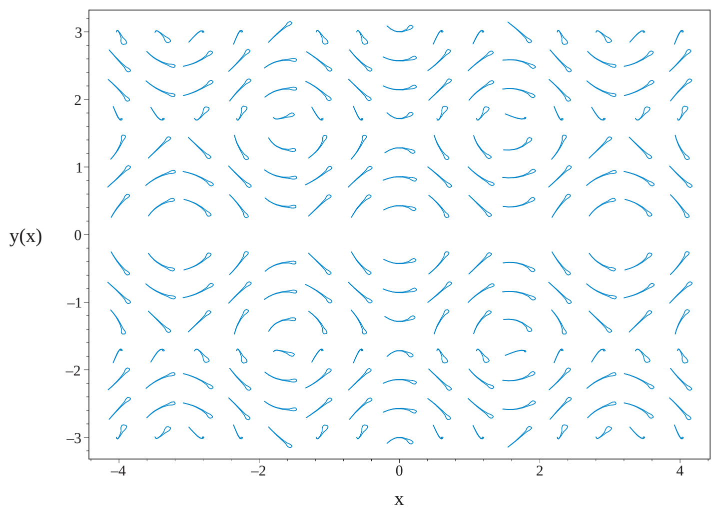


Figure 2.30: Slope field plot
$\sec(x)^2 \tan(y)\, y' + \sec(y)^2 \tan(x) = 0$

Summary of solutions found

$$y = \pi - \arccos\left(\sqrt{-\cos(x)^2 - 2c_1}\right)$$

$$y = \arccos\left(\sqrt{-\cos(x)^2 - 2c_1}\right)$$

Maple step by step solution

Let's solve

$\sec(x)^2 \tan(y) y' + \sec(y)^2 \tan(x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $y' = -\frac{\sec(y)^2 \tan(x)}{\sec(x)^2 \tan(y)}$

- Separate variables

  $\frac{y' \tan(y)}{\sec(y)^2} = -\frac{\tan(x)}{\sec(x)^2}$

- Integrate both sides with respect to x

  $\int \frac{y' \tan(y)}{\sec(y)^2} dx = \int -\frac{\tan(x)}{\sec(x)^2} dx + C1$

- Evaluate integral

  $-\frac{1}{2\sec(y)^2} = \frac{1}{2\sec(x)^2} + C1$

- Solve for y

  $\left\{y = \pi - \operatorname{arcsec}\left(\frac{1}{\sqrt{-\cos(x)^2 - 2C1}}\right), y = \operatorname{arcsec}\left(\frac{1}{\sqrt{-\cos(x)^2 - 2C1}}\right)\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 41

```
dsolve(sec(x)^2*tan(y(x))*diff(y(x),x)+sec(y(x))^2*tan(x) = 0,y(x),singsol=all)
```

$$y(x) = \operatorname{arcsec}\left(\frac{2}{\sqrt{-2\cos(2x)+8c_1}}\right)$$
$$y(x) = \frac{\pi}{2} + \operatorname{arccsc}\left(\frac{2}{\sqrt{-2\cos(2x)+8c_1}}\right)$$

Mathematica DSolve solution

Solving time : 0.504 (sec)
Leaf size : 41

```
DSolve[{Sec[x]^2*Tan[y[x]]*D[y[x],x]+Sec[y[x]]^2*Tan[x]==0,{}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to -\frac{1}{2}\arccos(-\cos(2x) - 2c_1)$$
$$y(x) \to \frac{1}{2}\arccos(-\cos(2x) - 2c_1)$$

2.3.5 Problem 5

Internal problem ID [18549]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 29. Problems at page 81
Problem number : 5
Date solved : `Tuesday, January 28, 2025 at 11:57:50 AM`
CAS classification :
`[[_homogeneous, ‘class A‘], _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

Solve

$$x + yy' = my$$

Solved as first order homogeneous class A ode

Time used: 0.557 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{my - x}{y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = my - x$ and $N = y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dx}x + u &= m - \frac{1}{u} \\ \frac{du}{dx} &= \frac{m - \frac{1}{u(x)} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{m - \frac{1}{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\, u(x)\, x + u(x)^2 - mu(x) + 1 = 0$$

Which is now solved as separable in $u(x)$.

The ode

$$u'(x) = -\frac{u(x)^2 - mu(x) + 1}{u(x)\, x} \tag{2.27}$$

is separable as it can be written as

$$\begin{aligned}u'(x) &= -\frac{u(x)^2 - mu(x) + 1}{u(x)\, x} \\ &= f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= -\frac{1}{x} \\ g(u) &= \frac{-mu + u^2 + 1}{u}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{u}{-mu + u^2 + 1}\, du &= \int -\frac{1}{x}\, dx\end{aligned}$$

$$\frac{\ln \left(u(x)^2 - mu(x) + 1\right)}{2} + \frac{m \,\mathrm{arctanh}\left(\frac{m-2u(x)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{-mu + u^2 + 1}{u} = 0$$

for $u(x)$ gives

$$u(x) = \frac{m}{2} - \frac{\sqrt{m^2-4}}{2}$$
$$u(x) = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln \left(u(x)^2 - mu(x) + 1\right)}{2} + \frac{m \,\mathrm{arctanh}\left(\frac{m-2u(x)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right) + c_1$$
$$u(x) = \frac{m}{2} - \frac{\sqrt{m^2-4}}{2}$$
$$u(x) = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}$$

Converting $\frac{\ln\left(u(x)^2-mu(x)+1\right)}{2} + \frac{m \,\mathrm{arctanh}\left(\frac{m-2u(x)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right) + c_1$ back to y gives

$$\frac{\mathrm{arctanh}\left(\frac{mx-2y}{x\sqrt{(m-2)(m+2)}}\right) m}{\sqrt{(m-2)(m+2)}} + \frac{\ln\left(\frac{-myx+y^2+x^2}{x^2}\right)}{2} = \ln\left(\frac{1}{x}\right) + c_1$$

Converting $u(x) = \frac{m}{2} - \frac{\sqrt{m^2-4}}{2}$ back to y gives

$$y = x\left(\frac{m}{2} - \frac{\sqrt{m^2-4}}{2}\right)$$

Converting $u(x) = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}$ back to y gives

$$y = x\left(\frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\right)$$

Summary of solutions found

$$\frac{\operatorname{arctanh}\left(\frac{mx-2y}{x\sqrt{(m-2)(m+2)}}\right)m}{\sqrt{(m-2)(m+2)}} + \frac{\ln\left(\frac{-myx+y^2+x^2}{x^2}\right)}{2} = \ln\left(\frac{1}{x}\right) + c_1$$

$$y = x\left(\frac{m}{2} - \frac{\sqrt{m^2-4}}{2}\right)$$

$$y = x\left(\frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\right)$$

Solved as first order homogeneous class D2 ode

Time used: 0.274 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$x + u(x)\,x(u'(x)\,x + u(x)) = mu(x)\,x$$

Which is now solved The ode

$$u'(x) = -\frac{u(x)^2 - mu(x) + 1}{u(x)\,x} \tag{2.28}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)^2 - mu(x) + 1}{u(x)\,x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{-mu + u^2 + 1}{u} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$

$$\int \frac{u}{-mu+u^2+1}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln\left(u(x)^2-mu(x)+1\right)}{2}+\frac{m \operatorname{arctanh}\left(\frac{m-2u(x)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$\frac{-mu+u^2+1}{u}=0$$

for $u(x)$ gives

$$\begin{aligned} u(x) &= \frac{m}{2}-\frac{\sqrt{m^2-4}}{2} \\ u(x) &= \frac{m}{2}+\frac{\sqrt{m^2-4}}{2} \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(x)^2-mu(x)+1\right)}{2}+\frac{m \operatorname{arctanh}\left(\frac{m-2u(x)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right)+c_1$$

$$u(x) = \frac{m}{2}-\frac{\sqrt{m^2-4}}{2}$$

$$u(x) = \frac{m}{2}+\frac{\sqrt{m^2-4}}{2}$$

Converting $\frac{\ln\left(u(x)^2-mu(x)+1\right)}{2}+\frac{m \operatorname{arctanh}\left(\frac{m-2u(x)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right)+c_1$ back to y gives

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{my}{x}+1\right)}{2}+\frac{m \operatorname{arctanh}\left(\frac{m-\frac{2y}{x}}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right)+c_1$$

Converting $u(x) = \frac{m}{2} - \frac{\sqrt{m^2-4}}{2}$ back to y gives

$$y = x\left(\frac{m}{2} - \frac{\sqrt{m^2-4}}{2}\right)$$

Converting $u(x) = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}$ back to y gives

$$y = x\left(\frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\right)$$

Summary of solutions found

$$\frac{\ln\left(\frac{y^2}{x^2} - \frac{my}{x} + 1\right)}{2} + \frac{m \operatorname{arctanh}\left(\frac{m-\frac{2y}{x}}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right) + c_1$$

$$y = x\left(\frac{m}{2} - \frac{\sqrt{m^2-4}}{2}\right)$$

$$y = x\left(\frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\right)$$

Solved as first order homogeneous class Maple C ode

Time used: 0.563 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{m(Y(X) + y_0) - x_0 - X}{Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{mY(X) - X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{mY - X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = mY - X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= m - \frac{1}{u} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{m - \frac{1}{u(X)} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{m - \frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) u(X)\, X + u(X)^2 - mu(X) + 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{u(X)^2 - mu(X) + 1}{u\,(X)\,X} \tag{2.29}$$

is separable as it can be written as

$$\begin{aligned} \frac{d}{dX}u(X) &= -\frac{u(X)^2 - mu(X) + 1}{u\,(X)\,X} \\ &= f(X)g(u) \end{aligned}$$

Where

$$
\begin{aligned}
f(X) &= -\frac{1}{X} \\
g(u) &= \frac{-mu + u^2 + 1}{u}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(X)\, dX \\
\int \frac{u}{-mu + u^2 + 1}\, du &= \int -\frac{1}{X}\, dX
\end{aligned}
$$

$$
\frac{\ln\left(u(X)^2 - mu(X) + 1\right)}{2} + \frac{m \operatorname{arctanh}\left(\frac{m-2u(X)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{X}\right) + c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$
\frac{-mu + u^2 + 1}{u} = 0
$$

for $u(X)$ gives

$$
\begin{aligned}
u(X) &= \frac{m}{2} - \frac{\sqrt{m^2-4}}{2} \\
u(X) &= \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\frac{\ln\left(u(X)^2 - mu(X) + 1\right)}{2} + \frac{m \operatorname{arctanh}\left(\frac{m-2u(X)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{X}\right) + c_1
$$

$$
u(X) = \frac{m}{2} - \frac{\sqrt{m^2-4}}{2}
$$

$$
u(X) = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}
$$

Converting $\frac{\ln\left(u(X)^2-mu(X)+1\right)}{2}+\frac{m\,\mathrm{arctanh}\left(\frac{m-2u(X)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}}=\ln\left(\frac{1}{X}\right)+c_1$ back to $Y(X)$ gives

$$\frac{\mathrm{arctanh}\left(\frac{mX-2Y(X)}{X\sqrt{(m-2)(m+2)}}\right)m}{\sqrt{(m-2)\,(m+2)}}+\frac{\ln\left(\frac{-mY(X)X+Y(X)^2+X^2}{X^2}\right)}{2}=\ln\left(\frac{1}{X}\right)+c_1$$

Converting $u(X)=\frac{m}{2}-\frac{\sqrt{m^2-4}}{2}$ back to $Y(X)$ gives

$$Y(X)=X\left(\frac{m}{2}-\frac{\sqrt{m^2-4}}{2}\right)$$

Converting $u(X)=\frac{m}{2}+\frac{\sqrt{m^2-4}}{2}$ back to $Y(X)$ gives

$$Y(X)=X\left(\frac{m}{2}+\frac{\sqrt{m^2-4}}{2}\right)$$

Using the solution for $Y(X)$

$$\frac{\mathrm{arctanh}\left(\frac{mX-2Y(X)}{X\sqrt{(m-2)(m+2)}}\right)m}{\sqrt{(m-2)\,(m+2)}}+\frac{\ln\left(\frac{-mY(X)X+Y(X)^2+X^2}{X^2}\right)}{2}=\ln\left(\frac{1}{X}\right)+c_1 \qquad \text{(A)}$$

And replacing back terms in the above solution using

$$Y=y+y_0$$
$$X=x+x_0$$

Or

$$Y=y$$
$$X=x$$

Then the solution in y becomes using EQ (A)

$$\frac{\mathrm{arctanh}\left(\frac{mx-2y}{x\sqrt{(m-2)(m+2)}}\right)m}{\sqrt{(m-2)\,(m+2)}}+\frac{\ln\left(\frac{-myx+y^2+x^2}{x^2}\right)}{2}=\ln\left(\frac{1}{x}\right)+c_1$$

Using the solution for $Y(X)$

$$Y(X)=X\left(\frac{m}{2}-\frac{\sqrt{m^2-4}}{2}\right) \qquad \text{(A)}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y = x\left(\frac{m}{2} - \frac{\sqrt{m^2-4}}{2}\right)
$$

Using the solution for $Y(X)$

$$
Y(X) = X\left(\frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\right) \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y = x\left(\frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\right)
$$

Solved as first order isobaric ode

Time used: 0.246 (sec)

Solving for y' gives

$$
y' = \frac{my - x}{y} \tag{1}
$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{my - x}{y} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = \frac{mxu(x) - x}{xu(x)}$$

The ode

$$u'(x) = -\frac{u(x)^2 - u(x)\, m + 1}{u(x)\, x} \tag{2.30}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)^2 - u(x)\, m + 1}{u(x)\, x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{-um + u^2 + 1}{u} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$

$$\int \frac{u}{-um + u^2 + 1}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln\left(u(x)^2-u(x)\,m+1\right)}{2}+\frac{m\,\operatorname{arctanh}\left(\frac{m-2u(x)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}}=\ln\left(\frac{1}{x}\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$\frac{-um+u^2+1}{u}=0$$

for $u(x)$ gives

$$u(x)=\frac{m}{2}-\frac{\sqrt{m^2-4}}{2}$$
$$u(x)=\frac{m}{2}+\frac{\sqrt{m^2-4}}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(x)^2-u(x)\,m+1\right)}{2}+\frac{m\,\operatorname{arctanh}\left(\frac{m-2u(x)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}}=\ln\left(\frac{1}{x}\right)+c_1$$
$$u(x)=\frac{m}{2}-\frac{\sqrt{m^2-4}}{2}$$
$$u(x)=\frac{m}{2}+\frac{\sqrt{m^2-4}}{2}$$

Converting $\frac{\ln\left(u(x)^2-u(x)m+1\right)}{2}+\frac{m\,\operatorname{arctanh}\left(\frac{m-2u(x)}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}}=\ln\left(\frac{1}{x}\right)+c_1$ back to y gives

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{my}{x}+1\right)}{2}+\frac{m\,\operatorname{arctanh}\left(\frac{m-\frac{2y}{x}}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}}=\ln\left(\frac{1}{x}\right)+c_1$$

Converting $u(x)=\frac{m}{2}-\frac{\sqrt{m^2-4}}{2}$ back to y gives

$$\frac{y}{x}=\frac{m}{2}-\frac{\sqrt{m^2-4}}{2}$$

Converting $u(x) = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}$ back to y gives

$$\frac{y}{x} = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}$$

Solving for y gives

$$\frac{\ln\left(\frac{y^2}{x^2} - \frac{my}{x} + 1\right)}{2} + \frac{m \operatorname{arctanh}\left(\frac{m-\frac{2y}{x}}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right) + c_1$$

$$y = -\frac{x\left(-m+\sqrt{m^2-4}\right)}{2}$$

$$y = \frac{x\left(m+\sqrt{m^2-4}\right)}{2}$$

<u>Summary of solutions found</u>

$$\frac{\ln\left(\frac{y^2}{x^2} - \frac{my}{x} + 1\right)}{2} + \frac{m \operatorname{arctanh}\left(\frac{m-\frac{2y}{x}}{\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = \ln\left(\frac{1}{x}\right) + c_1$$

$$y = -\frac{x\left(-m+\sqrt{m^2-4}\right)}{2}$$

$$y = \frac{x\left(m+\sqrt{m^2-4}\right)}{2}$$

Solved using Lie symmetry for first order ode

Time used: 7.026 (sec)

Writing the ode as

$$y' = \frac{my - x}{y}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(my-x)(b_3-a_2)}{y} - \frac{(my-x)^2 a_3}{y^2} + \frac{xa_2+ya_3+a_1}{y} - \left(\frac{m}{y} - \frac{my-x}{y^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{m^2y^2a_3 - 2mxya_3 + m\,y^2a_2 - m\,y^2b_3 + x^2a_3 + x^2b_2 - 2xya_2 + 2xyb_3 - y^2a_3 - b_2y^2 + xb_1 - ya_1}{y^2} = 0$$

Setting the numerator to zero gives

$$-m^2y^2a_3 + 2mxya_3 - m\,y^2a_2 + m\,y^2b_3 - x^2a_3 - x^2b_2 + 2xya_2 - 2xyb_3 + y^2a_3 + b_2y^2 - xb_1 + ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-m^2a_3v_2^2 - ma_2v_2^2 + 2ma_3v_1v_2 + mb_3v_2^2 + 2a_2v_1v_2 - a_3v_1^2 + a_3v_2^2 - b_2v_1^2 + b_2v_2^2 - 2b_3v_1v_2 + a_1v_2 - b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_3 - b_2)\, v_1^2 + (2ma_3 + 2a_2 - 2b_3)\, v_1 v_2 - b_1 v_1 \\ + \left(-m^2 a_3 - ma_2 + mb_3 + a_3 + b_2\right) v_2^2 + a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_1 &= 0 \\ -b_1 &= 0 \\ -a_3 - b_2 &= 0 \\ 2ma_3 + 2a_2 - 2b_3 &= 0 \\ -m^2 a_3 - ma_2 + mb_3 + a_3 + b_2 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\ a_2 &= mb_2 + b_3 \\ a_3 &= -b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x \\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{my - x}{y}\right)(x) \\ &= \frac{-myx + x^2 + y^2}{y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-myx+x^2+y^2}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(-myx + x^2 + y^2)}{2} - \frac{mx \operatorname{arctanh}\left(\frac{-mx+2y}{\sqrt{m^2x^2-4x^2}}\right)}{\sqrt{m^2x^2 - 4x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{my - x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{my - x}{myx - x^2 - y^2} \\ S_y &= -\frac{y}{myx - x^2 - y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0\, dR + c_2$$
$$S(R) = c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(-myx + y^2 + x^2)}{2} + \frac{m \,\mathrm{arctanh}\left(\frac{mx-2y}{x\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = c_2$$

Summary of solutions found

$$\frac{\ln(-myx + y^2 + x^2)}{2} + \frac{m \,\mathrm{arctanh}\left(\frac{mx-2y}{x\sqrt{m^2-4}}\right)}{\sqrt{m^2-4}} = c_2$$

Maple step by step solution

Let's solve

$x + yy' = my$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $y' = \frac{my-x}{y}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.085 (sec)
Leaf size : 57

```
dsolve(x+diff(y(x),x)*y(x) = m*y(x),y(x),singsol=all)
```

$$y(x) = \operatorname{RootOf}\left(\_Z^2 - e^{\operatorname{RootOf}\left(\left(4\,e^{-\_Z}\cosh\left(\frac{\sqrt{m^2-4}\,(2c_1+\_Z+2\ln(x))}{2m}\right)^2+m^2-4\right)x^2\right)} + 1 - \_Zm\right)x$$

Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 72

```
DSolve[{x+y[x]*D[y[x],x]==m*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{m\arctan\left(\frac{\frac{2y(x)}{x}-m}{\sqrt{4-m^2}}\right)}{\sqrt{4-m^2}} + \frac{1}{2}\log\left(-\frac{my(x)}{x}+\frac{y(x)^2}{x^2}+1\right) = -\log(x)+c_1, y(x)\right]$$

2.3.6 Problem 6

Internal problem ID [18550]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 29. Problems at page 81
Problem number : 6
Date solved : `Tuesday, January 28, 2025 at 11:57:59 AM`
CAS classification :
`[[_homogeneous, ‘class A‘], _exact, _rational, _dAlembert]`

Solve

$$\frac{2x}{y^3} + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right) y' = 0$$

Solved as first order homogeneous class A ode

Time used: 0.458 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{2yx}{-3x^2+y^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = 2yx$ and $N = 3x^2 - y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dx}x + u &= -\frac{2u}{u^2 - 3} \\ \frac{du}{dx} &= \frac{-\frac{2u(x)}{u(x)^2-3} - u(x)}{x} \end{aligned}$$

Or

$$u'(x) - \frac{-\frac{2u(x)}{u(x)^2-3} - u(x)}{x} = 0$$

Or

$$u'(x)\,u(x)^2\,x + u(x)^3 - 3u'(x)\,x - u(x) = 0$$

Or

$$x\left(u(x)^2 - 3\right)u'(x) + u(x)^3 - u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode

$$u'(x) = -\frac{u(x)\left(u(x)^2 - 1\right)}{x\left(u\left(x\right)^2 - 3\right)} \tag{2.31}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)\left(u(x)^2 - 1\right)}{x\left(u\left(x\right)^2 - 3\right)} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{u(u^2 - 1)}{u^2 - 3} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{u^2 - 3}{u\,(u^2 - 1)}\,du &= \int -\frac{1}{x}\,dx \end{aligned}$$

$$-\ln\left(\frac{(u(x)-1)\,(u(x)+1)}{u\,(x)^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u(u^2-1)}{u^2-3} = 0$$

for $u(x)$ gives

$$\begin{aligned} u(x) &= -1 \\ u(x) &= 0 \\ u(x) &= 1 \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln\left(\frac{(u(x)-1)\,(u(x)+1)}{u\,(x)^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$
$$u(x) = -1$$
$$u(x) = 0$$
$$u(x) = 1$$

Converting $-\ln\left(\frac{(u(x)-1)(u(x)+1)}{u(x)^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$ back to y gives

$$-\ln\left(-\frac{x(-y+x)\,(y+x)}{y^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$

Converting $u(x) = -1$ back to y gives

$$y = -x$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = 1$ back to y gives

$$y = x$$

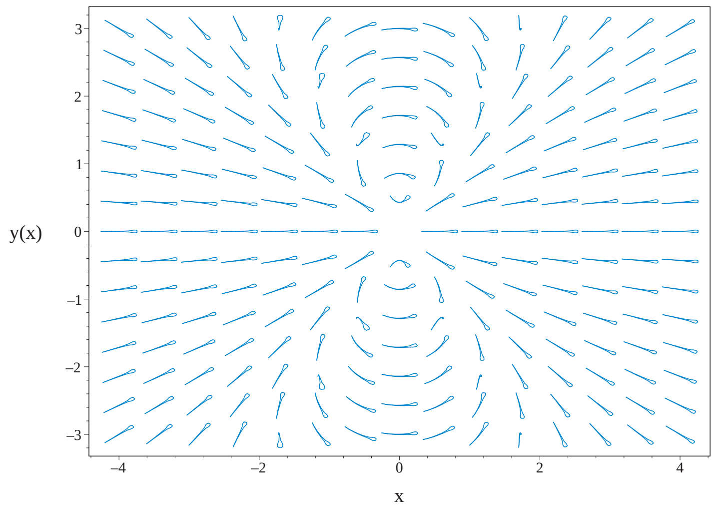


Figure 2.31: Slope field plot
$\frac{2x}{y^3}+\left(\frac{1}{y^2}-\frac{3x^2}{y^4}\right)y'=0$

Summary of solutions found

$$-\ln\left(-\frac{x(-y+x)\,(y+x)}{y^3}\right)=\ln\left(\frac{1}{x}\right)+c_1$$
$$y=0$$
$$y=x$$
$$y=-x$$

Solved as first order homogeneous class D2 ode

Time used: 0.221 (sec)

Applying change of variables $y=u(x)\,x$, then the ode becomes

$$\frac{2}{x^2u\left(x\right)^3}+\left(\frac{1}{u\left(x\right)^2x^2}-\frac{3}{x^2u\left(x\right)^4}\right)\left(u'(x)\,x+u(x)\right)=0$$

Which is now solved The ode

$$u'(x)=-\frac{u(x)\left(u(x)^2-1\right)}{x\left(u\left(x\right)^2-3\right)} \tag{2.32}$$

is separable as it can be written as

$$\begin{aligned}u'(x)&=-\frac{u(x)\left(u(x)^2-1\right)}{x\left(u\left(x\right)^2-3\right)}\\&=f(x)g(u)\end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= -\frac{1}{x} \\
g(u) &= \frac{u(u^2-1)}{u^2-3}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\
\int \frac{u^2-3}{u\,(u^2-1)}\, du &= \int -\frac{1}{x}\, dx
\end{aligned}
$$

$$
-\ln\left(\frac{(u(x)-1)\,(u(x)+1)}{u\,(x)^3}\right) = \ln\left(\frac{1}{x}\right) + c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$
\frac{u(u^2-1)}{u^2-3} = 0
$$

for $u(x)$ gives

$$
\begin{aligned}
u(x) &= -1 \\
u(x) &= 0 \\
u(x) &= 1
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
-\ln\left(\frac{(u(x)-1)\,(u(x)+1)}{u\,(x)^3}\right) &= \ln\left(\frac{1}{x}\right) + c_1 \\
u(x) &= -1 \\
u(x) &= 0 \\
u(x) &= 1
\end{aligned}
$$

Converting $-\ln\left(\frac{(u(x)-1)(u(x)+1)}{u(x)^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$ back to y gives

$$-\ln\left(\frac{\left(\frac{y}{x}-1\right)x^3\left(\frac{y}{x}+1\right)}{y^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$

Converting $u(x) = -1$ back to y gives

$$y = -x$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = 1$ back to y gives

$$y = x$$

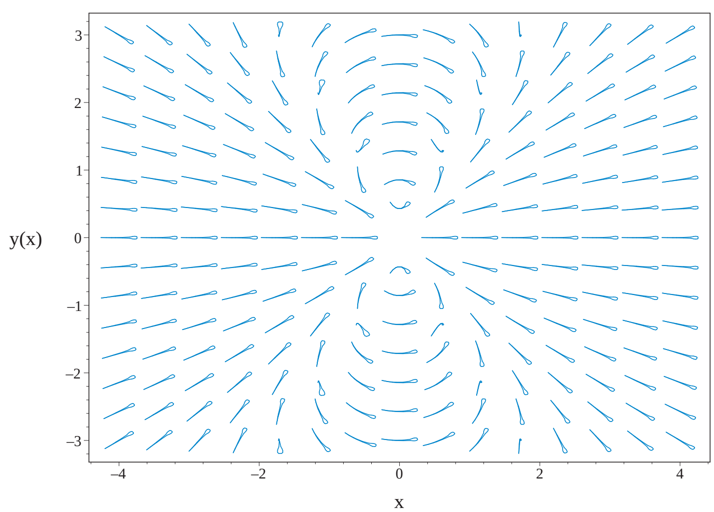


Figure 2.32: Slope field plot
$\frac{2x}{y^3} + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right) y' = 0$

Summary of solutions found

$$-\ln\left(\frac{\left(\frac{y}{x}-1\right)x^3\left(\frac{y}{x}+1\right)}{y^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$

$$y = 0$$
$$y = x$$
$$y = -x$$

Solved as first order homogeneous class Maple C ode

Time used: 0.523 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2(Y(X) + y_0)(x_0 + X)}{(Y(X) + y_0)^2 - 3(x_0 + X)^2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= 0 \\ y_0 &= 0 \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2Y(X)X}{-3X^2 + Y(X)^2}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{2YX}{-3X^2 + Y^2} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 2YX$ and $N = 3X^2 - Y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= -\frac{2u}{u^2 - 3} \\ \frac{du}{dX} &= \frac{-\frac{2u(X)}{u(X)^2-3} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{-\frac{2u(X)}{u(X)^2-3} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) u(X)^2 X + u(X)^3 - 3\left(\frac{d}{dX}u(X)\right) X - u(X) = 0$$

Or

$$X\left(u(X)^2 - 3\right)\left(\frac{d}{dX}u(X)\right) + u(X)^3 - u(X) = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{u(X)\left(u(X)^2 - 1\right)}{X\left(u\left(X\right)^2 - 3\right)} \tag{2.33}$$

is separable as it can be written as

$$\begin{aligned}
\frac{d}{dX}u(X) &= -\frac{u(X)\left(u(X)^2 - 1\right)}{X\left(u\left(X\right)^2 - 3\right)} \\
&= f(X)g(u)
\end{aligned}$$

Where

$$\begin{aligned}
f(X) &= -\frac{1}{X} \\
g(u) &= \frac{u(u^2-1)}{u^2-3}
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(X)\,dX \\
\int \frac{u^2-3}{u\left(u^2-1\right)}\,du &= \int -\frac{1}{X}\,dX
\end{aligned}$$

$$-\ln\left(\frac{\left(u(X)-1\right)\left(u(X)+1\right)}{u\left(X\right)^3}\right) = \ln\left(\frac{1}{X}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u(u^2-1)}{u^2-3} = 0$$

for $u(X)$ gives

$$
\begin{aligned}
u(X) &= -1 \\
u(X) &= 0 \\
u(X) &= 1
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
-\ln\left(\frac{(u(X)-1)\left(u(X)+1\right)}{u\left(X\right)^{3}}\right) &= \ln\left(\frac{1}{X}\right)+c_1 \\
u(X) &= -1 \\
u(X) &= 0 \\
u(X) &= 1
\end{aligned}
$$

Converting $-\ln\left(\frac{(u(X)-1)(u(X)+1)}{u(X)^3}\right) = \ln\left(\frac{1}{X}\right) + c_1$ back to $Y(X)$ gives

$$
-\ln\left(-\frac{X(-Y(X)+X)\left(Y(X)+X\right)}{Y\left(X\right)^{3}}\right) = \ln\left(\frac{1}{X}\right)+c_1
$$

Converting $u(X) = -1$ back to $Y(X)$ gives

$$
Y(X) = -X
$$

Converting $u(X) = 0$ back to $Y(X)$ gives

$$
Y(X) = 0
$$

Converting $u(X) = 1$ back to $Y(X)$ gives

$$
Y(X) = X
$$

Using the solution for $Y(X)$

$$
-\ln\left(-\frac{X(-Y(X)+X)\left(Y(X)+X\right)}{Y\left(X\right)^{3}}\right) = \ln\left(\frac{1}{X}\right)+c_1 \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$-\ln\left(-\frac{x(-y+x)\,(y+x)}{y^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$

Using the solution for $Y(X)$

$$Y(X) = 0 \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$y = 0$$

Using the solution for $Y(X)$

$$Y(X) = X \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$y = x$$

Using the solution for $Y(X)$

$$Y(X) = -X \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = -x$$

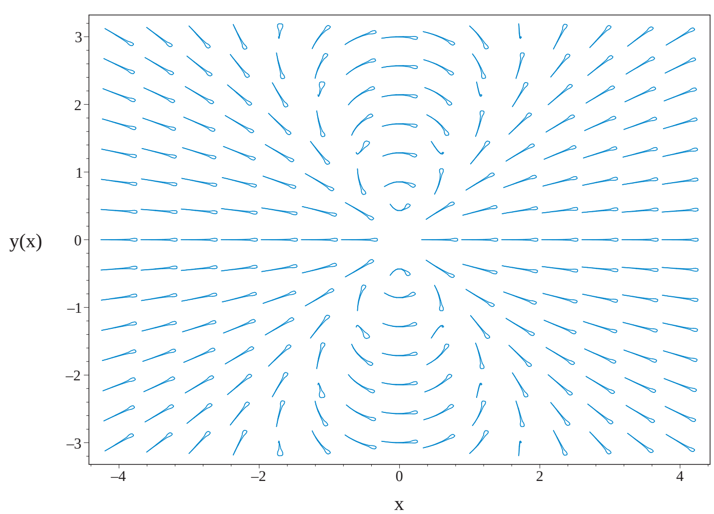


Figure 2.33: Slope field plot
$\frac{2x}{y^3} + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right) y' = 0$

Solved as first order Exact ode

Time used: 0.112 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)\mathrm{d}y &= \left(-\frac{2x}{y^3}\right)\mathrm{d}x\\ \left(\frac{2x}{y^3}\right)\mathrm{d}x + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \frac{2x}{y^3}\\ N(x,y) &= \frac{1}{y^2} - \frac{3x^2}{y^4}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x}{y^3}\right)\\ &= -\frac{6x}{y^4}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)\\ &= -\frac{6x}{y^4}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2x}{y^3}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x^2}{y^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{3x^2}{y^4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2} - \frac{3x^2}{y^4}$. Therefore equation (4) becomes

$$\frac{1}{y^2} - \frac{3x^2}{y^4} = -\frac{3x^2}{y^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{y^3} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{x^2}{y^3} - \frac{1}{y}$$

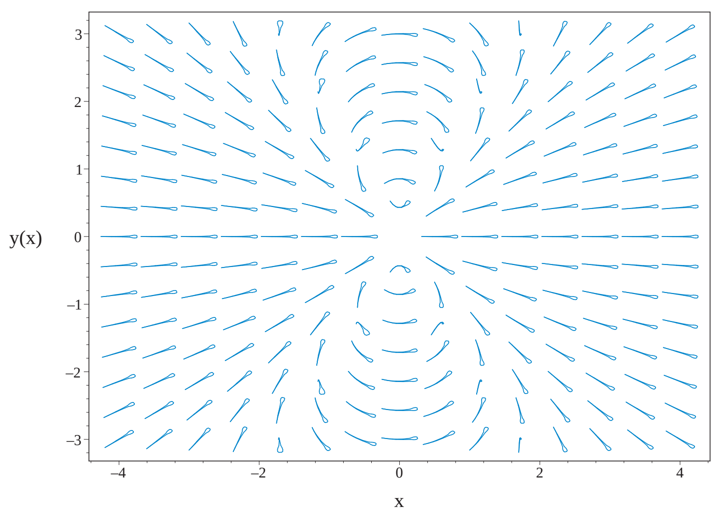


Figure 2.34: Slope field plot
$\frac{2x}{y^3} + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right) y' = 0$

Summary of solutions found

$$\frac{x^2}{y^3} - \frac{1}{y} = c_1$$

Solved as first order isobaric ode

Time used: 0.197 (sec)

Solving for y' gives

$$y' = -\frac{2yx}{y^2 - 3x^2} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = -\frac{2yx}{y^2 - 3x^2} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = -\frac{2x^2 u(x)}{x^2 u(x)^2 - 3x^2}$$

The ode

$$u'(x) = -\frac{u(x)\left(u(x)^2 - 1\right)}{\left(u(x)^2 - 3\right)x} \tag{2.34}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)\left(u(x)^2 - 1\right)}{\left(u(x)^2 - 3\right)x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$f(x) = -\frac{1}{x}$$
$$g(u) = \frac{u(u^2-1)}{u^2-3}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{u^2-3}{u\,(u^2-1)}\,du = \int -\frac{1}{x}\,dx$$

$$-\ln\left(\frac{(u(x)-1)\,(u(x)+1)}{u\,(x)^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u(u^2-1)}{u^2-3} = 0$$

for $u(x)$ gives

$$u(x) = -1$$
$$u(x) = 0$$
$$u(x) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln\left(\frac{(u(x)-1)\,(u(x)+1)}{u\,(x)^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$
$$u(x) = -1$$
$$u(x) = 0$$
$$u(x) = 1$$

Converting $-\ln\left(\frac{(u(x)-1)(u(x)+1)}{u(x)^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$ back to y gives

$$-\ln\left(\frac{\left(\frac{y}{x}-1\right)x^3\left(\frac{y}{x}+1\right)}{y^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$

Converting $u(x) = -1$ back to y gives

$$\frac{y}{x} = -1$$

Converting $u(x) = 0$ back to y gives

$$\frac{y}{x} = 0$$

Converting $u(x) = 1$ back to y gives

$$\frac{y}{x} = 1$$

Solving for y gives

$$-\ln\left(\frac{\left(\frac{y}{x}-1\right)x^3\left(\frac{y}{x}+1\right)}{y^3}\right) = \ln\left(\frac{1}{x}\right) + c_1$$
$$y = 0$$
$$y = x$$
$$y = -x$$

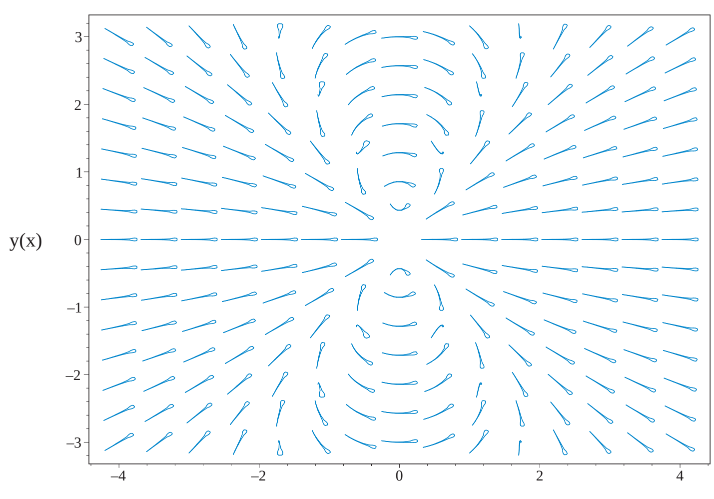


Figure 2.35: Slope field plot
$\frac{2x}{y^3} + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)y' = 0$

Summary of solutions found

$$-\ln\left(\frac{\left(\frac{y}{x}-1\right)x^3\left(\frac{y}{x}+1\right)}{y^3}\right)=\ln\left(\frac{1}{x}\right)+c_1$$

$$y=0$$

$$y=x$$

$$y=-x$$

Solved using Lie symmetry for first order ode

Time used: 0.821 (sec)

Writing the ode as

$$y'=-\frac{2yx}{-3x^2+y^2}$$

$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$

$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}b_2&-\frac{2yx(b_3-a_2)}{-3x^2+y^2}-\frac{4y^2x^2a_3}{(-3x^2+y^2)^2}\\&-\left(-\frac{2y}{-3x^2+y^2}-\frac{12y\,x^2}{(-3x^2+y^2)^2}\right)(xa_2+ya_3+a_1)\\&-\left(-\frac{2x}{-3x^2+y^2}+\frac{4y^2x}{(-3x^2+y^2)^2}\right)(xb_2+yb_3+b_1)=0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^4b_2 + 2y^2x^2a_3 - 8x^2y^2b_2 + 4x\,y^3a_2 - 4x\,y^3b_3 + 2y^4a_3 + y^4b_2 - 6x^3b_1 + 6x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1}{(3x^2 - y^2)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &3x^4b_2 + 2y^2x^2a_3 - 8x^2y^2b_2 + 4x\,y^3a_2 - 4x\,y^3b_3 + 2y^4a_3 \\ &+ y^4b_2 - 6x^3b_1 + 6x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &4a_2v_1v_2^3 + 2a_3v_1^2v_2^2 + 2a_3v_2^4 + 3b_2v_1^4 - 8b_2v_1^2v_2^2 + b_2v_2^4 \\ &- 4b_3v_1v_2^3 + 6a_1v_1^2v_2 + 2a_1v_2^3 - 6b_1v_1^3 - 2b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &3b_2v_1^4 - 6b_1v_1^3 + (2a_3 - 8b_2)\,v_1^2v_2^2 + 6a_1v_1^2v_2 \\ &+ (4a_2 - 4b_3)\,v_1v_2^3 - 2b_1v_1v_2^2 + (2a_3 + b_2)\,v_2^4 + 2a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 &= 0 \\
6a_1 &= 0 \\
-6b_1 &= 0 \\
-2b_1 &= 0 \\
3b_2 &= 0 \\
4a_2 - 4b_3 &= 0 \\
2a_3 - 8b_2 &= 0 \\
2a_3 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{2yx}{-3x^2 + y^2}\right)(x) \\
&= \frac{y\,x^2 - y^3}{3x^2 - y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{\frac{y\,x^2-y^3}{3x^2-y^2}}dy\end{aligned}$$

Which results in

$$S=3\ln(y)-\ln(y-x)-\ln(x+y)$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=-\frac{2yx}{-3x^2+y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=-\frac{2x}{x^2-y^2}\\S_y&=\frac{3}{y}+\frac{1}{x-y}-\frac{1}{x+y}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

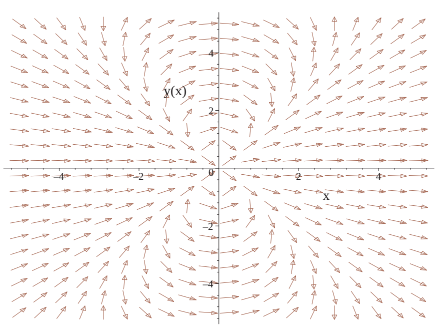
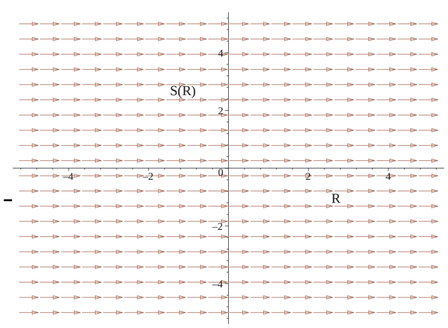
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0\,dR + c_2$$
$$S(R) = c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$3\ln(y) - \ln(y - x) - \ln(y + x) = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2yx}{-3x^2+y^2}$ 	$R = x$ $S = 3\ln(y) - \ln(y - x)$ -	$\frac{dS}{dR} = 0$ 

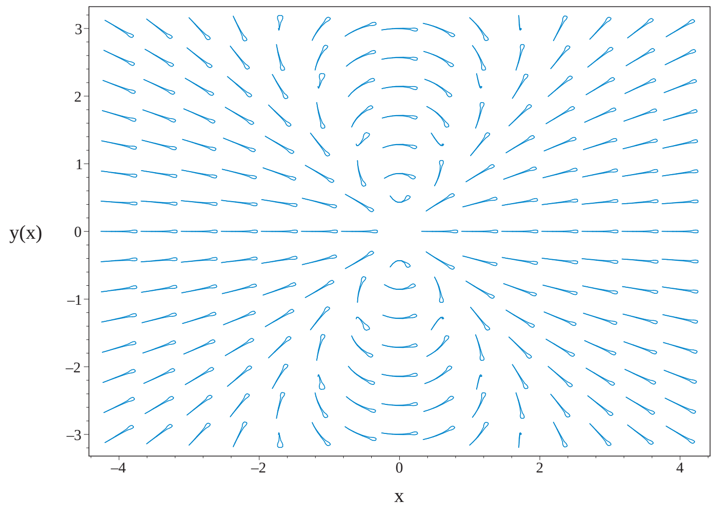


Figure 2.36: Slope field plot
$\frac{2x}{y^3}+\left(\frac{1}{y^2}-\frac{3x^2}{y^4}\right)y'=0$

Summary of solutions found

$$3\ln(y)-\ln(y-x)-\ln(y+x)=c_2$$

Maple step by step solution

Let's solve
$\frac{2x}{y^3}+\left(\frac{1}{y^2}-\frac{3x^2}{y^4}\right)y'=0$

- Highest derivative means the order of the ODE is 1
  y'
- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function
    $F'(x,y)=0$
  - Compute derivative of lhs
    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - Evaluate derivatives
    $-\frac{6x}{y^4}=-\frac{6x}{y^4}$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form
  $\left[F(x,y)=C1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x
  $F(x,y) = \int \frac{2x}{y^3}dx + \_F1(y)$
- Evaluate integral
  $F(x,y) = \frac{x^2}{y^3} + \_F1(y)$
- Take derivative of $F(x,y)$ with respect to y
  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$
- Compute derivative
  $\frac{1}{y^2} - \frac{3x^2}{y^4} = -\frac{3x^2}{y^4} + \frac{d}{dy}\_F1(y)$
- Isolate for $\frac{d}{dy}\_F1(y)$
  $\frac{d}{dy}\_F1(y) = \frac{1}{y^2}$
- Solve for $\_F1(y)$
  $\_F1(y) = -\frac{1}{y}$
- Substitute $\_F1(y)$ into equation for $F(x,y)$
  $F(x,y) = \frac{x^2}{y^3} - \frac{1}{y}$
- Substitute $F(x,y)$ into the solution of the ODE
  $\frac{x^2}{y^3} - \frac{1}{y} = C1$
- Solve for y
  $\left\{y = \frac{\left(12\sqrt{3}\,x\sqrt{27x^2 C1^2-4}\,C1+108x^2 C1^2-8\right)^{1/3}}{6C1} + \frac{2}{3C1\left(12\sqrt{3}\,x\sqrt{27x^2 C1^2-4}\,C1+108x^2 C1^2-8\right)^{1/3}} - \frac{1}{3C1}, y = -\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 313

```
dsolve(2*x/y(x)^3+(1/y(x)^2-3*x^2/y(x)^4)*diff(y(x),x) = 0,y(x),singsol=all)
```

$$y(x) = \frac{1+\frac{\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{1/3}}{2}+\frac{2}{\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{1/3}}}{3c_1}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{2/3}-4i\sqrt{3}-4\Big(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2}{12\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{1/3}c_1}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{2/3}-4i\sqrt{3}+4\Big(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+}{12\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1-108x^2c_1^2+8\right)^{1/3}c_1}$$

Mathematica DSolve solution

Solving time : `60.18` (sec)
Leaf size : 458

```
DSolve[{2*x/y[x]^3+(1/y[x]^2-3*x^2/y[x]^4)*D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{1}{3}\left(\frac{\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}e^{2c_1}}{\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}} - e^{c_1}\right)$$

$$y(x) \to \frac{i\left(\sqrt{3}+i\right)\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}}{6\sqrt[3]{2}} - \frac{i\left(\sqrt{3}-i\right)e^{2c_1}}{3\ 2^{2/3}\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}} - \frac{e^{c_1}}{3}$$

$$y(x) \to -\frac{i\left(\sqrt{3}-i\right)\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}}{6\sqrt[3]{2}} + \frac{i\left(\sqrt{3}+i\right)e^{2c_1}}{3\ 2^{2/3}\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}} - \frac{e^{c_1}}{3}$$

2.3.7 Problem 8

Internal problem ID [18551]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 29. Problems at page 81
Problem number : 8
Date solved : `Tuesday, January 28, 2025 at 11:58:04 AM`
CAS classification : `[_exact]`

Solve

$$\left(T+\frac{1}{\sqrt{t^2-T^2}}\right)T'=\frac{T}{t\sqrt{t^2-T^2}}-t$$

Solved as first order Exact ode

Time used: 0.459 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,T)\,\mathrm{d}t+N(t,T)\,\mathrm{d}T=0 \tag{1A}$$

Therefore

$$\left(T+\frac{1}{\sqrt{-T^2+t^2}}\right)\mathrm{d}T=\left(\frac{T}{t\sqrt{-T^2+t^2}}-t\right)\mathrm{d}t$$

$$\left(-\frac{T}{t\sqrt{-T^2+t^2}}+t\right)\mathrm{d}t+\left(T+\frac{1}{\sqrt{-T^2+t^2}}\right)\mathrm{d}T=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,T)&=-\frac{T}{t\sqrt{-T^2+t^2}}+t\\N(t,T)&=T+\frac{1}{\sqrt{-T^2+t^2}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial T}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial T}&=\frac{\partial}{\partial T}\left(-\frac{T}{t\sqrt{-T^2+t^2}}+t\right)\\&=-\frac{t}{(-T^2+t^2)^{3/2}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t}&=\frac{\partial}{\partial t}\left(T+\frac{1}{\sqrt{-T^2+t^2}}\right)\\&=-\frac{t}{(-T^2+t^2)^{3/2}}\end{aligned}$$

Since $\frac{\partial M}{\partial T} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, T)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial T} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int M \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int -\frac{T}{t\sqrt{-T^2+t^2}} + t \,\mathrm{d}t$$

$$\phi = \frac{t^2\sqrt{-T^2} + 2T\ln\left(\frac{\sqrt{-T^2}\,\sqrt{-T^2+t^2}-T^2}{t}\right) + 2T\ln(2)}{2\sqrt{-T^2}} + f(T) \tag{3}$$

Where $f(T)$ is used for the constant of integration since ϕ is a function of both t and T. Taking derivative of equation (3) w.r.t T gives

$$\begin{aligned}
\frac{\partial \phi}{\partial T} &= \frac{-\frac{t^2 T}{\sqrt{-T^2}} + 2\ln\left(\frac{\sqrt{-T^2}\,\sqrt{-T^2+t^2}-T^2}{t}\right) + \frac{2T\left(-\frac{\sqrt{-T^2+t^2}\,T}{\sqrt{-T^2}} - \frac{\sqrt{-T^2}\,T}{\sqrt{-T^2+t^2}} - 2T\right)}{\sqrt{-T^2}\,\sqrt{-T^2+t^2}-T^2} + 2\ln(2)}{2\sqrt{-T^2}} \\
&\quad + \frac{\left(t^2\sqrt{-T^2} + 2T\ln\left(\frac{\sqrt{-T^2}\,\sqrt{-T^2+t^2}-T^2}{t}\right) + 2T\ln(2)\right)T}{2\left(-T^2\right)^{3/2}} + f'(T) \\
&= \frac{2\sqrt{-T^2}\,\sqrt{-T^2+t^2} - 2T^2 + t^2}{\sqrt{-T^2+t^2}\,\left(\sqrt{-T^2}\,\sqrt{-T^2+t^2} - T^2\right)} + f'(T)
\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial T} = T + \frac{1}{\sqrt{-T^2+t^2}}$. Therefore equation (4) becomes

$$T + \frac{1}{\sqrt{-T^2+t^2}} = \frac{2\sqrt{-T^2}\,\sqrt{-T^2+t^2} - 2T^2 + t^2}{\sqrt{-T^2+t^2}\,\left(\sqrt{-T^2}\,\sqrt{-T^2+t^2} - T^2\right)} + f'(T) \tag{5}$$

Solving equation (5) for $f'(T)$ gives

$$\begin{aligned}
f'(T) &= -\frac{\sqrt{-T^2+t^2}\,T^3 + \sqrt{-T^2}\,T^3 - \sqrt{-T^2}\,T\,t^2 + \sqrt{-T^2}\,\sqrt{-T^2+t^2} - T^2 + t^2}{\sqrt{-T^2+t^2}\,\left(\sqrt{-T^2}\,\sqrt{-T^2+t^2} - T^2\right)} \\
&= \frac{\left(T^3 + \sqrt{-T^2}\right)\sqrt{-T^2+t^2} + (T-t)(T+t)\left(\sqrt{-T^2}\,T - 1\right)}{\sqrt{-T^2+t^2}\,\left(T^2 - \sqrt{-T^2}\,\sqrt{-T^2+t^2}\right)}
\end{aligned}$$

Integrating the above w.r.t T results in

$$\int f'(T)\,\mathrm{d}T = \int \left(\frac{\left(T^3+\sqrt{-T^2}\right)\sqrt{-T^2+t^2}+(T-t)\,(T+t)\left(\sqrt{-T^2}\,T-1\right)}{\sqrt{-T^2+t^2}\,\left(T^2-\sqrt{-T^2}\,\sqrt{-T^2+t^2}\right)}\right)\mathrm{d}T$$

$$f(T) = \frac{\sqrt{-T^2}\,\ln(T)}{T}+\frac{T^2}{2}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(T)$ into equation (3) gives ϕ

$$\phi = \frac{t^2\sqrt{-T^2}+2T\ln\left(\frac{\sqrt{-T^2}\,\sqrt{-T^2+t^2}-T^2}{t}\right)+2T\ln(2)}{2\sqrt{-T^2}}+\frac{\sqrt{-T^2}\,\ln(T)}{T}+\frac{T^2}{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{t^2\sqrt{-T^2}+2T\ln\left(\frac{\sqrt{-T^2}\,\sqrt{-T^2+t^2}-T^2}{t}\right)+2T\ln(2)}{2\sqrt{-T^2}}+\frac{\sqrt{-T^2}\,\ln(T)}{T}+\frac{T^2}{2}$$

Summary of solutions found

$$\frac{t^2\sqrt{-T^2}+2T\ln\left(\frac{\sqrt{-T^2}\,\sqrt{t^2-T^2}-T^2}{t}\right)+2T\ln(2)}{2\sqrt{-T^2}}+\frac{\sqrt{-T^2}\,\ln(T)}{T}+\frac{T^2}{2} = c_1$$

Maple step by step solution

Let's solve

$\left(T+\frac{1}{\sqrt{t^2-T^2}}\right)T' = \frac{T}{t\sqrt{t^2-T^2}}-t$

- Highest derivative means the order of the ODE is 1

  T'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(t,T) = 0$
  - Compute derivative of lhs

    $F'(t,T)+\left(\frac{\partial}{\partial T}F(t,T)\right)T' = 0$

- Evaluate derivatives
  $-\frac{1}{t\sqrt{-T^2+t^2}}-\frac{T^2}{t(-T^2+t^2)^{3/2}}=-\frac{t}{(-T^2+t^2)^{3/2}}$
- Simplify
  $-\frac{t}{(-T^2+t^2)^{3/2}}=-\frac{t}{(-T^2+t^2)^{3/2}}$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form
  $\left[F(t,T)=C1, M(t,T)=F'(t,T), N(t,T)=\frac{\partial}{\partial T}F(t,T)\right]$
- Solve for $F(t,T)$ by integrating $M(t,T)$ with respect to t
  $F(t,T)=\int\left(-\frac{T}{t\sqrt{-T^2+t^2}}+t\right)dt+\_F1(T)$
- Evaluate integral
  $F(t,T)=\frac{t^2}{2}+\frac{T\ln\left(\frac{-2T^2+2\sqrt{-T^2}\sqrt{-T^2+t^2}}{t}\right)}{\sqrt{-T^2}}+\_F1(T)$
- Take derivative of $F(t,T)$ with respect to T
  $N(t,T)=\frac{\partial}{\partial T}F(t,T)$
- Compute derivative
  $T+\frac{1}{\sqrt{-T^2+t^2}}=\frac{\ln\left(\frac{-2T^2+2\sqrt{-T^2}\sqrt{-T^2+t^2}}{t}\right)}{\sqrt{-T^2}}+\frac{T^2\ln\left(\frac{-2T^2+2\sqrt{-T^2}\sqrt{-T^2+t^2}}{t}\right)}{(-T^2)^{3/2}}+\frac{T\left(-4T-\frac{2\sqrt{-T^2+t^2}\,T}{\sqrt{-T^2}}-\frac{2\sqrt{-T^2}\,T}{\sqrt{-T^2+t^2}}\right)}{\sqrt{-T^2}\left(-2T^2+2\sqrt{-T^2}\sqrt{-T^2+t^2}\right)}$
- Isolate for $\frac{d}{dT}\_F1(T)$
  $\frac{d}{dT}\_F1(T)=T+\frac{1}{\sqrt{-T^2+t^2}}-\frac{\ln\left(\frac{-2T^2+2\sqrt{-T^2}\sqrt{-T^2+t^2}}{t}\right)}{\sqrt{-T^2}}-\frac{T^2\ln\left(\frac{-2T^2+2\sqrt{-T^2}\sqrt{-T^2+t^2}}{t}\right)}{(-T^2)^{3/2}}-\frac{T\left(-4T-\frac{2\sqrt{-T^2+t^2}}{\sqrt{-T^2}}\right.}{\sqrt{-T^2}\left(-2T^2+2\sqrt{}\right.}$
- Solve for $\_F1(T)$
  $\_F1(T)=\frac{(-T^2)^{3/2}T^4+T^6\sqrt{-T^2}-2\sqrt{-T^2}\,T^4t^2+2\arctan\left(\frac{T}{\sqrt{-T^2+t^2}}\right)(-T^2)^{3/2}t^2+2\arctan\left(\frac{T}{\sqrt{-T^2+t^2}}\right)\sqrt{-T^2}\,T^2t^2+4\ln(T}{4(-T^2)^{3/2}t^2}$
- Substitute $\_F1(T)$ into equation for $F(t,T)$
  $F(t,T)=\frac{t^2}{2}+\frac{T\ln\left(\frac{-2T^2+2\sqrt{-T^2}\sqrt{-T^2+t^2}}{t}\right)}{\sqrt{-T^2}}+\frac{(-T^2)^{3/2}T^4+T^6\sqrt{-T^2}-2\sqrt{-T^2}\,T^4t^2+2\arctan\left(\frac{T}{\sqrt{-T^2+t^2}}\right)(-T^2)^{3/2}t^2}{}$
- Substitute $F(t,T)$ into the solution of the ODE
  $\frac{t^2}{2}+\frac{T\ln\left(\frac{-2T^2+2\sqrt{-T^2}\sqrt{-T^2+t^2}}{t}\right)}{\sqrt{-T^2}}+\frac{(-T^2)^{3/2}T^4+T^6\sqrt{-T^2}-2\sqrt{-T^2}\,T^4t^2+2\arctan\left(\frac{T}{\sqrt{-T^2+t^2}}\right)(-T^2)^{3/2}t^2+2\arctan\left(\frac{}{\sqrt{-}}\right.}{4(-T^2)^{3/}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 79

```
dsolve((T(t)+1/(t^2-T(t)^2)^(1/2))*diff(T(t),t) = T(t)/t/(t^2-T(t)^2)^(1/2)-t,T(t),sin
```

$$\frac{\left(\frac{t^2}{2}+\frac{T^2}{2}+c_1\right)\sqrt{-T^2}+T\left(\ln\left(\frac{\sqrt{-T^2}\,\sqrt{t^2-T^2}-T^2}{t}\right)+\ln(2)-\ln(T)\right)}{\sqrt{-T^2}}=0$$

Mathematica DSolve solution

Solving time : 1.592 (sec)
Leaf size : 44

```
DSolve[{(T[t]+1/Sqrt[t^2-T[t]^2])*D[T[t],t]== T[t]/(t*Sqrt[t^2-T[t]^2])-t,{}},T[t],t,Include
```

$$\text{Solve}\left[-\arctan\left(\frac{\sqrt{t^2-T(t)^2}}{T(t)}\right)+\frac{t^2}{2}+\frac{T(t)^2}{2}=c_1,T(t)\right]$$

2.4 Chapter IV. Methods of solution: First order equations. section 31. Problems at page 85

2.4.1 Problem 1

Internal problem ID [18552]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 31. Problems at page 85
Problem number : 1
Date solved : `Tuesday, January 28, 2025 at 11:58:07 AM`
CAS classification : `[_separable]`

Solve

$$y' + xy = x$$

Solved as first order linear ode

Time used: 0.234 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = x$$
$$p(x) = x$$

The integrating factor μ is

$$\mu = e^{\int x dx}$$

Therefore the solution is

$$y = \left(\int x\, e^{\int x dx} dx + c_1\right) e^{-\int x dx}$$

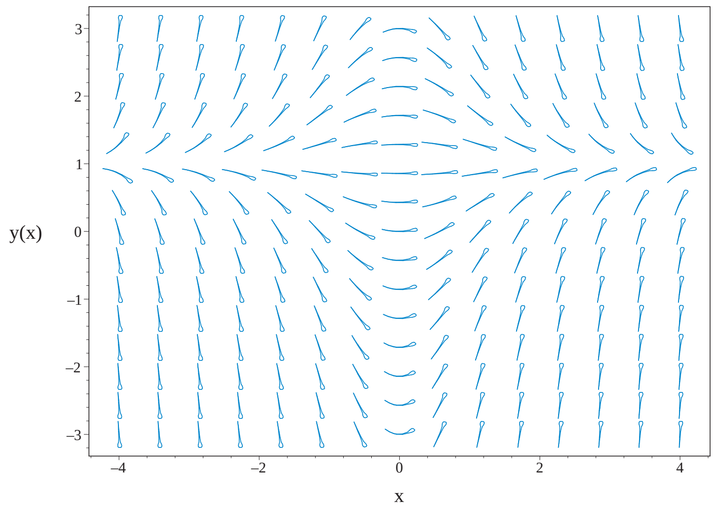


Figure 2.37: Slope field plot
$y' + xy = x$

Summary of solutions found

$$y = \left(\int x\,\mathrm{e}^{\int x dx} dx + c_1\right) \mathrm{e}^{-\int x dx}$$

Solved as first order separable ode

Time used: 0.100 (sec)

The ode

$$y' = -xy + x \tag{2.35}$$

is separable as it can be written as

$$\begin{aligned} y' &= -xy + x \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= x \\ g(y) &= -y + 1 \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

$$\int \frac{1}{-y+1}\, dy = \int x\, dx$$

$$-\ln(y-1) = \frac{x^2}{2} + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or

$$-y + 1 = 0$$

for y gives

$$y = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln(y-1) = \frac{x^2}{2} + c_1$$
$$y = 1$$

Solving for y gives

$$y = 1$$
$$y = e^{-\frac{x^2}{2}-c_1} + 1$$

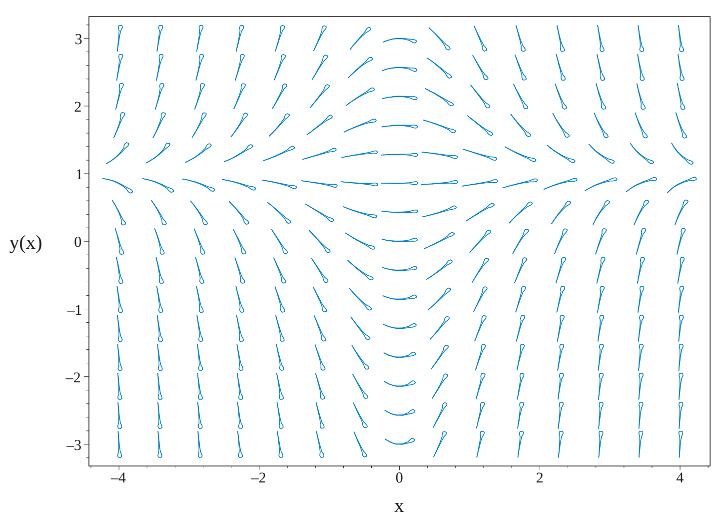


Figure 2.38: Slope field plot
$y' + xy = x$

Summary of solutions found

$$y = 1$$
$$y = e^{-\frac{x^2}{2}-c_1} + 1$$

Solved as first order Exact ode

Time used: 0.107 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-xy + x)\,\mathrm{d}x$$
$$(xy - x)\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= xy - x \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(xy - x) \\ &= x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((x) - (0)) \\ &= x \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int x \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\frac{x^2}{2}} \\ &= \mathrm{e}^{\frac{x^2}{2}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= e^{\frac{x^2}{2}}(xy - x) \\ &= x(y-1)\,e^{\frac{x^2}{2}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= e^{\frac{x^2}{2}}(1) \\ &= e^{\frac{x^2}{2}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\ \left(x(y-1)\,e^{\frac{x^2}{2}}\right) + \left(e^{\frac{x^2}{2}}\right)\frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y}\,dy &= \int \overline{N}\,dy \\ \int \frac{\partial \phi}{\partial y}\,dy &= \int e^{\frac{x^2}{2}}\,dy\end{aligned}$$

$$\phi = e^{\frac{x^2}{2}} y + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = x\,e^{\frac{x^2}{2}} y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = x(y-1)\,\mathrm{e}^{\frac{x^2}{2}}$. Therefore equation (4) becomes

$$x(y-1)\,\mathrm{e}^{\frac{x^2}{2}} = x\,\mathrm{e}^{\frac{x^2}{2}} y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -x\,\mathrm{e}^{\frac{x^2}{2}}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-x\,\mathrm{e}^{\frac{x^2}{2}}\right)\mathrm{d}x$$

$$f(x) = -\mathrm{e}^{\frac{x^2}{2}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{\frac{x^2}{2}} y - \mathrm{e}^{\frac{x^2}{2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{\frac{x^2}{2}} y - \mathrm{e}^{\frac{x^2}{2}}$$

Solving for y gives

$$y = \left(\mathrm{e}^{\frac{x^2}{2}} + c_1\right)\mathrm{e}^{-\frac{x^2}{2}}$$

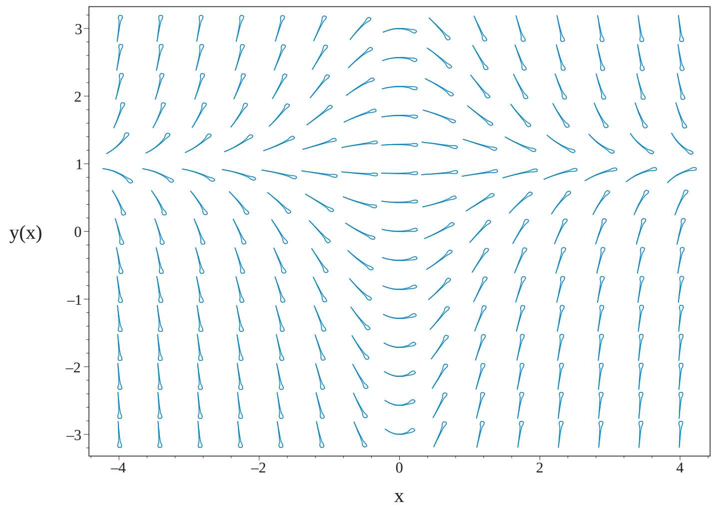


Figure 2.39: Slope field plot
$y' + xy = x$

Summary of solutions found

$$y = \left(\mathrm{e}^{\frac{x^2}{2}} + c_1\right) \mathrm{e}^{-\frac{x^2}{2}}$$

Solved using Lie symmetry for first order ode

Time used: 0.354 (sec)

Writing the ode as

$$\begin{aligned} y' &= -xy + x \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2+(-xy+x)(b_3-a_2)-(-xy+x)^2 a_3 \\
& \quad -(-y+1)(xa_2+ya_3+a_1)+x(xb_2+yb_3+b_1)=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\begin{aligned}
& -x^2y^2a_3+2x^2ya_3-x^2a_3+x^2b_2+2xya_2+y^2a_3 \\
& \quad -2xa_2+xb_1+xb_3+ya_1-ya_3-a_1+b_2=0
\end{aligned}
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -x^2y^2a_3+2x^2ya_3-x^2a_3+x^2b_2+2xya_2+y^2a_3 \\
& \quad -2xa_2+xb_1+xb_3+ya_1-ya_3-a_1+b_2=0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\{x,y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\{x=v_1, y=v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -a_3v_1^2v_2^2+2a_3v_1^2v_2+2a_2v_1v_2-a_3v_1^2+a_3v_2^2+b_2v_1^2 \\
& \quad +a_1v_2-2a_2v_1-a_3v_2+b_1v_1+b_3v_1-a_1+b_2=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
& -a_3v_1^2v_2^2+2a_3v_1^2v_2+(-a_3+b_2)v_1^2+2a_2v_1v_2 \\
& \quad +(-2a_2+b_1+b_3)v_1+a_3v_2^2+(a_1-a_3)v_2-a_1+b_2=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
2a_2 &= 0 \\
-a_3 &= 0 \\
2a_3 &= 0 \\
-a_1 + b_2 &= 0 \\
a_1 - a_3 &= 0 \\
-a_3 + b_2 &= 0 \\
-2a_2 + b_1 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= -b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= y - 1
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$S = \int \frac{1}{\eta} dy$$
$$= \int \frac{1}{y-1} dy$$

Which results in

$$S = \ln (y-1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -xy + x$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = 0$$
$$S_y = \frac{1}{y-1}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -R\,dR$$
$$S(R) = -\frac{R^2}{2} + c_2$$

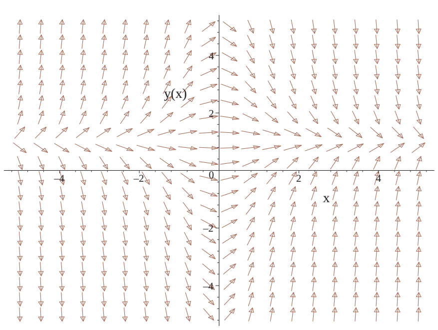
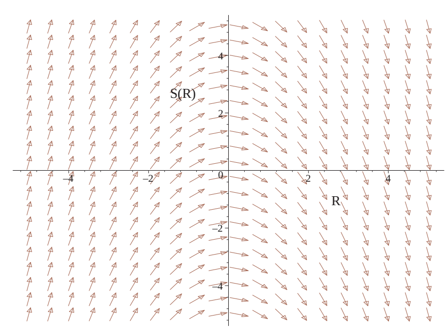
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y-1) = -\frac{x^2}{2} + c_2$$

Which gives

$$y = \mathrm{e}^{-\frac{x^2}{2}+c_2} + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -xy + x$ 	$R = x$ $S = \ln(y-1)$	$\frac{dS}{dR} = -R$ 

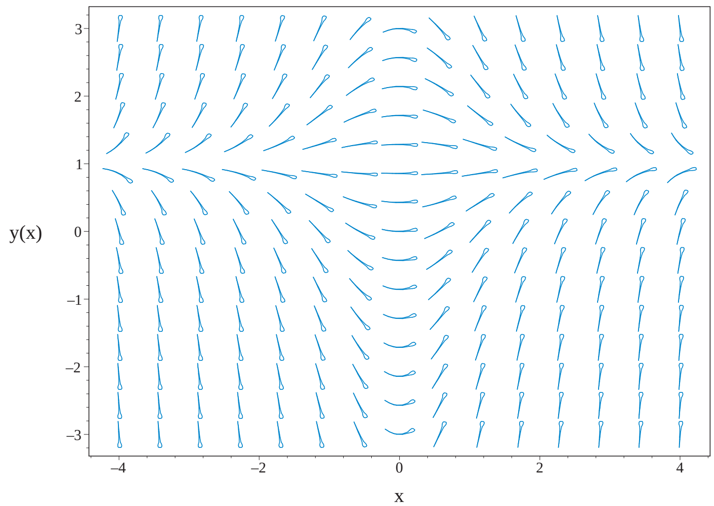


Figure 2.40: Slope field plot
$y' + xy = x$

Summary of solutions found

$$y = e^{-\frac{x^2}{2}+c_2} + 1$$

Maple step by step solution

Let's solve
$y' + xy = x$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = -xy + x$
- Separate variables
  $\frac{y'}{y-1} = -x$
- Integrate both sides with respect to x
  $\int \frac{y'}{y-1}dx = \int -xdx + C1$
- Evaluate integral
  $\ln(y-1) = -\frac{x^2}{2} + C1$
- Solve for y
  $y = e^{-\frac{x^2}{2}+C1} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 14

```
dsolve(diff(y(x),x)+x*y(x) = x,y(x),singsol=all)
```

$$y(x) = 1 + e^{-\frac{x^2}{2}} c_1$$

Mathematica DSolve solution

Solving time : 0.062 (sec)
Leaf size : 24

```
DSolve[{D[y[x],x]+x*y[x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 1 + c_1 e^{-\frac{x^2}{2}}$$
$$y(x) \to 1$$

2.4.2 Problem 2

Internal problem ID [18553]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 31. Problems at page 85
Problem number : 2
Date solved : `Tuesday, January 28, 2025 at 11:58:10 AM`
CAS classification : `[_linear]`

Solve

$$y' + \frac{y}{x} = \sin(x)$$

Solved as first order linear ode

Time used: 0.071 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{1}{x}$$
$$p(x) = \sin(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)(\sin(x))$$
$$\frac{d}{dx}(yx) = (x)(\sin(x))$$
$$d(yx) = (\sin(x)\, x)\, dx$$

Integrating gives

$$yx = \int \sin(x)\, x\, dx$$
$$= \sin(x) - \cos(x)\, x + c_1$$

Dividing throughout by the integrating factor x gives the final solution

$$y = \frac{\sin(x) - \cos(x)\, x + c_1}{x}$$

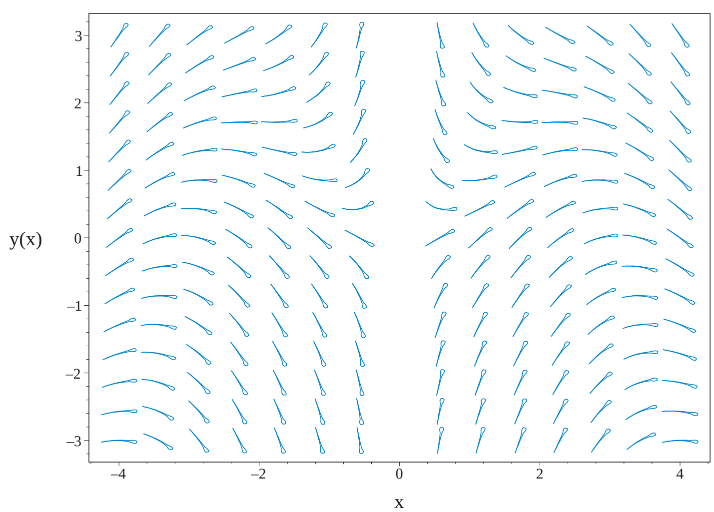


Figure 2.41: Slope field plot
$y' + \frac{y}{x} = \sin(x)$

Summary of solutions found

$$y = \frac{\sin(x) - \cos(x)\, x + c_1}{x}$$

Solved as first order Exact ode

Time used: 0.125 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(\sin(x)\,x-y)\,\mathrm{d}x\\ (-\sin(x)\,x+y)\,\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\sin(x)\,x+y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x)\,x + y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int N\,\mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int x\,\mathrm{d}y$$

$$\phi = yx + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -\sin(x)\,x + y$. Therefore equation (4) becomes

$$-\sin(x)\,x + y = y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\sin(x)\,x$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (-\sin(x)\,x)\,\mathrm{d}x$$
$$f(x) = \cos(x)\,x - \sin(x) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = yx + \cos(x)\,x - \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = yx + \cos(x)\,x - \sin(x)$$

Solving for y gives

$$y = -\frac{\cos(x)\,x - \sin(x) - c_1}{x}$$

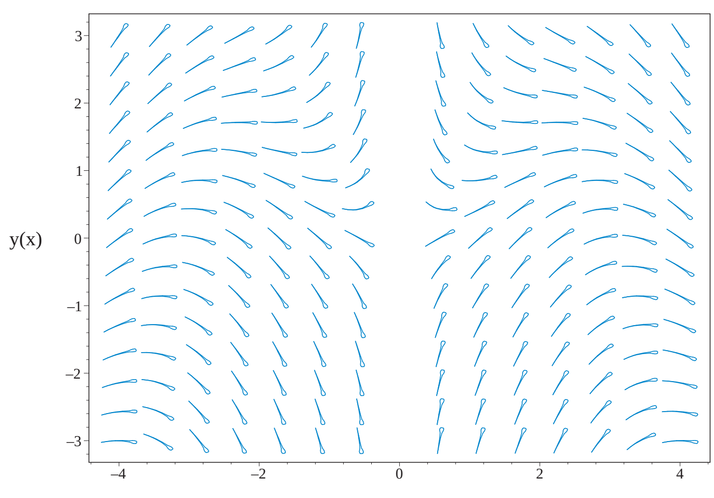


Figure 2.42: Slope field plot
$y' + \frac{y}{x} = \sin(x)$

Summary of solutions found

$$y = -\frac{\cos(x)\,x - \sin(x) - c_1}{x}$$

Maple step by step solution

Let's solve
$y' + \frac{y}{x} = \sin(x)$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = -\frac{y}{x} + \sin(x)$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
  $y' + \frac{y}{x} = \sin(x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)\sin(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y\mu(x))$
  $\mu(x)\left(y' + \frac{y}{x}\right) = y'\mu(x) + y\mu'(x)$
- Isolate $\mu'(x)$
  $\mu'(x) = \frac{\mu(x)}{x}$
- Solve to find the integrating factor
  $\mu(x) = x$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y\mu(x))\right) dx = \int \mu(x)\sin(x)\,dx + C1$
- Evaluate the integral on the lhs
  $y\mu(x) = \int \mu(x)\sin(x)\,dx + C1$
- Solve for y
  $y = \frac{\int \mu(x)\sin(x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = x$
  $y = \frac{\int \sin(x)x dx + C1}{x}$
- Evaluate the integrals on the rhs
  $y = \frac{\sin(x) - \cos(x)x + C1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 17

```
dsolve(diff(y(x),x)+y(x)/x = sin(x),y(x),singsol=all)
```

$$y(x) = \frac{\sin(x) - x\cos(x) + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 19

```
DSolve[{D[y[x],x]+y[x]/x==Sin[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sin(x) - x\cos(x) + c_1}{x}$$

2.4.3 Problem 3

Internal problem ID [18554]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 31. Problems at page 85
Problem number : 3
Date solved : `Tuesday, January 28, 2025 at 11:58:11 AM`
CAS classification : `[_Bernoulli]`

Solve

$$y' + \frac{y}{x} = \frac{\sin(x)}{y^3}$$

Solved as first order Bernoulli ode

Time used: 0.569 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{-y^4 + \sin(x)\, x}{x\, y^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \left(-\frac{1}{x}\right) y + (\sin(x)) \frac{1}{y^3} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= -\frac{1}{x} \\ f_1 &= \sin(x) \end{aligned}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \sin(x) \\ n &= -3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^3}$ gives

$$y'y^3 = -\frac{y^4}{x} + \sin(x) \tag{4}$$

Let

$$\begin{aligned} v &= y^{1-n} \\ &= y^4 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = 4y^3y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{v'(x)}{4} &= -\frac{v(x)}{x} + \sin(x) \\ v' &= -\frac{4v}{x} + 4\sin(x) \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

In canonical form a linear first order is

$$v'(x) + q(x)v(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{4}{x} \\ p(x) &= 4\sin(x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \frac{4}{x}dx} \\ &= x^4\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= (\mu)\,(4\sin(x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(v\,x^4\right) &= \left(x^4\right)(4\sin(x)) \\ \mathrm{d}\left(v\,x^4\right) &= \left(4\sin(x)\,x^4\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}v\,x^4 &= \int 4\sin(x)\,x^4\,dx \\ &= -4\cos(x)\,x^4 + 16\sin(x)\,x^3 + 48\cos(x)\,x^2 - 96\cos(x) - 96\sin(x)\,x + c_1\end{aligned}$$

Dividing throughout by the integrating factor x^4 gives the final solution

$$v(x) = \frac{4(-x^4+12x^2-24)\cos(x) + 16(x^3-6x)\sin(x) + c_1}{x^4}$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$y^4 = \frac{4(-x^4+12x^2-24)\cos(x) + 16(x^3-6x)\sin(x) + c_1}{x^4}$$

Solving for y gives

$$y = \frac{\left(-4\cos(x)\,x^4 + 16\sin(x)\,x^3 + 48\cos(x)\,x^2 - 96\cos(x) - 96\sin(x)\,x + c_1\right)^{1/4}}{x}$$

$$y = -\frac{i\left(-4\cos(x)\,x^4 + 16\sin(x)\,x^3 + 48\cos(x)\,x^2 - 96\cos(x) - 96\sin(x)\,x + c_1\right)^{1/4}}{x}$$

$$y = \frac{i\left(-4\cos(x)\,x^4 + 16\sin(x)\,x^3 + 48\cos(x)\,x^2 - 96\cos(x) - 96\sin(x)\,x + c_1\right)^{1/4}}{x}$$

$$y = -\frac{\left(-4\cos(x)\,x^4 + 16\sin(x)\,x^3 + 48\cos(x)\,x^2 - 96\cos(x) - 96\sin(x)\,x + c_1\right)^{1/4}}{x}$$

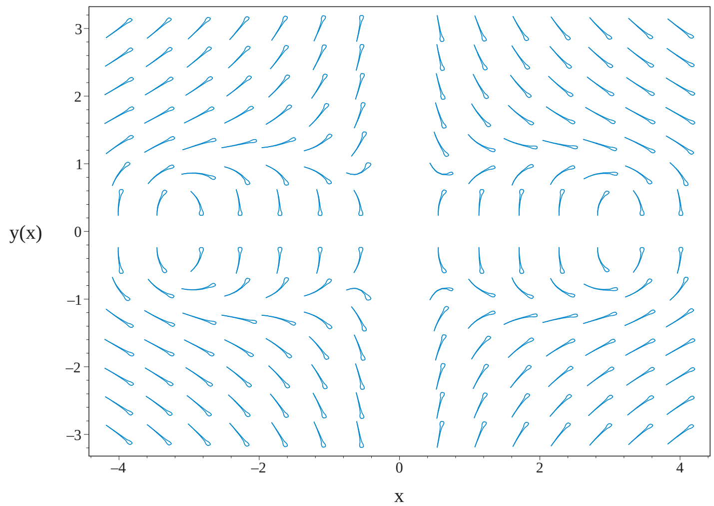


Figure 2.43: Slope field plot
$y' + \frac{y}{x} = \frac{\sin(x)}{y^3}$

Summary of solutions found

$$y = \frac{\left(-4\cos\left(x\right)x^4 + 16\sin\left(x\right)x^3 + 48\cos\left(x\right)x^2 - 96\cos\left(x\right) - 96\sin\left(x\right)x + c_1\right)^{1/4}}{x}$$

$$y = -\frac{i\left(-4\cos\left(x\right)x^4 + 16\sin\left(x\right)x^3 + 48\cos\left(x\right)x^2 - 96\cos\left(x\right) - 96\sin\left(x\right)x + c_1\right)^{1/4}}{x}$$

$$y = \frac{i\left(-4\cos\left(x\right)x^4 + 16\sin\left(x\right)x^3 + 48\cos\left(x\right)x^2 - 96\cos\left(x\right) - 96\sin\left(x\right)x + c_1\right)^{1/4}}{x}$$

$$y = -\frac{\left(-4\cos\left(x\right)x^4 + 16\sin\left(x\right)x^3 + 48\cos\left(x\right)x^2 - 96\cos\left(x\right) - 96\sin\left(x\right)x + c_1\right)^{1/4}}{x}$$

Solved as first order Exact ode

Time used: 0.532 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\,y^3\right)\mathrm{d}y &= \left(-y^4 + \sin\left(x\right)x\right)\mathrm{d}x\\ \left(y^4 - \sin\left(x\right)x\right)\mathrm{d}x + \left(x\,y^3\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^4 - \sin\left(x\right)x\\ N(x, y) &= x\,y^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^4 - \sin\left(x\right)x\right)\\ &= 4y^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\,y^3) \\ &= y^3\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x\,y^3}\left((4y^3) - (y^3)\right) \\ &= \frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \frac{3}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{3\ln(x)} \\ &= x^3\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^3\left(y^4 - \sin(x)\,x\right) \\ &= \left(y^4 - \sin(x)\,x\right)x^3\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x^3\left(x\,y^3\right) \\ &= x^4 y^3\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(y^4 - \sin(x)\,x\right)x^3\right) + \left(x^4 y^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \left(y^4 - \sin(x)\,x\right)x^3\,\mathrm{d}x$$

$$\phi = \left(x^4 - 12x^2 + 24\right)\cos(x) + 4\left(-x^3 + 6x\right)\sin(x) + \frac{y^4x^4}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = x^4y^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = x^4y^3$. Therefore equation (4) becomes

$$x^4y^3 = x^4y^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(x^4 - 12x^2 + 24\right)\cos(x) + 4\left(-x^3 + 6x\right)\sin(x) + \frac{y^4x^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \left(x^4 - 12x^2 + 24\right)\cos(x) + 4\left(-x^3 + 6x\right)\sin(x) + \frac{y^4x^4}{4}$$

Solving for y gives

$$y = \frac{\left(-4\cos\left(x\right)x^{4}+16\sin\left(x\right)x^{3}+48\cos\left(x\right)x^{2}-96\sin\left(x\right)x-96\cos\left(x\right)+4c_{1}\right)^{1/4}}{x}$$

$$y = -\frac{i\left(-4\cos\left(x\right)x^{4}+16\sin\left(x\right)x^{3}+48\cos\left(x\right)x^{2}-96\sin\left(x\right)x-96\cos\left(x\right)+4c_{1}\right)^{1/4}}{x}$$

$$y = \frac{i\left(-4\cos\left(x\right)x^{4}+16\sin\left(x\right)x^{3}+48\cos\left(x\right)x^{2}-96\sin\left(x\right)x-96\cos\left(x\right)+4c_{1}\right)^{1/4}}{x}$$

$$y = -\frac{\left(-4\cos\left(x\right)x^{4}+16\sin\left(x\right)x^{3}+48\cos\left(x\right)x^{2}-96\sin\left(x\right)x-96\cos\left(x\right)+4c_{1}\right)^{1/4}}{x}$$

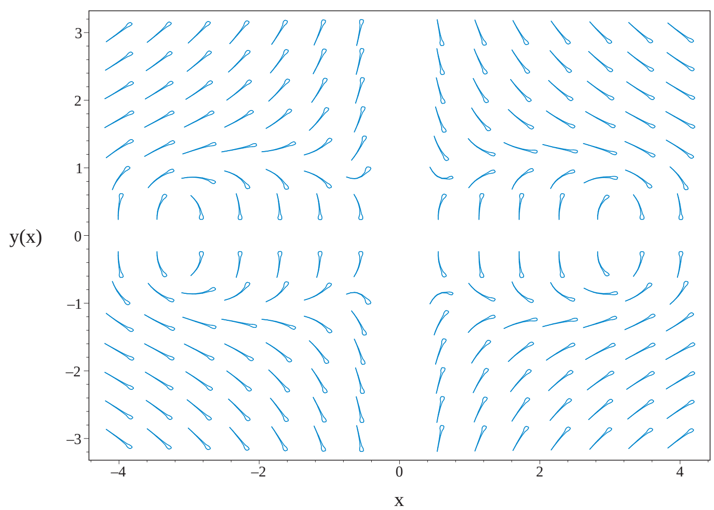


Figure 2.44: Slope field plot
$y' + \frac{y}{x} = \frac{\sin(x)}{y^3}$

Summary of solutions found

$$y = \frac{\left(-4\cos\left(x\right)x^{4}+16\sin\left(x\right)x^{3}+48\cos\left(x\right)x^{2}-96\sin\left(x\right)x-96\cos\left(x\right)+4c_{1}\right)^{1/4}}{x}$$

$$y = -\frac{i\left(-4\cos\left(x\right)x^{4}+16\sin\left(x\right)x^{3}+48\cos\left(x\right)x^{2}-96\sin\left(x\right)x-96\cos\left(x\right)+4c_{1}\right)^{1/4}}{x}$$

$$y = \frac{i\left(-4\cos\left(x\right)x^{4}+16\sin\left(x\right)x^{3}+48\cos\left(x\right)x^{2}-96\sin\left(x\right)x-96\cos\left(x\right)+4c_{1}\right)^{1/4}}{x}$$

$$y = -\frac{\left(-4\cos\left(x\right)x^{4}+16\sin\left(x\right)x^{3}+48\cos\left(x\right)x^{2}-96\sin\left(x\right)x-96\cos\left(x\right)+4c_{1}\right)^{1/4}}{x}$$

Maple step by step solution

Let's solve

$y' + \frac{y}{x} = \frac{\sin(x)}{y^3}$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $y' = -\frac{y}{x} + \frac{\sin(x)}{y^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 156

```
dsolve(diff(y(x),x)+y(x)/x = sin(x)/y(x)^3,y(x),singsol=all)
```

$$y(x) = \frac{(4(-x^4 + 12x^2 - 24)\cos(x) + 16(x^3 - 6x)\sin(x) + c_1)^{1/4}}{x}$$

$$y(x) = -\frac{(4(-x^4 + 12x^2 - 24)\cos(x) + 16(x^3 - 6x)\sin(x) + c_1)^{1/4}}{x}$$

$$y(x) = -\frac{i(4(-x^4 + 12x^2 - 24)\cos(x) + 16(x^3 - 6x)\sin(x) + c_1)^{1/4}}{x}$$

$$y(x) = \frac{i(4(-x^4 + 12x^2 - 24)\cos(x) + 16(x^3 - 6x)\sin(x) + c_1)^{1/4}}{x}$$

Mathematica DSolve solution

Solving time : `0.476` (sec)
Leaf size : 114

```
DSolve[{D[y[x],x]+y[x]/x==Sin[x]/y[x]^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{9\left(x^2-2\right)\sin(x)-3x\left(x^2-6\right)\cos(x)+c_1}}{x}$$
$$y(x) \to -\frac{\sqrt[3]{-1}\sqrt[3]{9\left(x^2-2\right)\sin(x)-3x\left(x^2-6\right)\cos(x)+c_1}}{x}$$
$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{9\left(x^2-2\right)\sin(x)-3x\left(x^2-6\right)\cos(x)+c_1}}{x}$$

2.4.4 Problem 4

Internal problem ID [18555]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 31. Problems at page 85
Problem number : 4
Date solved : `Tuesday, January 28, 2025 at 11:58:45 AM`
CAS classification : `[_linear]`

Solve

$$p' = \frac{p + a\,t^3 - 2pt^2}{t\,(-t^2+1)}$$

Solved as first order linear ode

Time used: 0.115 (sec)

In canonical form a linear first order is

$$p' + q(t)p = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = -\frac{2t^2-1}{t^3-t}$$
$$p(t) = -\frac{a\,t^2}{t^2-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q \, dt} \\ &= \mathrm{e}^{\int -\frac{2t^2-1}{t^3-t}dt} \\ &= \frac{1}{\sqrt{t-1}\,\sqrt{t+1}\,t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu p) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mu p) &= (\mu)\left(-\frac{a\,t^2}{t^2-1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\sqrt{t-1}\,\sqrt{t+1}\,t}\right) &= \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}\,t}\right)\left(-\frac{a\,t^2}{t^2-1}\right) \\ \mathrm{d}\left(\frac{p}{\sqrt{t-1}\,\sqrt{t+1}\,t}\right) &= \left(-\frac{at}{(t^2-1)\sqrt{t-1}\,\sqrt{t+1}}\right)\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{\sqrt{t-1}\,\sqrt{t+1}\,t} &= \int -\frac{at}{(t^2-1)\sqrt{t-1}\,\sqrt{t+1}}\,dt \\ &= \frac{\sqrt{t-1}\,\sqrt{t+1}\,a}{t^2-1} + c_1\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{\sqrt{t-1}\,\sqrt{t+1}\,t}$ gives the final solution

$$p = \frac{\left(\sqrt{t-1}\,\sqrt{t+1}\,a + c_1(t^2-1)\right)\sqrt{t-1}\,\sqrt{t+1}\,t}{t^2-1}$$

Summary of solutions found

$$p = \frac{\left(\sqrt{t-1}\,\sqrt{t+1}\,a + c_1(t^2-1)\right)\sqrt{t-1}\,\sqrt{t+1}\,t}{t^2-1}$$

Solved as first order homogeneous class D2 ode

Time used: 0.119 (sec)

Applying change of variables $p = u(t)\,t$, then the ode becomes

$$u'(t)\,t + u(t) = \frac{u(t)\,t + a\,t^3 - 2u(t)\,t^3}{t\,(-t^2+1)}$$

Which is now solved The ode

$$u'(t) = \frac{t(u(t) - a)}{t^2 - 1} \tag{2.36}$$

is separable as it can be written as

$$\begin{aligned} u'(t) &= \frac{t(u(t) - a)}{t^2 - 1} \\ &= f(t)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= \frac{t}{t^2 - 1} \\ g(u) &= u - a \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(t)\,dt \\ \int \frac{1}{u - a}\,du &= \int \frac{t}{t^2 - 1}\,dt \end{aligned}$$

$$\ln\left(-u(t) + a\right) = \ln\left(\sqrt{t^2 - 1}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$u - a = 0$$

for $u(t)$ gives

$$u(t) = a$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(-u(t)+a) = \ln\left(\sqrt{t^2-1}\right)+c_1$$
$$u(t) = a$$

Solving for $u(t)$ gives

$$u(t) = a$$
$$u(t) = -e^{c_1}\sqrt{t^2-1}+a$$

Converting $u(t) = a$ back to p gives

$$p = at$$

Converting $u(t) = -e^{c_1}\sqrt{t^2-1}+a$ back to p gives

$$p = \left(-e^{c_1}\sqrt{t^2-1}+a\right)t$$

Summary of solutions found

$$p = at$$
$$p = \left(-e^{c_1}\sqrt{t^2-1}+a\right)t$$

Solved as first order Exact ode

Time used: 0.125 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, p)\,\mathrm{d}t + N(t, p)\,\mathrm{d}p = 0 \tag{1A}$$

Therefore

$$\mathrm{d}p = \left(\frac{a\,t^3 - 2p\,t^2 + p}{t\,(-t^2 + 1)}\right)\mathrm{d}t$$
$$\left(-\frac{a\,t^3 - 2p\,t^2 + p}{t\,(-t^2 + 1)}\right)\mathrm{d}t + \mathrm{d}p = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, p) = -\frac{a\,t^3 - 2p\,t^2 + p}{t\,(-t^2 + 1)}$$
$$N(t, p) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial p} &= \frac{\partial}{\partial p}\left(-\frac{a\,t^3 - 2p\,t^2 + p}{t\,(-t^2 + 1)}\right)\\ &= \frac{-2t^2 + 1}{t^3 - t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial p} - \frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(-\frac{-2t^2+1}{t(-t^2+1)}\right) - (0)\right)\\ &= \frac{-2t^2+1}{t^3-t}\end{aligned}$$

Since A does not depend on p, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int \frac{-2t^2+1}{t^3-t}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}-\ln(t)}\\ &= \frac{1}{\sqrt{t-1}\,\sqrt{t+1}\,t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\sqrt{t-1}\,\sqrt{t+1}\,t}\left(-\frac{a\,t^3-2p\,t^2+p}{t(-t^2+1)}\right)\\ &= \frac{a\,t^3-2p\,t^2+p}{t^2(t^2-1)\sqrt{t-1}\,\sqrt{t+1}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{\sqrt{t-1}\,\sqrt{t+1}\,t}(1)\\ &= \frac{1}{\sqrt{t-1}\,\sqrt{t+1}\,t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}p}{\mathrm{d}t} = 0$$

$$\left(\frac{a\,t^3 - 2p\,t^2 + p}{t^2\,(t^2-1)\,\sqrt{t-1}\,\sqrt{t+1}}\right) + \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}\,t}\right)\frac{\mathrm{d}p}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t,p)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial p} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. p gives

$$\int \frac{\partial \phi}{\partial p}\,\mathrm{d}p = \int \overline{N}\,\mathrm{d}p$$

$$\int \frac{\partial \phi}{\partial p}\,\mathrm{d}p = \int \frac{1}{\sqrt{t-1}\,\sqrt{t+1}\,t}\,\mathrm{d}p$$

$$\phi = \frac{p}{\sqrt{t-1}\,\sqrt{t+1}\,t} + f(t) \tag{3}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and p. Taking derivative of equation (3) w.r.t t gives

$$\frac{\partial \phi}{\partial t} = -\frac{p}{2\,(t-1)^{3/2}\,\sqrt{t+1}\,t} - \frac{p}{2\sqrt{t-1}\,(t+1)^{3/2}\,t} - \frac{p}{\sqrt{t-1}\,\sqrt{t+1}\,t^2} + f'(t) \tag{4}$$

$$= -\frac{2\left(t^2 - \frac{1}{2}\right)p}{(t-1)^{3/2}\,(t+1)^{3/2}\,t^2} + f'(t)$$

But equation (1) says that $\frac{\partial \phi}{\partial t} = \frac{a\,t^3 - 2p\,t^2 + p}{t^2(t^2-1)\sqrt{t-1}\,\sqrt{t+1}}$. Therefore equation (4) becomes

$$\frac{a\,t^3 - 2p\,t^2 + p}{t^2\,(t^2-1)\,\sqrt{t-1}\,\sqrt{t+1}} = -\frac{2\left(t^2 - \frac{1}{2}\right)p}{(t-1)^{3/2}\,(t+1)^{3/2}\,t^2} + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$f'(t) = \frac{at}{(t-1)^{3/2}\,(t+1)^{3/2}}$$

Integrating the above w.r.t t gives

$$\int f'(t)\,\mathrm{d}t = \int \left(\frac{at}{(t-1)^{3/2}\,(t+1)^{3/2}}\right) \mathrm{d}t$$
$$f(t) = -\frac{a}{\sqrt{t-1}\,\sqrt{t+1}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(t)$ into equation (3) gives ϕ

$$\phi = \frac{p}{\sqrt{t-1}\,\sqrt{t+1}\,t} - \frac{a}{\sqrt{t-1}\,\sqrt{t+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{p}{\sqrt{t-1}\,\sqrt{t+1}\,t} - \frac{a}{\sqrt{t-1}\,\sqrt{t+1}}$$

Solving for p gives

$$p = c_1\sqrt{t-1}\,\sqrt{t+1}\,t + at$$

Summary of solutions found

$$p = c_1\sqrt{t-1}\,\sqrt{t+1}\,t + at$$

Solved using Lie symmetry for first order ode

Time used: 0.415 (sec)

Writing the ode as

$$p' = \frac{-a\,t^3 + 2p\,t^2 - p}{t\,(t^2-1)}$$
$$p' = \omega(t, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_p - \xi_t) - \omega^2\xi_p - \omega_t\xi - \omega_p\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ta_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + tb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(-a\,t^3 + 2p\,t^2 - p)\,(b_3 - a_2)}{t\,(t^2 - 1)} - \frac{(-a\,t^3 + 2p\,t^2 - p)^2\,a_3}{t^2\,(t^2 - 1)^2} \\ &- \left(\frac{-3a\,t^2 + 4pt}{t\,(t^2 - 1)} - \frac{-a\,t^3 + 2p\,t^2 - p}{t^2\,(t^2 - 1)} - \frac{2(-a\,t^3 + 2p\,t^2 - p)}{(t^2 - 1)^2}\right)(pa_3 \\ &+ ta_2 + a_1) - \frac{(2t^2 - 1)\,(pb_3 + tb_2 + b_1)}{t\,(t^2 - 1)} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{a^2t^6a_3 - 4ap\,t^5a_3 - a\,t^6a_2 + a\,t^6b_3 + 2p^2t^4a_3 + t^6b_2 + 4ap\,t^3a_3 + 3a\,t^4a_2 - a\,t^4b_3 - 2p\,t^4a_1 + 2t^5b_1 + 2a\,t}{t^2\,(t^2 - 1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-a^2t^6a_3 + 4ap\,t^5a_3 + a\,t^6a_2 - a\,t^6b_3 - 2p^2t^4a_3 - t^6b_2 - 4ap\,t^3a_3 - 3a\,t^4a_2 + a\,t^4b_3 \\ &+ 2p\,t^4a_1 - 2t^5b_1 - 2a\,t^3a_1 + 3p^2t^2a_3 + 2p\,t^3a_2 + t^4b_2 - p\,t^2a_1 + 3t^3b_1 + pa_1 - tb_1 \\ &= 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{p, t\}$ in them.

$$\{p, t\}$$

The following substitution is now made to be able to collect on all terms with $\{p, t\}$ in them

$$\{p = v_1, t = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a^2a_3v_2^6 + aa_2v_2^6 + 4aa_3v_1v_2^5 - ab_3v_2^6 - 2a_3v_1^2v_2^4 - b_2v_2^6 \\
&- 3aa_2v_2^4 - 4aa_3v_1v_2^3 + ab_3v_2^4 + 2a_1v_1v_2^4 - 2b_1v_2^5 - 2aa_1v_2^3 \\
&+ 2a_2v_1v_2^3 + 3a_3v_1^2v_2^2 + b_2v_2^4 - a_1v_1v_2^2 + 3b_1v_2^3 + a_1v_1 - b_1v_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-2a_3v_1^2v_2^4 + 3a_3v_1^2v_2^2 + 4aa_3v_1v_2^5 + 2a_1v_1v_2^4 + (-4aa_3 + 2a_2)\, v_1v_2^3 \\
&- a_1v_1v_2^2 + a_1v_1 + \left(-a^2a_3 + aa_2 - ab_3 - b_2\right) v_2^6 - 2b_1v_2^5 \\
&+ (-3aa_2 + ab_3 + b_2)\, v_2^4 + (-2aa_1 + 3b_1)\, v_2^3 - b_1v_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
-a_1 &= 0 \\
2a_1 &= 0 \\
-2a_3 &= 0 \\
3a_3 &= 0 \\
-2b_1 &= 0 \\
-b_1 &= 0 \\
4aa_3 &= 0 \\
-2aa_1 + 3b_1 &= 0 \\
-4aa_3 + 2a_2 &= 0 \\
-3aa_2 + ab_3 + b_2 &= 0 \\
-a^2a_3 + aa_2 - ab_3 - b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= -ab_3 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 0$$
$$\eta = -at + p$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial p}\right) S(t, p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-at + p} dy \end{aligned}$$

Which results in

$$S = \ln(-at + p)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, p) S_p}{R_t + \omega(t, p) R_p} \tag{2}$$

Where in the above R_t, R_p, S_t, S_p are all partial derivatives and $\omega(t, p)$ is the right hand side of the original ode given by

$$\omega(t, p) = \frac{-a\, t^3 + 2p\, t^2 - p}{t\,(t^2 - 1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_p &= 0 \\
S_t &= \frac{a}{at-p} \\
S_p &= \frac{1}{-at+p}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2t^2-1}{t^3-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R^2-1}{R^3-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int \frac{2R^2-1}{R(R^2-1)}\, dR \\
S(R) &= \frac{\ln(R-1)}{2} + \frac{\ln(R+1)}{2} + \ln(R) + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to t, p coordinates. This results in

$$\ln(-at+p) = \frac{\ln(t-1)}{2} + \frac{\ln(t+1)}{2} + \ln(t) + c_2$$

Which gives

$$p = t\left(e^{-\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}-c_2} a + 1\right) e^{\ln(\sqrt{t-1})+\ln(\sqrt{t+1})+c_2}$$

Summary of solutions found

$$p = t\left(e^{-\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}-c_2} a + 1\right) e^{\ln(\sqrt{t-1})+\ln(\sqrt{t+1})+c_2}$$

Maple step by step solution

Let's solve

$p' = \frac{p+a\,t^3-2pt^2}{t(-t^2+1)}$

- Highest derivative means the order of the ODE is 1

  p'

- Solve for the highest derivative

  $p' = \frac{p+a\,t^3-2pt^2}{t(-t^2+1)}$

- Collect w.r.t. p and simplify

  $p' = \frac{(2t^2-1)p}{t(t^2-1)} - \frac{a\,t^2}{t^2-1}$

- Group terms with p on the lhs of the ODE and the rest on the rhs of the ODE

  $p' - \frac{(2t^2-1)p}{t(t^2-1)} = -\frac{a\,t^2}{t^2-1}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(p' - \frac{(2t^2-1)p}{t(t^2-1)}\right) = -\frac{\mu(t)a\,t^2}{t^2-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(p\mu(t))$

  $\mu(t)\left(p' - \frac{(2t^2-1)p}{t(t^2-1)}\right) = p'\mu(t) + p\mu'(t)$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{\mu(t)(2t^2-1)}{t(t^2-1)}$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{t\sqrt{t+1}\,\sqrt{t-1}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(p\mu(t))\right) dt = \int -\frac{\mu(t)a\,t^2}{t^2-1} dt + C1$

- Evaluate the integral on the lhs

  $p\mu(t) = \int -\frac{\mu(t)a\,t^2}{t^2-1} dt + C1$

- Solve for p

  $p = \frac{\int -\frac{\mu(t)a\,t^2}{t^2-1} dt + C1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{t\sqrt{t+1}\,\sqrt{t-1}}$

  $p = t\sqrt{t+1}\,\sqrt{t-1}\left(\int -\frac{at}{(t^2-1)\sqrt{t+1}\,\sqrt{t-1}} dt + C1\right)$

- Evaluate the integrals on the rhs

  $p = t\sqrt{t+1}\,\sqrt{t-1}\left(\frac{\sqrt{t-1}\,\sqrt{t+1}\,a}{t^2-1} + C1\right)$

- Simplify

$$p = \frac{t\left(\sqrt{t-1}\,\sqrt{t+1}\,a + C1\,(t^2-1)\right)\sqrt{t-1}\,\sqrt{t+1}}{t^2-1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 20

```
dsolve(diff(p(t),t) = (p(t)+a*t^3-2*p(t)*t^2)/t/(-t^2+1),p(t),singsol=all)
```

$$p = t\left(\sqrt{t+1}\,\sqrt{t-1}\,c_1 + a\right)$$

Mathematica DSolve solution

Solving time : 0.048 (sec)
Leaf size : 23

```
DSolve[{D[p[t],t]==(p[t]+a*t^3-2*p[t]*t^2 )/(t*(1-t^2)),{}},p[t],t,IncludeSingularSolutions-
```

$$p(t) \to t\left(a + c_1\sqrt{1-t^2}\right)$$

2.4.5 Problem 5

Internal problem ID [18556]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 31. Problems at page 85
Problem number : 5
Date solved : `Tuesday, January 28, 2025 at 11:58:47 AM`
CAS classification : `[_Bernoulli]`

Solve

$$(T\ln(t)-1)\,T = tT'$$

Solved as first order Bernoulli ode

Time used: 0.110 (sec)

In canonical form, the ODE is

$$\begin{aligned} T' &= F(t,T) \\ &= \frac{(T\ln(t)-1)\,T}{t} \end{aligned}$$

This is a Bernoulli ODE.

$$T' = \left(-\frac{1}{t}\right)T + \left(\frac{\ln(t)}{t}\right)T^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$T' = f_0(t)T + f_1(t)T^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= -\frac{1}{t} \\ f_1 &= \frac{\ln(t)}{t} \end{aligned}$$

The first step is to divide the above equation by T^n which gives

$$\frac{T'}{T^n} = f_0(t)T^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $v = T^{1-n}$ in equation (3) which generates a new ODE in $v(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $T(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= -\frac{1}{t} \\ f_1(t) &= \frac{\ln(t)}{t} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $T^n = T^2$ gives

$$T'\frac{1}{T^2} = -\frac{1}{tT} + \frac{\ln(t)}{t} \tag{4}$$

Let

$$\begin{aligned} v &= T^{1-n} \\ &= \frac{1}{T} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$v' = -\frac{1}{T^2}T' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -v'(t) &= -\frac{v(t)}{t} + \frac{\ln(t)}{t} \\ v' &= \frac{v}{t} - \frac{\ln(t)}{t} \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(t)$ which is now solved.

In canonical form a linear first order is

$$v'(t) + q(t)v(t) = p(t)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{1}{t} \\
p(t) &= -\frac{\ln(t)}{t}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dt} \\
&= \mathrm{e}^{\int -\frac{1}{t}dt} \\
&= \frac{1}{t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu v) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu v) &= (\mu)\left(-\frac{\ln(t)}{t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v}{t}\right) &= \left(\frac{1}{t}\right)\left(-\frac{\ln(t)}{t}\right) \\
\mathrm{d}\left(\frac{v}{t}\right) &= \left(-\frac{\ln(t)}{t^2}\right)\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{v}{t} &= \int -\frac{\ln(t)}{t^2}\,dt \\
&= \frac{\ln(t)}{t} + \frac{1}{t} + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{t}$ gives the final solution

$$
v(t) = c_1 t + \ln(t) + 1
$$

The substitution $v = T^{1-n}$ is now used to convert the above solution back to T which results in

$$
\frac{1}{T} = c_1 t + \ln(t) + 1
$$

Solving for T gives

$$
T = \frac{1}{c_1 t + \ln(t) + 1}
$$

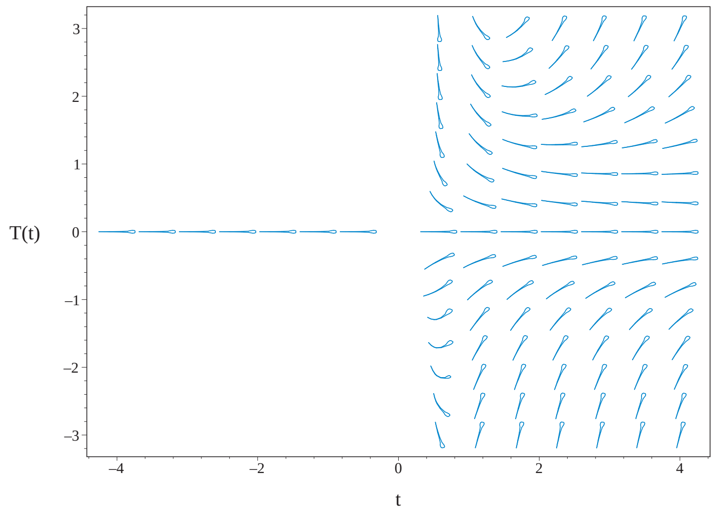


Figure 2.45: Slope field plot
$(T\ln(t)-1)\,T = tT'$

Summary of solutions found

$$T = \frac{1}{c_1 t + \ln(t) + 1}$$

Solved as first order Exact ode

Time used: 0.125 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,T)\,\mathrm{d}t + N(t,T)\,\mathrm{d}T = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-t)\,\mathrm{d}T &= (-(T\ln(t)-1)\,T)\,\mathrm{d}t\\ ((T\ln(t)-1)\,T)\,\mathrm{d}t + (-t)\,\mathrm{d}T &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,T) &= (T\ln(t)-1)\,T\\ N(t,T) &= -t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial T} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial T} &= \frac{\partial}{\partial T}((T\ln(t)-1)\,T)\\ &= -1 + 2T\ln(t)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(-t)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial T} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial T} - \frac{\partial N}{\partial t}\right)\\ &= -\frac{1}{t}((-1+2T\ln(t)) - (-1))\\ &= -\frac{2T\ln(t)}{t}\end{aligned}$$

Since A depends on T, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial T}\right) \\ &= \frac{1}{(T\ln(t) - 1)\,T}((-1) - (-1 + 2T\ln(t))) \\ &= -\frac{2\ln(t)}{T\ln(t) - 1} \end{aligned}$$

Since B depends on t, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial T}}{xM - yN}$$

R is now checked to see if it is a function of only $t = tT$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial T}}{xM - yN} \\ &= \frac{(-1) - (-1 + 2T\ln(t))}{t\,((T\ln(t) - 1)\,T) - T\,(-t)} \\ &= -\frac{2}{tT} \end{aligned}$$

Replacing all powers of terms tT by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R\,\mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right)\mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(t)} \\ &= \frac{1}{t^2} \end{aligned}$$

Now t is replaced back with tT giving

$$\mu = \frac{1}{T^2 t^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{T^2 t^2}((T\ln(t)-1)\,T) \\ &= \frac{T\ln(t)-1}{T\,t^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{T^2 t^2}(-t) \\ &= -\frac{1}{t\,T^2}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}T}{\mathrm{d}t} &= 0 \\ \left(\frac{T\ln(t)-1}{T\,t^2}\right)+\left(-\frac{1}{t\,T^2}\right)\frac{\mathrm{d}T}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,T)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial T} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int\overline{M}\,\mathrm{d}t \\ \int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int\frac{T\ln(t)-1}{T\,t^2}\,\mathrm{d}t\end{aligned}$$

$$\phi = \frac{-T\ln(t)-T+1}{Tt}+f(T) \tag{3}$$

Where $f(T)$ is used for the constant of integration since ϕ is a function of both t and T. Taking derivative of equation (3) w.r.t T gives

$$\frac{\partial\phi}{\partial T} = \frac{-\ln(t)-1}{Tt}-\frac{-T\ln(t)-T+1}{T^2 t}+f'(T) \tag{4}$$

$$= -\frac{1}{t\,T^2}+f'(T)$$

But equation (2) says that $\frac{\partial \phi}{\partial T} = -\frac{1}{t T^2}$. Therefore equation (4) becomes

$$-\frac{1}{t T^2} = -\frac{1}{t T^2} + f'(T) \tag{5}$$

Solving equation (5) for $f'(T)$ gives

$$f'(T) = 0$$

Therefore

$$f(T) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(T)$ into equation (3) gives ϕ

$$\phi = \frac{-T \ln(t) - T + 1}{Tt} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{-T \ln(t) - T + 1}{Tt}$$

Solving for T gives

$$T = \frac{1}{c_1 t + \ln(t) + 1}$$

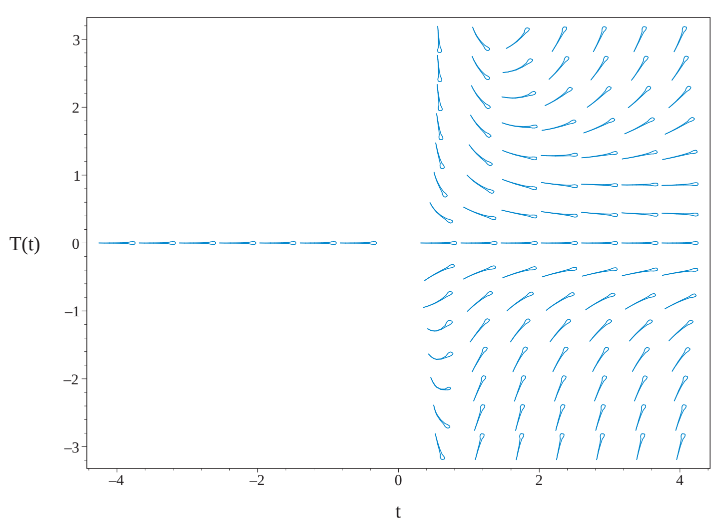


Figure 2.46: Slope field plot
$(T \ln(t) - 1) T = tT'$

Summary of solutions found

$$T = \frac{1}{c_1 t + \ln(t) + 1}$$

Solved using Lie symmetry for first order ode

Time used: 1.017 (sec)

Writing the ode as

$$T' = \frac{(T\ln(t) - 1)\,T}{t}$$
$$T' = \omega(t, T)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_T - \xi_t) - \omega^2 \xi_T - \omega_t \xi - \omega_T \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = T^2 a_6 + T t a_5 + t^2 a_4 + T a_3 + t a_2 + a_1 \tag{1E}$$
$$\eta = T^2 b_6 + T t b_5 + t^2 b_4 + T b_3 + t b_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &T b_5 + 2t b_4 + b_2 + \frac{(T\ln(t) - 1)\,T(-T a_5 + 2T b_6 - 2t a_4 + t b_5 - a_2 + b_3)}{t} \\ &- \frac{(T\ln(t) - 1)^2\,T^2(2T a_6 + t a_5 + a_3)}{t^2} \\ &- \left(\frac{T^2}{t^2} - \frac{(T\ln(t) - 1)\,T}{t^2}\right)(T^2 a_6 + T t a_5 + t^2 a_4 + T a_3 + t a_2 + a_1) \\ &- \left(\frac{T\ln(t)}{t} + \frac{T\ln(t) - 1}{t}\right)(T^2 b_6 + T t b_5 + t^2 b_4 + T b_3 + t b_2 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-T b_5 t^2 + T^3 t a_5 + T^2 t^2 a_4 + T^2 t a_5 + T^2 t b_6 - T t^2 a_4 + 2\ln(t)^2 T^5 a_6 - 5\ln(t)\,T^4 a_6 + \ln(t)^2 T^4 t a_5 - 2\ln}{}$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&Tb_5t^2 - T^3ta_5 - T^2t^2a_4 - T^2ta_5 - T^2tb_6 + T\,t^2a_4 - 2\ln(t)^2 T^5a_6 \\
&+ 5\ln(t)\,T^4a_6 - \ln(t)^2 T^4ta_5 + 2\ln(t)\,T^3ta_5 - \ln(t)\,T^2t^2a_4 \\
&- \ln(t)\,T^2t^2b_5 - 2\ln(t)\,T\,t^3b_4 - 2T^2a_3 - T^2ta_2 - \ln(t)^2 T^4a_3 \\
&+ 3\ln(t)\,T^3a_3 + \ln(t)\,T^2a_1 - \ln(t)\,T^2tb_3 - 2\ln(t)\,T\,t^2b_2 - 2\ln(t)\,Ttb_1 \\
&+ 3t^3b_4 - T^4a_6 - 3T^3a_6 + 2b_2t^2 - T^3a_3 - T^2a_1 - Ta_1 + tb_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{T, t\}$ in them.

$$\{T, t, \ln(t)\}$$

The following substitution is now made to be able to collect on all terms with $\{T, t\}$ in them

$$\{T = v_1, t = v_2, \ln(t) = v_3\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-v_3^2v_1^4v_2a_5 - 2v_3^2v_1^5a_6 - v_3^2v_1^4a_3 - v_3v_1^2v_2^2a_4 + 2v_3v_1^3v_2a_5 + 5v_3v_1^4a_6 \\
&- 2v_3v_1v_2^3b_4 - v_3v_1^2v_2^2b_5 + 3v_3v_1^3a_3 - v_1^2v_2^2a_4 - v_1^3v_2a_5 - v_1^4a_6 - 2v_3v_1v_2^2b_2 \\
&- v_3v_1^2v_2b_3 + v_3v_1^2a_1 - v_1^2v_2a_2 - v_1^3a_3 + v_1v_2^2a_4 - v_1^2v_2a_5 - 3v_1^3a_6 - 2v_3v_1v_2b_1 \\
&+ 3v_2^3b_4 + v_1b_5v_2^2 - v_1^2v_2b_6 - v_1^2a_1 - 2v_1^2a_3 + 2b_2v_2^2 - v_1a_1 + v_2b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2v_3^2v_1^5a_6 - v_3^2v_1^4v_2a_5 - v_3^2v_1^4a_3 + 5v_3v_1^4a_6 - v_1^4a_6 + 2v_3v_1^3v_2a_5 - v_1^3v_2a_5 \\
&+ 3v_3v_1^3a_3 + (-a_3 - 3a_6)\,v_1^3 + (-a_4 - b_5)\,v_1^2v_2^2v_3 - v_1^2v_2^2a_4 - v_3v_1^2v_2b_3 \\
&+ (-a_2 - a_5 - b_6)\,v_1^2v_2 + v_3v_1^2a_1 + (-a_1 - 2a_3)\,v_1^2 - 2v_3v_1v_2^3b_4 - 2v_3v_1v_2^2b_2 \\
&+ (a_4 + b_5)\,v_1v_2^2 - 2v_3v_1v_2b_1 - v_1a_1 + 3v_2^3b_4 + 2b_2v_2^2 + v_2b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
-a_1 &= 0 \\
-a_3 &= 0 \\
3a_3 &= 0 \\
-a_4 &= 0 \\
-a_5 &= 0 \\
2a_5 &= 0 \\
-2a_6 &= 0 \\
-a_6 &= 0 \\
5a_6 &= 0 \\
-2b_1 &= 0 \\
-2b_2 &= 0 \\
2b_2 &= 0 \\
-b_3 &= 0 \\
-2b_4 &= 0 \\
3b_4 &= 0 \\
-a_1 - 2a_3 &= 0 \\
-a_3 - 3a_6 &= 0 \\
-a_4 - b_5 &= 0 \\
a_4 + b_5 &= 0 \\
-a_2 - a_5 - b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_6 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -t \\
\eta &= T^2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t, T)\, \xi \\
&= T^2 - \left(\frac{(T \ln(t) - 1)\, T}{t}\right)(-t) \\
&= T^2 \ln(t) + T^2 - T \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, T) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dT}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial T}\right) S(t, T) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{T^2 \ln(t) + T^2 - T} dy \end{aligned}$$

Which results in

$$S = \ln(T \ln(t) + T - 1) - \ln(T)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, T) S_T}{R_t + \omega(t, T) R_T} \tag{2}$$

Where in the above R_t, R_T, S_t, S_T are all partial derivatives and $\omega(t, T)$ is the right hand side of the original ode given by

$$\omega(t, T) = \frac{(T \ln(t) - 1) T}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_T &= 0 \\ S_t &= \frac{T}{t (T \ln(t) + T - 1)} \\ S_T &= \frac{1}{T (T \ln(t) + T - 1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, T in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{1}{R}\,dR$$
$$S(R) = \ln(R) + c_2$$

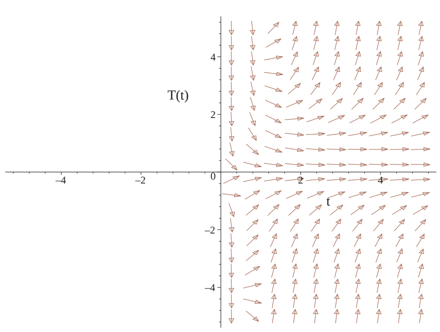
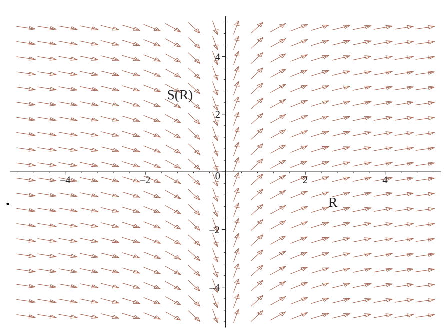
To complete the solution, we just need to transform the above back to t, T coordinates. This results in

$$\ln(T\ln(t) + T - 1) - \ln(T) = \ln(t) + c_2$$

Which gives

$$T = \frac{1}{1 - e^{c_2}t + \ln(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, T coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dT}{dt} = \frac{(T\ln(t)-1)T}{t}$ 	$R = t$ $S = \ln(T\ln(t) + T - 1)$	$\frac{dS}{dR} = \frac{1}{R}$ 

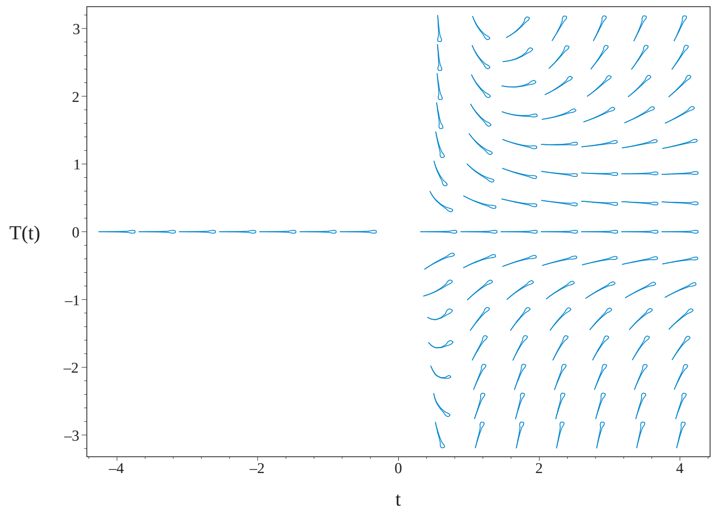


Figure 2.47: Slope field plot
$(T\ln(t)-1)T = tT'$

Summary of solutions found

$$T = \frac{1}{1-e^{c_2}t+\ln(t)}$$

Maple step by step solution

Let's solve
$(T\ln(t)-1)T = tT'$

- Highest derivative means the order of the ODE is 1
  T'
- Solve for the highest derivative
  $T' = \frac{(T\ln(t)-1)T}{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 13

```
dsolve((T(t)*ln(t)-1)*T(t) = diff(T(t),t)*t,T(t),singsol=all)
```

$$T = \frac{1}{1 + c_1 t + \ln(t)}$$

Mathematica DSolve solution

Solving time : `0.136` (sec)
Leaf size : 20

```
DSolve[{(T[t]*Log[t]-1)*T[t]==t*D[T[t],t],{}},T[t],t,IncludeSingularSolutions->True]
```

$$T(t) \to \frac{1}{\log(t) + c_1 t + 1}$$
$$T(t) \to 0$$

2.4.6 Problem 6

Internal problem ID [18557]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 31. Problems at page 85
Problem number : 6
Date solved : `Tuesday, January 28, 2025 at 11:58:49 AM`
CAS classification : `[_linear]`

Solve

$$y' + y\cos(x) = \frac{\sin(2x)}{2}$$

Solved as first order linear ode

Time used: 0.128 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \cos(x) \\ p(x) &= \frac{\sin(2x)}{2} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int \cos(x)dx} \\ &= e^{\sin(x)} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\sin(2x)}{2}\right)$$
$$\frac{d}{dx}\left(y\,e^{\sin(x)}\right) = \left(e^{\sin(x)}\right)\left(\frac{\sin(2x)}{2}\right)$$
$$d\left(y\,e^{\sin(x)}\right) = \left(\frac{\sin(2x)\,e^{\sin(x)}}{2}\right) dx$$

Integrating gives

$$y\,e^{\sin(x)} = \int \frac{\sin(2x)\,e^{\sin(x)}}{2}\,dx$$
$$= \sin(x)\,e^{\sin(x)} - e^{\sin(x)} + c_1$$

Dividing throughout by the integrating factor $e^{\sin(x)}$ gives the final solution

$$y = \sin(x) + e^{-\sin(x)}c_1 - 1$$

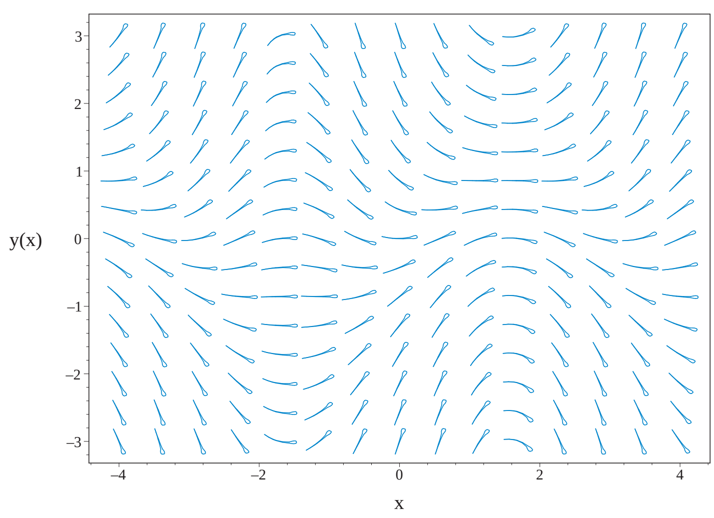


Figure 2.48: Slope field plot
$y' + y\cos(x) = \frac{\sin(2x)}{2}$

Summary of solutions found

$$y = \sin(x) + e^{-\sin(x)}c_1 - 1$$

Solved as first order Exact ode

Time used: 0.177 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-y\cos(x) + \frac{\sin(2x)}{2}\right)\mathrm{d}x$$

$$\left(y\cos(x) - \frac{\sin(2x)}{2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= y\cos(x) - \frac{\sin(2x)}{2} \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\cos(x) - \frac{\sin(2x)}{2}\right) \\ &= \cos(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cos(x)) - (0)) \\ &= \cos(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cos(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\sin(x)} \\ &= e^{\sin(x)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{\sin(x)}\left(y\cos(x) - \frac{\sin(2x)}{2}\right)\\ &= \cos(x)(-\sin(x)+y)\,\mathrm{e}^{\sin(x)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{\sin(x)}(1)\\ &= \mathrm{e}^{\sin(x)}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\cos(x)(-\sin(x)+y)\,\mathrm{e}^{\sin(x)}\right) + \left(\mathrm{e}^{\sin(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int\overline{N}\,\mathrm{d}y\\ \int\frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int\mathrm{e}^{\sin(x)}\,\mathrm{d}y\end{aligned}$$

$$\phi = y\,\mathrm{e}^{\sin(x)} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = \mathrm{e}^{\sin(x)}\cos(x)\,y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \cos(x)(-\sin(x) + y)\,\mathrm{e}^{\sin(x)}$. Therefore equation (4) becomes

$$\cos(x)(-\sin(x) + y)\,\mathrm{e}^{\sin(x)} = \mathrm{e}^{\sin(x)}\cos(x)\,y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\cos(x)\,\mathrm{e}^{\sin(x)}\sin(x)$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\cos(x)\,\mathrm{e}^{\sin(x)}\sin(x)\right)\mathrm{d}x$$

$$f(x) = -\sin(x)\,\mathrm{e}^{\sin(x)} + \mathrm{e}^{\sin(x)} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\,\mathrm{e}^{\sin(x)} - \sin(x)\,\mathrm{e}^{\sin(x)} + \mathrm{e}^{\sin(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{\sin(x)} - \sin(x)\,\mathrm{e}^{\sin(x)} + \mathrm{e}^{\sin(x)}$$

Solving for y gives

$$y = \mathrm{e}^{-\sin(x)}\left(\sin(x)\,\mathrm{e}^{\sin(x)} - \mathrm{e}^{\sin(x)} + c_1\right)$$

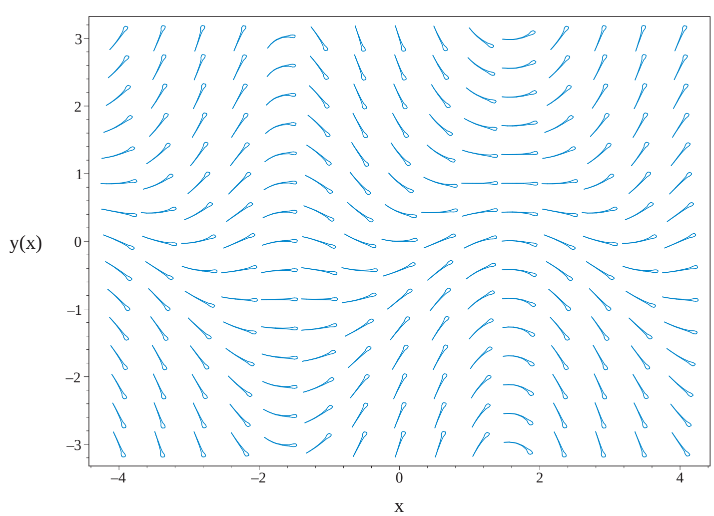


Figure 2.49: Slope field plot
$y' + y\cos(x) = \frac{\sin(2x)}{2}$

Summary of solutions found

$$y = e^{-\sin(x)}\left(\sin\left(x\right) e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

Maple step by step solution

Let's solve

$y' + y\cos\left(x\right) = \frac{\sin(2x)}{2}$

- Highest derivative means the order of the ODE is 1

  y'
- Solve for the highest derivative

  $y' = -y\cos\left(x\right) + \frac{\sin(2x)}{2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\cos\left(x\right) = \frac{\sin(2x)}{2}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\cos\left(x\right)\right) = \frac{\mu(x)\sin(2x)}{2}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y\mu(x))$

  $\mu(x)\left(y' + y\cos\left(x\right)\right) = y'\mu(x) + y\mu'(x)$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\cos\left(x\right)$
- Solve to find the integrating factor

  $\mu(x) = e^{\sin(x)}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(y\mu(x))\right) dx = \int \frac{\mu(x)\sin(2x)}{2} dx + C1$
- Evaluate the integral on the lhs

  $y\mu(x) = \int \frac{\mu(x)\sin(2x)}{2} dx + C1$
- Solve for y

  $y = \frac{\int \frac{\mu(x)\sin(2x)}{2} dx + C1}{\mu(x)}$
- Substitute $\mu(x) = e^{\sin(x)}$

  $y = \frac{\int \frac{\sin(2x) e^{\sin(x)}}{2} dx + C1}{e^{\sin(x)}}$
- Evaluate the integrals on the rhs

  $y = \frac{\sin(x) e^{\sin(x)} - e^{\sin(x)} + C1}{e^{\sin(x)}}$
- Simplify

$$y = \sin(x) + e^{-\sin(x)} C1 - 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(diff(y(x),x)+y(x)*cos(x) = 1/2*sin(2*x),y(x),singsol=all)
```

$$y(x) = \sin(x) - 1 + e^{-\sin(x)} c_1$$

Mathematica DSolve solution

Solving time : 0.052 (sec)
Leaf size : 18

```
DSolve[{D[y[x],x]+y[x]*Cos[x]==1/2*Sin[2*x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sin(x) + c_1 e^{-\sin(x)} - 1$$

2.4.7 Problem 7

Internal problem ID [18558]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 31. Problems at page 85
Problem number : 7
Date solved : `Tuesday, January 28, 2025 at 11:58:52 AM`
CAS classification : `[_Bernoulli]`

Solve

$$y - \cos(x)\, y' = y^2 \cos(x)\,(1 - \sin(x))$$

Solved as first order Bernoulli ode

Time used: 0.239 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(\cos(x)\sin(x)\, y - \cos(x)\, y + 1)}{\cos(x)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \left(\frac{1}{\cos(x)}\right) y + \left(\frac{\cos(x)\sin(x) - \cos(x)}{\cos(x)}\right) y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= \frac{1}{\cos(x)} \\ f_1 &= \frac{\cos(x)\sin(x) - \cos(x)}{\cos(x)} \end{aligned}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{\cos(x)} \\ f_1(x) &= \frac{\cos(x)\sin(x) - \cos(x)}{\cos(x)} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{\cos(x)\,y} + \frac{\cos(x)\sin(x) - \cos(x)}{\cos(x)} \tag{4}$$

Let

$$\begin{aligned} v &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -v'(x) &= \frac{v(x)}{\cos(x)} + \frac{\cos(x)\sin(x) - \cos(x)}{\cos(x)} \\ v' &= -\frac{v}{\cos(x)} - \frac{\cos(x)\sin(x) - \cos(x)}{\cos(x)} \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

In canonical form a linear first order is

$$v'(x) + q(x)v(x) = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \sec(x) \\
p(x) &= 1 - \sin(x)
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= \mathrm{e}^{\int \sec(x) dx} \\
&= \sec(x) + \tan(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= (\mu)(1 - \sin(x)) \\
\frac{\mathrm{d}}{\mathrm{d}x}(v(\sec(x) + \tan(x))) &= (\sec(x) + \tan(x))(1 - \sin(x)) \\
\mathrm{d}(v(\sec(x) + \tan(x))) &= ((1 - \sin(x))(\sec(x) + \tan(x))) \, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
v(\sec(x) + \tan(x)) &= \int (1 - \sin(x))(\sec(x) + \tan(x)) \, dx \\
&= \sin(x) + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor $\sec(x) + \tan(x)$ gives the final solution

$$
v(x) = \frac{(\sin(x) + c_1)(\cos(x) - \sin(x) + 1)}{\cos(x) + \sin(x) + 1}
$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$
\frac{1}{y} = \frac{(\sin(x) + c_1)(\cos(x) - \sin(x) + 1)}{\cos(x) + \sin(x) + 1}
$$

Solving for y gives

$$
y = \frac{\cos(x) + \sin(x) + 1}{\cos(x)\sin(x) + \cos(x) c_1 - \sin(x)^2 - c_1 \sin(x) + \sin(x) + c_1}
$$

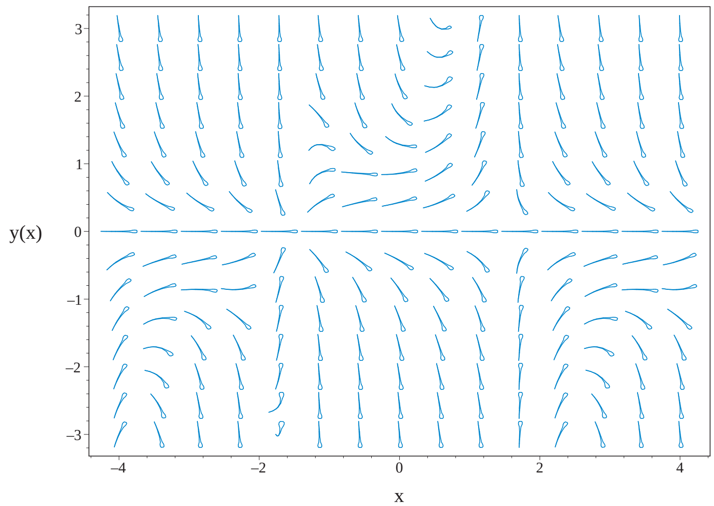


Figure 2.50: Slope field plot
$y - \cos(x) y' = y^2 \cos(x) (1 - \sin(x))$

Summary of solutions found

$$y = \frac{\cos(x) + \sin(x) + 1}{\cos(x)\sin(x) + \cos(x) c_1 - \sin(x)^2 - c_1 \sin(x) + \sin(x) + c_1}$$

Maple step by step solution

Let's solve
$y - \cos(x) y' = y^2 \cos(x) (1 - \sin(x))$

- Highest derivative means the order of the ODE is 1
y'
- Solve for the highest derivative
$y' = -\frac{-y + y^2 \cos(x)(1-\sin(x))}{\cos(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 27

```
dsolve(y(x)-cos(x)*diff(y(x),x) = y(x)^2*cos(x)*(1-sin(x)),y(x),singsol=all)
```

$$y(x) = \frac{\cos(x) + \sin(x) + 1}{(c_1 + \sin(x))(-\sin(x) + \cos(x) + 1)}$$

Mathematica DSolve solution

Solving time : 0.419 (sec)
Leaf size : 41

```
DSolve[{y[x]-Cos[x]*D[y[x],x]==y[x]^2*Cos[x]*(1-Sin[x]),{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{2\operatorname{arctanh}\left(\tan\left(\frac{x}{2}\right)\right)}}{\cos(x)e^{2\operatorname{arctanh}\left(\tan\left(\frac{x}{2}\right)\right)} + c_1}$$
$$y(x) \to 0$$

2.5 Chapter IV. Methods of solution: First order equations. section 32. Problems at page 89

2.5.1 Problem 2

Internal problem ID [18559]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 32. Problems at page 89
Problem number : 2
Date solved : `Tuesday, January 28, 2025 at 11:58:57 AM`
CAS classification : `[_rational, _dAlembert]`

Solve

$$xy'^2 - y + 2y' = 0$$

Solved as first order ode of type dAlembert

Time used: 0.112 (sec)

Let $p = y'$ the ode becomes

$$x\,p^2 + 2p - y = 0$$

Solving for y from the above results in

$$y = x\,p^2 + 2p \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = p^2$$
$$g = 2p$$

Hence (2) becomes

$$-p^2 + p = (2xp + 2)\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 + p = 0$$

Solving the above for p results in

$$p_1 = 0$$
$$p_2 = 1$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 0$$
$$y = x + 2$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 + p(x)}{2p(x)\, x + 2} \tag{3}$$

Inverting the above ode gives

$$\frac{d}{dp} x(p) = \frac{2x(p)\, p + 2}{-p^2 + p} \tag{4}$$

This ODE is now solved for $x(p)$. The integrating factor is

$$\mu = \mathrm{e}^{\int \frac{2}{p-1} dp}$$
$$\mu = (p-1)^2$$
$$\mu = (p-1)^2 \tag{5}$$

Integrating gives

$$\begin{aligned} x(p) &= \frac{1}{\mu}\left(\int \mu\left(-\frac{2}{p\,(p-1)}\right) dp + c_1\right) \\ &= \frac{1}{\mu}\left(\frac{-2p + 2\ln(p) + c_1}{(p-1)^2} + c_1\right) \\ &= \frac{-2p + 2\ln(p) + c_1}{(p-1)^2} \end{aligned} \tag{5}$$

Now we need to eliminate p between the above solution and (1A). The first method is to solve for p from Eq. (1A) and substitute the result into Eq. (5). The Second method is to solve for p from Eq. (5) and substitute the result into (1A).

Eliminating p from the following two equations

$$x = \frac{-2p + 2\ln(p) + c_1}{(p-1)^2}$$
$$y = x\,p^2 + 2p$$

results in

$$p = e^{\mathrm{RootOf}\left(-x\,e^{2\_Z}+2x\,e^{\_Z}-2\,e^{\_Z}+c_1+2\_Z-x\right)}$$

Substituting the above into Eq (1A) and simplifying gives

$$y = x\,e^{2\,\mathrm{RootOf}\left(-x\,e^{2\_Z}+2x\,e^{\_Z}-2\,e^{\_Z}+c_1+2\_Z-x\right)} + 2\,e^{\mathrm{RootOf}\left(-x\,e^{2\_Z}+2x\,e^{\_Z}-2\,e^{\_Z}+c_1+2\_Z-x\right)}$$

Summary of solutions found

$$y = 0$$
$$y = x + 2$$
$$y = x\,e^{2\,\mathrm{RootOf}\left(-x\,e^{2\_Z}+2x\,e^{\_Z}-2\,e^{\_Z}+c_1+2\_Z-x\right)} + 2\,e^{\mathrm{RootOf}\left(-x\,e^{2\_Z}+2x\,e^{\_Z}-2\,e^{\_Z}+c_1+2\_Z-x\right)}$$

Maple step by step solution

Let's solve
$xy'^2 - y + 2y' = 0$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $\left[y' = \frac{-1+\sqrt{xy+1}}{x}, y' = -\frac{1+\sqrt{xy+1}}{x}\right]$
- Solve the equation $y' = \frac{-1+\sqrt{xy+1}}{x}$
- Solve the equation $y' = -\frac{1+\sqrt{xy+1}}{x}$
- Set of solutions
  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 65

```
dsolve(x*diff(y(x),x)^2-y(x)+2*diff(y(x),x) = 0,y(x),singsol=all)
```

$$y(x) = 2x\,\mathrm{e}^{\mathrm{RootOf}\left(-\mathrm{e}^{2\_Z}x+2x\,\mathrm{e}^{\_Z}-2\,\mathrm{e}^{\_Z}+c_1+2\_Z-x\right)} + 2\,\mathrm{RootOf}\left(-\mathrm{e}^{2\_Z}x+2x\,\mathrm{e}^{\_Z}-2\,\mathrm{e}^{\_Z}+c_1+2\_Z-x\right)+c_1-x$$

Mathematica DSolve solution

Solving time : 12.594 (sec)
Leaf size : 50

```
DSolve[{x*D[y[x],x]^2-y[x]+2*D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$\mathrm{Solve}\left[\left\{x = \frac{2\log(K[1])-2K[1]}{(K[1]-1)^2}+\frac{c_1}{(K[1]-1)^2}, y(x) = xK[1]^2+2K[1]\right\}, \{y(x), K[1]\}\right]$$

2.5.2 Problem 3

Internal problem ID [18560]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 32. Problems at page 89
Problem number : 3
Date solved : `Tuesday, January 28, 2025 at 11:58:59 AM`
CAS classification : `[_quadrature]`

Solve

$$2{y'}^3 + {y'}^2 - y = 0$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} + \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} \tag{1}$$

$$y' = -\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} + \frac{i\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} - \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}\right)}{2} \tag{2}$$

$$y' = -\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} - \frac{i\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} - \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}\right)}{2} \tag{3}$$

Now each of the above is solved separately.

Solving Eq. (1)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{6\left(-1+54\tau+6\sqrt{81\tau^{2}-3\tau}\right)^{1/3}}{\left(-1+54\tau+6\sqrt{81\tau^{2}-3\tau}\right)^{2/3}-\left(-1+54\tau+6\sqrt{81\tau^{2}-3\tau}\right)^{1/3}+1}d\tau = x+c_{1}$$

We now need to find the singular solutions, these are found by finding for what values $(\frac{\left(-1+54y+6\sqrt{81y^{2}-3y}\right)^{1/3}}{6}+\frac{1}{6\left(-1+54y+6\sqrt{81y^{2}-3y}\right)^{1/3}}-\frac{1}{6})$ is zero. These give

$$y = \text{RootOf}\left(-\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^{2}-\_Z}\right)^{2/3} + \left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^{2}-\_Z}\right)^{1/3}-1\right)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $\text{RootOf}\left(-\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^{2}-\_Z}\right)^{2/3}+\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^{2}-\_Z}\right)^{1/}\right.$ will not be used

Solving Eq. (2)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{12\left(-1+54\tau+6\sqrt{81\tau^{2}-3\tau}\right)^{1/3}}{i\sqrt{3}\,\left(-1+54\tau+6\sqrt{81\tau^{2}-3\tau}\right)^{2/3}-i\sqrt{3}-\left(-1+54\tau+6\sqrt{81\tau^{2}-3\tau}\right)^{2/3}-2\left(-1+54\tau+6\sqrt{81\tau^{2}\right.}}$$

We now need to find the singular solutions, these are found by finding for what values $(-\frac{\left(-1+54y+6\sqrt{81y^{2}-3y}\right)^{1/3}}{12}-\frac{1}{12\left(-1+54y+6\sqrt{81y^{2}-3y}\right)^{1/3}}-\frac{1}{6}+\frac{i\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{81y^{2}-3y}\right)^{1/3}}{6}-\frac{1}{6\left(-1+54y+6\sqrt{81y^{2}-3y}\right)^{1/3}}\right)}{2})$ is zero. These give

$$y = \text{RootOf}\left(i\sqrt{3}\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^{2}-\_Z}\right)^{2/3}-i\sqrt{3} - \left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^{2}-\_Z}\right)^{2/3} - 2\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^{2}-\_Z}\right)^{1/3}-1\right)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $\mathrm{RootOf}\left(i\sqrt{3}\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^2-\_Z}\right)^{2/3}-i\sqrt{3}-\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z\right.\right.$ will not be used

Solving Eq. (3)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y}-\frac{12\left(-1+54\tau+6\sqrt{81\tau^2-3\tau}\right)^{1/3}}{i\sqrt{3}\left(-1+54\tau+6\sqrt{81\tau^2-3\tau}\right)^{2/3}+\left(-1+54\tau+6\sqrt{81\tau^2-3\tau}\right)^{2/3}-i\sqrt{3}+2\left(-1+54\tau+6\sqrt{81\right.}}$$

We now need to find the singular solutions, these are found by finding for what values

$$\left(-\frac{\left(-1+54y+6\sqrt{81y^2-3y}\right)^{1/3}}{12}-\frac{1}{12\left(-1+54y+6\sqrt{81y^2-3y}\right)^{1/3}}-\frac{1}{6}-\frac{i\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{81y^2-3y}\right)^{1/3}}{6}-\frac{1}{6\left(-1+54y+6\sqrt{81y^2-3y}\right)^{1/3}}\right)}{2}\right)$$

is zero. These give

$$\begin{aligned} y = \mathrm{RootOf}\Bigg(& i\sqrt{3}\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^2-\_Z}\right)^{2/3} \\ &+\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^2-\_Z}\right)^{2/3}-i\sqrt{3} \\ &+2\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^2-\_Z}\right)^{1/3}+1\Bigg)\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $\mathrm{RootOf}\left(i\sqrt{3}\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^2-\_Z}\right)^{2/3}+\left(-1+54\_Z+6\sqrt{3}\,\sqrt{27\_Z^2-\_Z\right.\right.$ will not be used

Maple step by step solution

Let's solve

$2y'^3+y'^2-y=0$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

$$\left[y' = \frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} + \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6}, y' = -\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{}{12(}\right.$$

□ Solve the equation $y' = \frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} + \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6}$

- Separate variables

$$\frac{y'}{\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} + \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} + \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6}} dx = \int 1dx + \_C1$$

- Cannot compute integral

$$\int \frac{y'}{\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} + \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6}} dx = x + \_C1$$

□ Solve the equation $y' = -\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} - \frac{I\sqrt{3}\left(\frac{\left(-1+54y+6\right.}{}\right.}{}$

- Separate variables

$$\frac{y'}{-\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} - \frac{I\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} - \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}\right)}{2}}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{-\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} - \frac{I\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} - \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}\right.}{2}}$$

- Cannot compute integral

$$\int \frac{y'}{-\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} - \frac{I\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} - \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}\right.}{2}}$$

☐ Solve the equation $y' = -\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} + \frac{I\sqrt{3}\left(\frac{\left(-1+54y+6\cdots\right.}{\cdots}\right.}{\cdots}$

- Separate variables

$$\frac{y'}{-\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} + \frac{I\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} - \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}\right)}{2}\cdots}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{-\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} + \frac{I\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} - \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}\cdots\right.}{2}\cdots}$$

- Cannot compute integral

$$\int \frac{y'}{-\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{1}{12\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6} + \frac{I\sqrt{3}\left(\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} - \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}\cdots\right.}{2}\cdots}$$

- Set of solutions

$$\left\{\int \frac{y'}{\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{6} + \frac{1}{6\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}} - \frac{1}{6}}dx = x + C1, \int \frac{\cdots}{-\frac{\left(-1+54y+6\sqrt{-3y+81y^2}\right)^{1/3}}{12} - \frac{\cdots}{12\left(-1+54\cdots\right.}\cdots}\right.$$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
```

```
   <- differential order: 1; missing  x  successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 385

```
dsolve(2*diff(y(x),x)^3+diff(y(x),x)^2-y(x) = 0,y(x),singsol=all)
```

$$y(x) = 0$$

$$-6\sqrt{3}\left(\int^{y(x)} \frac{\left(18\sqrt{27\_a^2-\_a}+(54\_a-1)\sqrt{3}\right)^{1/3}}{3^{2/3}-\sqrt{3}\left(18\sqrt{27\_a^2-\_a}+(54\_a-1)\sqrt{3}\right)^{1/3}+3^{1/3}\left(18\sqrt{27\_a^2-\_a}+(54\_a-1)\sqrt{\right.}}\right.$$

$$\frac{-72\left(\int^{y(x)} \frac{\left(18\sqrt{27\_a^2-\_a}+(54\_a-1)\sqrt{3}\right)^{1/3}}{\left(i3^{5/6}+3^{1/3}-2\,3^{1/6}\left(18\sqrt{27\_a^2-\_a}+(54\_a-1)\sqrt{3}\right)^{1/3}\right)\left(3^{1/3}+3^{1/6}\left(18\sqrt{27\_a^2-\_a}+(54\_a-1)\sqrt{3}\right)^{1/3}\right)}d\_a\right)+}{\sqrt{3}+3i}$$

$$\frac{72\left(\int^{y(x)} \frac{\left(18\sqrt{27\_a^2-\_a}+(54\_a-1)\sqrt{3}\right)^{1/3}}{\left(-i3^{5/6}+3^{1/3}-2\,3^{1/6}\left(18\sqrt{27\_a^2-\_a}+(54\_a-1)\sqrt{3}\right)^{1/3}\right)\left(3^{1/3}+3^{1/6}\left(18\sqrt{27\_a^2-\_a}+(54\_a-1)\sqrt{3}\right)^{1/3}\right)}d\_a\right)+}{-\sqrt{3}+3i}$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{2*D[y[x],x]^3+D[y[x],x]^2-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

Timed out

2.5.3 Problem 4

Internal problem ID [18561]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 32. Problems at page 89
Problem number : 4
Date solved : `Tuesday, January 28, 2025 at 12:00:07 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = \mathrm{e}^{z-y'}$$

Solved as first order quadrature ode

Time used: 0.081 (sec)

Since the ode has the form $y' = f(z)$, then we only need to integrate $f(z)$.

$$\int dy = \int \operatorname{LambertW}\left(\mathrm{e}^{z}\right) dz$$

$$y = \operatorname{LambertW}\left(\mathrm{e}^{z}\right) + \frac{\operatorname{LambertW}\left(\mathrm{e}^{z}\right)^{2}}{2} + c_1$$

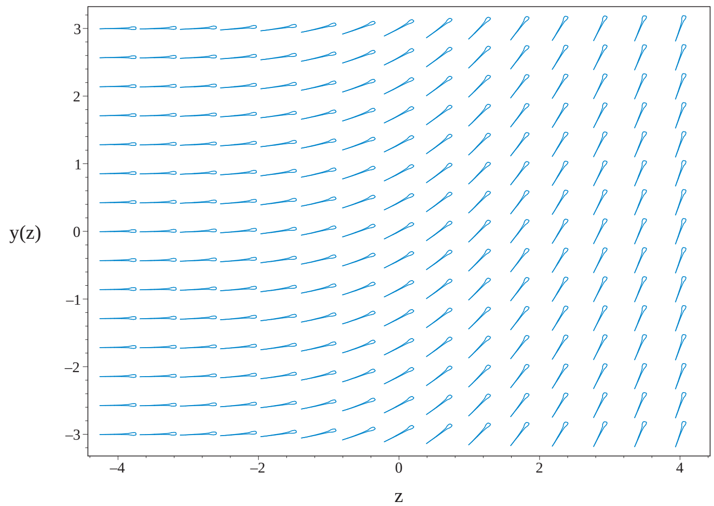


Figure 2.51: Slope field plot
$y' = e^{z-y'}$

Summary of solutions found

$$y = \mathrm{LambertW}\left(e^{z}\right) + \frac{\mathrm{LambertW}\left(e^{z}\right)^{2}}{2} + c_1$$

Maple step by step solution

Let's solve
$y' = e^{z-y'}$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = \mathit{LambertW}(e^{z})$
- Integrate both sides with respect to z
  $\int y' dz = \int \mathit{LambertW}(e^{z})\, dz + C1$
- Evaluate integral
  $y = \frac{\mathit{LambertW}(e^{z})^{2}}{2} + \mathit{LambertW}(e^{z}) + C1$
- Solve for y
  $y = \frac{\mathit{LambertW}(e^{z})^{2}}{2} + \mathit{LambertW}(e^{z}) + C1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(diff(y(z),z) = exp(z-diff(y(z),z)),y(z),singsol=all)
```

$$y = \frac{\mathrm{LambertW}\left(\mathrm{e}^{z}\right)^{2}}{2} + \mathrm{LambertW}\left(\mathrm{e}^{z}\right) + c_1$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 22

```
DSolve[{D[y[z],z]==Exp[z-D[y[z],z]],{}},y[z],z,IncludeSingularSolutions->True]
```

$$y(z) \to \frac{1}{2}W(e^{z})^{2} + W(e^{z}) + c_1$$

2.5.4 Problem 5

Internal problem ID [18562]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 32. Problems at page 89
Problem number : 5
Date solved : `Tuesday, January 28, 2025 at 12:00:08 PM`
CAS classification : `[[_homogeneous, 'class G']]`

Solve

$$\sqrt{t^2+T} = T'$$

Solved as first order isobaric ode

Time used: 1.718 (sec)

Solving for T' gives

$$T' = \sqrt{t^2+T} \tag{1}$$

Each of the above ode's is now solved An ode $T' = f(t,T)$ is isobaric if

$$f(tt, t^m T) = t^{m-1} f(t,T) \tag{1}$$

Where here

$$f(t,T) = \sqrt{t^2+T} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 2$$

Since the ode is isobaric of order $m = 2$, then the substitution

$$\begin{aligned} T &= ut^m \\ &= u\,t^2 \end{aligned}$$

Converts the ODE to a separable in $u(t)$. Performing this substitution gives

$$2tu(t) + t^2u'(t) = \sqrt{t^2 + t^2u(t)}$$

The ode

$$u'(t) = \frac{\sqrt{1 + u(t)} - 2u(t)}{t} \tag{2.37}$$

is separable as it can be written as

$$\begin{aligned} u'(t) &= \frac{\sqrt{1 + u(t)} - 2u(t)}{t} \\ &= f(t)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= \frac{1}{t} \\ g(u) &= \sqrt{u + 1} - 2u \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(t)\,dt$$

$$\int \frac{1}{\sqrt{u + 1} - 2u}\,du = \int \frac{1}{t}\,dt$$

$$-\frac{\ln\left(2u(t) - \sqrt{1 + u(t)}\right)}{2} + \frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(4\sqrt{1+u(t)}-1\right)\sqrt{17}}{17}\right)}{17} = \ln(t) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\sqrt{u + 1} - 2u = 0$$

for $u(t)$ gives

$$u(t) = \frac{1}{8} + \frac{\sqrt{17}}{8}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln\left(2u(t)-\sqrt{1+u\left(t\right)}\right)}{2}+\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(4\sqrt{1+u(t)}-1\right)\sqrt{17}}{17}\right)}{17}=\ln\left(t\right)+c_1$$

$$u(t)=\frac{1}{8}+\frac{\sqrt{17}}{8}$$

Converting $-\frac{\ln\left(2u(t)-\sqrt{1+u(t)}\right)}{2}+\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(4\sqrt{1+u(t)}-1\right)\sqrt{17}}{17}\right)}{17}=\ln\left(t\right)+c_1$ back to T gives

$$-\frac{\ln\left(\frac{2T}{t^2}-\sqrt{1+\frac{T}{t^2}}\right)}{2}+\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(4\sqrt{1+\frac{T}{t^2}}-1\right)\sqrt{17}}{17}\right)}{17}=\ln\left(t\right)+c_1$$

Converting $u(t)=\frac{1}{8}+\frac{\sqrt{17}}{8}$ back to T gives

$$\frac{T}{t^2}=\frac{1}{8}+\frac{\sqrt{17}}{8}$$

Solving for T gives

$$-\frac{\ln\left(\frac{2T}{t^2}-\sqrt{1+\frac{T}{t^2}}\right)}{2}+\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(4\sqrt{1+\frac{T}{t^2}}-1\right)\sqrt{17}}{17}\right)}{17}=\ln\left(t\right)+c_1$$

$$T=\frac{\left(1+\sqrt{17}\right)t^2}{8}$$

Summary of solutions found

$$-\frac{\ln\left(\frac{2T}{t^2}-\sqrt{1+\frac{T}{t^2}}\right)}{2}+\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(4\sqrt{1+\frac{T}{t^2}}-1\right)\sqrt{17}}{17}\right)}{17}=\ln\left(t\right)+c_1$$

$$T=\frac{\left(1+\sqrt{17}\right)t^2}{8}$$

Solved using Lie symmetry for first order ode

Time used: 1.231 (sec)

Writing the ode as

$$
\begin{aligned}
T' &= \sqrt{t^2+T} \\
T' &= \omega(t,T)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_T - \xi_t) - \omega^2 \xi_T - \omega_t \xi - \omega_T \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = T a_3 + t a_2 + a_1 \tag{1E}
$$

$$
\eta = T b_3 + t b_2 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
b_2 + \sqrt{t^2+T}\,(b_3 - a_2) - \left(t^2+T\right) a_3 - \frac{t(T a_3 + t a_2 + a_1)}{\sqrt{t^2+T}} - \frac{T b_3 + t b_2 + b_1}{2\sqrt{t^2+T}} = 0 \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{2\sqrt{t^2+T}\,t^2 a_3 + 2\sqrt{t^2+T}\,T a_3 + 2T t a_3 + 4t^2 a_2 - 2t^2 b_3 - 2b_2\sqrt{t^2+T} + 2T a_2 - T b_3 + 2t a_1 + t b_2 + b_1}{2\sqrt{t^2+T}} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2\sqrt{t^2+T}\,t^2 a_3 - 2\sqrt{t^2+T}\,T a_3 - 2T t a_3 - 4t^2 a_2 + 2t^2 b_3 \\
&+ 2b_2\sqrt{t^2+T} - 2T a_2 + T b_3 - 2t a_1 - t b_2 - b_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-2\sqrt{t^2+T}\,t^2a_3 - 2(t^2+T)\,a_2 + 2(t^2+T)\,b_3 - 2\sqrt{t^2+T}\,Ta_3 \\
&\quad - 2Tta_3 - 2t^2a_2 + 2b_2\sqrt{t^2+T} - Tb_3 - 2ta_1 - tb_2 - b_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2\sqrt{t^2+T}\,t^2a_3 - 2\sqrt{t^2+T}\,Ta_3 - 2Tta_3 - 4t^2a_2 + 2t^2b_3 \\
&\quad + 2b_2\sqrt{t^2+T} - 2Ta_2 + Tb_3 - 2ta_1 - tb_2 - b_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{T, t\}$ in them.

$$
\left\{T, t, \sqrt{t^2+T}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{T, t\}$ in them

$$
\left\{T = v_1, t = v_2, \sqrt{t^2+T} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_3v_2^2a_3 - 4v_2^2a_2 - 2v_1v_2a_3 - 2v_3v_1a_3 + 2v_2^2b_3 \\
&\quad - 2v_2a_1 - 2v_1a_2 - v_2b_2 + 2b_2v_3 + v_1b_3 - b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2v_1v_2a_3 - 2v_3v_1a_3 + (-2a_2 + b_3)\,v_1 - 2v_3v_2^2a_3 \\
&\quad + (-4a_2 + 2b_3)\,v_2^2 + (-2a_1 - b_2)\,v_2 + 2b_2v_3 - b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_3 &= 0 \\
-b_1 &= 0 \\
2b_2 &= 0 \\
-2a_1 - b_2 &= 0 \\
-4a_2 + 2b_3 &= 0 \\
-2a_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= t \\
\eta &= 2T
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t, T)\,\xi \\
&= 2T - \left(\sqrt{t^2+T}\right)(t) \\
&= -\sqrt{t^2+T}\,t + 2T \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, T) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dT}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial T}\right) S(t, T) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{t^2+T}\,t + 2T} dy
\end{aligned}
$$

Which results in

$$S = -\frac{\ln\left(\sqrt{t^2+T}\,t+2T\right)}{4} - \frac{\sqrt{17}\,\text{arctanh}\left(\frac{\left(4\sqrt{t^2+T}+t\right)\sqrt{17}}{17t}\right)}{34} + \frac{\ln\left(-\sqrt{t^2+T}\,t+2T\right)}{4} - \frac{\sqrt{17}\,\text{arctanh}\left(\right.}{3}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t+\omega(t,T)S_T}{R_t+\omega(t,T)R_T} \tag{2}$$

Where in the above R_t, R_T, S_t, S_T are all partial derivatives and $\omega(t,T)$ is the right hand side of the original ode given by

$$\omega(t,T) = \sqrt{t^2+T}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_T &= 0 \\ S_t &= \frac{t^6+2T\,t^4-3T^2t^2-4T^3}{\left(\sqrt{t^2+T}\,t-2T\right)^2\left(\sqrt{t^2+T}\,t+2T\right)\sqrt{t^2+T}} \\ S_T &= \frac{\left(t+2\sqrt{t^2+T}\right)T+t^3}{\left(-t^4-T\,t^2+4T^2\right)\sqrt{t^2+T}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t,T in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R,S.

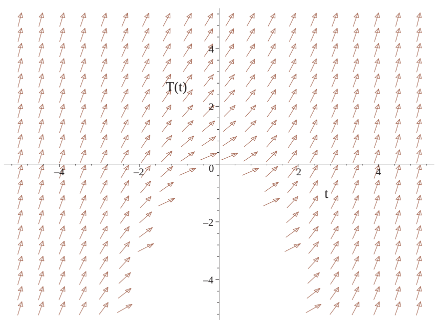
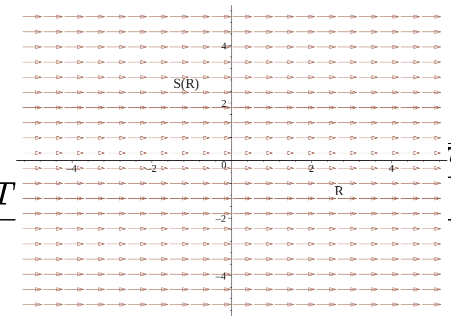
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 0\,dR + c_2 \\ S(R) &= c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to t, T coordinates. This results in

$$-\frac{\ln\left(\sqrt{t^2+T}\,t+2T\right)}{4}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(4\sqrt{t^2+T}+t\right)\sqrt{17}}{17t}\right)}{34}+\frac{\ln\left(-\sqrt{t^2+T}\,t+2T\right)}{4}+\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(t-4\sqrt{\ }\right.}{}\right.}{34}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, T coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dT}{dt}=\sqrt{t^2+T}$ 	$R = t$ $S = -\frac{\ln\left(\sqrt{t^2+T}\,t+2T\right)}{4}$	$\frac{dS}{dR}=0$ 

<u>Summary of solutions found</u>

$$-\frac{\ln\left(\sqrt{t^2+T}\,t+2T\right)}{4}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(4\sqrt{t^2+T}+t\right)\sqrt{17}}{17t}\right)}{34}+\frac{\ln\left(-\sqrt{t^2+T}\,t+2T\right)}{4}+\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(t-4\sqrt{t^2+T}\right)\sqrt{17}}{17t}\right)}{34}+\frac{\ln\left(-t^4-T t^2+4T^2\right)}{4}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(-t^2+8T\right)\sqrt{17}}{17t^2}\right)}{34}=c_2$$

Maple step by step solution

Let's solve

$\sqrt{t^2+T} = T'$

- Highest derivative means the order of the ODE is 1

  T'

- Solve for the highest derivative

  $T' = \sqrt{t^2+T}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 136

```
dsolve((t^2+T(t))^(1/2) = diff(T(t),t),T(t),singsol=all)
```

$$17\ln\left(-t^4-t^2T+4T^2\right)+17\ln\left(-\sqrt{t^2+T}\,t+2T\right)-17\ln\left(\sqrt{t^2+T}\,t+2T\right)+\left(2\operatorname{arctanh}\left(\frac{(t^2-8T)\sqrt{17}}{17t^2}\right)+2\operatorname{arctanh}\left(\frac{\left(t-4\sqrt{t^2+T}\right)\sqrt{17}}{17t}\right)-2\operatorname{arctanh}\left(\frac{\left(4\sqrt{t^2+T}+t\right)\sqrt{17}}{17t}\right)\right)\sqrt{17}-c_1=0$$

Mathematica DSolve solution

Solving time : 0.277 (sec)
Leaf size : 135

```
DSolve[{Sqrt[t^2+T[t]]==D[T[t],t],{}},T[t],t,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{1}{34}\left(-34\log\left(\sqrt{t^2+T(t)}-t\right)-\left(\sqrt{17}-17\right)\log\left(2\left(\sqrt{17}-4\right)t\sqrt{t^2+T(t)}-2\left(\sqrt{17}-4\right)t^2-\left(\sqrt{17}-3\right)T(t)\right)+\left(17+\sqrt{17}\right)\log\left(2\left(4+\sqrt{17}\right)t\sqrt{t^2+T(t)}-2\left(4+\sqrt{17}\right)t^2-\left(3+\sqrt{17}\right)T(t)\right)\right)=c_1,T(t)\right]$$

2.5.5 Problem 7

Internal problem ID [18563]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 32. Problems at page 89
Problem number : 7
Date solved : `Tuesday, January 28, 2025 at 12:00:12 PM`
CAS classification : `[_quadrature]`

Solve

$$\left(x^2-1\right) {y'}^2 = 1$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{1}{\sqrt{x^2-1}} \tag{1}$$

$$y' = -\frac{1}{\sqrt{x^2-1}} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{1}{\sqrt{x^2-1}}\, dx$$

$$y = \ln\left(x+\sqrt{x^2-1}\right) + c_1$$

Solving Eq. (2)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -\frac{1}{\sqrt{x^2-1}}\, dx$$

$$y = -\ln\left(x+\sqrt{x^2-1}\right) + c_2$$

Maple step by step solution

Let's solve

$\left(x^2-1\right) y'^2=1$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $\left[y'=\frac{1}{\sqrt{x^2-1}}, y'=-\frac{1}{\sqrt{x^2-1}}\right]$

□ Solve the equation $y'=\frac{1}{\sqrt{x^2-1}}$

  - Integrate both sides with respect to x

    $\int y' dx=\int \frac{1}{\sqrt{x^2-1}}dx+\_C1$

  - Evaluate integral

    $y=\ln\left(x+\sqrt{x^2-1}\right)+\_C1$

  - Solve for y

    $y=\ln\left(x+\sqrt{x^2-1}\right)+\_C1$

□ Solve the equation $y'=-\frac{1}{\sqrt{x^2-1}}$

  - Integrate both sides with respect to x

    $\int y' dx=\int -\frac{1}{\sqrt{x^2-1}}dx+\_C1$

  - Evaluate integral

    $y=-\ln\left(x+\sqrt{x^2-1}\right)+\_C1$

  - Solve for y

    $y=-\ln\left(x+\sqrt{x^2-1}\right)+\_C1$

- Set of solutions

  $\left\{y=-\ln\left(x+\sqrt{x^2-1}\right)+C1, y=\ln\left(x+\sqrt{x^2-1}\right)+C1\right\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

Maple dsolve solution

Solving time : `0.016` (sec)
Leaf size : 33

```
dsolve((x^2-1)*diff(y(x),x)^2 = 1,y(x),singsol=all)
```

$$y(x) = \ln\left(x + \sqrt{x^2 - 1}\right) + c_1$$
$$y(x) = -\ln\left(x + \sqrt{x^2 - 1}\right) + c_1$$

Mathematica DSolve solution

Solving time : `0.004` (sec)
Leaf size : 41

```
DSolve[{(x^2-1)*D[y[x],x]^2==1,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\log\left(\sqrt{x^2 - 1} + x\right) + c_1$$
$$y(x) \to \log\left(\sqrt{x^2 - 1} + x\right) + c_1$$

2.5.6 Problem 8

Internal problem ID [18564]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 32. Problems at page 89
Problem number : 8
Date solved : `Tuesday, January 28, 2025 at 12:00:12 PM`
CAS classification : `[[_homogeneous, ‘class C‘], _Riccati]`

Solve

$$y' = (x+y)^2$$

Solved as first order homogeneous class C ode

Time used: 0.079 (sec)

Let

$$z = x + y \tag{1}$$

Then

$$z'(x) = 1 + y'$$

Therefore

$$y' = z'(x) - 1$$

Hence the given ode can now be written as

$$z'(x) - 1 = z^2$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^2+1} dz$$

$$x + c_1 = \arctan(z)$$

Replacing z back by its value from (1) then the above gives the solution as Solving for y gives

$$y = -x + \tan(x + c_1)$$

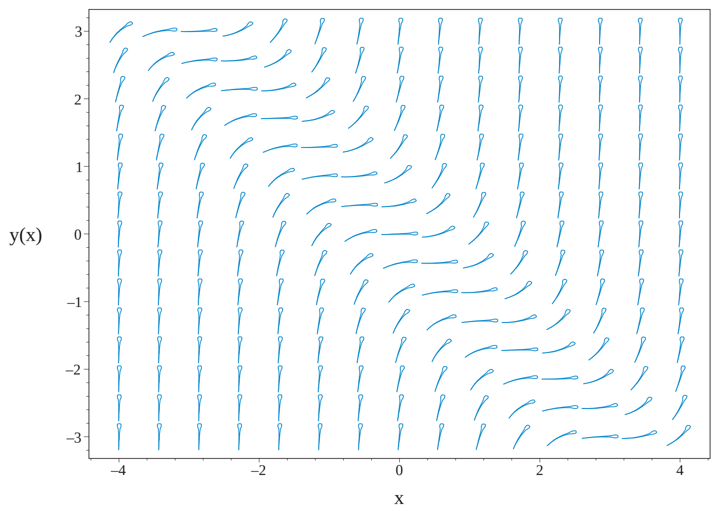


Figure 2.52: Slope field plot
$y' = (x+y)^2$

<u>Summary of solutions found</u>

$$y = -x + \tan(x + c_1)$$

Solved using Lie symmetry for first order ode

Time used: 0.515 (sec)

Writing the ode as

$$y' = (x+y)^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + (x+y)^2 (b_3 - a_2) - (x+y)^4 a_3 \\ &- (2x+2y)(x a_2 + y a_3 + a_1) - (2x+2y)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} &-x^4 a_3 - 4x^3 y a_3 - 6x^2 y^2 a_3 - 4x\, y^3 a_3 - y^4 a_3 - 3x^2 a_2 - 2x^2 b_2 + x^2 b_3 - 4xy a_2 \\ &- 2xy a_3 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 - y^2 b_3 - 2x a_1 - 2x b_1 - 2y a_1 - 2y b_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^4 a_3 - 4x^3 y a_3 - 6x^2 y^2 a_3 - 4x\, y^3 a_3 - y^4 a_3 - 3x^2 a_2 - 2x^2 b_2 + x^2 b_3 - 4xy a_2 \\ &- 2xy a_3 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 - y^2 b_3 - 2x a_1 - 2x b_1 - 2y a_1 - 2y b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3v_1^4-4a_3v_1^3v_2-6a_3v_1^2v_2^2-4a_3v_1v_2^3-a_3v_2^4-3a_2v_1^2-4a_2v_1v_2-a_2v_2^2-2a_3v_1v_2 \\ -2a_3v_2^2-2b_2v_1^2-2b_2v_1v_2+b_3v_1^2-b_3v_2^2-2a_1v_1-2a_1v_2-2b_1v_1-2b_1v_2+b_2=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &-a_3v_1^4-4a_3v_1^3v_2-6a_3v_1^2v_2^2+(-3a_2-2b_2+b_3)\,v_1^2 \\ &-4a_3v_1v_2^3+(-4a_2-2a_3-2b_2)\,v_1v_2+(-2a_1-2b_1)\,v_1 \\ &-a_3v_2^4+(-a_2-2a_3-b_3)\,v_2^2+(-2a_1-2b_1)\,v_2+b_2=0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -6a_3 &= 0 \\ -4a_3 &= 0 \\ -a_3 &= 0 \\ -2a_1-2b_1 &= 0 \\ -4a_2-2a_3-2b_2 &= 0 \\ -3a_2-2b_2+b_3 &= 0 \\ -a_2-2a_3-b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= 1 - \left((x+y)^2\right)(-1) \\ &= x^2 + 2xy + y^2 + 1 \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2 + 2xy + y^2 + 1} dy\end{aligned}$$

Which results in

$$S = \arctan(x+y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (x+y)^2$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{1 + (x + y)^2} \\
S_y &= \frac{1}{1 + (x + y)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int 1\,dR \\
S(R) &= R + c_2
\end{aligned}
$$

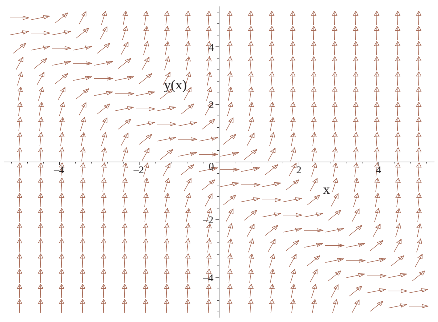
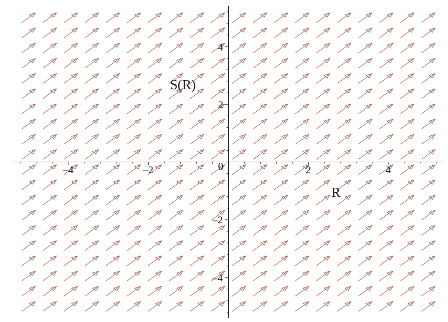
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
\arctan(x + y) = x + c_2
$$

Which gives

$$
y = -x + \tan(x + c_2)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (x+y)^2$ 	$R = x$ $S = \arctan(x+y)$	$\frac{dS}{dR} = 1$ 

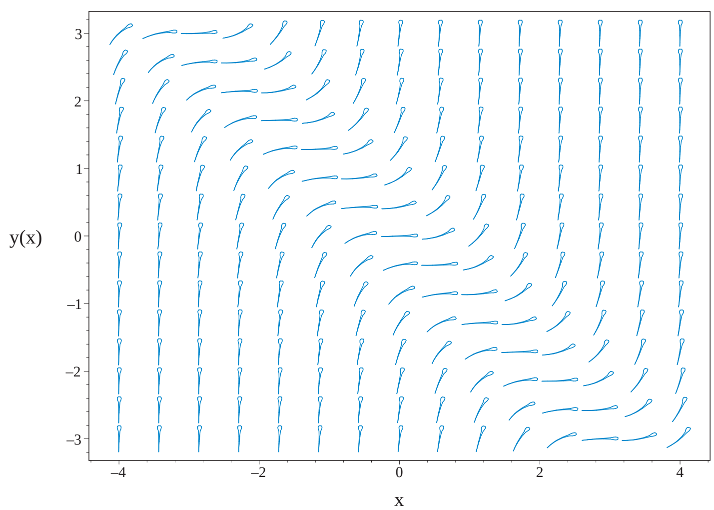


Figure 2.53: Slope field plot
$y' = (x+y)^2$

Summary of solutions found

$$y = -x + \tan(x + c_2)$$

Solved as first order ode of type Riccati

Time used: 0.548 (sec)

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= (x+y)^2
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = x^2 + 2xy + y^2
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = x^2$, $f_1(x) = 2x$ and $f_2(x) = 1$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{u} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= 2x \\
f_2^2 f_0 &= x^2
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
u''(x) - 2xu'(x) + x^2 u(x) = 0
$$

In normal form the given ode is written as

$$
\frac{d^2 u}{dx^2} + p(x)\left(\frac{du}{dx}\right) + q(x)\, u = 0 \qquad (2)
$$

Where

$$
\begin{aligned}
p(x) &= -2x \\
q(x) &= x^2
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= x^2 - \frac{(-2x)'}{2} - \frac{(-2x)^2}{4} \\
&= x^2 - \frac{(-2)}{2} - \frac{(4x^2)}{4} \\
&= x^2 - (-1) - x^2 \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
u = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\
&= e^{-\int \frac{-2x}{2}} \\
&= \mathrm{e}^{\frac{x^2}{2}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
u = v(x)\, \mathrm{e}^{\frac{x^2}{2}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
\mathrm{e}^{\frac{x^2}{2}} \left(\frac{d^2}{dx^2} v(x) + v(x) \right) = 0
$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$
\frac{d^2}{dx^2} v(x) + v(x) = 0
$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} u &= v(x)\, z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x))\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{\frac{x^2}{2}}$$

Hence (7) becomes

$$u = (c_1 \cos(x) + c_2 \sin(x))\, \mathrm{e}^{\frac{x^2}{2}}$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$u'(x) = (-c_1 \sin(x) + c_2 \cos(x))\, \mathrm{e}^{\frac{x^2}{2}} + (c_1 \cos(x) + c_2 \sin(x))\, x\, \mathrm{e}^{\frac{x^2}{2}}$$

Doing change of constants, the solution becomes

$$y = -\frac{\left((-c_3 \sin(x) + \cos(x))\, \mathrm{e}^{\frac{x^2}{2}} + (c_3 \cos(x) + \sin(x))\, x\, \mathrm{e}^{\frac{x^2}{2}}\right) \mathrm{e}^{-\frac{x^2}{2}}}{c_3 \cos(x) + \sin(x)}$$

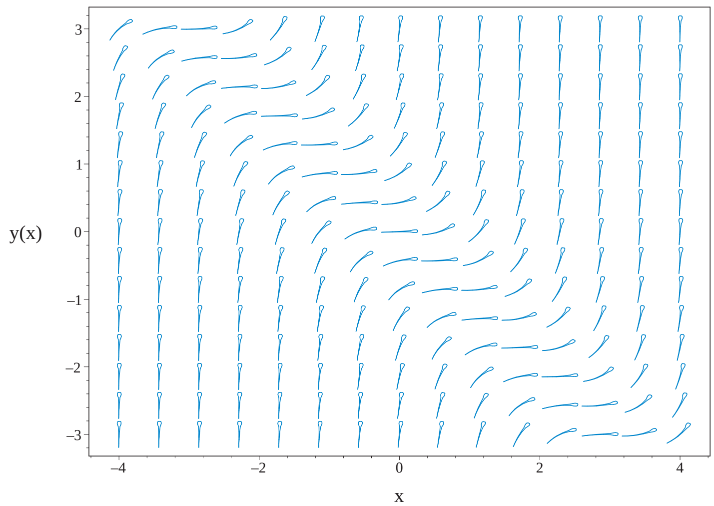


Figure 2.54: Slope field plot
$y' = (x + y)^2$

Summary of solutions found

$$y = -\frac{\left(\left(-c_3 \sin\left(x\right)+\cos\left(x\right)\right) \mathrm{e}^{\frac{x^2}{2}}+\left(c_3 \cos\left(x\right)+\sin\left(x\right)\right) x \,\mathrm{e}^{\frac{x^2}{2}}\right) \mathrm{e}^{-\frac{x^2}{2}}}{c_3 \cos\left(x\right)+\sin\left(x\right)}$$

Maple step by step solution

Let's solve

$y' = (x+y)^2$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $y' = (x+y)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 16

```
dsolve(diff(y(x),x) = (x+y(x))^2,y(x),singsol=all)
```

$$y(x) = -x-\tan\left(-x+c_1\right)$$

Mathematica DSolve solution

Solving time : 0.509 (sec)
Leaf size : 14

```
DSolve[{D[y[x],x]==(x+y[x])^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -x + \tan(x + c_1)$$

2.6 Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91

2.6.1 Problem 1

Internal problem ID [18565]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 1
Date solved : `Tuesday, January 28, 2025 at 12:00:14 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$\theta'' = -p^2\theta$$

Solved as second order linear constant coeff ode

Time used: 0.086 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\theta''(t) + B\theta'(t) + C\theta(t) = 0$$

Where in the above $A = 1, B = 0, C = p^2$. Let the solution be $\theta = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} + p^2 e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + p^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = p^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(p^2)} \\ &= \pm\sqrt{-p^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-p^2} \\ \lambda_2 &= -\sqrt{-p^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= ip \\ \lambda_2 &= -ip\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = p$. Therefore the final solution, when using Euler relation, can be written as

$$\theta = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$\theta = e^{0}(c_1 \cos(pt) + c_2 \sin(pt))$$

Or

$$\theta = c_1 \cos(pt) + c_2 \sin(pt)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\theta = c_1 \cos(pt) + c_2 \sin(pt)$$

Solved as second order can be made integrable

Time used: 0.877 (sec)

Multiplying the ode by θ' gives

$$\theta'\theta'' + p^2\theta'\theta = 0$$

Integrating the above w.r.t t gives

$$\int \left(\theta'\theta'' + p^2\theta'\theta\right) dt = 0$$
$$\frac{\theta'^2}{2} + \frac{p^2\theta^2}{2} = c_1$$

Which is now solved for θ. Solving for the derivative gives these ODE's to solve

$$\theta' = \sqrt{-p^2\theta^2 + 2c_1} \tag{1}$$
$$\theta' = -\sqrt{-p^2\theta^2 + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{-p^2\theta^2 + 2c_1}} d\theta = dt$$
$$\frac{\arctan\left(\frac{p\theta}{\sqrt{-p^2\theta^2+2c_1}}\right)}{p} = t + c_2$$

Singular solutions are found by solving

$$\sqrt{-p^2\theta^2 + 2c_1} = 0$$

for θ. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$\theta = \frac{\sqrt{2}\,\sqrt{c_1}}{p}$$
$$\theta = -\frac{\sqrt{2}\,\sqrt{c_1}}{p}$$

Solving for θ gives

$$\theta = \frac{\sqrt{2}\,\sqrt{c_1}}{p}$$

$$\theta = \frac{\tan\left(c_2 p + p t\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(c_2 p + p t)^2 + 1}}}{p}$$

$$\theta = -\frac{\sqrt{2}\,\sqrt{c_1}}{p}$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{-p^2\theta^2 + 2c_1}} d\theta = dt$$

$$-\frac{\arctan\left(\frac{p\theta}{\sqrt{-p^2\theta^2+2c_1}}\right)}{p} = t + c_3$$

Singular solutions are found by solving

$$-\sqrt{-p^2\theta^2 + 2c_1} = 0$$

for θ. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$\theta = \frac{\sqrt{2}\,\sqrt{c_1}}{p}$$

$$\theta = -\frac{\sqrt{2}\,\sqrt{c_1}}{p}$$

Solving for θ gives

$$\theta = \frac{\sqrt{2}\,\sqrt{c_1}}{p}$$

$$\theta = -\frac{\sqrt{2}\,\sqrt{c_1}}{p}$$

$$\theta = -\frac{\tan\left(c_3 p + p t\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(c_3 p + p t)^2 + 1}}}{p}$$

Will add steps showing solving for IC soon.

The solution

$$\theta = \frac{\sqrt{2}\,\sqrt{c_1}}{p}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$\theta = -\frac{\sqrt{2}\,\sqrt{c_1}}{p}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\theta = \frac{\tan\left(c_2 p + pt\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(c_2 p+pt)^2+1}}}{p}$$

$$\theta = -\frac{\tan\left(c_3 p + pt\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(c_3 p+pt)^2+1}}}{p}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.063 (sec)

Writing the ode as

$$\theta'' + p^2\theta = 0 \tag{1}$$

$$A\theta'' + B\theta' + C\theta = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= p^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = \theta e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-p^2}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -p^2 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(-p^2\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then θ is found using the inverse transformation

$$
\theta = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.31: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -p^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\sqrt{-p^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in θ is found from

$$\theta_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} \theta_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-p^2}\, t} \end{aligned}$$

Which simplifies to

$$\theta_1 = \mathrm{e}^{\sqrt{-p^2}\, t}$$

The second solution θ_2 to the original ode is found using reduction of order

$$\theta_2 = \theta_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{\theta_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} \theta_2 &= \theta_1 \int \frac{1}{\theta_1^2}\, dt \\ &= \mathrm{e}^{\sqrt{-p^2}\, t} \int \frac{1}{\mathrm{e}^{2\sqrt{-p^2}\, t}}\, dt \\ &= \mathrm{e}^{\sqrt{-p^2}\, t} \left(\frac{\sqrt{-p^2}\, \mathrm{e}^{-2\sqrt{-p^2}\, t}}{2p^2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
\theta &= c_1\theta_1 + c_2\theta_2 \\
&= c_1\left(\mathrm{e}^{\sqrt{-p^2}\,t}\right) + c_2\left(\mathrm{e}^{\sqrt{-p^2}\,t}\left(\frac{\sqrt{-p^2}\,\mathrm{e}^{-2\sqrt{-p^2}\,t}}{2p^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\theta = c_1\,\mathrm{e}^{\sqrt{-p^2}\,t} + \frac{c_2\,\mathrm{e}^{-\sqrt{-p^2}\,t}\sqrt{-p^2}}{2p^2}
$$

Solved as second order ode adjoint method

Time used: 0.599 (sec)

In normal form the ode

$$
\theta'' = -p^2\theta \tag{1}
$$

Becomes

$$
\theta'' + p(t)\,\theta' + q(t)\,\theta = r(t) \tag{2}
$$

Where

$$
\begin{aligned}
p(t) &= 0 \\
q(t) &= p^2 \\
r(t) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (0)' + \left(p^2\xi(t)\right) &= 0 \\
\xi''(t) + p^2\xi(t) &= 0
\end{aligned}
$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
A\xi''(t) + B\xi'(t) + C\xi(t) = 0
$$

Where in the above $A = 1, B = 0, C = p^2$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} + p^2 e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + p^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = p^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(p^2)} \\ &= \pm\sqrt{-p^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-p^2} \\ \lambda_2 &= -\sqrt{-p^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= ip \\ \lambda_2 &= -ip\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = p$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$\xi = e^0(c_1 \cos(pt) + c_2 \sin(pt))$$

Or

$$\xi = c_1 \cos(pt) + c_2 \sin(pt)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(t)\,\theta' - \theta\xi'(t) + \xi(t)\,p(t)\,\theta = \int \xi(t)\,r(t)\,dt$$
$$\theta' + \theta\left(p(t) - \frac{\xi'(t)}{\xi(t)}\right) = \frac{\int \xi(t)\,r(t)\,dt}{\xi(t)}$$

Or

$$\theta' - \frac{\theta(-c_1 p \sin(pt) + c_2 p \cos(pt))}{c_1 \cos(pt) + c_2 \sin(pt)} = 0$$

Which is now a first order ode. This is now solved for θ. In canonical form a linear first order is

$$\theta' + q(t)\theta = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = \frac{p(\sin(pt)\,c_1 - \cos(pt)\,c_2)}{c_1 \cos(pt) + c_2 \sin(pt)}$$
$$p(t) = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt}\\ &= \mathrm{e}^{\int \frac{p(\sin(pt)c_1 - \cos(pt)c_2)}{c_1 \cos(pt) + c_2 \sin(pt)}dt}\\ &= \frac{1}{c_1 \cos(pt) + c_2 \sin(pt)}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu\theta = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\theta}{c_1 \cos(pt) + c_2 \sin(pt)}\right) = 0$$

Integrating gives

$$\begin{aligned}\frac{\theta}{c_1 \cos(pt) + c_2 \sin(pt)} &= \int 0\,dt + c_3\\ &= c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(pt)+c_2 \sin(pt)}$ gives the final solution

$$\theta = (c_1 \cos(pt) + c_2 \sin(pt))\, c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$\theta = (c_1 \cos(pt) + c_2 \sin(pt))\, c_3$$

The constants can be merged to give

$$\theta = c_1 \cos(pt) + c_2 \sin(pt)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\theta = c_1 \cos(pt) + c_2 \sin(pt)$$

Maple step by step solution

Let's solve
$\theta'' = -p^2\theta$

- Highest derivative means the order of the ODE is 2
  θ''
- Group terms with θ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\theta'' + p^2\theta = 0$
- Characteristic polynomial of ODE
  $p^2 + r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0\pm\left(\sqrt{-4p^2}\right)}{2}$
- Roots of the characteristic polynomial
  $r = \left(\sqrt{-p^2}, -\sqrt{-p^2}\right)$
- 1st solution of the ODE
  $\theta_1(t) = e^{t\sqrt{-p^2}}$
- 2nd solution of the ODE
  $\theta_2(t) = e^{-t\sqrt{-p^2}}$
- General solution of the ODE

$$\theta = C1\theta_1(t) + C2\theta_2(t)$$

- Substitute in solutions

$$\theta = C1\,\mathrm{e}^{t\sqrt{-p^2}} + C2\,\mathrm{e}^{-t\sqrt{-p^2}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(diff(diff(theta(t),t),t) = -p^2*theta(t),theta(t),singsol=all)
```

$$\theta = c_1 \sin(pt) + c_2 \cos(pt)$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 20

```
DSolve[{D[theta[t],{t,2}]==-p^2*theta[t],{}},theta[t],t,IncludeSingularSolutions->True]
```

$$\theta(t) \to c_1 \cos(pt) + c_2 \sin(pt)$$

2.6.2 Problem 2 (eq 39)

Internal problem ID [18566]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 2 (eq 39)
Date solved : `Tuesday, January 28, 2025 at 12:00:16 PM`
CAS classification : `[_quadrature]`

Solve

$$\sec(\theta)^2 = \frac{m s'}{k}$$

Solved as first order quadrature ode

Time used: 0.106 (sec)

Since the ode has the form $s' = f(\theta)$, then we only need to integrate $f(\theta)$.

$$\int ds = \int \frac{\sec(\theta)^2 k}{m}\, d\theta$$
$$s = \frac{k\tan(\theta)}{m} + c_1$$

Summary of solutions found

$$s = \frac{k\tan(\theta)}{m} + c_1$$

Solved as first order Exact ode

Time used: 0.063 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta,s)\,\mathrm{d}\theta+N(\theta,s)\,\mathrm{d}s=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{m}{k}\right)\mathrm{d}s&=\left(-\sec(\theta)^2\right)\mathrm{d}\theta\\ \left(\sec(\theta)^2\right)\mathrm{d}\theta+\left(-\frac{m}{k}\right)\mathrm{d}s&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(\theta,s)&=\sec(\theta)^2\\ N(\theta,s)&=-\frac{m}{k}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial s} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial s} &= \frac{\partial}{\partial s}\left(\sec(\theta)^2\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(-\frac{m}{k}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial s} = \frac{\partial N}{\partial \theta}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(\theta, s)$

$$\frac{\partial \phi}{\partial \theta} = M \tag{1}$$

$$\frac{\partial \phi}{\partial s} = N \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta &= \int M \, \mathrm{d}\theta\\ \int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta &= \int \sec(\theta)^2 \, \mathrm{d}\theta\\ \phi &= \tan(\theta) + f(s) \end{aligned}\tag{3}$$

Where $f(s)$ is used for the constant of integration since ϕ is a function of both θ and s. Taking derivative of equation (3) w.r.t s gives

$$\frac{\partial \phi}{\partial s} = 0 + f'(s) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial s} = -\frac{m}{k}$. Therefore equation (4) becomes

$$-\frac{m}{k} = 0 + f'(s) \tag{5}$$

Solving equation (5) for $f'(s)$ gives

$$f'(s) = -\frac{m}{k}$$

Integrating the above w.r.t s gives

$$\int f'(s)\,\mathrm{d}s = \int \left(-\frac{m}{k}\right) \mathrm{d}s$$
$$f(s) = -\frac{ms}{k} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(s)$ into equation (3) gives ϕ

$$\phi = \tan(\theta) - \frac{ms}{k} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \tan(\theta) - \frac{ms}{k}$$

Solving for s gives

$$s = \frac{(-c_1 + \tan(\theta))\,k}{m}$$

Summary of solutions found

$$s = \frac{(-c_1 + \tan(\theta))\,k}{m}$$

Maple step by step solution

Let's solve
$\sec(\theta)^2 = \frac{ms'}{k}$

- Highest derivative means the order of the ODE is 1
  s'
- Separate variables

$s' = \frac{\sec(\theta)^2 k}{m}$

- Integrate both sides with respect to θ

  $\int s' d\theta = \int \frac{\sec(\theta)^2 k}{m} d\theta + C1$

- Evaluate integral

  $s = \frac{k \tan(\theta)}{m} + C1$

- Solve for s

  $s = \frac{k \tan(\theta) + C1 m}{m}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.000 (sec)
Leaf size : 13

```
dsolve(sec(theta)^2 = m/k*diff(s(theta),theta),s(theta),singsol=all)
```

$$s(\theta) = \frac{k \tan(\theta)}{m} + c_1$$

Mathematica DSolve solution

Solving time : 0.01 (sec)
Leaf size : 15

```
DSolve[{Sec[theta]^2==m/k*D[s[theta],theta],{}},s[theta],theta,IncludeSingularSolutions->Tru
```

$$s(\theta) \to \frac{k \tan(\theta)}{m} + c_1$$

2.6.3 Problem 3 (eq 41)

Internal problem ID [18567]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 3 (eq 41)
Date solved : `Tuesday, January 28, 2025 at 12:00:17 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' = \frac{m\sqrt{1+y'^2}}{k}$$

Solved as second order missing x ode

Time used: 8.183 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) = \frac{m\sqrt{1+p(y)^2}}{k}$$

Which is now solved as first order ode for $p(y)$.

Integrating gives

$$\int \frac{pk}{m\sqrt{p^2+1}}dp = dy$$
$$\frac{\sqrt{p^2+1}\,k}{m} = y + c_1$$

Singular solutions are found by solving

$$\frac{m\sqrt{p^2+1}}{pk} = 0$$

for p. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p = -i$$
$$p = i$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{\sqrt{1+y'^2}\,k}{m} = y + c_1$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{\sqrt{c_1^2m^2 + 2yc_1\,m^2 + y^2m^2 - k^2}}{k} \tag{1}$$
$$y' = -\frac{\sqrt{c_1^2m^2 + 2yc_1\,m^2 + y^2m^2 - k^2}}{k} \tag{2}$$

Now each of the above is solved separately.

<u>Solving Eq. (1)</u>

Integrating gives

$$\int \frac{k}{\sqrt{c_1^2m^2 + 2c_1\,m^2y + m^2y^2 - k^2}}dy = dx$$
$$\frac{k\ln\left(\frac{c_1\,m^2+m^2y}{\sqrt{m^2}} + \sqrt{c_1^2m^2 + 2c_1\,m^2y + m^2y^2 - k^2}\right)}{\sqrt{m^2}} = x + c_2$$

Singular solutions are found by solving

$$\frac{\sqrt{c_1^2m^2 + 2c_1\,m^2y + m^2y^2 - k^2}}{k} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1m - k}{m}$$
$$y = -\frac{c_1m + k}{m}$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{k}{\sqrt{c_1^2m^2 + 2c_1\,m^2y + m^2y^2 - k^2}}dy = dx$$
$$-\frac{k\ln\left(\frac{c_1\,m^2+m^2y}{\sqrt{m^2}} + \sqrt{c_1^2m^2 + 2c_1\,m^2y + m^2y^2 - k^2}\right)}{\sqrt{m^2}} = x + c_3$$

Singular solutions are found by solving

$$-\frac{\sqrt{c_1^2m^2 + 2c_1\,m^2y + m^2y^2 - k^2}}{k} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1m - k}{m}$$
$$y = -\frac{c_1m + k}{m}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -i$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -i\,dx$$
$$y = -ix + c_4$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = i$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int i\,dx$$
$$y = ix + c_5$$

Will add steps showing solving for IC soon.

Solving for y from the above solution(s) gives (after possible removing of solutions that do not verify)

$$y = -\frac{c_1 m - k}{m}$$
$$y = -\frac{c_1 m + k}{m}$$
$$y = \frac{\left(-2c_1\, m^2 \mathrm{e}^{\frac{\sqrt{m^2}\,(x+c_2)}{k}} + \sqrt{m^2}\, k^2 + \mathrm{e}^{\frac{2\sqrt{m^2}\,(x+c_2)}{k}} \sqrt{m^2}\right) \mathrm{e}^{-\frac{\sqrt{m^2}\,(x+c_2)}{k}}}{2m^2}$$
$$y = \frac{\left(-2c_1\, m^2 \mathrm{e}^{-\frac{\sqrt{m^2}\,(x+c_3)}{k}} + \sqrt{m^2}\, k^2 + \mathrm{e}^{-\frac{2\sqrt{m^2}\,(x+c_3)}{k}} \sqrt{m^2}\right) \mathrm{e}^{\frac{\sqrt{m^2}\,(x+c_3)}{k}}}{2m^2}$$
$$y = -ix + c_4$$
$$y = ix + c_5$$

The solution

$$y = -\frac{c_1 m - k}{m}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{c_1 m + k}{m}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = \frac{\left(-2c_1\, m^2 \mathrm{e}^{\frac{\sqrt{m^2}\,(x+c_2)}{k}} + \sqrt{m^2}\, k^2 + \mathrm{e}^{\frac{2\sqrt{m^2}\,(x+c_2)}{k}} \sqrt{m^2}\right) \mathrm{e}^{-\frac{\sqrt{m^2}\,(x+c_2)}{k}}}{2m^2}$$

$$y = \frac{\left(-2c_1\, m^2 \mathrm{e}^{-\frac{\sqrt{m^2}\,(x+c_3)}{k}} + \sqrt{m^2}\, k^2 + \mathrm{e}^{-\frac{2\sqrt{m^2}\,(x+c_3)}{k}} \sqrt{m^2}\right) \mathrm{e}^{\frac{\sqrt{m^2}\,(x+c_3)}{k}}}{2m^2}$$

$$y = -ix + c_4$$

$$y = ix + c_5$$

Solved as second order missing y ode

Time used: 0.327 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) - \frac{m\sqrt{1 + u(x)^2}}{k} = 0$$

Which is now solved for $u(x)$ as first order ode.

Integrating gives

$$\int \frac{k}{m\sqrt{u^2+1}} du = dx$$

$$\frac{k \operatorname{arcsinh}(u)}{m} = x + c_1$$

Singular solutions are found by solving

$$\frac{m\sqrt{u^2+1}}{k} = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$\begin{aligned} u(x) &= -i \\ u(x) &= i \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= -i \\ u(x) &= i \\ u(x) &= \sinh\left(\frac{m(x+c_1)}{k}\right) \end{aligned}$$

In summary, these are the solution found for $u(x)$

$$\begin{aligned} u(x) &= -i \\ u(x) &= i \\ u(x) &= \sinh\left(\frac{m(x+c_1)}{k}\right) \end{aligned}$$

For solution $u(x) = -i$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = -i$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dy &= \int -i \, dx \\ y &= -ix + c_2 \end{aligned}$$

For solution $u(x) = i$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = i$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dy &= \int i \, dx \\ y &= ix + c_3 \end{aligned}$$

For solution $u(x) = \sinh\left(\frac{m(x+c_1)}{k}\right)$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = \sinh\left(\frac{m(x+c_1)}{k}\right)$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \sinh\left(\frac{m(x+c_1)}{k}\right) dx$$
$$y = \frac{k\cosh\left(\frac{mx}{k} + \frac{c_1 m}{k}\right)}{m} + c_4$$

$$y = \frac{k\cosh\left(\frac{m(x+c_1)}{k}\right)}{m} + c_4$$

In summary, these are the solution found for (y)

$$y = -ix + c_2$$
$$y = ix + c_3$$
$$y = \frac{k\cosh\left(\frac{m(x+c_1)}{k}\right)}{m} + c_4$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -ix + c_2$$
$$y = ix + c_3$$
$$y = \frac{k\cosh\left(\frac{m(x+c_1)}{k}\right)}{m} + c_4$$

Maple step by step solution

Let's solve

$y'' = \frac{m\sqrt{1+y'^2}}{k}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) = \frac{m\sqrt{1+u(x)^2}}{k}$

- Solve for the highest derivative

  $u'(x) = \frac{m\sqrt{1+u(x)^2}}{k}$

- Separate variables

  $\frac{u'(x)}{\sqrt{1+u(x)^2}} = \frac{m}{k}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{\sqrt{1+u(x)^2}}dx = \int \frac{m}{k}dx + C1$

- Evaluate integral

  $\mathrm{arcsinh}(u(x)) = \frac{mx}{k} + C1$

- Solve for $u(x)$

  $u(x) = \sinh\left(\frac{C1k+xm}{k}\right)$

- Solve 1st ODE for $u(x)$

  $u(x) = \sinh\left(\frac{C1k+xm}{k}\right)$

- Make substitution $u = y'$

  $y' = \sinh\left(\frac{C1k+xm}{k}\right)$

- Integrate both sides to solve for y

  $\int y'dx = \int \sinh\left(\frac{C1k+xm}{k}\right)dx + C2$

- Compute integrals

  $y = \frac{k\cosh\left(\frac{mx}{k}+C1\right)}{m} + C2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)-m^2*(diff(y(x), x))/
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = m*(_b(_a)^2+1)^(1/2)/k, _b(_a), H
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

Maple dsolve solution

Solving time : 0.284 (sec)
Leaf size : 36

```
dsolve(diff(diff(y(x),x),x) = m/k*(diff(y(x),x)^2+1)^(1/2),y(x),singsol=all)
```

$$y(x) = -ix + c_1$$
$$y(x) = ix + c_1$$
$$y(x) = \frac{k \cosh\left(\frac{m(x+c_1)}{k}\right)}{m} + c_2$$

Mathematica DSolve solution

Solving time : 0.357 (sec)
Leaf size : 23

```
DSolve[{D[y[x],{x,2}]==m/k*Sqrt[1+D[y[x],x]^2],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{k \cosh\left(\frac{mx}{k} + c_1\right)}{m} + c_2$$

2.6.4 Problem 4 (eq 50)

Internal problem ID [18568]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 4 (eq 50)
Date solved : `Tuesday, January 28, 2025 at 12:00:26 PM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$\phi'' = \frac{4\pi nc}{\sqrt{v_0^2 + \frac{2e(\phi - V_0)}{m}}}$$

Solved as second order missing x ode

Time used: 3.440 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable ϕ an independent variable. Using

$$\phi' = p$$

Then

$$\begin{aligned}\phi'' &= \frac{dp}{dx}\\ &= \frac{dp}{d\phi}\frac{d\phi}{dx}\\ &= p\frac{dp}{d\phi}\end{aligned}$$

Hence the ode becomes

$$p(\phi)\left(\frac{d}{d\phi}p(\phi)\right)=\frac{4\pi nc}{\sqrt{v_0^2+\frac{2e(\phi-V_0)}{m}}}$$

Which is now solved as first order ode for $p(\phi)$.

The ode

$$p'=\frac{4\pi nc}{p\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}} \tag{2.38}$$

is separable as it can be written as

$$\begin{aligned}p'&=\frac{4\pi nc}{p\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}}\\&=f(\phi)g(p)\end{aligned}$$

Where

$$\begin{aligned}f(\phi)&=\frac{4\pi nc}{\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}}\\g(p)&=\frac{1}{p}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int\frac{1}{g(p)}\,dp&=\int f(\phi)\,d\phi\\\int p\,dp&=\int\frac{4\pi nc}{\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}}\,d\phi\end{aligned}$$

$$\frac{p^2}{2}=\frac{4\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}\,\pi ncm}{e}+c_1$$

Solving for p gives

$$p=\frac{\sqrt{2}\sqrt{e\left(4\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}\,\pi ncm+c_1e\right)}}{e}$$

$$p=-\frac{\sqrt{2}\sqrt{e\left(4\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}\,\pi ncm+c_1e\right)}}{e}$$

For solution (1) found earlier, since $p = \phi'$ then we now have a new first order ode to solve which is

$$\phi' = \frac{\sqrt{2}\,\sqrt{e\left(4\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\,\pi ncm+c_1e\right)}}{e}$$

Integrating gives

$$\int \frac{e\sqrt{2}}{2\sqrt{e\left(4\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}\,\pi ncm+c_1e\right)}}d\phi = dx$$

$$\frac{\sqrt{2}\,\sqrt{4ecmn\pi\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}+c_1\,e^2}\left(2\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}\,\pi ncm-c_1e\right)}{24e\,n^2m\,c^2\pi^2} = x+c_2$$

For solution (2) found earlier, since $p = \phi'$ then we now have a new first order ode to solve which is

$$\phi' = -\frac{\sqrt{2}\,\sqrt{e\left(4\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\,\pi ncm+c_1e\right)}}{e}$$

Integrating gives

$$\int -\frac{e\sqrt{2}}{2\sqrt{e\left(4\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}\,\pi ncm+c_1e\right)}}d\phi = dx$$

$$-\frac{\left(\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}\,\pi ncm-\frac{c_1e}{2}\right)\sqrt{2}\,\sqrt{4ecmn\pi\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}+c_1\,e^2}}{12e\,n^2m\,c^2\pi^2} = x+c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\frac{\sqrt{2}\,\sqrt{4ecmn\pi\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}+c_1\,e^2}\left(2\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}\,\pi ncm-c_1e\right)}{24e\,n^2m\,c^2\pi^2} = x+c_2$$

$$-\frac{\left(\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}\,\pi ncm-\frac{c_1e}{2}\right)\sqrt{2}\,\sqrt{4ecmn\pi\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}+c_1\,e^2}}{12e\,n^2m\,c^2\pi^2} = x+c_3$$

Solved as second order can be made integrable

Time used: 3.571 (sec)

Multiplying the ode by ϕ' gives

$$\phi'\phi'' - \frac{4\phi'\pi nc}{\sqrt{\frac{v_0^2m+2e\phi-2eV_0}{m}}} = 0$$

Integrating the above w.r.t x gives

$$\int \left(\phi'\phi'' - \frac{4\phi'\pi nc}{\sqrt{\frac{v_0^2m+2e\phi-2eV_0}{m}}}\right) dx = 0$$

$$\frac{\phi'^2}{2} - \frac{4\pi nc\sqrt{\frac{2e\phi}{m} + \frac{v_0^2m-2eV_0}{m}}\, m}{e} = c_1$$

Which is now solved for ϕ. Solving for the derivative gives these ODE's to solve

$$\phi' = \frac{\sqrt{2}\sqrt{e\left(4\sqrt{\frac{v_0^2m+2e\phi-2eV_0}{m}}\,\pi ncm + c_1e\right)}}{e} \tag{1}$$

$$\phi' = -\frac{\sqrt{2}\sqrt{e\left(4\sqrt{\frac{v_0^2m+2e\phi-2eV_0}{m}}\,\pi ncm + c_1e\right)}}{e} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{e\sqrt{2}}{2\sqrt{e\left(4\sqrt{\frac{v_0^2m-2eV_0+2e\phi}{m}}\,\pi ncm + c_1e\right)}} d\phi = dx$$

$$\frac{\sqrt{2}\sqrt{4ecmn\pi\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}} + c_1\, e^2}\left(2\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}\,\pi ncm - c_1e\right)}{24e\, n^2m\, c^2\pi^2} = x + c_2$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{e\sqrt{2}}{2\sqrt{e\left(4\sqrt{\frac{v_0^2m-2eV_0+2e\phi}{m}}\,\pi ncm+c_1e\right)}}d\phi=dx$$

$$-\frac{\left(\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}\,\pi ncm-\frac{c_1e}{2}\right)\sqrt{2}\,\sqrt{4ecmn\pi\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}+c_1\,e^2}}{12e\,n^2m\,c^2\pi^2}=x+c_3$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$\frac{\sqrt{2}\,\sqrt{4ecmn\pi\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}+c_1\,e^2}\left(2\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}\,\pi ncm-c_1e\right)}{24e\,n^2m\,c^2\pi^2}=x+c_2$$

$$-\frac{\left(\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}\,\pi ncm-\frac{c_1e}{2}\right)\sqrt{2}\,\sqrt{4ecmn\pi\sqrt{\frac{(-2V_0+2\phi)e+v_0^2m}{m}}+c_1\,e^2}}{12e\,n^2m\,c^2\pi^2}=x+c_3$$

Maple step by step solution

Let's solve
$\phi''=\frac{4\pi nc}{\sqrt{v_0^2+\frac{2e(\phi-V_0)}{m}}}$

- Highest derivative means the order of the ODE is 2
  ϕ''
- Define new dependent variable u
  $u(x)=\phi'$
- Compute ϕ''
  $u'(x)=\phi''$
- Use chain rule on the lhs
  $\phi'\left(\frac{d}{d\phi}u(\phi)\right)=\phi''$
- Substitute in the definition of u
  $u(\phi)\left(\frac{d}{d\phi}u(\phi)\right)=\phi''$
- Make substitutions $\phi'=u(\phi)\,,\phi''=u(\phi)\left(\frac{d}{d\phi}u(\phi)\right)$ to reduce order of ODE

$$u(\phi)\left(\frac{d}{d\phi}u(\phi)\right) = \frac{4\pi n c}{\sqrt{v_0^2+\frac{2e(\phi-V_0)}{m}}}$$

- Integrate both sides with respect to ϕ

$$\int u(\phi)\left(\frac{d}{d\phi}u(\phi)\right)d\phi = \int \frac{4\pi n c}{\sqrt{v_0^2+\frac{2e(\phi-V_0)}{m}}}d\phi + C1$$

- Evaluate integral

$$\frac{u(\phi)^2}{2} = -\frac{4(-v_0^2 m+2eV_0-2e\phi)\pi n c}{e\sqrt{-\frac{-v_0^2 m+2eV_0-2e\phi}{m}}} + C1$$

- Solve for $u(\phi)$

$$\left\{u(\phi) = \frac{\sqrt{-2e\sqrt{-\frac{-v_0^2 m+2eV_0-2e\phi}{m}}\left(-4\pi n c m\, v_0^2+8V_0\pi n c e-8\phi\pi n c e-C1\, e\sqrt{-\frac{-v_0^2 m+2eV_0-2e\phi}{m}}\right)}}{e\sqrt{-\frac{-v_0^2 m+2eV_0-2e\phi}{m}}}, u(\phi) = -\cdots\right.$$

- Solve 1st ODE for $u(\phi)$

$$u(\phi) = \frac{\sqrt{-2e\sqrt{-\frac{-v_0^2 m+2eV_0-2e\phi}{m}}\left(-4\pi n c m\, v_0^2+8V_0\pi n c e-8\phi\pi n c e-C1\, e\sqrt{-\frac{-v_0^2 m+2eV_0-2e\phi}{m}}\right)}}{e\sqrt{-\frac{-v_0^2 m+2eV_0-2e\phi}{m}}}$$

- Revert to original variables with substitution $u(\phi) = \phi', \phi = \phi$

$$\phi' = \frac{\sqrt{-2e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}\left(-4\pi n c m\, v_0^2+8V_0\pi n c e-8\phi\pi n c e-C1\, e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}\right)}}{e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}}$$

- Solve for the highest derivative

$$\phi' = \frac{\sqrt{-2e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}\left(-4\pi n c m\, v_0^2+8V_0\pi n c e-8\phi\pi n c e-C1\, e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}\right)}}{e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}}$$

- Separate variables

$$\frac{\phi'\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}}{\sqrt{-2e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}\left(-4\pi n c m\, v_0^2+8V_0\pi n c e-8\phi\pi n c e-C1\, e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}\right)}} = \frac{1}{e}$$

- Integrate both sides with respect to x

$$\int \frac{\phi'\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}}{\sqrt{-2e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}\left(-4\pi n c m\, v_0^2+8V_0\pi n c e-8\phi\pi n c e-C1\, e\sqrt{-\frac{-v_0^2 m-2e\phi+2eV_0}{m}}\right)}}dx = \int \frac{1}{e}dx + C2$$

- Evaluate integral

$$-\frac{\left(-4\pi ncm\,v_0^2+8V_0\pi nce-8\phi\pi nce-C1\,e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\right)\sqrt{\frac{\left(-v_0^2m-2e\phi+2eV_0\right)\left(4\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\,\pi ncm+C1\right.}{m}}}{12\sqrt{-2e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\left(-4\pi ncm\,v_0^2+8V_0\pi nce-8\phi\pi nce-C1\,e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\right)}\,e\sqrt{\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\left(-4\pi ncm\,v_0^2-\right.}}$$

- Solve for ϕ

$$\phi=\frac{-v_0^2m^2n^2c^2+2eV_0n^2m\,c^2+\left(\frac{\left(e\left(576\pi^4\,C2^2c^4e^2m^2n^4+1152\pi^4\,C2\,c^4e\,m^2n^4x+576\pi^4c^4m^2n^4x^2+24\sqrt{2}\,c^2m\,n^2\sqrt{288\,C2^4\pi^4c^4e^4m^2n^4+1152\right.\right.}{}\right.}{}$$

- Solve 2nd ODE for $u(\phi)$

$$u(\phi)=-\frac{\sqrt{-2e\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}\left(-4\pi ncm\,v_0^2+8V_0\pi nce-8\phi\pi nce-C1\,e\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}\right)}}{e\sqrt{-\frac{-v_0^2m+2eV_0-2e\phi}{m}}}$$

- Revert to original variables with substitution $u(\phi)=\phi',\phi=\phi$

$$\phi'=-\frac{\sqrt{-2e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\left(-4\pi ncm\,v_0^2+8V_0\pi nce-8\phi\pi nce-C1\,e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\right)}}{e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}}$$

- Solve for the highest derivative

$$\phi'=-\frac{\sqrt{-2e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\left(-4\pi ncm\,v_0^2+8V_0\pi nce-8\phi\pi nce-C1\,e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\right)}}{e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}}$$

- Separate variables

$$\frac{\phi'\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}}{\sqrt{-2e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\left(-4\pi ncm\,v_0^2+8V_0\pi nce-8\phi\pi nce-C1\,e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\right)}}=-\frac{1}{e}$$

- Integrate both sides with respect to x

$$\int\frac{\phi'\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}}{\sqrt{-2e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\left(-4\pi ncm\,v_0^2+8V_0\pi nce-8\phi\pi nce-C1\,e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\right)}}dx=\int-\frac{1}{e}dx+C2$$

- Evaluate integral

$$-\frac{\left(-4\pi ncm\,v_0^2+8V_0\pi nce-8\phi\pi nce-C1\,e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\right)\sqrt{\frac{\left(-v_0^2m-2e\phi+2eV_0\right)\left(4\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\,\pi ncm+C1\right.}{m}}}{12\sqrt{-2e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\left(-4\pi ncm\,v_0^2+8V_0\pi nce-8\phi\pi nce-C1\,e\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\right)}\,e\sqrt{\sqrt{-\frac{-v_0^2m-2e\phi+2eV_0}{m}}\left(-4\pi ncm\,v_0^2-\right.}}$$

- Solve for ϕ

$$\phi = \frac{-v_0^2 m^2 n^2 c^2 + 2eV_0 n^2 m c^2 + \left(\frac{\left(e\left(576\pi^4 C2^2 c^4 e^2 m^2 n^4 - 1152\pi^4 C2\, c^4 e\, m^2 n^4 x + 576\pi^4 c^4 m^2 n^4 x^2 + 24\sqrt{2}\, c^2 m\, n^2 \sqrt{288 C2^4 \pi^4 c^4 e^4 m^2 n^4 - 1152}\right.\right.}{}\right.}{}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, _b(_a)*(diff(_b(_a), _a))-4*Pi*n*c/(-(-m*v__0^2+2*V_
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[-2/3*(-m*v__0^2+2*V__
```

Maple dsolve solution

Solving time : 0.073 (sec)
Leaf size : 210

```
dsolve(diff(diff(phi(x),x),x) = 4*Pi*n*c/(v__0^2+2*e/m*(phi(x)-V__0))^(1/2),phi(x),sin
```

$$e\left(\int^{\phi(x)} \frac{\sqrt{\frac{(-2V_0+2\_\_a)e+v_0^2 m}{m}}}{4\sqrt{e\left(\frac{c_1\sqrt{(2V_0-2\_\_a)e-v_0^2 m}}{16}+\left((\_a-V_0)\,e+\frac{v_0^2 m}{2}\right)\pi c n\right)\sqrt{\frac{(-2V_0+2\_\_a)e+v_0^2 m}{m}}}}d\_a\right) - x - c_2 = 0$$

$$-e\left(\int^{\phi(x)} \frac{\sqrt{\frac{(-2V_0+2\_\_a)e+v_0^2 m}{m}}}{4\sqrt{e\left(\frac{c_1\sqrt{(2V_0-2\_\_a)e-v_0^2 m}}{16}+\left((\_a-V_0)\,e+\frac{v_0^2 m}{2}\right)\pi c n\right)\sqrt{\frac{(-2V_0+2\_\_a)e+v_0^2 m}{m}}}}d\_a\right) - x - c_2 = 0$$

Mathematica DSolve solution

Solving time : 79.952 (sec)
Leaf size : 2754

```
DSolve[{D[phi[x],{x,2}]==4*Pi*n*c/Sqrt[v0^2+2*e/m*(phi[x]-V0)],{}},phi[x],x,IncludeSingularS
```

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2.6.5 Problem 8 (eq 68)

Internal problem ID [18569]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 8 (eq 68)
Date solved : `Tuesday, January 28, 2025 at 12:00:34 PM`
CAS classification : `[_separable]`

Solve

$$y' = x(a y^2 + b)$$

Solved as first order separable ode

Time used: 0.266 (sec)

The ode

$$y' = x(a y^2 + b) \tag{2.39}$$

is separable as it can be written as

$$\begin{aligned} y' &= x(a y^2 + b) \\ &= f(x) g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= x \\ g(y) &= a\, y^2 + b \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

$$\int \frac{1}{a\,y^2+b}\,dy = \int x\,dx$$

$$\frac{\arctan\left(\frac{ay}{\sqrt{ba}}\right)}{\sqrt{ba}} = \frac{x^2}{2}+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y)=0$ or

$$a\,y^2+b=0$$

for y gives

$$y = \frac{\sqrt{-ba}}{a}$$
$$y = -\frac{\sqrt{-ba}}{a}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\arctan\left(\frac{ay}{\sqrt{ba}}\right)}{\sqrt{ba}} = \frac{x^2}{2}+c_1$$

$$y = \frac{\sqrt{-ba}}{a}$$

$$y = -\frac{\sqrt{-ba}}{a}$$

Solving for y gives

$$y = \frac{\sqrt{-ba}}{a}$$

$$y = \frac{\tan\left(\frac{x^2\sqrt{ba}}{2} + c_1\sqrt{ba}\right)\sqrt{ba}}{a}$$

$$y = -\frac{\sqrt{-ba}}{a}$$

Summary of solutions found

$$y = \frac{\sqrt{-ba}}{a}$$

$$y = \frac{\tan\left(\frac{x^2\sqrt{ba}}{2} + c_1\sqrt{ba}\right)\sqrt{ba}}{a}$$

$$y = -\frac{\sqrt{-ba}}{a}$$

Solved as first order Exact ode

Time used: 0.163 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(x\left(a\,y^2 + b\right)\right)\mathrm{d}x\\ \left(-x\left(a\,y^2 + b\right)\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x\left(a\,y^2 + b\right)\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x\left(a\,y^2 + b\right)\right)\\ &= -2xay\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-2xay) - (0))\\ &= -2xay\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{x\,(a\,y^2+b)}((0) - (-2xay)) \\
&= -\frac{2ay}{a\,y^2+b}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{2ay}{a\,y^2+b}\,\mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\ln(a\,y^2+b)} \\
&= \frac{1}{a\,y^2+b}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{a\,y^2+b}\left(-x\left(a\,y^2+b\right)\right) \\
&= -x
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{a\,y^2+b}(1) \\
&= \frac{1}{a\,y^2+b}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-x) + \left(\frac{1}{a\,y^2+b}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -x \,\mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{a\,y^2+b}$. Therefore equation (4) becomes

$$\frac{1}{a\,y^2 + b} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{a\,y^2 + b}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{a\,y^2 + b}\right) \mathrm{d}y$$

$$f(y) = \frac{\arctan\left(\frac{ay}{\sqrt{ba}}\right)}{\sqrt{ba}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\arctan\left(\frac{ay}{\sqrt{ba}}\right)}{\sqrt{ba}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\arctan\left(\frac{ay}{\sqrt{ba}}\right)}{\sqrt{ba}}$$

Solving for y gives

$$y = \frac{\tan\left(\frac{x^2\sqrt{ba}}{2} + c_1\sqrt{ba}\right)\sqrt{ba}}{a}$$

Summary of solutions found

$$y = \frac{\tan\left(\frac{x^2\sqrt{ba}}{2} + c_1\sqrt{ba}\right)\sqrt{ba}}{a}$$

Solved using Lie symmetry for first order ode

Time used: 0.726 (sec)

Writing the ode as

$$y' = x\left(a\,y^2 + b\right)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &2xb_4 + yb_5 + b_2 + x\left(a\,y^2+b\right)\left(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3\right) \\ &- x^2\left(a\,y^2+b\right)^2\left(xa_5 + 2ya_6 + a_3\right) \\ &- \left(a\,y^2+b\right)\left(x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1\right) \\ &- 2xay\left(x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} &-a^2x^3y^4a_5 - 2a^2x^2y^5a_6 - a^2x^2y^4a_3 - 2ab\,x^3y^2a_5 - 4ab\,x^2y^3a_6 - 2ab\,x^2y^2a_3 \\ &- 2a\,x^3yb_4 - 3a\,x^2y^2a_4 - a\,x^2y^2b_5 - 2ax\,y^3a_5 - a\,y^4a_6 - b^2x^3a_5 - 2b^2x^2ya_6 \\ &- 2a\,x^2yb_2 - 2ax\,y^2a_2 - ax\,y^2b_3 - a\,y^3a_3 - b^2x^2a_3 - 2axyb_1 - a\,y^2a_1 - 3b\,x^2a_4 \\ &+ b\,x^2b_5 - 2bxya_5 + 2bxyb_6 - b\,y^2a_6 - 2bxa_2 + bxb_3 - bya_3 - ba_1 + 2xb_4 + yb_5 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} &-a^2x^3y^4a_5 - 2a^2x^2y^5a_6 - a^2x^2y^4a_3 - 2ab\,x^3y^2a_5 - 4ab\,x^2y^3a_6 \\ &- 2ab\,x^2y^2a_3 - 2a\,x^3yb_4 - 3a\,x^2y^2a_4 - a\,x^2y^2b_5 - 2ax\,y^3a_5 - a\,y^4a_6 \\ &- b^2x^3a_5 - 2b^2x^2ya_6 - 2a\,x^2yb_2 - 2ax\,y^2a_2 - ax\,y^2b_3 - a\,y^3a_3 \\ &- b^2x^2a_3 - 2axyb_1 - a\,y^2a_1 - 3b\,x^2a_4 + b\,x^2b_5 - 2bxya_5 + 2bxyb_6 \\ &- b\,y^2a_6 - 2bxa_2 + bxb_3 - bya_3 - ba_1 + 2xb_4 + yb_5 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a^2a_5v_1^3v_2^4 - 2a^2a_6v_1^2v_2^5 - a^2a_3v_1^2v_2^4 - 2aba_5v_1^3v_2^2 - 4aba_6v_1^2v_2^3 \\ &- 2aba_3v_1^2v_2^2 - 3aa_4v_1^2v_2^2 - 2aa_5v_1v_2^3 - aa_6v_2^4 - 2ab_4v_1^3v_2 - ab_5v_1^2v_2^2 \\ &- b^2a_5v_1^3 - 2b^2a_6v_1^2v_2 - 2aa_2v_1v_2^2 - aa_3v_2^3 - 2ab_2v_1^2v_2 - ab_3v_1v_2^2 \\ &- b^2a_3v_1^2 - aa_1v_2^2 - 2ab_1v_1v_2 - 3ba_4v_1^2 - 2ba_5v_1v_2 - ba_6v_2^2 + bb_5v_1^2 \\ &+ 2bb_6v_1v_2 - 2ba_2v_1 - ba_3v_2 + bb_3v_1 - ba_1 + 2b_4v_1 + b_5v_2 + b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-a^2 a_5 v_1^3 v_2^4 - 2aba_5 v_1^3 v_2^2 - 2ab_4 v_1^3 v_2 - b^2 a_5 v_1^3 - 2a^2 a_6 v_1^2 v_2^5 - a^2 a_3 v_1^2 v_2^4 \\
&- 4aba_6 v_1^2 v_2^3 + (-2aba_3 - 3aa_4 - ab_5) v_1^2 v_2^2 + \left(-2b^2 a_6 - 2ab_2\right) v_1^2 v_2 \\
&+ \left(-b^2 a_3 - 3ba_4 + bb_5\right) v_1^2 - 2aa_5 v_1 v_2^3 + (-2aa_2 - ab_3) v_1 v_2^2 \\
&+ (-2ab_1 - 2ba_5 + 2bb_6) v_1 v_2 + (-2ba_2 + bb_3 + 2b_4) v_1 - aa_6 v_2^4 \\
&- aa_3 v_2^3 + (-aa_1 - ba_6) v_2^2 + (-ba_3 + b_5) v_2 - ba_1 + b_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-aa_3 &= 0 \\
-2aa_5 &= 0 \\
-aa_6 &= 0 \\
-2ab_4 &= 0 \\
-a^2 a_3 &= 0 \\
-a^2 a_5 &= 0 \\
-2a^2 a_6 &= 0 \\
-b^2 a_5 &= 0 \\
-2aba_5 &= 0 \\
-4aba_6 &= 0 \\
-2aa_2 - ab_3 &= 0 \\
-ba_1 + b_2 &= 0 \\
-ba_3 + b_5 &= 0 \\
-aa_1 - ba_6 &= 0 \\
-2b^2 a_6 - 2ab_2 &= 0 \\
-2ba_2 + bb_3 + 2b_4 &= 0 \\
-b^2 a_3 - 3ba_4 + bb_5 &= 0 \\
-2aba_3 - 3aa_4 - ab_5 &= 0 \\
-2ab_1 - 2ba_5 + 2bb_6 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= \frac{a b_1}{b}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= \frac{a\, y^2 + b}{b}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{a\, y^2 + b}{b}} dy
\end{aligned}
$$

Which results in

$$S = \frac{b \arctan\left(\frac{ay}{\sqrt{ba}}\right)}{\sqrt{ba}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x\left(a\,y^2 + b\right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 0 \\
S_y &= \frac{b}{a\,y^2 + b}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = bx \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = bR$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int bR\,dR \\
S(R) &= \frac{b\,R^2}{2} + c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\sqrt{b}\,\arctan\left(\frac{\sqrt{a}\,y}{\sqrt{b}}\right)}{\sqrt{a}} = \frac{b\,x^2}{2} + c_2$$

Which gives

$$y = \frac{\sqrt{b}\,\tan\left(\frac{\sqrt{a}\,(b\,x^2+2c_2)}{2\sqrt{b}}\right)}{\sqrt{a}}$$

Summary of solutions found

$$y = \frac{\sqrt{b}\,\tan\left(\frac{\sqrt{a}\,(b\,x^2+2c_2)}{2\sqrt{b}}\right)}{\sqrt{a}}$$

Solved as first order ode of type Riccati

Time used: 1.102 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x\left(a\,y^2 + b\right) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = xa\,y^2 + bx$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bx$, $f_1(x) = 0$ and $f_2(x) = ax$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{uax} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$
\begin{aligned}
f_2' &= a \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= a^2 x^3 b
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
axu''(x) - au'(x) + a^2 x^3 b u(x) = 0
$$

In normal form the ode

$$
ax\left(\frac{d^2u}{dx^2}\right) - a\left(\frac{du}{dx}\right) + a^2 x^3 b u = 0 \tag{1}
$$

Becomes

$$
\frac{d^2u}{dx^2} + p(x)\left(\frac{du}{dx}\right) + q(x)\, u = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= b\, x^2 a
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2} u(\tau) + p_1\left(\frac{d}{d\tau} u(\tau)\right) + q_1 u(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\frac{d^2}{dx^2}\tau(x) + p(x)\left(\frac{d}{dx}\tau(x)\right)}{\left(\frac{d}{dx}\tau(x)\right)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\left(\frac{d}{dx}\tau(x)\right)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\frac{d^2}{dx^2}\tau(x) + p(x)\left(\frac{d}{dx}\tau(x)\right) = 0
$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\int p(x)dx} dx \\ &= \int e^{-\int -\frac{1}{x} dx} dx \\ &= \int e^{\ln(x)}\, dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\left(\frac{d}{dx}\tau\,(x)\right)^2} \\ &= \frac{b\,x^2 a}{x^2} \\ &= ba \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}u(\tau) + q_1 u(\tau) &= 0 \\ \frac{d^2}{d\tau^2}u(\tau) + bau(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $u(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(\tau) + Bu'(\tau) + Cu(\tau) = 0$$

Where in the above $A = 1, B = 0, C = ba$. Let the solution be $u(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\tau\lambda} + ba\, e^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$ba + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = ba$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(ba)}\\ &= \pm\sqrt{-ba}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-ba}\\ \lambda_2 &= -\sqrt{-ba}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{\left((-1+i)\sqrt{\operatorname{signum}(ba)} + 1 + i\right)\sqrt{ba}}{2}\\ \lambda_2 &= -\frac{\left((-1+i)\sqrt{\operatorname{signum}(ba)} + 1 + i\right)\sqrt{ba}}{2}\end{aligned}$$

The roots are complex but they are not conjugate of each others. Hence simplification using Euler relation is not possible here. Therefore the final solution is

$$\begin{aligned}u(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}\\ &= c_1 e^{\frac{\tau\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{2}} + c_2 e^{-\frac{\tau\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{2}}\end{aligned}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to u using (6) which results in

$$u = c_1 \mathrm{e}^{\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}} + c_2 \mathrm{e}^{-\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}}$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$\begin{aligned}u'(x) = &\frac{c_1 x\left((-1+i)\sqrt{\operatorname{signum}(ba)} + 1 + i\right)\sqrt{ba}\,\mathrm{e}^{\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}}}{2}\\ &- \frac{c_2 x\left((-1+i)\sqrt{\operatorname{signum}(ba)} + 1 + i\right)\sqrt{ba}\,\mathrm{e}^{-\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}}}{2}\end{aligned}$$

Doing change of constants, the solution becomes

$$y = -\frac{\frac{c_3 x\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}\,\mathrm{e}^{\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}}}{2} - \frac{x\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}\,\mathrm{e}^{-\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}}}{2}}{ax\left(c_3\,\mathrm{e}^{\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}} + \mathrm{e}^{-\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}}\right)}$$

Summary of solutions found

$$y = -\frac{\frac{c_3 x\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}\,\mathrm{e}^{\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}}}{2} - \frac{x\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}\,\mathrm{e}^{-\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}}}{2}}{ax\left(c_3\,\mathrm{e}^{\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}} + \mathrm{e}^{-\frac{x^2\left((-1+i)\sqrt{\operatorname{signum}(ba)}+1+i\right)\sqrt{ba}}{4}}\right)}$$

Maple step by step solution

Let's solve
$y' = x(ay^2 + b)$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = x(ay^2 + b)$
- Separate variables
  $\frac{y'}{ay^2+b} = x$
- Integrate both sides with respect to x
  $\int \frac{y'}{ay^2+b}dx = \int xdx + C1$
- Evaluate integral
  $\frac{\arctan\left(\frac{ay}{\sqrt{ba}}\right)}{\sqrt{ba}} = \frac{x^2}{2} + C1$
- Solve for y
  $y = \frac{\tan\left(\frac{x^2\sqrt{ba}}{2}+C1\sqrt{ba}\right)\sqrt{ba}}{a}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 28

```
dsolve(diff(y(x),x) = x*(a*y(x)^2+b),y(x),singsol=all)
```

$$y(x) = \frac{\tan\left(\frac{\sqrt{ba}\,(x^2+2c_1)}{2}\right)\sqrt{ba}}{a}$$

Mathematica DSolve solution

Solving time : 8.114 (sec)
Leaf size : 75

```
DSolve[{D[y[x],x]==x*(a*y[x]^2+b),{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{b}\tan\left(\frac{1}{2}\sqrt{a}\sqrt{b}(x^2+2c_1)\right)}{\sqrt{a}}$$
$$y(x) \to -\frac{i\sqrt{b}}{\sqrt{a}}$$
$$y(x) \to \frac{i\sqrt{b}}{\sqrt{a}}$$

2.6.6 Problem 8 (eq 69)

Internal problem ID [18570]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 8 (eq 69)
Date solved : `Tuesday, January 28, 2025 at 12:00:36 PM`
CAS classification : `[_separable]`

Solve

$$n' = (n^2 + 1)\, x$$

Solved as first order separable ode

Time used: 0.150 (sec)

The ode

$$n' = (n^2 + 1)\, x \tag{2.40}$$

is separable as it can be written as

$$\begin{aligned} n' &= (n^2 + 1)\, x \\ &= f(x)g(n) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= x \\ g(n) &= n^2 + 1 \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(n)}\,dn = \int f(x)\,dx$$
$$\int \frac{1}{n^2+1}\,dn = \int x\,dx$$

$$\arctan(n) = \frac{x^2}{2} + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(n)$ is zero, since we had to divide by this above. Solving $g(n) = 0$ or

$$n^2 + 1 = 0$$

for n gives

$$n = -i$$
$$n = i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\arctan(n) = \frac{x^2}{2} + c_1$$
$$n = -i$$
$$n = i$$

Solving for n gives

$$n = -i$$
$$n = i$$
$$n = \tan\left(\frac{x^2}{2} + c_1\right)$$

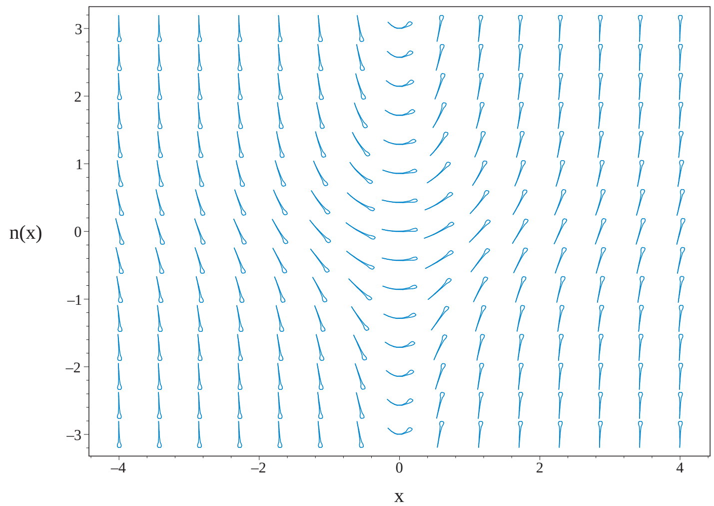


Figure 2.55: Slope field plot
$n' = (n^2 + 1)\, x$

Summary of solutions found

$$n = -i$$
$$n = i$$
$$n = \tan\left(\frac{x^2}{2} + c_1\right)$$

Solved as first order Exact ode

Time used: 0.101 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\, \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, n)\,\mathrm{d}x + N(x, n)\,\mathrm{d}n = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}n &= \left(\left(n^2+1\right)x\right)\mathrm{d}x\\ \left(-\left(n^2+1\right)x\right)\mathrm{d}x + \mathrm{d}n &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, n) &= -\left(n^2+1\right)x\\ N(x, n) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial n} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial n} &= \frac{\partial}{\partial n}\left(-\left(n^2+1\right)x\right)\\ &= -2nx\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial n} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial n} - \frac{\partial N}{\partial x}\right)\\ &= 1((-2nx) - (0))\\ &= -2nx\end{aligned}$$

Since A depends on n, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial n}\right) \\ &= -\frac{1}{(n^2+1)\,x}((0) - (-2nx)) \\ &= -\frac{2n}{n^2+1} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \,\mathrm{d}n} \\ &= e^{\int -\frac{2n}{n^2+1} \,\mathrm{d}n} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(n^2+1)} \\ &= \frac{1}{n^2+1} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{n^2+1}\left(-\left(n^2+1\right)x\right) \\ &= -x \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{n^2+1}(1) \\ &= \frac{1}{n^2+1} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}n}{\mathrm{d}x} &= 0 \\ (-x) + \left(\frac{1}{n^2+1}\right)\frac{\mathrm{d}n}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,n)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial n} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -x \,\mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(n) \tag{3}$$

Where $f(n)$ is used for the constant of integration since ϕ is a function of both x and n. Taking derivative of equation (3) w.r.t n gives

$$\frac{\partial \phi}{\partial n} = 0 + f'(n) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial n} = \frac{1}{n^2+1}$. Therefore equation (4) becomes

$$\frac{1}{n^2+1} = 0 + f'(n) \tag{5}$$

Solving equation (5) for $f'(n)$ gives

$$f'(n) = \frac{1}{n^2+1}$$

Integrating the above w.r.t n gives

$$\int f'(n) \,\mathrm{d}n = \int \left(\frac{1}{n^2+1}\right) \mathrm{d}n$$

$$f(n) = \arctan(n) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(n)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \arctan(n) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \arctan(n)$$

Solving for n gives

$$n = \tan\left(\frac{x^2}{2} + c_1\right)$$

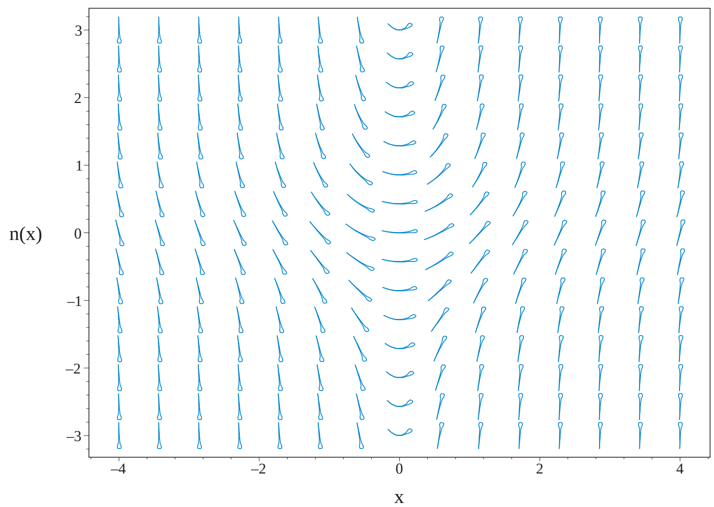


Figure 2.56: Slope field plot
$n' = (n^2 + 1)\, x$

Summary of solutions found

$$n = \tan\left(\frac{x^2}{2} + c_1\right)$$

Solved using Lie symmetry for first order ode

Time used: 0.611 (sec)

Writing the ode as

$$n' = (n^2 + 1)\, x$$
$$n' = \omega(x, n)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_n - \xi_x) - \omega^2 \xi_n - \omega_x \xi - \omega_n \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = n^2 a_6 + n x a_5 + x^2 a_4 + n a_3 + x a_2 + a_1 \tag{1E}$$

$$\eta = n^2 b_6 + n x b_5 + x^2 b_4 + n b_3 + x b_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &n b_5 + 2 x b_4 + b_2 + \left(n^2+1\right) x(-n a_5 + 2 n b_6 - 2 x a_4 + x b_5 - a_2 + b_3) \\ &- \left(n^2+1\right)^2 x^2 (2 n a_6 + x a_5 + a_3) \\ &- \left(n^2+1\right) \left(n^2 a_6 + n x a_5 + x^2 a_4 + n a_3 + x a_2 + a_1\right) \\ &- 2 n x \left(n^2 b_6 + n x b_5 + x^2 b_4 + n b_3 + x b_2 + b_1\right) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} &-2 n^5 x^2 a_6 - n^4 x^3 a_5 - n^4 x^2 a_3 - 4 n^3 x^2 a_6 - 2 n^2 x^3 a_5 - n^4 a_6 - 2 n^3 x a_5 \\ &- 2 n^2 x^2 a_3 - 3 n^2 x^2 a_4 - n^2 x^2 b_5 - 2 n\, x^3 b_4 - n^3 a_3 - 2 n^2 x a_2 - n^2 x b_3 \\ &- 2 n\, x^2 a_6 - 2 n\, x^2 b_2 - x^3 a_5 - n^2 a_1 - n^2 a_6 - 2 n x a_5 - 2 n x b_1 + 2 n x b_6 \\ &- x^2 a_3 - 3 x^2 a_4 + x^2 b_5 - n a_3 + n b_5 - 2 x a_2 + x b_3 + 2 x b_4 - a_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} &-2 n^5 x^2 a_6 - n^4 x^3 a_5 - n^4 x^2 a_3 - 4 n^3 x^2 a_6 - 2 n^2 x^3 a_5 - n^4 a_6 - 2 n^3 x a_5 \\ &- 2 n^2 x^2 a_3 - 3 n^2 x^2 a_4 - n^2 x^2 b_5 - 2 n\, x^3 b_4 - n^3 a_3 - 2 n^2 x a_2 - n^2 x b_3 \\ &- 2 n\, x^2 a_6 - 2 n\, x^2 b_2 - x^3 a_5 - n^2 a_1 - n^2 a_6 - 2 n x a_5 - 2 n x b_1 + 2 n x b_6 \\ &- x^2 a_3 - 3 x^2 a_4 + x^2 b_5 - n a_3 + n b_5 - 2 x a_2 + x b_3 + 2 x b_4 - a_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{n, x\}$ in them.

$$\{n, x\}$$

The following substitution is now made to be able to collect on all terms with $\{n, x\}$ in them

$$\{n = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_5v_1^4v_2^3 - 2a_6v_1^5v_2^2 - a_3v_1^4v_2^2 - 2a_5v_1^2v_2^3 - 4a_6v_1^3v_2^2 - 2a_3v_1^2v_2^2 - 3a_4v_1^2v_2^2 \\
&- 2a_5v_1^3v_2 - a_6v_1^4 - 2b_4v_1v_2^3 - b_5v_1^2v_2^2 - 2a_2v_1^2v_2 - a_3v_1^3 - a_5v_2^3 - 2a_6v_1v_2^2 \\
&- 2b_2v_1v_2^2 - b_3v_1^2v_2 - a_1v_1^2 - a_3v_2^2 - 3a_4v_2^2 - 2a_5v_1v_2 - a_6v_1^2 - 2b_1v_1v_2 \\
&+ b_5v_2^2 + 2b_6v_1v_2 - 2a_2v_2 - a_3v_1 + b_3v_2 + 2b_4v_2 + b_5v_1 - a_1 + b_2 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2a_6v_1^5v_2^2 - a_5v_1^4v_2^3 - a_3v_1^4v_2^2 - a_6v_1^4 - 4a_6v_1^3v_2^2 - 2a_5v_1^3v_2 - a_3v_1^3 \\
&- 2a_5v_1^2v_2^3 + (-2a_3 - 3a_4 - b_5)\,v_1^2v_2^2 + (-2a_2 - b_3)\,v_1^2v_2 + (-a_1 - a_6)\,v_1^2 \\
&- 2b_4v_1v_2^3 + (-2a_6 - 2b_2)\,v_1v_2^2 + (-2a_5 - 2b_1 + 2b_6)\,v_1v_2 + (-a_3 + b_5)\,v_1 \\
&- a_5v_2^3 + (-a_3 - 3a_4 + b_5)\,v_2^2 + (-2a_2 + b_3 + 2b_4)\,v_2 - a_1 + b_2 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a_3 &= 0 \\
-2a_5 &= 0 \\
-a_5 &= 0 \\
-4a_6 &= 0 \\
-2a_6 &= 0 \\
-a_6 &= 0 \\
-2b_4 &= 0 \\
-a_1 - a_6 &= 0 \\
-a_1 + b_2 &= 0 \\
-2a_2 - b_3 &= 0 \\
-a_3 + b_5 &= 0 \\
-2a_6 - 2b_2 &= 0 \\
-2a_2 + b_3 + 2b_4 &= 0 \\
-2a_3 - 3a_4 - b_5 &= 0 \\
-a_3 - 3a_4 + b_5 &= 0 \\
-2a_5 - 2b_1 + 2b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= b_6 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= n^2 + 1
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, n) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dn}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial n}\right) S(x, n) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{n^2 + 1} dy
\end{aligned}
$$

Which results in

$$S = \arctan(n)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,n)S_n}{R_x + \omega(x,n)R_n} \tag{2}$$

Where in the above R_x, R_n, S_x, S_n are all partial derivatives and $\omega(x,n)$ is the right hand side of the original ode given by

$$\omega(x,n) = (n^2+1)\,x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_n &= 0 \\ S_x &= 0 \\ S_n &= \frac{1}{n^2+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, n in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int R\,dR \\ S(R) &= \frac{R^2}{2} + c_2 \end{aligned}$$

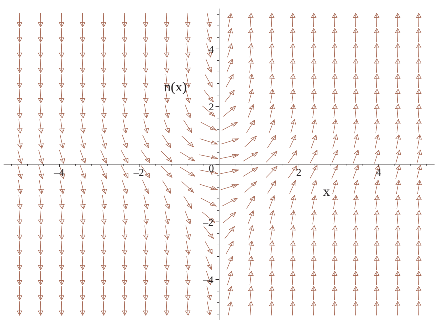
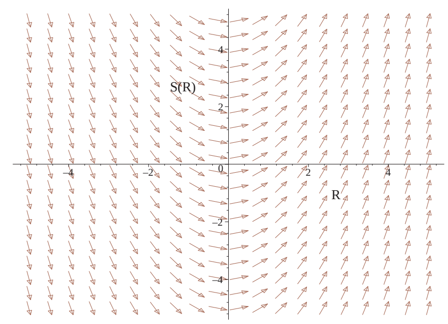
To complete the solution, we just need to transform the above back to x, n coordinates. This results in

$$\arctan(n) = \frac{x^2}{2} + c_2$$

Which gives

$$n = \tan\left(\frac{x^2}{2} + c_2\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, n coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dn}{dx} = (n^2+1)\, x$ 	$R = x$ $S = \arctan(n)$	$\frac{dS}{dR} = R$ 

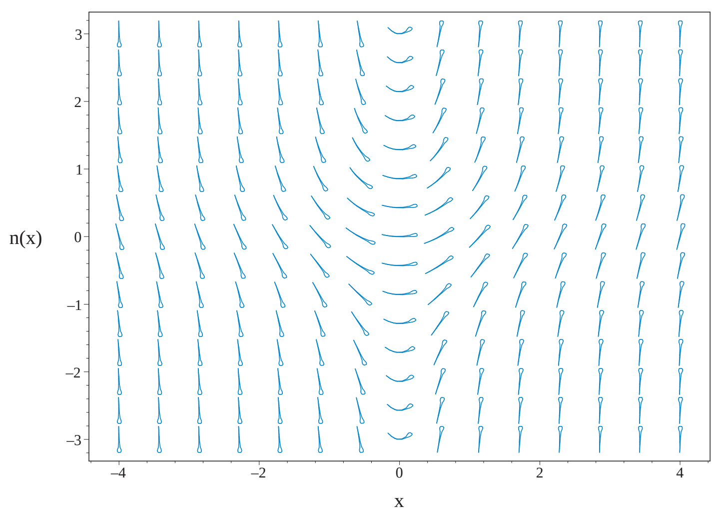


Figure 2.57: Slope field plot
$n' = (n^2+1)\, x$

Summary of solutions found

$$n = \tan\left(\frac{x^2}{2} + c_2\right)$$

Solved as first order ode of type Riccati

Time used: 0.425 (sec)

In canonical form the ODE is

$$\begin{aligned} n' &= F(x, n) \\ &= (n^2 + 1)\, x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$n' = n^2 x + x$$

With Riccati ODE standard form

$$n' = f_0(x) + f_1(x)n + f_2(x)n^2$$

Shows that $f_0(x) = x$, $f_1(x) = 0$ and $f_2(x) = x$. Let

$$\begin{aligned} n &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{ux} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^3 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x u''(x) - u'(x) + x^3 u(x) = 0$$

In normal form the ode

$$x\left(\frac{d^2u}{dx^2}\right) - \frac{du}{dx} + x^3u = 0 \tag{1}$$

Becomes

$$\frac{d^2u}{dx^2} + p(x)\left(\frac{du}{dx}\right) + q(x)\,u = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}u(\tau) + p_1\left(\frac{d}{d\tau}u(\tau)\right) + q_1u(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\frac{d^2}{dx^2}\tau(x) + p(x)\left(\frac{d}{dx}\tau(x)\right)}{\left(\frac{d}{dx}\tau(x)\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\left(\frac{d}{dx}\tau(x)\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\frac{d^2}{dx^2}\tau(x) + p(x)\left(\frac{d}{dx}\tau(x)\right) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\int p(x)dx}dx \\ &= \int e^{-\int -\frac{1}{x}dx}dx \\ &= \int e^{\ln(x)}\,dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\left(\frac{d}{dx}\tau(x)\right)^2} \\ &= \frac{x^2}{x^2} \\ &= 1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}u(\tau) + q_1 u(\tau) &= 0 \\ \frac{d^2}{d\tau^2}u(\tau) + u(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $u(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(\tau) + Bu'(\tau) + Cu(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $u(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\tau\lambda} + \mathrm{e}^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i \\ \lambda_2 &= -i \end{aligned}$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$u(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$u(\tau) = e^{0}(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$u(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to u using (6) which results in

$$u = c_1 \cos\left(\frac{x^2}{2}\right) + c_2 \sin\left(\frac{x^2}{2}\right)$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$u'(x) = -c_1 x \sin\left(\frac{x^2}{2}\right) + c_2 x \cos\left(\frac{x^2}{2}\right)$$

Doing change of constants, the solution becomes

$$n = -\frac{-c_3 x \sin\left(\frac{x^2}{2}\right) + x \cos\left(\frac{x^2}{2}\right)}{x\left(c_3 \cos\left(\frac{x^2}{2}\right) + \sin\left(\frac{x^2}{2}\right)\right)}$$

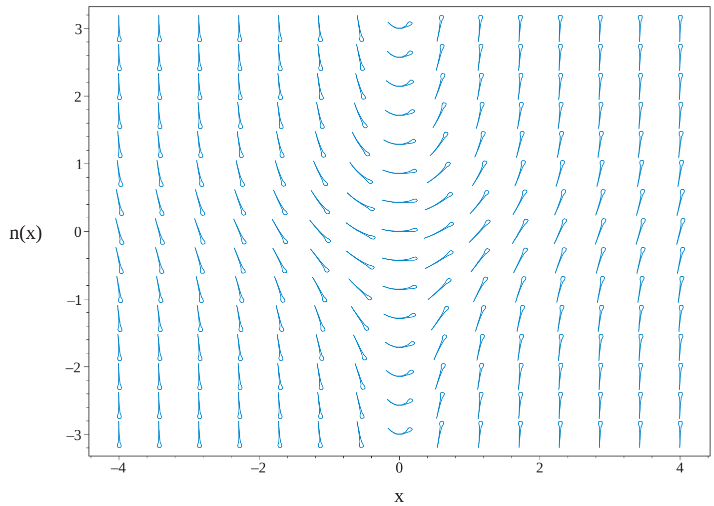


Figure 2.58: Slope field plot
$n' = (n^2 + 1)\, x$

Summary of solutions found

$$n = -\frac{-c_3 x \sin\left(\frac{x^2}{2}\right) + x \cos\left(\frac{x^2}{2}\right)}{x \left(c_3 \cos\left(\frac{x^2}{2}\right) + \sin\left(\frac{x^2}{2}\right)\right)}$$

Maple step by step solution

Let's solve
$n' = (n^2 + 1)\, x$

- Highest derivative means the order of the ODE is 1
  n'
- Solve for the highest derivative
  $n' = (n^2 + 1)\, x$
- Separate variables
  $\frac{n'}{n^2+1} = x$
- Integrate both sides with respect to x
  $\int \frac{n'}{n^2+1} dx = \int x dx + C1$
- Evaluate integral
  $\arctan(n) = \frac{x^2}{2} + C1$
- Solve for n

$$n = \tan\left(\frac{x^2}{2} + C1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 12

```
dsolve(diff(n(x),x) = (n(x)^2+1)*x,n(x),singsol=all)
```

$$n(x) = \tan\left(\frac{x^2}{2} + c_1\right)$$

Mathematica DSolve solution

Solving time : `0.191` (sec)
Leaf size : 30

```
DSolve[{D[n[x],x]==(n[x]^2+1)*x,{}},n[x],x,IncludeSingularSolutions->True]
```

$$n(x) \to \tan\left(\frac{x^2}{2} + c_1\right)$$
$$n(x) \to -i$$
$$n(x) \to i$$

2.6.7 Problem 9 (a)

Internal problem ID [18571]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 9 (a)
Date solved : `Tuesday, January 28, 2025 at 12:00:39 PM`
CAS classification : `[_separable]`

Solve

$$v' + \frac{2v}{u} = 3v$$

Solved as first order linear ode

Time used: 0.054 (sec)

In canonical form a linear first order is

$$v' + q(u)v = p(u)$$

Comparing the above to the given ode shows that

$$q(u) = -\frac{3u-2}{u}$$
$$p(u) = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,du} \\ &= e^{\int -\frac{3u-2}{u}du} \\ &= u^2 e^{-3u}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}u}\mu v = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}u}\left(v\,u^2\mathrm{e}^{-3u}\right) = 0$$

Integrating gives

$$v\,u^2\mathrm{e}^{-3u} = \int 0\,du + c_1$$
$$= c_1$$

Dividing throughout by the integrating factor $u^2\mathrm{e}^{-3u}$ gives the final solution

$$v = \frac{\mathrm{e}^{3u}c_1}{u^2}$$

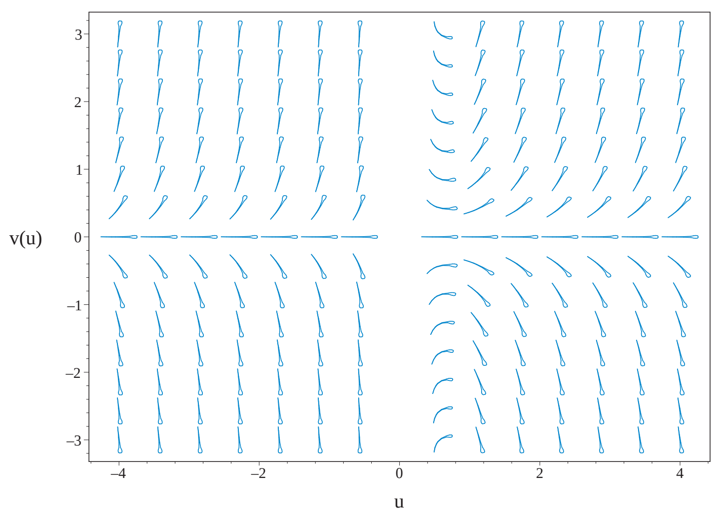


Figure 2.59: Slope field plot
$v' + \frac{2v}{u} = 3v$

Summary of solutions found

$$v = \frac{\mathrm{e}^{3u}c_1}{u^2}$$

Solved as first order separable ode

Time used: 0.098 (sec)

The ode

$$v' = \frac{v(3u-2)}{u} \tag{2.41}$$

is separable as it can be written as

$$\begin{aligned} v' &= \frac{v(3u-2)}{u} \\ &= f(u)g(v) \end{aligned}$$

Where

$$\begin{aligned} f(u) &= \frac{3u-2}{u} \\ g(v) &= v \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(v)}\, dv &= \int f(u)\, du \\ \int \frac{1}{v}\, dv &= \int \frac{3u-2}{u}\, du \end{aligned}$$

$$\ln(v) = 3u + \ln\left(\frac{1}{u^2}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(v)$ is zero, since we had to divide by this above. Solving $g(v) = 0$ or

$$v = 0$$

for v gives

$$v = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(v) = 3u + \ln\left(\frac{1}{u^2}\right) + c_1$$

$$v = 0$$

Solving for v gives

$$v = 0$$

$$v = \frac{e^{3u+c_1}}{u^2}$$

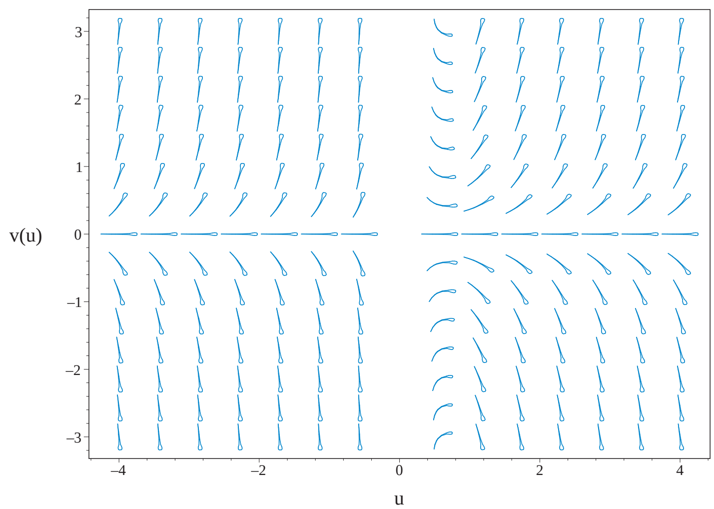


Figure 2.60: Slope field plot
$v' + \frac{2v}{u} = 3v$

Summary of solutions found

$$v = 0$$

$$v = \frac{e^{3u+c_1}}{u^2}$$

Solved as first order homogeneous class D2 ode

Time used: 0.091 (sec)

Applying change of variables $v = u(u)\, u$, then the ode becomes

$$u'(u)\, u + 3u(u) = 3u(u)\, u$$

Which is now solved The ode

$$u'(u) = \frac{3u(u)\,(u-1)}{u} \tag{2.42}$$

is separable as it can be written as

$$\begin{aligned} u'(u) &= \frac{3u(u)\,(u-1)}{u} \\ &= f(u)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(u) &= \frac{3u-3}{u} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(u)\, du \\ \int \frac{1}{u}\, du &= \int \frac{3u-3}{u}\, du \end{aligned}$$

$$\ln\left(u(u)\right) = 3u + \ln\left(\frac{1}{u^3}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$u = 0$$

for $u(u)$ gives

$$u(u) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(u(u)\right) = 3u + \ln\left(\frac{1}{u^3}\right) + c_1$$
$$u(u) = 0$$

Solving for $u(u)$ gives

$$u(u) = 0$$
$$u(u) = \frac{e^{3u+c_1}}{u^3}$$

Converting $u(u) = 0$ back to v gives

$$v = 0$$

Converting $u(u) = \frac{e^{3u+c_1}}{u^3}$ back to v gives

$$v = \frac{e^{3u+c_1}}{u^2}$$

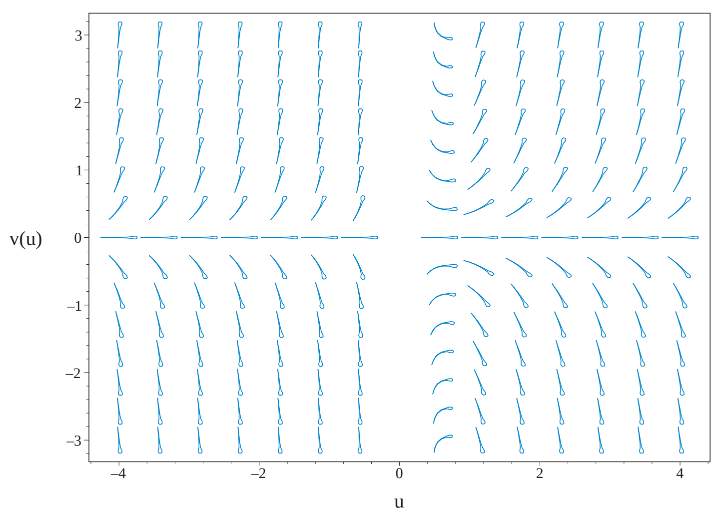


Figure 2.61: Slope field plot
$v' + \frac{2v}{u} = 3v$

Summary of solutions found

$$v = 0$$
$$v = \frac{e^{3u+c_1}}{u^2}$$

Solved as first order Exact ode

Time used: 0.145 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(u,v)\,\mathrm{d}u + N(u,v)\,\mathrm{d}v = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
\mathrm{d}v &= \left(3v - \frac{2v}{u}\right) \mathrm{d}u \\
\left(\frac{2v}{u} - 3v\right) \mathrm{d}u + \mathrm{d}v &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(u,v) &= \frac{2v}{u} - 3v \\
N(u,v) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}\left(\frac{2v}{u} - 3v\right) \\
&= \frac{2}{u} - 3
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial u} &= \frac{\partial}{\partial u}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial u}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u}\right) \\
&= 1\left(\left(\frac{2}{u} - 3\right) - (0)\right) \\
&= \frac{2}{u} - 3
\end{aligned}
$$

Since A does not depend on v, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}u} \\
&= e^{\int \frac{2}{u} - 3 \, \mathrm{d}u}
\end{aligned}
$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3u+2\ln(u)}\\ &= u^2 \mathrm{e}^{-3u}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= u^2\mathrm{e}^{-3u}\left(\frac{2v}{u} - 3v\right)\\ &= -v(3u-2)\, u\, \mathrm{e}^{-3u}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= u^2\mathrm{e}^{-3u}(1)\\ &= u^2\mathrm{e}^{-3u}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}v}{\mathrm{d}u} &= 0\\ \left(-v(3u-2)\, u\, \mathrm{e}^{-3u}\right) + \left(u^2\mathrm{e}^{-3u}\right)\frac{\mathrm{d}v}{\mathrm{d}u} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(u, v)$

$$\frac{\partial\phi}{\partial u} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial v} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. v gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial v}\,\mathrm{d}v &= \int \overline{N}\,\mathrm{d}v\\ \int \frac{\partial\phi}{\partial v}\,\mathrm{d}v &= \int u^2\mathrm{e}^{-3u}\,\mathrm{d}v\\ \phi &= v\,u^2\mathrm{e}^{-3u} + f(u)\end{aligned} \tag{3}$$

Where $f(u)$ is used for the constant of integration since ϕ is a function of both u and v. Taking derivative of equation (3) w.r.t u gives

$$\begin{aligned}\frac{\partial\phi}{\partial u} &= 2vu\,\mathrm{e}^{-3u} - 3v\,u^2\mathrm{e}^{-3u} + f'(u) \qquad (4)\\ &= -v(3u-2)\,u\,\mathrm{e}^{-3u} + f'(u)\end{aligned}$$

But equation (1) says that $\frac{\partial\phi}{\partial u} = -v(3u-2)\,u\,\mathrm{e}^{-3u}$. Therefore equation (4) becomes

$$-v(3u-2)\,u\,\mathrm{e}^{-3u} = -v(3u-2)\,u\,\mathrm{e}^{-3u} + f'(u) \tag{5}$$

Solving equation (5) for $f'(u)$ gives

$$f'(u) = 0$$

Therefore

$$f(u) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(u)$ into equation (3) gives ϕ

$$\phi = v\,u^2\mathrm{e}^{-3u} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = v\,u^2\mathrm{e}^{-3u}$$

Solving for v gives

$$v = \frac{\mathrm{e}^{3u}c_1}{u^2}$$

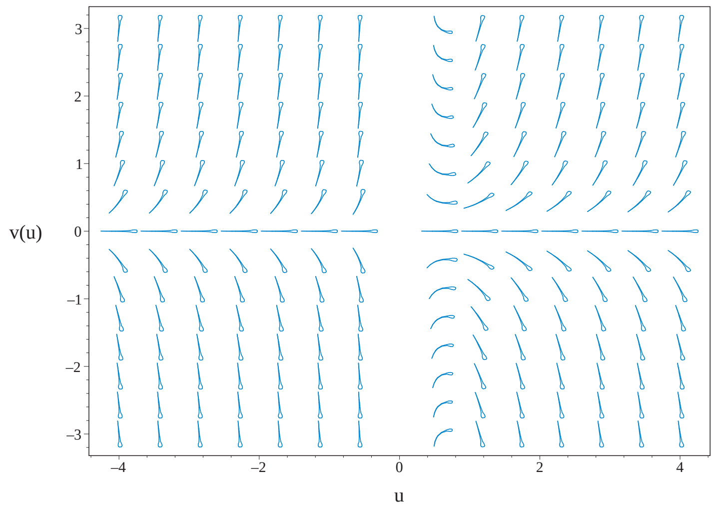


Figure 2.62: Slope field plot
$v' + \frac{2v}{u} = 3v$

Summary of solutions found

$$v = \frac{e^{3u} c_1}{u^2}$$

Solved using Lie symmetry for first order ode

Time used: 0.432 (sec)

Writing the ode as

$$v' = \frac{v(3u - 2)}{u}$$
$$v' = \omega(u, v)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_u + \omega(\eta_v - \xi_u) - \omega^2 \xi_v - \omega_u \xi - \omega_v \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = u a_2 + v a_3 + a_1 \tag{1E}$$
$$\eta = u b_2 + v b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{v(3u-2)(b_3-a_2)}{u} - \frac{v^2(3u-2)^2 a_3}{u^2} - \left(\frac{3v}{u} - \frac{v(3u-2)}{u^2}\right)(ua_2+va_3+a_1) - \frac{(3u-2)(ub_2+vb_3+b_1)}{u} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{9u^2v^2a_3 + 3u^3b_2 + 3u^2va_2 - 12u\,v^2a_3 + 3u^2b_1 - 3b_2u^2 + 6v^2a_3 - 2ub_1 + 2va_1}{u^2} = 0$$

Setting the numerator to zero gives

$$-9u^2v^2a_3 - 3u^3b_2 - 3u^2va_2 + 12u\,v^2a_3 - 3u^2b_1 + 3b_2u^2 - 6v^2a_3 + 2ub_1 - 2va_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{u, v\}$ in them.

$$\{u, v\}$$

The following substitution is now made to be able to collect on all terms with $\{u, v\}$ in them

$$\{u = v_1, v = v_2\}$$

The above PDE (6E) now becomes

$$-9a_3v_1^2v_2^2 - 3a_2v_1^2v_2 + 12a_3v_1v_2^2 - 3b_2v_1^3 - 6a_3v_2^2 - 3b_1v_1^2 + 3b_2v_1^2 - 2a_1v_2 + 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-3b_2v_1^3 - 9a_3v_1^2v_2^2 - 3a_2v_1^2v_2 + (-3b_1 + 3b_2)v_1^2 + 12a_3v_1v_2^2 + 2b_1v_1 - 6a_3v_2^2 - 2a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
-3a_2 &= 0 \\
-9a_3 &= 0 \\
-6a_3 &= 0 \\
12a_3 &= 0 \\
2b_1 &= 0 \\
-3b_2 &= 0 \\
-3b_1 + 3b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= v
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(u, v) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{du}{\xi} = \frac{dv}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v}\right) S(u, v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = u
$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{v} dy \end{aligned}$$

Which results in

$$S = \ln(v)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_u + \omega(u,v) S_v}{R_u + \omega(u,v) R_v} \tag{2}$$

Where in the above R_u, R_v, S_u, S_v are all partial derivatives and $\omega(u,v)$ is the right hand side of the original ode given by

$$\omega(u,v) = \frac{v(3u-2)}{u}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_u &= 1 \\ R_v &= 0 \\ S_u &= 0 \\ S_v &= \frac{1}{v} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3u-2}{u} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for u, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3R-2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{3R-2}{R}\, dR$$
$$S(R) = 3R - 2\ln(R) + c_2$$

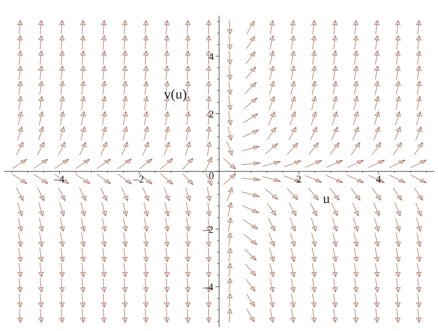
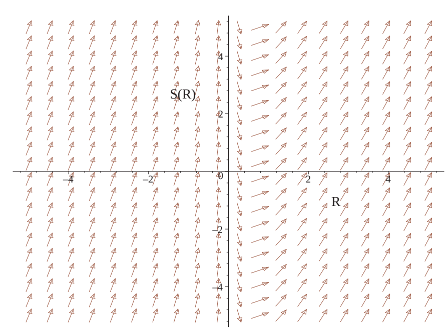
To complete the solution, we just need to transform the above back to u, v coordinates. This results in

$$\ln(v) = 3u - 2\ln(u) + c_2$$

Which gives

$$v = \frac{e^{3u+c_2}}{u^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in u, v coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dv}{du} = \frac{v(3u-2)}{u}$ 	$R = u$ $S = \ln(v)$	$\frac{dS}{dR} = \frac{3R-2}{R}$ 

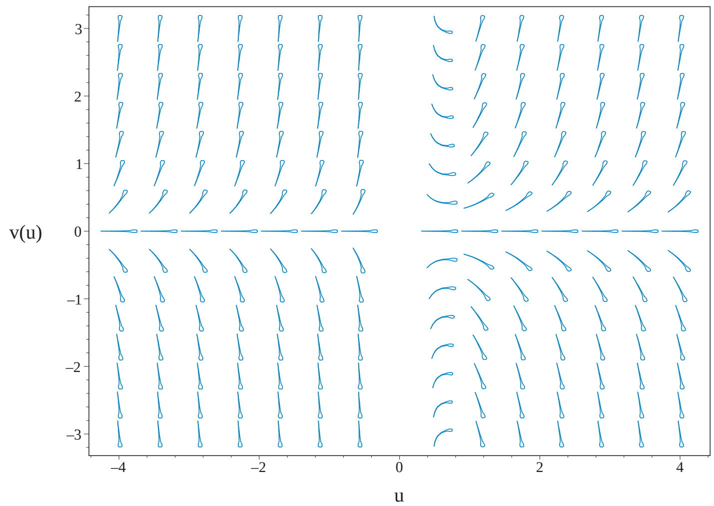


Figure 2.63: Slope field plot
$v' + \frac{2v}{u} = 3v$

Summary of solutions found

$$v = \frac{e^{3u+c_2}}{u^2}$$

Maple step by step solution

Let's solve
$v' + \frac{2v}{u} = 3v$

- Highest derivative means the order of the ODE is 1
  v'
- Solve for the highest derivative
  $v' = 3v - \frac{2v}{u}$
- Separate variables
  $\frac{v'}{v} = \frac{3u-2}{u}$
- Integrate both sides with respect to u
  $\int \frac{v'}{v} du = \int \frac{3u-2}{u} du + C1$
- Evaluate integral
  $\ln(v) = 3u - 2\ln(u) + C1$
- Solve for v
  $v = \frac{e^{3u+C1}}{u^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 13

```
dsolve(diff(v(u),u)+2*v(u)/u = 3*v(u),v(u),singsol=all)
```

$$v = \frac{c_1 \mathrm{e}^{3u}}{u^2}$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 21

```
DSolve[{D[v[u],u]+2*v[u]/u==3*v[u],{}},v[u],u,IncludeSingularSolutions->True]
```

$$v(u) \to \frac{c_1 e^{3u}}{u^2}$$
$$v(u) \to 0$$

2.6.8 Problem 9 (b)

Internal problem ID [18572]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 9 (b)
Date solved : `Tuesday, January 28, 2025 at 12:00:41 PM`
CAS classification : `[_separable]`

Solve

$$\sqrt{-u^2+1}\, v' = 2u\sqrt{1-v^2}$$

Solved as first order separable ode

Time used: 0.127 (sec)

The ode

$$v' = \frac{2u\sqrt{1-v^2}}{\sqrt{-u^2+1}} \tag{2.43}$$

is separable as it can be written as

$$\begin{aligned} v' &= \frac{2u\sqrt{1-v^2}}{\sqrt{-u^2+1}} \\ &= f(u)g(v) \end{aligned}$$

Where

$$\begin{aligned} f(u) &= \frac{2u}{\sqrt{-u^2+1}} \\ g(v) &= \sqrt{-v^2+1} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(v)}\,dv = \int f(u)\,du$$

$$\int \frac{1}{\sqrt{-v^2+1}}\,dv = \int \frac{2u}{\sqrt{-u^2+1}}\,du$$

$$\arcsin(v) = -2\sqrt{-u^2+1} + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(v)$ is zero, since we had to divide by this above. Solving $g(v) = 0$ or

$$\sqrt{-v^2+1} = 0$$

for v gives

$$v = -1$$
$$v = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\arcsin(v) = -2\sqrt{-u^2+1} + c_1$$
$$v = -1$$
$$v = 1$$

Solving for v gives

$$v = -1$$
$$v = 1$$
$$v = \sin\left(-2\sqrt{-u^2+1} + c_1\right)$$

Summary of solutions found

$$v = -1$$
$$v = 1$$
$$v = \sin\left(-2\sqrt{-u^2+1} + c_1\right)$$

Maple step by step solution

Let's solve

$\sqrt{-u^2+1}\, v' = 2u\sqrt{1-v^2}$

- Highest derivative means the order of the ODE is 1

  v'

- Solve for the highest derivative

  $v' = \frac{2u\sqrt{1-v^2}}{\sqrt{-u^2+1}}$

- Separate variables

  $\frac{v'}{\sqrt{1-v^2}} = \frac{2u}{\sqrt{-u^2+1}}$

- Integrate both sides with respect to u

  $\int \frac{v'}{\sqrt{1-v^2}}du = \int \frac{2u}{\sqrt{-u^2+1}}du + C1$

- Evaluate integral

  $\arcsin(v) = \frac{2(u-1)(u+1)}{\sqrt{-u^2+1}} + C1$

- Solve for v

  $v = \sin\left(\frac{C1\sqrt{-u^2+1}+2u^2-2}{\sqrt{-u^2+1}}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 32

```
dsolve((-u^2+1)^(1/2)*diff(v(u),u) = 2*u*(1-v(u)^2)^(1/2),v(u),singsol=all)
```

$$v = \sin\left(\frac{2c_1\sqrt{-u^2+1}+2u^2-2}{\sqrt{-u^2+1}}\right)$$

Mathematica DSolve solution

Solving time : 0.257 (sec)
Leaf size : 44

```
DSolve[{Sqrt[1-u^2]*D[v[u],u]==2*u*Sqrt[1-v[u]^2],{}},v[u],u,IncludeSingularSolutions->True]
```

$$v(u) \to -\sin\left(2\sqrt{1-u^2}-c_1\right)$$
$$v(u) \to -1$$
$$v(u) \to 1$$
$$v(u) \to \text{Interval}[\{-1,1\}]$$

2.6.9 Problem 9 (c)

Internal problem ID [18573]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 9 (c)
Date solved : `Tuesday, January 28, 2025 at 12:00:43 PM`
CAS classification : `[_quadrature]`

Solve

$$\sqrt{1+v'} = \frac{e^{u}}{2}$$

Solved as first order quadrature ode

Time used: 0.075 (sec)

Since the ode has the form $v' = f(u)$, then we only need to integrate $f(u)$.

$$\int dv = \int \frac{e^{2u}}{4} - 1\, du$$

$$v = -u + \frac{e^{2u}}{8} + c_1$$

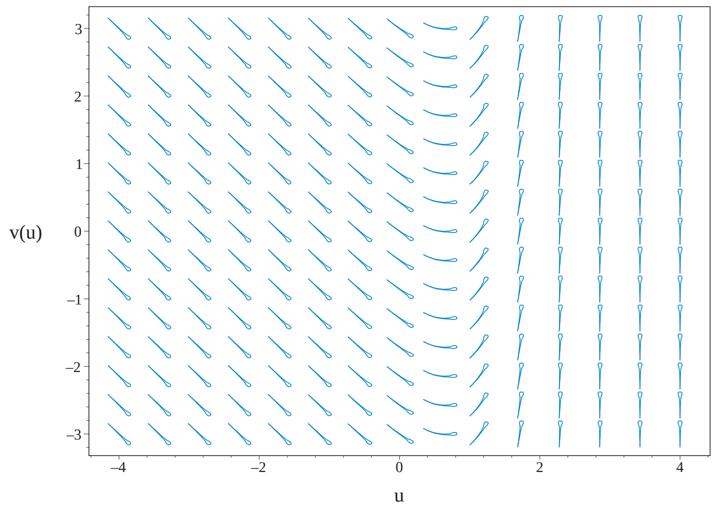


Figure 2.64: Slope field plot
$\sqrt{1+v'} = \frac{e^u}{2}$

Summary of solutions found

$$v = -u + \frac{e^{2u}}{8} + c_1$$

Maple step by step solution

Let's solve
$\sqrt{1+v'} = \frac{e^u}{2}$

- Highest derivative means the order of the ODE is 1
  v'
- Solve for the highest derivative
  $v' = \frac{(e^u)^2}{4} - 1$
- Integrate both sides with respect to u
  $\int v'du = \int \left(\frac{(e^u)^2}{4} - 1\right) du + C1$
- Evaluate integral
  $v = -u + \frac{(e^u)^2}{8} + C1$
- Solve for v
  $v = -u + \frac{(e^u)^2}{8} + C1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 17

```
dsolve((1+diff(v(u),u))^(1/2) = 1/2*exp(u),v(u),singsol=all)
```

$$v = \frac{e^{2u}}{8} - \ln\left(e^{u}\right) + c_1$$

Mathematica DSolve solution

Solving time : 0.015 (sec)
Leaf size : 20

```
DSolve[{Sqrt[1+D[v[u],u]]==Exp[u]/2,{}},v[u],u,IncludeSingularSolutions->True]
```

$$v(u) \to -u + \frac{e^{2u}}{8} + c_1$$

2.6.10 Problem 9 (d)

Internal problem ID [18574]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 9 (d)
Date solved : `Tuesday, January 28, 2025 at 12:00:44 PM`
CAS classification : `[_separable]`

Solve

$$\frac{y'}{x} = y \sin\left(x^2-1\right) - \frac{2y}{\sqrt{x}}$$

Solved as first order linear ode

Time used: 0.189 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x}dx} \\ &= \mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}\right) = 0$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}} &= \int 0\,dx + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}$ gives the final solution

$$y = \mathrm{e}^{-\frac{\cos\left(x^2-1\right)}{2}-\frac{4x^{3/2}}{3}} c_1$$

Summary of solutions found

$$y = \mathrm{e}^{-\frac{\cos\left(x^2-1\right)}{2}-\frac{4x^{3/2}}{3}} c_1$$

Solved as first order separable ode

Time used: 0.162 (sec)

The ode

$$y' = y\left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x} \tag{2.44}$$

is separable as it can be written as

$$\begin{aligned} y' &= y\left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x} \\ g(y) &= y \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\,dy &= \int f(x)\,dx \\ \int \frac{1}{y}\,dy &= \int \left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x}\,dx \end{aligned}$$

$$\ln(y) = -\frac{\cos(x^2-1)}{2} - \frac{4x^{3/2}}{3} + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or

$$y = 0$$

for y gives

$$y = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(y) = -\frac{\cos(x^2-1)}{2} - \frac{4x^{3/2}}{3} + c_1$$
$$y = 0$$

Solving for y gives

$$y = 0$$
$$y = e^{-\frac{\cos(x^2-1)}{2} - \frac{4x^{3/2}}{3} + c_1}$$

Summary of solutions found

$$y = 0$$
$$y = e^{-\frac{\cos(x^2-1)}{2} - \frac{4x^{3/2}}{3} + c_1}$$

Solved as first order homogeneous class D2 ode

Time used: 0.242 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$\frac{u'(x)\, x + u(x)}{x} = u(x)\, x \sin(x^2-1) - 2u(x)\sqrt{x}$$

Which is now solved The ode

$$u'(x) = -\frac{u(x)\left(-\sin(x^2-1)\, x^2 + 2x^{3/2} + 1\right)}{x} \tag{2.45}$$

is separable as it can be written as

$$
\begin{aligned}
u'(x) &= -\frac{u(x)\left(-\sin\left(x^2-1\right)x^2+2x^{3/2}+1\right)}{x} \\
&= f(x)g(u)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= -\frac{-\sin\left(x^2-1\right)x^2+2x^{3/2}+1}{x} \\
g(u) &= u
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{-\sin\left(x^2-1\right)x^2+2x^{3/2}+1}{x}\,dx
\end{aligned}
$$

$$
\ln\left(u(x)\right) = -\frac{\cos\left(x^2-1\right)}{2}+\ln\left(\frac{1}{x}\right)-\frac{4x^{3/2}}{3}+c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$
u = 0
$$

for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(u(x)\right) &= -\frac{\cos\left(x^2-1\right)}{2}+\ln\left(\frac{1}{x}\right)-\frac{4x^{3/2}}{3}+c_1 \\
u(x) &= 0
\end{aligned}
$$

Solving for $u(x)$ gives

$$u(x) = 0$$

$$u(x) = \frac{e^{-\frac{\cos(x^2-1)}{2}-\frac{4x^{3/2}}{3}+c_1}}{x}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{-\frac{\cos(x^2-1)}{2}-\frac{4x^{3/2}}{3}+c_1}}{x}$ back to y gives

$$y = e^{-\frac{\cos(x^2-1)}{2}-\frac{4x^{3/2}}{3}+c_1}$$

Summary of solutions found

$$y = 0$$

$$y = e^{-\frac{\cos(x^2-1)}{2}-\frac{4x^{3/2}}{3}+c_1}$$

Solved as first order Exact ode

Time used: 0.268 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{x}\right)\mathrm{d}y = \left(y\sin\left(x^2-1\right) - \frac{2y}{\sqrt{x}}\right)\mathrm{d}x$$

$$\left(-y\sin\left(x^2-1\right) + \frac{2y}{\sqrt{x}}\right)\mathrm{d}x + \left(\frac{1}{x}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -y\sin\left(x^2-1\right) + \frac{2y}{\sqrt{x}}$$
$$N(x, y) = \frac{1}{x}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y\sin\left(x^2-1\right) + \frac{2y}{\sqrt{x}}\right)\\ &= -\sin\left(x^2-1\right) + \frac{2}{\sqrt{x}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{x}\right)\\ &= -\frac{1}{x^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= x\left(\left(-\sin\left(x^2-1\right) + \frac{2}{\sqrt{x}}\right) - \left(-\frac{1}{x^2}\right)\right) \\
&= x\left(-\sin\left(x^2-1\right) + \frac{2}{\sqrt{x}} + \frac{1}{x^2}\right)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int x\left(-\sin\left(x^2-1\right)+\frac{2}{\sqrt{x}}+\frac{1}{x^2}\right) \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{\cos\left(x^2-1\right)}{2}+\ln(x)+\frac{4x^{3/2}}{3}} \\
&= x\,\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x\,\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}\left(-y\sin\left(x^2-1\right) + \frac{2y}{\sqrt{x}}\right) \\
&= \left(-\sin\left(x^2-1\right)\sqrt{x}+2\right)\sqrt{x}\,y\,\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x\,\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}\left(\frac{1}{x}\right) \\
&= \mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(-\sin\left(x^2-1\right)\sqrt{x}+2\right)\sqrt{x}\,y\,\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}\right) + \left(\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \, dy = \int \overline{N} \, dy$$
$$\int \frac{\partial \phi}{\partial y} \, dy = \int e^{\frac{\cos(x^2-1)}{2}+\frac{4x^{3/2}}{3}} \, dy$$
$$\phi = y \, e^{\frac{\cos(x^2-1)}{2}+\frac{4x^{3/2}}{3}} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = y\left(-\sin\left(x^2-1\right)x + 2\sqrt{x}\right) e^{\frac{\cos(x^2-1)}{2}+\frac{4x^{3/2}}{3}} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \left(-\sin\left(x^2-1\right)\sqrt{x}+2\right)\sqrt{x}\, y \, e^{\frac{\cos(x^2-1)}{2}+\frac{4x^{3/2}}{3}}$. Therefore equation (4) becomes

$$\begin{aligned}&\left(-\sin\left(x^2-1\right)\sqrt{x}\right.\\ &\left.+2\right)\sqrt{x}\, y \, e^{\frac{\cos(x^2-1)}{2}+\frac{4x^{3/2}}{3}} = y\left(-\sin\left(x^2-1\right)x + 2\sqrt{x}\right) e^{\frac{\cos(x^2-1)}{2}+\frac{4x^{3/2}}{3}} + f'(x)\end{aligned} \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = y \, e^{\frac{\cos(x^2-1)}{2}+\frac{4x^{3/2}}{3}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{\frac{\cos\left(x^2-1\right)}{2}+\frac{4x^{3/2}}{3}}$$

Solving for y gives

$$y = \mathrm{e}^{-\frac{\cos\left(x^2-1\right)}{2}-\frac{4x^{3/2}}{3}} c_1$$

Summary of solutions found

$$y = \mathrm{e}^{-\frac{\cos\left(x^2-1\right)}{2}-\frac{4x^{3/2}}{3}} c_1$$

Solved using Lie symmetry for first order ode

Time used: 0.579 (sec)

Writing the ode as

$$\begin{aligned}
y' &= y\left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + y\left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x}\,(b_3-a_2) - y^2\left(\sin\left(x^2-1\right)\sqrt{x}-2\right)^2 xa_3 \\
&- \left(y\left(2x^{3/2}\cos\left(x^2-1\right)+\frac{\sin\left(x^2-1\right)}{2\sqrt{x}}\right)\sqrt{x}\right. \\
&\left.+\frac{y\left(\sin\left(x^2-1\right)\sqrt{x}-2\right)}{2\sqrt{x}}\right)(xa_2+ya_3+a_1) \\
&- \left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x}\,(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\sin\left(x^2-1\right)^2 x^{5/2}y^2a_3+2x^{7/2}\cos\left(x^2-1\right)ya_2+2x^{5/2}\cos\left(x^2-1\right)y^2a_3+2x^{5/2}\cos\left(x^2-1\right)ya_1-4\sin(x}{}$$

Setting the numerator to zero gives

$$\begin{aligned}&-\sin\left(x^2-1\right)^2 x^{5/2}y^2a_3-2x^{7/2}\cos\left(x^2-1\right)ya_2 \\ &-2x^{5/2}\cos\left(x^2-1\right)y^2a_3-2x^{5/2}\cos\left(x^2-1\right)ya_1-\sin\left(x^2-1\right)x^{5/2}b_2-2\sin\left(x^2-1\right)x^{3/2}ya_2+4\sin\left(x^2-1\right)x^2\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}&-4y^2x^{3/2}a_3+2x^2b_2+3xya_2+y^2a_3+b_2\sqrt{x}+2xb_1+ya_1-\frac{x^{5/2}y^2a_3}{2}+\frac{x^{5/2}y^2a_3\cos\left(2x^2-2\right)}{2} \\ &-2x^{7/2}\cos\left(x^2-1\right)ya_2-2x^{5/2}\cos\left(x^2-1\right)y^2a_3-2x^{5/2}\cos\left(x^2-1\right)ya_1-\sin\left(x^2-1\right)x^{5/2}b_2-2\sin\left(x^2-1\right)x\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{x},x^{3/2},x^{5/2},x^{7/2},\cos\left(x^2-1\right),\cos\left(2x^2-2\right),\sin\left(x^2-1\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,\sqrt{x}=v_3,x^{3/2}=v_4,x^{5/2}=v_5,x^{7/2}=v_6,\cos\left(x^2-1\right)=v_7,\cos\left(2x^2-2\right)=v_8,\sin\left(x^2-1\right)=v_9\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-4v_2^2v_4a_3+2v_1^2b_2+3v_1v_2a_2+v_2^2a_3+b_2v_3+2v_1b_1+v_2a_1-\frac{1}{2}v_5v_2^2a_3 \\ &+\frac{1}{2}v_5v_2^2a_3v_8-2v_6v_7v_2a_2-2v_5v_7v_2^2a_3-2v_5v_7v_2a_1-v_9v_5b_2 \\ &-2v_9v_4v_2a_2+4v_9v_1^2v_2^2a_3-v_9v_4b_1-v_9v_3v_2^2a_3-v_9v_3v_2a_1=0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-4v_2^2v_4a_3 + 2v_1^2b_2 + 3v_1v_2a_2 + v_2^2a_3 + b_2v_3 + 2v_1b_1 + v_2a_1 - \frac{1}{2}v_5v_2^2a_3 \\
&+ \frac{1}{2}v_5v_2^2a_3v_8 - 2v_6v_7v_2a_2 - 2v_5v_7v_2^2a_3 - 2v_5v_7v_2a_1 - v_9v_5b_2 \\
&- 2v_9v_4v_2a_2 + 4v_9v_1^2v_2^2a_3 - v_9v_4b_1 - v_9v_3v_2^2a_3 - v_9v_3v_2a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
b_2 &= 0 \\
-2a_1 &= 0 \\
-a_1 &= 0 \\
-2a_2 &= 0 \\
3a_2 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
-a_3 &= 0 \\
4a_3 &= 0 \\
-\frac{a_3}{2} &= 0 \\
\frac{a_3}{2} &= 0 \\
-b_1 &= 0 \\
2b_1 &= 0 \\
-b_2 &= 0 \\
2b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 0$$
$$\eta = y$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y} dy \end{aligned}$$

Which results in

$$S = \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y\left(\sin\left(x^2 - 1\right) \sqrt{x} - 2\right) \sqrt{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 0 \\
S_y &= \frac{1}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \left(\sin\left(R^2-1\right)\sqrt{R}-2\right)\sqrt{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int \left(\sin\left(R^2-1\right)\sqrt{R}-2\right)\sqrt{R}\,dR \\
S(R) &= -\frac{\cos\left(R^2-1\right)}{2} - \frac{4R^{3/2}}{3} + c_2
\end{aligned}
$$

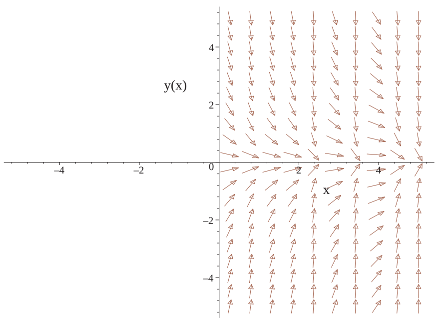
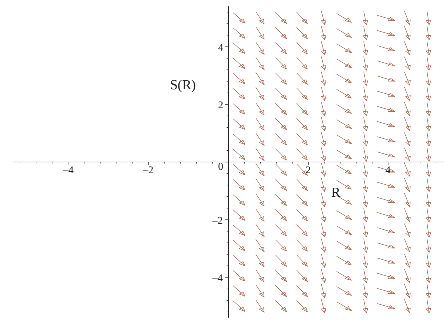
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y) = -\frac{\cos\left(x^2-1\right)}{2} - \frac{4x^{3/2}}{3} + c_2$$

Which gives

$$y = \mathrm{e}^{-\frac{\cos\left(x^2-1\right)}{2}-\frac{4x^{3/2}}{3}+c_2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y\left(\sin\left(x^2-1\right)\sqrt{x}-2\right)\sqrt{x}$ 	$R = x$ $S = \ln\left(y\right)$	$\frac{dS}{dR} = \left(\sin\left(R^2-1\right)\sqrt{R}-2\right)\sqrt{R}$ 

Summary of solutions found

$$y = \mathrm{e}^{-\frac{\cos\left(x^2-1\right)}{2}-\frac{4x^{3/2}}{3}+c_2}$$

Maple step by step solution

Let's solve

$\frac{y'}{x} = y\sin\left(x^2-1\right) - \frac{2y}{\sqrt{x}}$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $y' = \left(y\sin\left(x^2-1\right) - \frac{2y}{\sqrt{x}}\right)x$

- Separate variables

  $\frac{y'}{y} = \left(-2+\sqrt{x}\,\sin\left(\left(x-1\right)\left(1+x\right)\right)\right)\sqrt{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \left(-2+\sqrt{x}\,\sin\left(\left(x-1\right)\left(1+x\right)\right)\right)\sqrt{x}dx + C1$

- Evaluate integral

  $\ln\left(y\right) = -\frac{\cos\left(\left(x-1\right)\left(1+x\right)\right)}{2} - \frac{4x^{3/2}}{3} + C1$

- Solve for y

$$y = \mathrm{e}^{-\frac{\cos((x-1)(1+x))}{2}-\frac{4x^{3/2}}{3}+C1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.000 (sec)
Leaf size : 21

```
dsolve(1/x*diff(y(x),x) = y(x)*sin(x^2-1)-2*y(x)/x^(1/2),y(x),singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{\cos\left(x^2-1\right)}{2}-\frac{4x^{3/2}}{3}}$$

Mathematica DSolve solution

Solving time : 0.102 (sec)
Leaf size : 37

```
DSolve[{1/x*D[y[x],x]==y[x]*Sin[x^2-1]-2*y[x]/Sqrt[x],{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to c_1 e^{\frac{1}{6}\left(-8x^{3/2}-3\cos\left(1-x^2\right)\right)}$$
$$y(x) \to 0$$

2.6.11 Problem 9 (e)

Internal problem ID [18575]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 9 (e)
Date solved : `Tuesday, January 28, 2025 at 12:00:46 PM`
CAS classification :
`[[_homogeneous, 'class A'], _rational, [_Abel, '2nd type', 'class A']]`

Solve

$$y' = 1 + \frac{2y}{x - y}$$

Solved as first order polynomial type ode

Time used: 0.480 (sec)

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = 1, b_1 = 1, c_1 = 0, a_2 = 1, b_2 = -1, c_2 = 0$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE.

The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1 x + b_1 y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$
\begin{aligned}
X &= x - x_0 \\
Y &= y - y_0
\end{aligned}
$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$
\begin{aligned}
a_1 x_0 + b_1 y_0 + c_1 &= 0 \\
a_2 x_0 + b_2 y_0 + c_2 &= 0
\end{aligned}
$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$
\begin{aligned}
x_0 + y_0 &= 0 \\
x_0 - y_0 &= 0
\end{aligned}
$$

Solving for x_0, y_0 from the above gives

$$
\begin{aligned}
x_0 &= 0 \\
y_0 &= 0
\end{aligned}
$$

Therefore the transformation becomes

$$
\begin{aligned}
X &= x - 0 \\
Y &= y - 0
\end{aligned}
$$

Using this transformation in $y' = 1 + \frac{2y}{x-y}$ result in

$$
\frac{dY}{dX} = \frac{X + Y}{X - Y}
$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{X + Y}{-X + Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = X+Y$ and $N = X-Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u-1}{u-1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) - \left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 1 = 0$$

Or

$$X(u(X)-1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{u(X)^2 + 1}{X\left(u\left(X\right)-1\right)} \tag{2.46}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{u(X)^2 + 1}{X\left(u\left(X\right)-1\right)}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= -\frac{1}{X}\\ g(u) &= \frac{u^2+1}{u-1}\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(X)\, dX$$
$$\int \frac{u-1}{u^2+1}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_2$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2+1}{u-1} = 0$$

for $u(X)$ gives

$$u(X) = -i$$
$$u(X) = i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_2$$
$$u(X) = -i$$
$$u(X) = i$$

Converting $\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_2$ back to $Y(X)$ gives

$$\frac{\ln\left(\frac{Y(X)^2+X^2}{X^2}\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) = \ln\left(\frac{1}{X}\right) + c_2$$

Converting $u(X) = -i$ back to $Y(X)$ gives

$$Y(X) = -iX$$

Converting $u(X) = i$ back to $Y(X)$ gives

$$Y(X) = iX$$

The solution is implicit $\frac{\ln\left(\frac{Y(X)^2+X^2}{X^2}\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) = \ln\left(\frac{1}{X}\right) + c_2$. Replacing $Y = y - y_0, X = x - x_0$ gives

$$\frac{\ln\left(\frac{x^2+y^2}{x^2}\right)}{2} - \arctan\left(\frac{y}{x}\right) = \ln\left(\frac{1}{x}\right) + c_2$$

The solution is

$$Y(X) = -iX$$

Replacing $Y = y - y_0, X = x - x_0$ gives

$$y = -ix$$

The solution is

$$Y(X) = iX$$

Replacing $Y = y - y_0, X = x - x_0$ gives

$$y = ix$$

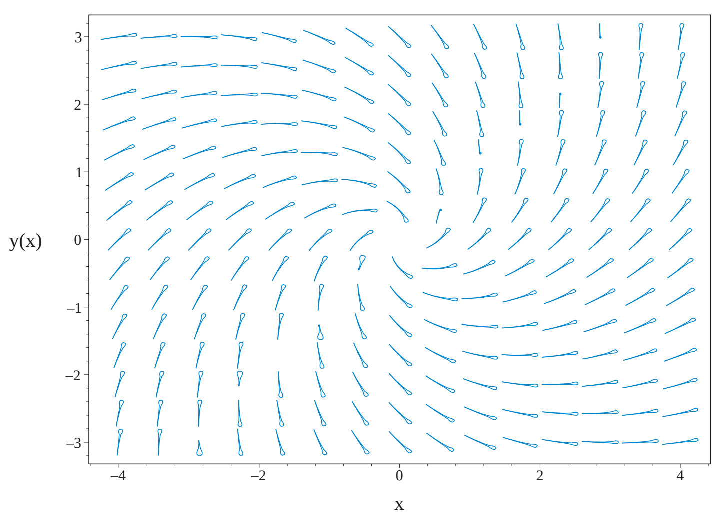


Figure 2.65: Slope field plot
$y' = 1 + \frac{2y}{x-y}$

Summary of solutions found

$$\frac{\ln\left(\frac{x^2+y^2}{x^2}\right)}{2} - \arctan\left(\frac{y}{x}\right) = \ln\left(\frac{1}{x}\right) + c_2$$
$$y = -ix$$
$$y = ix$$

Solved as first order homogeneous class A ode

Time used: 0.279 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x+y}{-x+y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = x+y$ and $N = x-y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = \frac{-u-1}{u-1}$$
$$\frac{du}{dx} = \frac{\frac{-u(x)-1}{u(x)-1} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{-u(x)-1}{u(x)-1} - u(x)}{x} = 0$$

Or

$$u'(x)\, xu(x) - u'(x)\, x + u(x)^2 + 1 = 0$$

Or

$$x(u(x)-1)\,u'(x)+u(x)^2+1=0$$

Which is now solved as separable in $u(x)$.

The ode

$$u'(x)=-\frac{u(x)^2+1}{x\,(u\,(x)-1)} \tag{2.47}$$

is separable as it can be written as

$$\begin{aligned}u'(x)&=-\frac{u(x)^2+1}{x\,(u\,(x)-1)}\\&=f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x)&=-\frac{1}{x}\\g(u)&=\frac{u^2+1}{u-1}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int\frac{1}{g(u)}\,du&=\int f(x)\,dx\\\int\frac{u-1}{u^2+1}\,du&=\int-\frac{1}{x}\,dx\end{aligned}$$

$$\frac{\ln\left(u(x)^2+1\right)}{2}-\arctan\left(u(x)\right)=\ln\left(\frac{1}{x}\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$\frac{u^2+1}{u-1}=0$$

for $u(x)$ gives

$$\begin{aligned}u(x)&=-i\\u(x)&=i\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(x)^2+1\right)}{2}-\arctan\left(u(x)\right)=\ln\left(\frac{1}{x}\right)+c_1$$

$$u(x)=-i$$

$$u(x)=i$$

Converting $\frac{\ln\left(u(x)^2+1\right)}{2}-\arctan\left(u(x)\right)=\ln\left(\frac{1}{x}\right)+c_1$ back to y gives

$$\frac{\ln\left(\frac{x^2+y^2}{x^2}\right)}{2}-\arctan\left(\frac{y}{x}\right)=\ln\left(\frac{1}{x}\right)+c_1$$

Converting $u(x)=-i$ back to y gives

$$y=-ix$$

Converting $u(x)=i$ back to y gives

$$y=ix$$

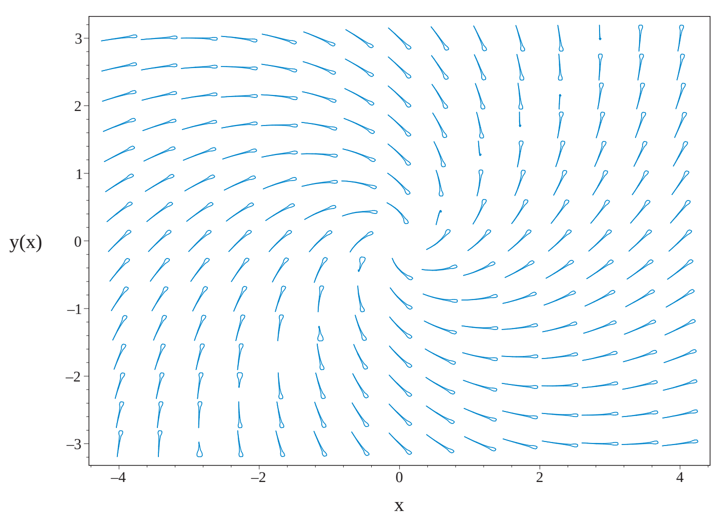


Figure 2.66: Slope field plot
$y'=1+\frac{2y}{x-y}$

Summary of solutions found

$$\frac{\ln\left(\frac{x^2+y^2}{x^2}\right)}{2} - \arctan\left(\frac{y}{x}\right) = \ln\left(\frac{1}{x}\right) + c_1$$
$$y = -ix$$
$$y = ix$$

Solved as first order homogeneous class D2 ode

Time used: 0.114 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 1 + \frac{2u(x)\,x}{x - u\,(x)\,x}$$

Which is now solved The ode

$$u'(x) = -\frac{u(x)^2 + 1}{(u\,(x) - 1)\,x} \tag{2.48}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)^2 + 1}{(u\,(x) - 1)\,x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{u^2+1}{u-1} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{u-1}{u^2+1}\,du &= \int -\frac{1}{x}\,dx \end{aligned}$$

$$\frac{\ln\left(u(x)^2 + 1\right)}{2} - \arctan\left(u(x)\right) = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2+1}{u-1} = 0$$

for $u(x)$ gives

$$u(x) = -i$$
$$u(x) = i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan\left(u(x)\right) = \ln\left(\frac{1}{x}\right) + c_1$$
$$u(x) = -i$$
$$u(x) = i$$

Converting $\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan\left(u(x)\right) = \ln\left(\frac{1}{x}\right) + c_1$ back to y gives

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) = \ln\left(\frac{1}{x}\right) + c_1$$

Converting $u(x) = -i$ back to y gives

$$y = -ix$$

Converting $u(x) = i$ back to y gives

$$y = ix$$

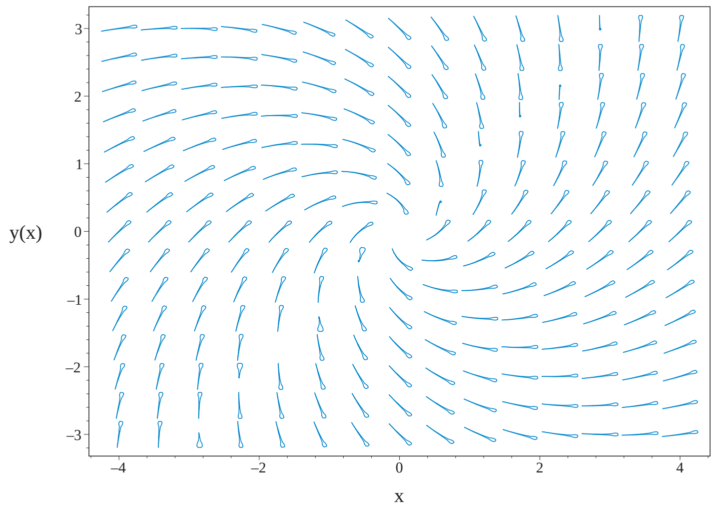


Figure 2.67: Slope field plot
$y' = 1 + \frac{2y}{x-y}$

Summary of solutions found

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) = \ln\left(\frac{1}{x}\right) + c_1$$
$$y = -ix$$
$$y = ix$$

Solved as first order homogeneous class Maple C ode

Time used: 0.384 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{x_0 + X + Y(X) + y_0}{-x_0 - X + Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X + Y(X)}{-X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{X+Y}{-X+Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = X+Y$ and $N = X-Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u-1}{u-1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X u(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0$$

Or

$$X(u(X)-1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{u(X)^2 + 1}{X\,(u\,(X)-1)} \tag{2.49}$$

is separable as it can be written as

$$\begin{aligned} \frac{d}{dX}u(X) &= -\frac{u(X)^2 + 1}{X\,(u\,(X)-1)} \\ &= f(X)g(u) \end{aligned}$$

Where

$$f(X) = -\frac{1}{X}$$
$$g(u) = \frac{u^2+1}{u-1}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(X)\, dX$$
$$\int \frac{u-1}{u^2+1}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2+1}{u-1} = 0$$

for $u(X)$ gives

$$u(X) = -i$$
$$u(X) = i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_1$$
$$u(X) = -i$$
$$u(X) = i$$

Converting $\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_1$ back to $Y(X)$ gives

$$\frac{\ln\left(\frac{Y(X)^2+X^2}{X^2}\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) = \ln\left(\frac{1}{X}\right) + c_1$$

Converting $u(X) = -i$ back to $Y(X)$ gives

$$Y(X) = -iX$$

Converting $u(X) = i$ back to $Y(X)$ gives

$$Y(X) = iX$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2+X^2}{X^2}\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) = \ln\left(\frac{1}{X}\right) + c_1 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$\frac{\ln\left(\frac{x^2+y^2}{x^2}\right)}{2} - \arctan\left(\frac{y}{x}\right) = \ln\left(\frac{1}{x}\right) + c_1$$

Using the solution for $Y(X)$

$$Y(X) = -iX \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = -ix$$

Using the solution for $Y(X)$

$$Y(X) = iX \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = ix$$

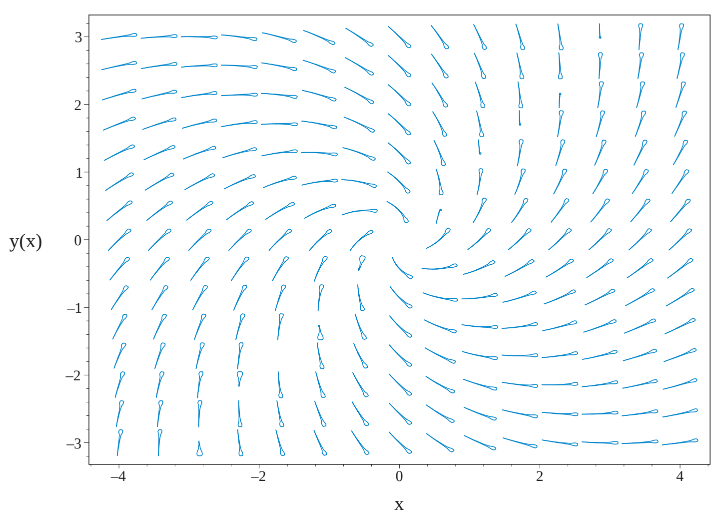


Figure 2.68: Slope field plot
$y' = 1 + \frac{2y}{x-y}$

Solved as first order Exact ode

Time used: 0.200 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-x+y)\,\mathrm{d}y = (-x-y)\,\mathrm{d}x$$
$$(x+y)\,\mathrm{d}x + (-x+y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = x+y$$
$$N(x,y) = -x+y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x+y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x+y)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x+y$ and $N = -x+y$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{x+y}{x^2+y^2}\\ N &= \frac{-x+y}{x^2+y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{x+y}{x^2+y^2}\right)\mathrm{d}x$$

$$\left(\frac{x+y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= \frac{x+y}{x^2+y^2} \\ N(x,y) &= \frac{-x+y}{x^2+y^2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{x^2+y^2}\right) \\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+y}{x^2+y^2}\right) \\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x+y}{x^2+y^2}\,\mathrm{d}x$$

$$\phi = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \tag{4}$$

$$= \frac{-x+y}{x^2+y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{-x+y}{x^2+y^2} = \frac{-x+y}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right)$$

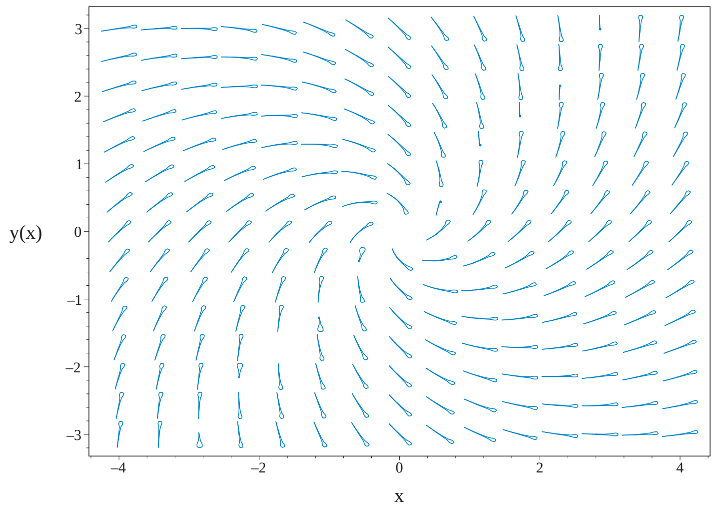


Figure 2.69: Slope field plot
$y' = 1 + \frac{2y}{x-y}$

Summary of solutions found

$$\frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) = c_1$$

Solved as first order isobaric ode

Time used: 0.187 (sec)

Solving for y' gives

$$y' = -\frac{x+y}{-x+y} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = -\frac{x+y}{-x+y} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= u x^m \\ &= u x \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = -\frac{x + xu(x)}{-x + xu(x)}$$

The ode

$$u'(x) = -\frac{u(x)^2 + 1}{(u(x) - 1)\, x} \tag{2.50}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)^2 + 1}{(u(x) - 1)\, x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{u^2 + 1}{u - 1} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{u - 1}{u^2 + 1}\, du &= \int -\frac{1}{x}\, dx \end{aligned}$$

$$\frac{\ln\left(u(x)^2 + 1\right)}{2} - \arctan\left(u(x)\right) = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2 + 1}{u - 1} = 0$$

for $u(x)$ gives

$$\begin{aligned} u(x) &= -i \\ u(x) &= i \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(x)^2+1\right)}{2}-\arctan\left(u(x)\right)=\ln\left(\frac{1}{x}\right)+c_1$$
$$u(x)=-i$$
$$u(x)=i$$

Converting $\frac{\ln\left(u(x)^2+1\right)}{2}-\arctan\left(u(x)\right)=\ln\left(\frac{1}{x}\right)+c_1$ back to y gives

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)=\ln\left(\frac{1}{x}\right)+c_1$$

Converting $u(x)=-i$ back to y gives

$$\frac{y}{x}=-i$$

Converting $u(x)=i$ back to y gives

$$\frac{y}{x}=i$$

Solving for y gives

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)=\ln\left(\frac{1}{x}\right)+c_1$$
$$y=-ix$$
$$y=ix$$

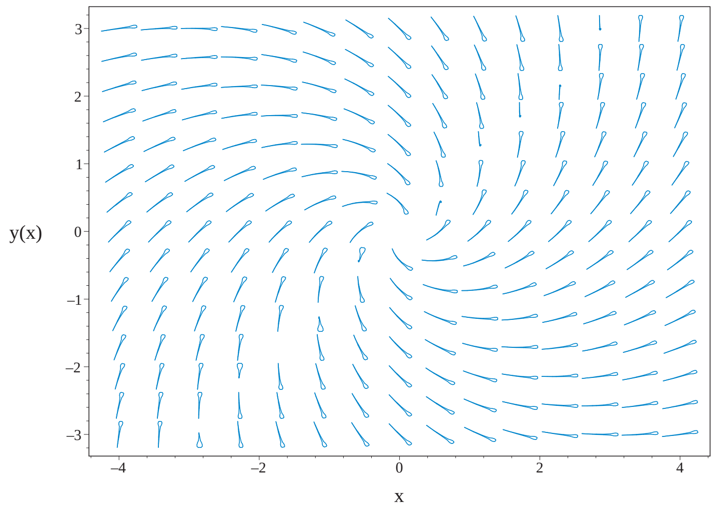


Figure 2.70: Slope field plot
$y' = 1 + \frac{2y}{x-y}$

Summary of solutions found

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) = \ln\left(\frac{1}{x}\right) + c_1$$
$$y = -ix$$
$$y = ix$$

Solved using Lie symmetry for first order ode

Time used: 0.503 (sec)

Writing the ode as

$$y' = -\frac{x+y}{-x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(x+y)(b_3-a_2)}{-x+y} - \frac{(x+y)^2 a_3}{(-x+y)^2} \\ & - \left(-\frac{1}{-x+y} - \frac{x+y}{(-x+y)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(-\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(x-y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 \\ & - 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 \\ & - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_2 - a_3 - b_2 + b_3) v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3) v_1 v_2 \\ - 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3) v_2^2 + 2a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -2b_1 &= 0 \\ -a_2 - a_3 - b_2 + b_3 &= 0 \\ a_2 + a_3 + b_2 - b_3 &= 0 \\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{x + y}{-x + y}\right)(x) \\ &= \frac{-x^2 - y^2}{x - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{\frac{-x^2-y^2}{x-y}}dy\end{aligned}$$

Which results in

$$S=\frac{\ln\left(x^2+y^2\right)}{2}-\arctan\left(\frac{y}{x}\right)$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=-\frac{x+y}{-x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=\frac{x+y}{x^2+y^2}\\S_y&=\frac{-x+y}{x^2+y^2}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

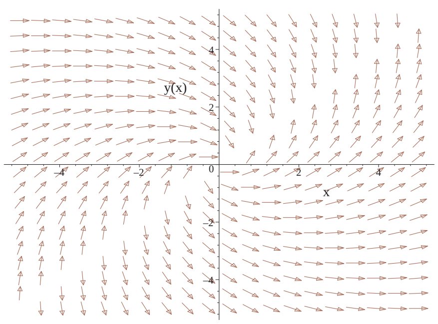
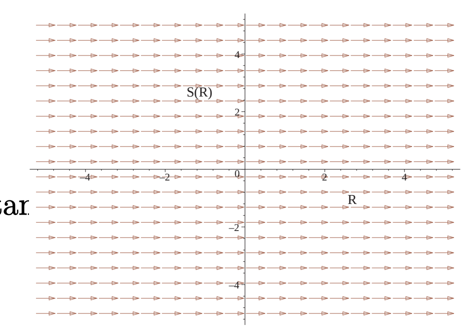
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}\int dS &= \int 0\, dR + c_2 \\ S(R) &= c_2\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{-x+y}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \arctan$	$\frac{dS}{dR} = 0$ 

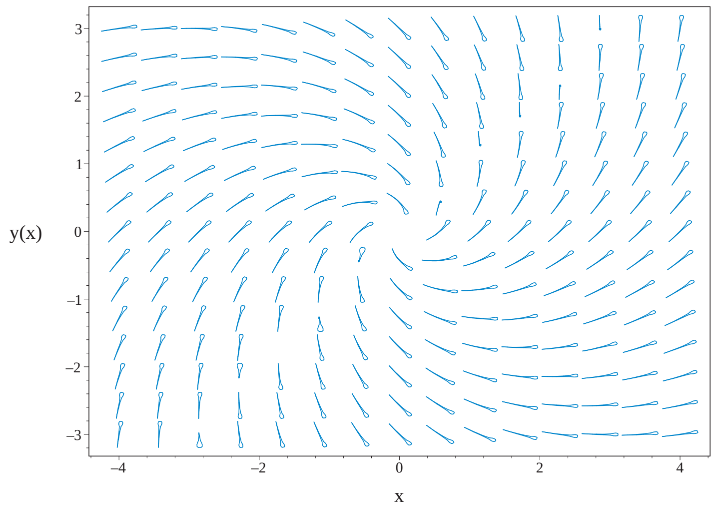


Figure 2.71: Slope field plot
$y' = 1 + \frac{2y}{x-y}$

Summary of solutions found

$$\frac{\ln\left(x^2+y^2\right)}{2} - \arctan\left(\frac{y}{x}\right) = c_2$$

Maple step by step solution

Let's solve
$y' = 1 + \frac{2y}{x-y}$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = 1 + \frac{2y}{x-y}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
```

```
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 24

```
dsolve(diff(y(x),x) = 1+2*y(x)/(x-y(x)),y(x),singsol=all)
```

$$y(x) = \tan\left(\text{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right) x$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 36

```
DSolve[{D[y[x],x]==1+2*y[x]/(x-y[x]),{}},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x) + c_1, y(x)\right]$$

2.6.12 Problem 10 (a)

Internal problem ID [18576]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 10 (a)
Date solved : `Tuesday, January 28, 2025 at 12:00:50 PM`
CAS classification : `[_separable]`

Solve

$$v' + 2uv = 2u$$

Solved as first order linear ode

Time used: 0.071 (sec)

In canonical form a linear first order is

$$v' + q(u)v = p(u)$$

Comparing the above to the given ode shows that

$$q(u) = 2u$$
$$p(u) = 2u$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, du}\\ &= e^{\int 2u du}\\ &= e^{u^2}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mu v) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}u}(\mu v) = (\mu)(2u)$$
$$\frac{\mathrm{d}}{\mathrm{d}u}\left(v\,\mathrm{e}^{u^2}\right) = \left(\mathrm{e}^{u^2}\right)(2u)$$
$$\mathrm{d}\left(v\,\mathrm{e}^{u^2}\right) = \left(2u\,\mathrm{e}^{u^2}\right)\,\mathrm{d}u$$

Integrating gives

$$\begin{aligned} v\,\mathrm{e}^{u^2} &= \int 2u\,\mathrm{e}^{u^2}\,du \\ &= \mathrm{e}^{u^2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor e^{u^2} gives the final solution

$$v = 1 + c_1\,\mathrm{e}^{-u^2}$$

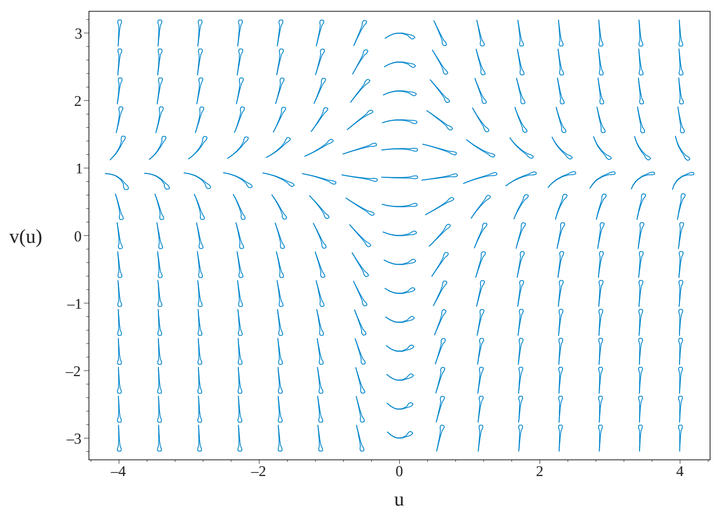


Figure 2.72: Slope field plot
$v' + 2uv = 2u$

Summary of solutions found

$$v = 1 + c_1\,\mathrm{e}^{-u^2}$$

Solved as first order separable ode

Time used: 0.096 (sec)

The ode

$$v' = -2uv + 2u \tag{2.51}$$

is separable as it can be written as

$$\begin{aligned} v' &= -2uv + 2u \\ &= f(u)g(v) \end{aligned}$$

Where

$$\begin{aligned} f(u) &= u \\ g(v) &= -2v + 2 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(v)}\, dv &= \int f(u)\, du \\ \int \frac{1}{-2v+2}\, dv &= \int u\, du \end{aligned}$$

$$-\frac{\ln(v-1)}{2} = \frac{u^2}{2} + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(v)$ is zero, since we had to divide by this above. Solving $g(v) = 0$ or

$$-2v + 2 = 0$$

for v gives

$$v = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} -\frac{\ln(v-1)}{2} &= \frac{u^2}{2} + c_1 \\ v &= 1 \end{aligned}$$

Solving for v gives

$$v = 1$$
$$v = e^{-u^2-2c_1} + 1$$

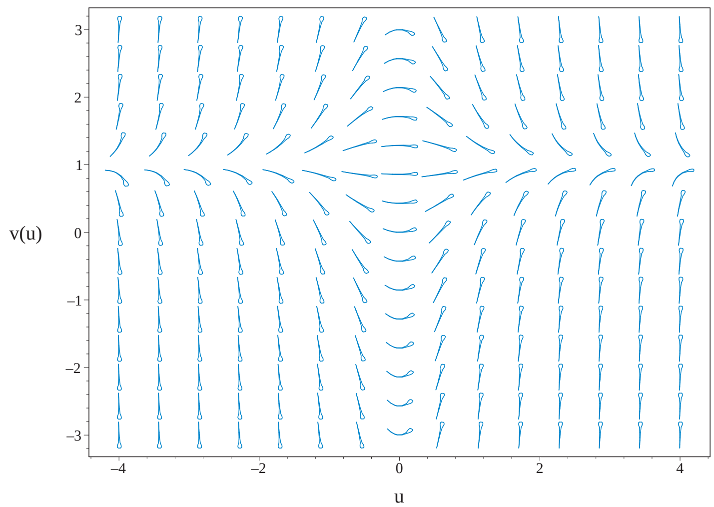


Figure 2.73: Slope field plot
$v' + 2uv = 2u$

<u>Summary of solutions found</u>

$$v = 1$$
$$v = e^{-u^2-2c_1} + 1$$

Solved as first order Exact ode

Time used: 0.149 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(u, v)\, \mathrm{d}u + N(u, v)\, \mathrm{d}v = 0 \tag{1A}$$

Therefore

$$\mathrm{d}v = (-2uv + 2u)\, \mathrm{d}u$$
$$(2uv - 2u)\, \mathrm{d}u + \mathrm{d}v = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(u, v) = 2uv - 2u$$
$$N(u, v) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}(2uv - 2u) \\ &= 2u\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial u} &= \frac{\partial}{\partial u}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial u}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u}\right) \\
&= 1((2u) - (0)) \\
&= 2u
\end{aligned}
$$

Since A does not depend on v, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}u} \\
&= e^{\int 2u \, \mathrm{d}u}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{u^2} \\
&= \mathrm{e}^{u^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{u^2}(2uv - 2u) \\
&= 2u(v-1)\,\mathrm{e}^{u^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{u^2}(1) \\
&= \mathrm{e}^{u^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}v}{\mathrm{d}u} &= 0 \\
\left(2u(v-1)\,\mathrm{e}^{u^2}\right) + \left(\mathrm{e}^{u^2}\right)\frac{\mathrm{d}v}{\mathrm{d}u} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(u, v)$

$$
\frac{\partial \phi}{\partial u} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial v} = \overline{N} \tag{2}
$$

Integrating (2) w.r.t. v gives

$$\int \frac{\partial \phi}{\partial v}\,\mathrm{d}v = \int \overline{N}\,\mathrm{d}v$$

$$\int \frac{\partial \phi}{\partial v}\,\mathrm{d}v = \int \mathrm{e}^{u^2}\,\mathrm{d}v$$

$$\phi = v\,\mathrm{e}^{u^2} + f(u) \tag{3}$$

Where $f(u)$ is used for the constant of integration since ϕ is a function of both u and v. Taking derivative of equation (3) w.r.t u gives

$$\frac{\partial \phi}{\partial u} = 2vu\,\mathrm{e}^{u^2} + f'(u) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial u} = 2u(v-1)\,\mathrm{e}^{u^2}$. Therefore equation (4) becomes

$$2u(v-1)\,\mathrm{e}^{u^2} = 2vu\,\mathrm{e}^{u^2} + f'(u) \tag{5}$$

Solving equation (5) for $f'(u)$ gives

$$f'(u) = -2u\,\mathrm{e}^{u^2}$$

Integrating the above w.r.t u gives

$$\int f'(u)\,\mathrm{d}u = \int \left(-2u\,\mathrm{e}^{u^2}\right)\mathrm{d}u$$

$$f(u) = -\mathrm{e}^{u^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(u)$ into equation (3) gives ϕ

$$\phi = v\,\mathrm{e}^{u^2} - \mathrm{e}^{u^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = v\,\mathrm{e}^{u^2} - \mathrm{e}^{u^2}$$

Solving for v gives

$$v = \mathrm{e}^{-u^2}\left(\mathrm{e}^{u^2} + c_1\right)$$

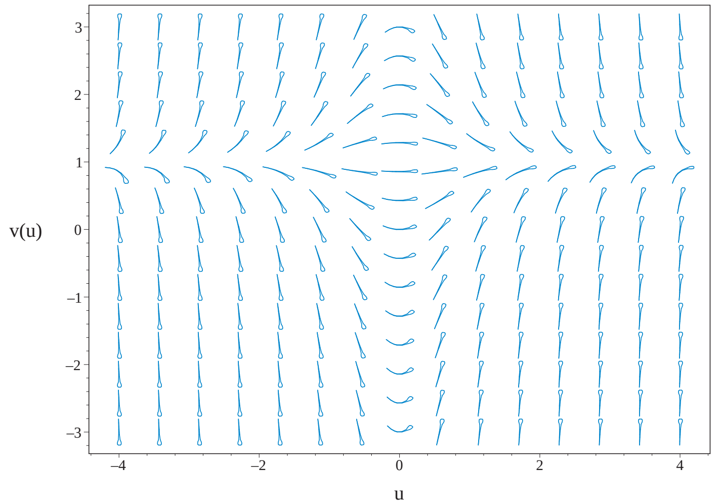


Figure 2.74: Slope field plot
$v' + 2uv = 2u$

Summary of solutions found

$$v = \mathrm{e}^{-u^2}\left(\mathrm{e}^{u^2} + c_1\right)$$

Solved using Lie symmetry for first order ode

Time used: 0.431 (sec)

Writing the ode as

$$v' = -2uv + 2u$$
$$v' = \omega(u, v)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_u + \omega(\eta_v - \xi_u) - \omega^2\xi_v - \omega_u\xi - \omega_v\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ua_2 + va_3 + a_1 \tag{1E}$$
$$\eta = ub_2 + vb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + (-2uv + 2u)(b_3 - a_2) - (-2uv + 2u)^2 a_3 \\ &- (-2v + 2)(ua_2 + va_3 + a_1) + 2u(ub_2 + vb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} &-4u^2v^2a_3 + 8u^2va_3 - 4u^2a_3 + 2u^2b_2 + 4uva_2 + 2v^2a_3 \\ &- 4ua_2 + 2ub_1 + 2ub_3 + 2va_1 - 2va_3 - 2a_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} &-4u^2v^2a_3 + 8u^2va_3 - 4u^2a_3 + 2u^2b_2 + 4uva_2 + 2v^2a_3 \\ &- 4ua_2 + 2ub_1 + 2ub_3 + 2va_1 - 2va_3 - 2a_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{u, v\}$ in them.

$$\{u, v\}$$

The following substitution is now made to be able to collect on all terms with $\{u, v\}$ in them

$$\{u = v_1, v = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-4a_3v_1^2v_2^2 + 8a_3v_1^2v_2 + 4a_2v_1v_2 - 4a_3v_1^2 + 2a_3v_2^2 + 2b_2v_1^2 \\ &+ 2a_1v_2 - 4a_2v_1 - 2a_3v_2 + 2b_1v_1 + 2b_3v_1 - 2a_1 + b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-4a_3v_1^2v_2^2 + 8a_3v_1^2v_2 + (-4a_3 + 2b_2)\,v_1^2 + 4a_2v_1v_2 \\ + (-4a_2 + 2b_1 + 2b_3)\,v_1 + 2a_3v_2^2 + (2a_1 - 2a_3)\,v_2 - 2a_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 4a_2 &= 0 \\ -4a_3 &= 0 \\ 2a_3 &= 0 \\ 8a_3 &= 0 \\ -2a_1 + b_2 &= 0 \\ 2a_1 - 2a_3 &= 0 \\ -4a_3 + 2b_2 &= 0 \\ -4a_2 + 2b_1 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= -b_3 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= v - 1 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(u, v) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{du}{\xi} = \frac{dv}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v}\right) S(u,v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = u$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{v-1} dy \end{aligned}$$

Which results in

$$S = \ln (v-1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_u + \omega(u,v) S_v}{R_u + \omega(u,v) R_v} \tag{2}$$

Where in the above R_u, R_v, S_u, S_v are all partial derivatives and $\omega(u,v)$ is the right hand side of the original ode given by

$$\omega(u,v) = -2uv + 2u$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_u &= 1 \\ R_v &= 0 \\ S_u &= 0 \\ S_v &= \frac{1}{v-1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2u \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for u, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -2R \, dR$$
$$S(R) = -R^2 + c_2$$

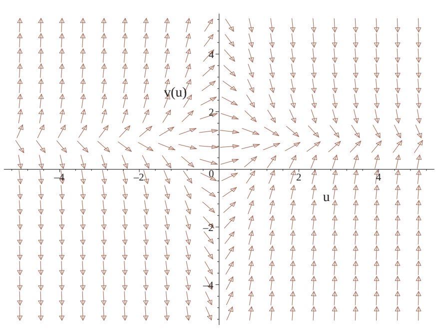
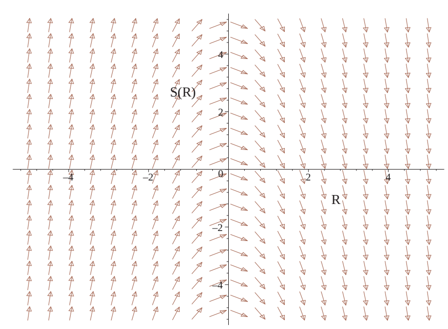
To complete the solution, we just need to transform the above back to u, v coordinates. This results in

$$\ln(v-1) = -u^2 + c_2$$

Which gives

$$v = \mathrm{e}^{-u^2+c_2} + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in u, v coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dv}{du} = -2uv + 2u$ 	$R = u$ $S = \ln(v-1)$	$\frac{dS}{dR} = -2R$ 

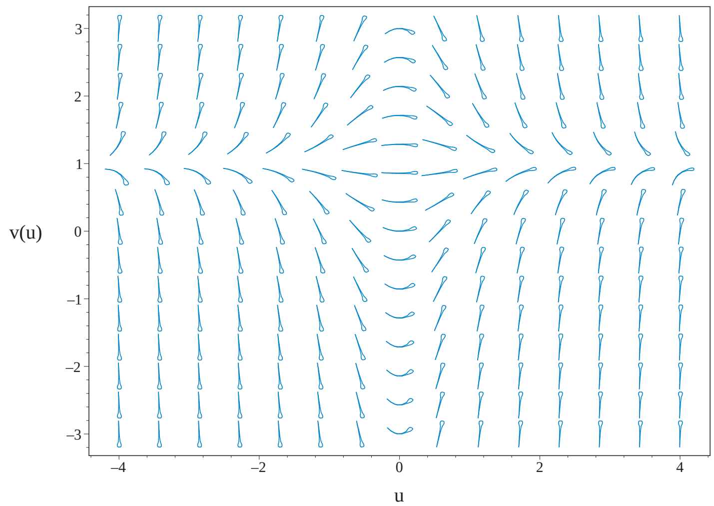


Figure 2.75: Slope field plot
$v' + 2uv = 2u$

Summary of solutions found

$$v = e^{-u^2+c_2} + 1$$

Maple step by step solution

Let's solve
$v' + 2vu = 2u$

- Highest derivative means the order of the ODE is 1
v'
- Solve for the highest derivative
$v' = -2vu + 2u$
- Separate variables
$\frac{v'}{v-1} = -2u$
- Integrate both sides with respect to u
$\int \frac{v'}{v-1} du = \int -2u du + C1$
- Evaluate integral
$\ln(v-1) = -u^2 + C1$
- Solve for v
$v = e^{-u^2+C1} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 14

```
dsolve(diff(v(u),u)+2*v(u)*u = 2*u,v(u),singsol=all)
```

$$v = 1 + e^{-u^2} c_1$$

Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 22

```
DSolve[{D[v[u],u]+2*u*v[u]==2*u,{}},v[u],u,IncludeSingularSolutions->True]
```

$$v(u) \to 1 + c_1 e^{-u^2}$$
$$v(u) \to 1$$

2.6.13 Problem 10 (b)

Internal problem ID [18577]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 10 (b)
Date solved : `Tuesday, January 28, 2025 at 12:00:52 PM`
CAS classification : `[_separable]`

Solve

$$1 + v^2 + \left(u^2 + 1\right) v v' = 0$$

Solved as first order separable ode

Time used: 0.237 (sec)

The ode

$$v' = -\frac{v^2 + 1}{(u^2 + 1)\, v} \tag{2.52}$$

is separable as it can be written as

$$\begin{aligned} v' &= -\frac{v^2 + 1}{(u^2 + 1)\, v} \\ &= f(u)g(v) \end{aligned}$$

Where

$$\begin{aligned} f(u) &= -\frac{1}{u^2 + 1} \\ g(v) &= \frac{v^2 + 1}{v} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(v)}\,dv = \int f(u)\,du$$
$$\int \frac{v}{v^2+1}\,dv = \int -\frac{1}{u^2+1}\,du$$

$$\frac{\ln\left(v^2+1\right)}{2} = -\arctan\left(u\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(v)$ is zero, since we had to divide by this above. Solving $g(v)=0$ or

$$\frac{v^2+1}{v} = 0$$

for v gives

$$v = -i$$
$$v = i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(v^2+1\right)}{2} = -\arctan\left(u\right)+c_1$$
$$v = -i$$
$$v = i$$

Solving for v gives

$$v = -i$$
$$v = i$$
$$v = \sqrt{-1+\mathrm{e}^{-2\arctan(u)+2c_1}}$$
$$v = -\sqrt{-1+\mathrm{e}^{-2\arctan(u)+2c_1}}$$

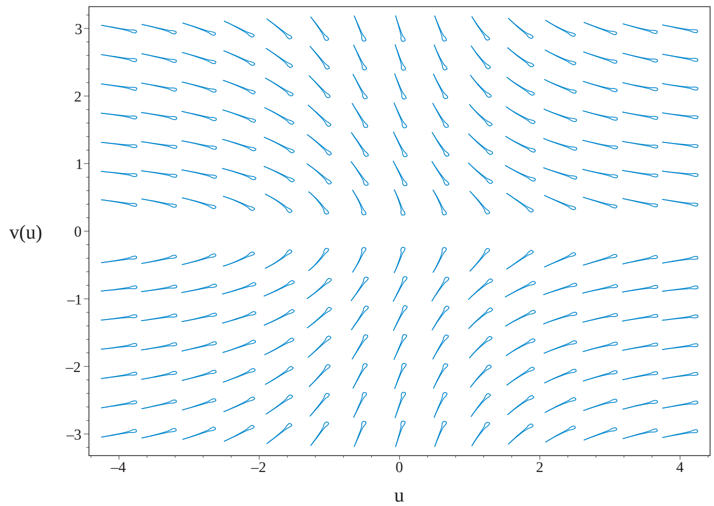


Figure 2.76: Slope field plot
$1+v^2+(u^2+1)\,vv'=0$

Summary of solutions found

$$v=-i$$
$$v=i$$
$$v=\sqrt{-1+\mathrm{e}^{-2\arctan(u)+2c_1}}$$
$$v=-\sqrt{-1+\mathrm{e}^{-2\arctan(u)+2c_1}}$$

Solved as first order Bernoulli ode

Time used: 0.191 (sec)

In canonical form, the ODE is

$$\begin{aligned}v' &= F(u,v)\\ &= -\frac{v^2+1}{(u^2+1)\,v}\end{aligned}$$

This is a Bernoulli ODE.

$$v'=\left(-\frac{1}{u^2+1}\right)v+\left(-\frac{1}{u^2+1}\right)\frac{1}{v} \tag{1}$$

The standard Bernoulli ODE has the form

$$v'=f_0(u)v+f_1(u)v^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= -\frac{1}{u^2+1} \\ f_1 &= -\frac{1}{u^2+1} \end{aligned}$$

The first step is to divide the above equation by v^n which gives

$$\frac{v'}{v^n} = f_0(u)v^{1-n} + f_1(u) \tag{3}$$

The next step is use the substitution $v = v^{1-n}$ in equation (3) which generates a new ODE in $v(u)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $v(u)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(u) &= -\frac{1}{u^2+1} \\ f_1(u) &= -\frac{1}{u^2+1} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $v^n = \frac{1}{v}$ gives

$$v'v = -\frac{v^2}{u^2+1} - \frac{1}{u^2+1} \tag{4}$$

Let

$$\begin{aligned} v &= v^{1-n} \\ &= v^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t u gives

$$v' = 2vv' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{v'(u)}{2} &= -\frac{v(u)}{u^2+1} - \frac{1}{u^2+1} \\ v' &= -\frac{2v}{u^2+1} - \frac{2}{u^2+1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(u)$ which is now solved.

In canonical form a linear first order is

$$v'(u) + q(u)v(u) = p(u)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(u) &= \frac{2}{u^2+1} \\ p(u) &= -\frac{2}{u^2+1} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,du} \\ &= \mathrm{e}^{\int \frac{2}{u^2+1}du} \\ &= \mathrm{e}^{2\arctan(u)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}(\mu v) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}u}(\mu v) &= (\mu)\left(-\frac{2}{u^2+1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}u}\left(v\,\mathrm{e}^{2\arctan(u)}\right) &= \left(\mathrm{e}^{2\arctan(u)}\right)\left(-\frac{2}{u^2+1}\right) \\ \mathrm{d}\left(v\,\mathrm{e}^{2\arctan(u)}\right) &= \left(-\frac{2\,\mathrm{e}^{2\arctan(u)}}{u^2+1}\right)\mathrm{d}u \end{aligned}$$

Integrating gives

$$\begin{aligned} v\,\mathrm{e}^{2\arctan(u)} &= \int -\frac{2\,\mathrm{e}^{2\arctan(u)}}{u^2+1}\,du \\ &= -\mathrm{e}^{2\arctan(u)} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{2\arctan(u)}$ gives the final solution

$$v(u) = -1 + c_1\,\mathrm{e}^{-2\arctan(u)}$$

The substitution $v = v^{1-n}$ is now used to convert the above solution back to v which results in

$$v^2 = -1 + c_1\,\mathrm{e}^{-2\arctan(u)}$$

Solving for v gives

$$v = \sqrt{-1 + c_1 \,\mathrm{e}^{-2 \arctan(u)}}$$

$$v = -\sqrt{-1 + c_1 \,\mathrm{e}^{-2 \arctan(u)}}$$

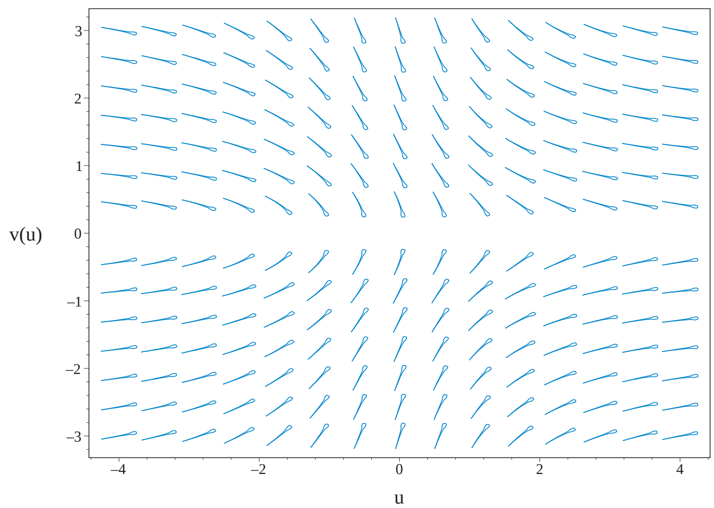


Figure 2.77: Slope field plot
$1 + v^2 + (u^2 + 1)\, v v' = 0$

Summary of solutions found

$$v = \sqrt{-1 + c_1 \,\mathrm{e}^{-2 \arctan(u)}}$$

$$v = -\sqrt{-1 + c_1 \,\mathrm{e}^{-2 \arctan(u)}}$$

Solved as first order Exact ode

Time used: 0.262 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(u, v)\,\mathrm{d}u + N(u, v)\,\mathrm{d}v = 0 \tag{1A}$$

Therefore

$$\left(\left(u^2+1\right)v\right)\mathrm{d}v = \left(-v^2-1\right)\mathrm{d}u$$
$$\left(v^2+1\right)\mathrm{d}u + \left(\left(u^2+1\right)v\right)\mathrm{d}v = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(u, v) = v^2 + 1$$
$$N(u, v) = \left(u^2+1\right)v$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}\left(v^2+1\right)\\ &= 2v\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial u} &= \frac{\partial}{\partial u}\left(\left(u^2+1\right)v\right)\\ &= 2uv\end{aligned}$$

Since $\frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial u}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u}\right) \\ &= \frac{1}{(u^2+1)\,v}((2v) - (2uv)) \\ &= \frac{-2u+2}{u^2+1} \end{aligned}$$

Since A does not depend on v, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}u} \\ &= e^{\int \frac{-2u+2}{u^2+1}\,\mathrm{d}u} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(u^2+1)+2\arctan(u)} \\ &= \frac{\mathrm{e}^{2\arctan(u)}}{u^2+1} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{\mathrm{e}^{2\arctan(u)}}{u^2+1}\left(v^2+1\right) \\ &= \frac{(v^2+1)\,\mathrm{e}^{2\arctan(u)}}{u^2+1} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{\mathrm{e}^{2\arctan(u)}}{u^2+1}\left(\left(u^2+1\right)v\right) \\ &= v\,\mathrm{e}^{2\arctan(u)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}v}{\mathrm{d}u} &= 0 \\ \left(\frac{(v^2+1)\,\mathrm{e}^{2\arctan(u)}}{u^2+1}\right) + \left(v\,\mathrm{e}^{2\arctan(u)}\right)\frac{\mathrm{d}v}{\mathrm{d}u} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(u, v)$

$$\frac{\partial\phi}{\partial u} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial v} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. u gives

$$\int \frac{\partial\phi}{\partial u}\, \mathrm{d}u = \int \overline{M}\, \mathrm{d}u$$
$$\int \frac{\partial\phi}{\partial u}\, \mathrm{d}u = \int \frac{(v^2+1)\, \mathrm{e}^{2\arctan(u)}}{u^2+1}\, \mathrm{d}u$$
$$\phi = \frac{(v^2+1)\, \mathrm{e}^{2\arctan(u)}}{2} + f(v) \tag{3}$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both u and v. Taking derivative of equation (3) w.r.t v gives

$$\frac{\partial\phi}{\partial v} = v\, \mathrm{e}^{2\arctan(u)} + f'(v) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial v} = v\, \mathrm{e}^{2\arctan(u)}$. Therefore equation (4) becomes

$$v\, \mathrm{e}^{2\arctan(u)} = v\, \mathrm{e}^{2\arctan(u)} + f'(v) \tag{5}$$

Solving equation (5) for $f'(v)$ gives

$$f'(v) = 0$$

Therefore

$$f(v) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(v)$ into equation (3) gives ϕ

$$\phi = \frac{(v^2+1)\, \mathrm{e}^{2\arctan(u)}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{(v^2+1)\, \mathrm{e}^{2\arctan(u)}}{2}$$

Solving for v gives

$$v = e^{-2\arctan(u)}\sqrt{-e^{2\arctan(u)}\left(e^{2\arctan(u)} - 2c_1\right)}$$

$$v = -e^{-2\arctan(u)}\sqrt{-e^{2\arctan(u)}\left(e^{2\arctan(u)} - 2c_1\right)}$$

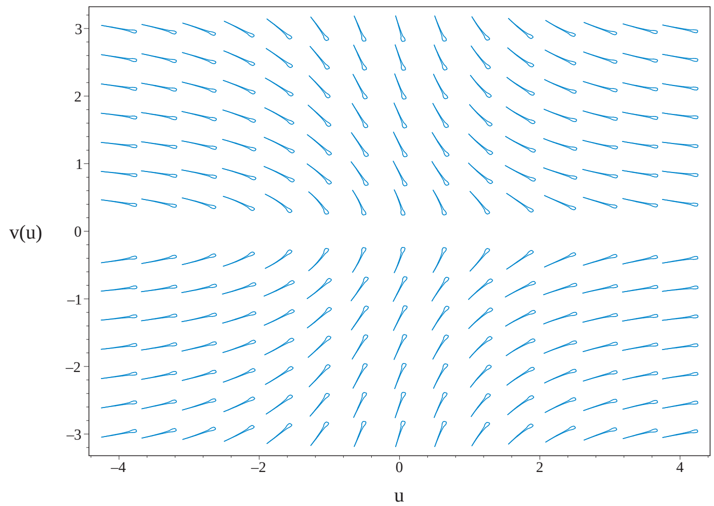


Figure 2.78: Slope field plot
$1 + v^2 + (u^2 + 1)\, vv' = 0$

Summary of solutions found

$$v = e^{-2\arctan(u)}\sqrt{-e^{2\arctan(u)}\left(e^{2\arctan(u)} - 2c_1\right)}$$

$$v = -e^{-2\arctan(u)}\sqrt{-e^{2\arctan(u)}\left(e^{2\arctan(u)} - 2c_1\right)}$$

Solved using Lie symmetry for first order ode

Time used: 0.920 (sec)

Writing the ode as

$$v' = -\frac{v^2 + 1}{(u^2 + 1)\, v}$$

$$v' = \omega(u, v)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_u + \omega(\eta_v - \xi_u) - \omega^2 \xi_v - \omega_u \xi - \omega_v \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = u^2 a_4 + uva_5 + v^2 a_6 + ua_2 + va_3 + a_1 \tag{1E}$$

$$\eta = u^2 b_4 + uvb_5 + v^2 b_6 + ub_2 + vb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &2ub_4 + vb_5 + b_2 - \frac{(v^2+1)(-2ua_4 + ub_5 - va_5 + 2vb_6 - a_2 + b_3)}{(u^2+1)v} \\ &- \frac{(v^2+1)^2 (ua_5 + 2va_6 + a_3)}{(u^2+1)^2 v^2} \\ &- \frac{2(v^2+1)u(u^2a_4 + uva_5 + v^2a_6 + ua_2 + va_3 + a_1)}{(u^2+1)^2 v} \\ &- \left(-\frac{2}{u^2+1} + \frac{v^2+1}{(u^2+1)v^2}\right)(u^2b_4 + uvb_5 + v^2b_6 + ub_2 + vb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2u^5v^2b_4 + u^4v^3b_5 + u^4v^2b_2 + u^4v^2b_4 - u^2v^4a_5 - u^2v^4b_6 - 2u\,v^5a_6 + u^3v^2b_2 + 4u^3v^2b_4 - u^2v^3a_2 + 2u^2v^3b_5}{}$$

$= 0$

Setting the numerator to zero gives

$$\begin{aligned} &2u^5v^2b_4 + u^4v^3b_5 + u^4v^2b_2 + u^4v^2b_4 - u^2v^4a_5 - u^2v^4b_6 - 2u\,v^5a_6 \\ &+ u^3v^2b_2 + 4u^3v^2b_4 - u^2v^3a_2 + 2u^2v^3b_5 - 2u\,v^4a_3 - u\,v^4a_5 - 2v^5a_6 \\ &- u^4b_4 - 2u^3vb_5 - u^2v^2a_5 + u^2v^2b_1 + 2u^2v^2b_2 + u^2v^2b_4 - 3u^2v^2b_6 \\ &- 2u\,v^3a_1 + 2u\,v^3a_4 - 2u\,v^3a_6 - v^4a_3 + v^4a_5 - v^4b_6 - u^3b_2 - u^2va_2 \\ &- 2u^2vb_3 - 2u\,v^2a_3 - 2u\,v^2a_5 + u\,v^2b_2 + 2ub_4v^2 + v^3a_2 - 4v^3a_6 \\ &+ v^3b_5 - u^2b_1 - u^2b_4 - 2uva_1 + 2uva_4 - 2uvb_5 - 2v^2a_3 + v^2a_5 + v^2b_1 \\ &+ b_2v^2 - 3v^2b_6 - ua_5 - ub_2 + va_2 - 2va_6 - 2vb_3 - a_3 - b_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{u, v\}$ in them.

$$\{u, v\}$$

The following substitution is now made to be able to collect on all terms with $\{u, v\}$ in them

$$\{u = v_1, v = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&2b_4v_1^5v_2^2 + b_5v_1^4v_2^3 - a_5v_1^2v_2^4 - 2a_6v_1v_2^5 + b_2v_1^4v_2^2 + b_4v_1^4v_2^2 - b_6v_1^2v_2^4 \\
&- a_2v_1^2v_2^3 - 2a_3v_1v_2^4 - a_5v_1v_2^4 - 2a_6v_2^5 + b_2v_1^3v_2^2 + 4b_4v_1^3v_2^2 + 2b_5v_1^2v_2^3 \\
&- 2a_1v_1v_2^3 - a_3v_2^4 + 2a_4v_1v_2^3 - a_5v_1^2v_2^2 + a_5v_2^4 - 2a_6v_1v_2^3 + b_1v_1^2v_2^2 \\
&+ 2b_2v_1^2v_2^2 - b_4v_1^4 + b_4v_1^2v_2^2 - 2b_5v_1^3v_2 - 3b_6v_1^2v_2^2 - b_6v_2^4 - a_2v_1^2v_2 + a_2v_2^3 \\
&- 2a_3v_1v_2^2 - 2a_5v_1v_2^2 - 4a_6v_2^3 - b_2v_1^3 + b_2v_1v_2^2 - 2b_3v_1^2v_2 + 2b_4v_1v_2^2 \\
&+ b_5v_2^3 - 2a_1v_1v_2 - 2a_3v_2^2 + 2a_4v_1v_2 + a_5v_2^2 - b_1v_1^2 + b_1v_2^2 + b_2v_2^2 - b_4v_1^2 \\
&- 2b_5v_1v_2 - 3b_6v_2^2 + a_2v_2 - a_5v_1 - 2a_6v_2 - b_2v_1 - 2b_3v_2 - a_3 - b_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(b_2 + b_4)\,v_1^4v_2^2 + (b_2 + 4b_4)\,v_1^3v_2^2 + (-a_5 - b_6)\,v_1^2v_2^4 + (-a_2 + 2b_5)\,v_1^2v_2^3 \\
&+ (-a_5 + b_1 + 2b_2 + b_4 - 3b_6)\,v_1^2v_2^2 + (-a_2 - 2b_3)\,v_1^2v_2 + (-2a_3 - a_5)\,v_1v_2^4 \\
&+ (-2a_1 + 2a_4 - 2a_6)\,v_1v_2^3 + (-2a_3 - 2a_5 + b_2 + 2b_4)\,v_1v_2^2 \\
&+ (-2a_1 + 2a_4 - 2b_5)\,v_1v_2 - 2a_6v_2^5 - b_4v_1^4 - b_2v_1^3 + (-b_1 - b_4)\,v_1^2 \\
&+ (-a_5 - b_2)\,v_1 + (-a_3 + a_5 - b_6)\,v_2^4 + (a_2 - 4a_6 + b_5)\,v_2^3 \\
&+ (-2a_3 + a_5 + b_1 + b_2 - 3b_6)\,v_2^2 + (a_2 - 2a_6 - 2b_3)\,v_2 \\
&- 2b_5v_1^3v_2 + 2b_4v_1^5v_2^2 + b_5v_1^4v_2^3 - 2a_6v_1v_2^5 - a_3 - b_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_5 &= 0 \\
-2a_6 &= 0 \\
-b_2 &= 0 \\
-b_4 &= 0 \\
2b_4 &= 0 \\
-2b_5 &= 0 \\
-a_2 - 2b_3 &= 0 \\
-a_2 + 2b_5 &= 0 \\
-2a_3 - a_5 &= 0 \\
-a_3 - b_1 &= 0 \\
-a_5 - b_2 &= 0 \\
-a_5 - b_6 &= 0 \\
-b_1 - b_4 &= 0 \\
b_2 + b_4 &= 0 \\
b_2 + 4b_4 &= 0 \\
-2a_1 + 2a_4 - 2a_6 &= 0 \\
-2a_1 + 2a_4 - 2b_5 &= 0 \\
a_2 - 4a_6 + b_5 &= 0 \\
a_2 - 2a_6 - 2b_3 &= 0 \\
-a_3 + a_5 - b_6 &= 0 \\
-2a_3 - 2a_5 + b_2 + 2b_4 &= 0 \\
-2a_3 + a_5 + b_1 + b_2 - 3b_6 &= 0 \\
-a_5 + b_1 + 2b_2 + b_4 - 3b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_4 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= a_4 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= u^2 + 1 \\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(u, v)\,\xi \\
&= 0 - \left(-\frac{v^2 + 1}{(u^2 + 1)\,v}\right)(u^2 + 1) \\
&= \frac{v^2 + 1}{v} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(u, v) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{du}{\xi} = \frac{dv}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v}\right) S(u,v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = u$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{v^2+1}{v}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(v^2+1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_u + \omega(u,v) S_v}{R_u + \omega(u,v) R_v} \tag{2}$$

Where in the above R_u, R_v, S_u, S_v are all partial derivatives and $\omega(u,v)$ is the right hand side of the original ode given by

$$\omega(u,v) = -\frac{v^2+1}{(u^2+1)\, v}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_u &= 1 \\ R_v &= 0 \\ S_u &= 0 \\ S_v &= \frac{v}{v^2+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{u^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for u, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

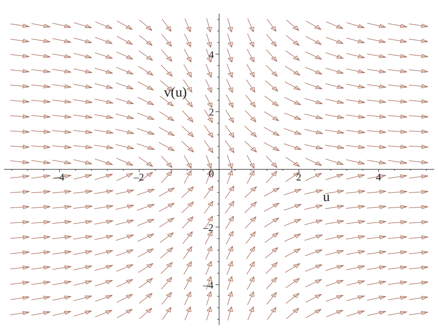
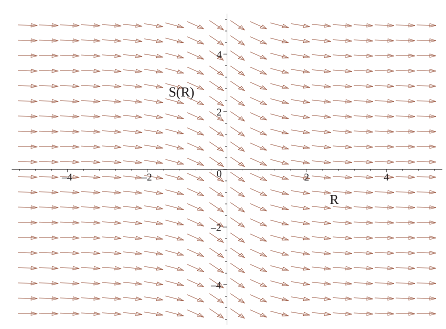
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -\frac{1}{R^2+1}\, dR$$
$$S(R) = -\arctan(R) + c_2$$

To complete the solution, we just need to transform the above back to u, v coordinates. This results in

$$\frac{\ln(v^2+1)}{2} = -\arctan(u) + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in u, v coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dv}{du} = -\frac{v^2+1}{(u^2+1)v}$ 	$R = u$ $S = \frac{\ln(v^2+1)}{2}$	$\frac{dS}{dR} = -\frac{1}{R^2+1}$ 

Solving for v gives

$$v = \sqrt{-1 + e^{-2\arctan(u)+2c_2}}$$
$$v = -\sqrt{-1 + e^{-2\arctan(u)+2c_2}}$$

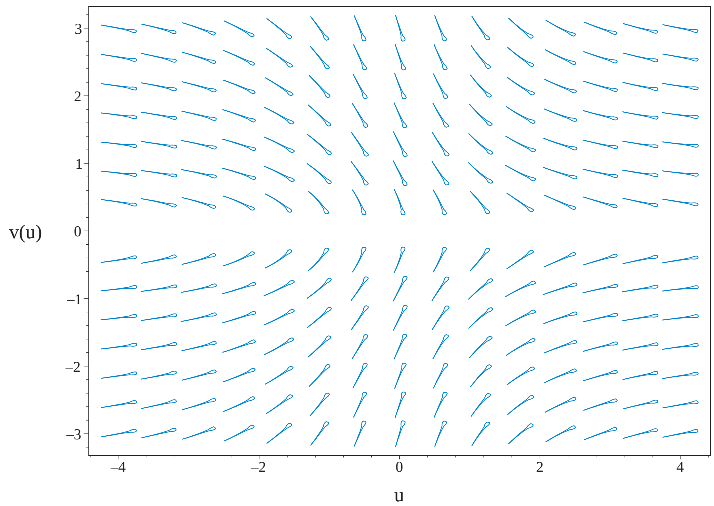


Figure 2.79: Slope field plot
$1 + v^2 + (u^2 + 1) vv' = 0$

Summary of solutions found

$$v = \sqrt{-1 + e^{-2\arctan(u) + 2c_2}}$$

$$v = -\sqrt{-1 + e^{-2\arctan(u) + 2c_2}}$$

Maple step by step solution

Let's solve
$1 + v^2 + (u^2 + 1) vv' = 0$

- Highest derivative means the order of the ODE is 1
  v'
- Solve for the highest derivative
  $v' = \frac{-v^2-1}{(u^2+1)v}$
- Separate variables
  $\frac{v'v}{-v^2-1} = \frac{1}{u^2+1}$
- Integrate both sides with respect to u
  $\int \frac{v'v}{-v^2-1} du = \int \frac{1}{u^2+1} du + C1$
- Evaluate integral
  $-\frac{\ln(v^2+1)}{2} = \arctan(u) + C1$
- Solve for v

$$\left\{v = \sqrt{-1+\mathrm{e}^{-2\arctan(u)-2C1}}, v = -\sqrt{-1+\mathrm{e}^{-2\arctan(u)-2C1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 31

```
dsolve(1+v(u)^2+(u^2+1)*v(u)*diff(v(u),u) = 0,v(u),singsol=all)
```

$$v = \sqrt{\mathrm{e}^{-2\arctan(u)}c_1 - 1}$$
$$v = -\sqrt{\mathrm{e}^{-2\arctan(u)}c_1 - 1}$$

Mathematica DSolve solution

Solving time : `2.538` (sec)
Leaf size : 59

```
DSolve[{(1+v[u]^2)+(1+u^2)*v[u]*D[v[u],u]==0,{}},v[u],u,IncludeSingularSolutions->True]
```

$$v(u) \to -\sqrt{-1+e^{-2\arctan(u)+2c_1}}$$
$$v(u) \to \sqrt{-1+e^{-2\arctan(u)+2c_1}}$$
$$v(u) \to -i$$
$$v(u) \to i$$

2.6.14 Problem 10 (c)

Internal problem ID [18578]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter IV. Methods of solution: First order equations. section 33. Problems at page 91
Problem number : 10 (c)
Date solved : `Tuesday, January 28, 2025 at 12:00:55 PM`
CAS classification : `[_separable]`

Solve

$$u \ln(u) v' + \sin(v)^2 = 1$$

Solved as first order separable ode

Time used: 0.183 (sec)

The ode

$$v' = -\frac{\sin(v)^2 - 1}{\ln(u) u} \tag{2.53}$$

is separable as it can be written as

$$\begin{aligned} v' &= -\frac{\sin(v)^2 - 1}{\ln(u) u} \\ &= f(u)g(v) \end{aligned}$$

Where

$$\begin{aligned} f(u) &= \frac{1}{\ln(u) u} \\ g(v) &= -\sin(v)^2 + 1 \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(v)} \, dv = \int f(u) \, du$$
$$\int \frac{1}{-\sin(v)^2 + 1} \, dv = \int \frac{1}{\ln(u) \, u} \, du$$

$$\tan(v) = \ln(\ln(u)) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(v)$ is zero, since we had to divide by this above. Solving $g(v) = 0$ or

$$-\sin(v)^2 + 1 = 0$$

for v gives

$$v = -\frac{\pi}{2}$$
$$v = \frac{\pi}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\tan(v) = \ln(\ln(u)) + c_1$$
$$v = -\frac{\pi}{2}$$
$$v = \frac{\pi}{2}$$

Solving for v gives

$$v = -\frac{\pi}{2}$$
$$v = \frac{\pi}{2}$$
$$v = \arctan(\ln(\ln(u)) + c_1)$$

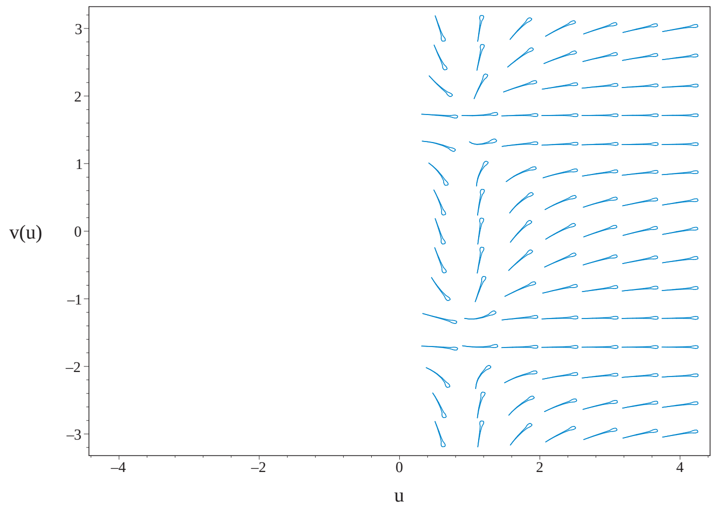


Figure 2.80: Slope field plot
$u \ln(u) v' + \sin(v)^2 = 1$

Summary of solutions found

$$v = -\frac{\pi}{2}$$

$$v = \frac{\pi}{2}$$

$$v = \arctan(\ln(\ln(u)) + c_1)$$

Maple step by step solution

Let's solve

$u \ln(u) v' + \sin(v)^2 = 1$

- Highest derivative means the order of the ODE is 1

  v'

- Solve for the highest derivative

  $v' = \frac{-\sin(v)^2+1}{u \ln(u)}$

- Separate variables

  $\frac{v'}{-\sin(v)^2+1} = \frac{1}{\ln(u)u}$

- Integrate both sides with respect to u

  $\int \frac{v'}{-\sin(v)^2+1} du = \int \frac{1}{\ln(u)u} du + C1$

- Evaluate integral

$$\tan(v) = \ln(\ln(u)) + C1$$

- Solve for v

$$v = \arctan(\ln(\ln(u)) + C1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 10

```
dsolve(u*ln(u)*diff(v(u),u)+sin(v(u))^2 = 1,v(u),singsol=all)
```

$$v = \arctan(\ln(\ln(u)) + c_1)$$

Mathematica DSolve solution

Solving time : 0.359 (sec)
Leaf size : 52

```
DSolve[{u*Log[u]*D[v[u],u]+Sin[v[u]]==0,{}},v[u],u,IncludeSingularSolutions->True]
```

$$v(u) \to -\arccos(-\tanh(-\log(\log(u)) + c_1))$$
$$v(u) \to \arccos(-\tanh(-\log(\log(u)) + c_1))$$
$$v(u) \to 0$$
$$v(u) \to -\pi$$
$$v(u) \to \pi$$

2.7 Chapter V. Singular solutions. section 36. Problems at page 99

2.7.1 Problem 1 (eq 98)

Internal problem ID [18579]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter V. Singular solutions. section 36. Problems at page 99
Problem number : 1 (eq 98)
Date solved : `Tuesday, January 28, 2025 at 12:00:58 PM`
CAS classification : `[[_1st_order, _with_linear_symmetries]]`

Solve

$$4yy'^3 - 2x^2y'^2 + 4xyy' + x^3 = 16y^2$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912y^6}\,y\right)^{1/3}}{6y} - \frac{x(-x^3 + 12y^2)}{6y\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912y^6}\,y\right)^{1/3}} + \frac{x^2}{6y} \tag{1}$$

$$y' = -\frac{\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912y^6}\,y\right)^{1/3}}{12y} + \frac{x(-x^3 + 12y^2)}{12y\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912y^6}\,y\right)^{1/3}} + \frac{x^2}{6y} + \frac{i\sqrt{3}\left(\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1/3}}{6y} + \frac{x(-x^3+12y^2)}{6y\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\sqrt{-2x^9+91y^2x^6-1376y^4}\right.}\right.}{2} \tag{2}$$

$$y' = -\frac{\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912y^6}\,y\right)^{1/3}}{12y} + \frac{x(-x^3 + 12y^2)}{12y\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912y^6}\,y\right)^{1/3}} + \frac{x^2}{6y} - \frac{i\sqrt{3}\left(\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1/3}}{6y} + \frac{x(-x^3+12y^2)}{6y\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376y^4\right.}}\right.}{2} \tag{3}$$

Now each of the above is solved separately.

Solving Eq. (1)

Solving for y' gives

$$y' = -\frac{-x^4 + 12y^2x - x^2\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912y^6}\,y\right)^{1/3} - \left(x^6 - \right.}{6y\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-2x^9 + 91y^2x^6 - 13\right.}} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = -\frac{-x^4 + 12y^2x - x^2\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912y^6}\,y\right)^{1/3} -}{6y\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-2x^9 + 91y^2\right.}} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = \frac{3}{2}$$

Since the ode is isobaric of order $m = \frac{3}{2}$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= u\,x^{3/2} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$\frac{3\sqrt{x}\,u(x)}{2} + x^{3/2}u'(x) = -\frac{-x^4 + 12x^4u(x)^2 - x^2\left(x^6 - 45x^6u(x)^2 + 432x^6u(x)^4 + 3\sqrt{3}\,\sqrt{-2x^9 + 91x^9u(\right.}{6x^{3/2}u(x)\left(x^6 - 45\right.}$$

The ode

$$u'(x) = \tag{2.54}$$
$$-\frac{\left(12u(x)^2\, 3^{2/3}+9\, 3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\, u(x)^4-45\sqrt{3}\, u(x)^2+9u(x)\sqrt{\left(27u\left(x\right)^2-2\right)\left(4u\left(x\right)-1\right)^2\left(4u\left(\right.\right.}\right.\right.\right.}{}$$

is separable as it can be written as

$$u'(x) = -\frac{\left(12u(x)^2\, 3^{2/3}+9\, 3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\, u(x)^4-45\sqrt{3}\, u(x)^2+9u(x)\sqrt{\left(27u\left(x\right)^2-2\right)\left(4u\left(x\right)-1\right)^2}\right.\right.\right.}{}$$
$$= f(x)g(u)$$

Where

$$f(x) = -\frac{3^{2/3}}{18x}$$
$$g(u) = \frac{12u^2 3^{2/3}+9\, 3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\, u^4-45\sqrt{3}\, u^2+9u\sqrt{\left(27u^2-2\right)\left(4u-1\right)^2\left(4u+1\right)^2}+\sqrt{3}\right)\right)^{1/3} u^2}{\left(\right.}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$
$$\int \frac{\left(\sqrt{3}\left(\right.\right.}{12u^2 3^{2/3}+9\, 3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\, u^4-45\sqrt{3}\, u^2+9u\sqrt{\left(27u^2-2\right)\left(4u-1\right)^2\left(4u+1\right)^2}+\sqrt{3}\right)\right)^{1/3} u^2-3^{2}}$$

$$\int^{u(x)} \frac{\left(\sqrt{\right.}{12\tau^2 3^{2/3}+9\, 3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\, \tau^4-45\sqrt{3}\, \tau^2+9\tau\sqrt{\left(27\tau^2-2\right)\left(4\tau-1\right)^2\left(4\tau+1\right)^2}+\sqrt{3}\right)\right)^{1/3} \tau^2-}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{12u^2 3^{2/3} + 9\,3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\,u^4 - 45\sqrt{3}\,u^2 + 9u\sqrt{(27u^2-2)(4u-1)^2(4u+1)^2} + \sqrt{3}\right)\right)^{1/3} u^2 - 3^{2/3} -}{\left(\sqrt{3}\left(432\right.\right.}$$

for $u(x)$ gives

$$u(x) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{u(x)} \frac{\left(\sqrt{\right.}{12\tau^2 3^{2/3} + 9\,3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\,\tau^4 - 45\sqrt{3}\,\tau^2 + 9\tau\sqrt{(27\tau^2-2)(4\tau-1)^2(4\tau+1)^2} + \sqrt{3}\right)\right)^{1/3} \tau^2 -}$$

$$u(x) = 1$$

Converting $\int^{u(x)} \frac{\left(\sqrt{3}\left(432\sqrt{3}\,\tau^4-45\sqrt{3}\,\tau^2+9\tau\sqrt{(\right.\right.}{12\tau^2 3^{2/3}+9\,3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\,\tau^4-45\sqrt{3}\,\tau^2+9\tau\sqrt{(27\tau^2-2)(4\tau-1)^2(4\tau+1)^2}+\sqrt{3}\right)\right)^{1/3}\tau^2-3^{2/3}-3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\,\tau^4-\right.\right.}$ $-\frac{3^{2/3}\ln(x)}{18} + c_1$ back to y gives

$$\int^{\frac{y}{x^{3/2}}} \frac{\left(\sqrt{\right.}{12\tau^2 3^{2/3} + 9\,3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\,\tau^4 - 45\sqrt{3}\,\tau^2 + 9\tau\sqrt{(27\tau^2-2)(4\tau-1)^2(4\tau+1)^2} + \sqrt{3}\right)\right)^{1/3} \tau^2 -}$$

Converting $u(x) = 1$ back to y gives

$$\frac{y}{x^{3/2}} = 1$$

Solving for y gives

$$\int^{\frac{y}{x^{3/2}}} \frac{\left(\sqrt{\right.}{12\tau^2 3^{2/3} + 9\,3^{1/3}\left(\sqrt{3}\left(432\sqrt{3}\,\tau^4 - 45\sqrt{3}\,\tau^2 + 9\tau\sqrt{(27\tau^2-2)(4\tau-1)^2(4\tau+1)^2} + \sqrt{3}\right)\right)^{1/3}\tau^2 -}$$

$$-\frac{3^{2/3}\ln(x)}{18} + c_1$$

$$y = x^{3/2}$$

We now need to find the singular solutions, these are found by finding for what values $\left(\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1/3}}{6y} - \frac{x(-x^3+12y^2)}{6y\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1}}\right.$ $\left.\frac{x^2}{6y}\right)$ is zero. These give

$$\begin{aligned} y = \operatorname{RootOf}\Big(&-x^4 - x^2\Big(x^6 - 45x^3\_Z^2 + 432\_Z^4 \\ &+ 3\sqrt{3}\sqrt{-2x^9 + 91\_Z^2x^6 - 1376\_Z^4x^3 + 6912\_Z^6}\,\_Z\Big)^{1/3} + 12\_Z^2x \\ &- \left(x^6-45x^3\_Z^2+432\_Z^4+3\sqrt{3}\sqrt{-2x^9 + 91\_Z^2x^6 - 1376\_Z^4x^3 + 6912\_Z^6}\,\_Z\right)^{2/3}\Big) \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $\operatorname{RootOf}\Big(-x^4 - x^2\Big(x^6 - 45x^3\_Z^2 + 432\_Z^4 + 3\sqrt{3}\sqrt{-2x^9 + 91\_Z^2x^6 - 1376\_Z^4x^3 + 691$ will not be used

Solving Eq. (2)

Writing the ode as

$$y' = \frac{-i\sqrt{3}\,x^4 + 12i\sqrt{3}\,y^2x + i\sqrt{3}\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912y^6}\,y\right)^{2}}{12}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\text{Expression too large to display} \tag{5E}$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-\left(2x^3-27y^2\right)\left(x^3-16y^2\right)^2}, \left(x^6-45x^3y^2+432y^4 + 3\sqrt{3}\sqrt{-\left(2x^3-27y^2\right)\left(x^3-16y^2\right)^2}\, y\right)^{1/3}, \left(x^6-45x^3y^2+432y^4+3\sqrt{3}\sqrt{-\left(2x^3-27y^2\right)\left(x^3-16y^2\right)^2}\, y\right)$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-\left(2x^3 - 27y^2\right)\left(x^3 - 16y^2\right)^2} = v_3, \left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-\left(2x^3 - 27y^2\right)\left(x^3 - 16y^2\right)^2}\,y\right)^{1/3} = v_4, \left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-\left(2x^3 - 27y^2\right)\left(x^3 - 16y^2\right)^2}\right.$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-20736a_1 &= 0\\
-48a_1 &= 0\\
2160a_1 &= 0\\
-7344a_3 &= 0\\
-12a_3 &= 0\\
588a_3 &= 0\\
20736a_3 &= 0\\
-1080b_1 &= 0\\
24b_1 &= 0\\
10368b_1 &= 0\\
-936b_2 &= 0\\
24b_2 &= 0\\
3888b_2 &= 0\\
62208b_2 &= 0\\
-995328\sqrt{3}\, a_1 &= 0\\
-13104\sqrt{3}\, a_1 &= 0\\
288\sqrt{3}\, a_1 &= 0\\
198144\sqrt{3}\, a_1 &= 0\\
-446976\sqrt{3}\, a_3 &= 0\\
-3564\sqrt{3}\, a_3 &= 0\\
72\sqrt{3}\, a_3 &= 0\\
62640\sqrt{3}\, a_3 &= 0\\
995328\sqrt{3}\, a_3 &= 0\\
-99072\sqrt{3}\, b_1 &= 0\\
-144\sqrt{3}\, b_1 &= 0\\
6552\sqrt{3}\, b_1 &= 0\\
497664\sqrt{3}\, b_1 &= 0\\
-96768\sqrt{3}\, b_2 &= 0\\
-59760\sqrt{3}\, b_2 &= 0\\
-144\sqrt{3}\, b_2 &= 0\\
5688\sqrt{3}\, b_2 &= 0\\
2985984\sqrt{3}\, b_2 &= 0\\
-31104a_2 + 20736b_3 &= 0\\
-72a_2 + 48b_3 &= 0\\
3240a_2 - 2160b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{2b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{2x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{\frac{2x}{3}} \\
&= \frac{3y}{2x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 \, x^{3/2}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^{3/2}}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{\frac{2x}{3}}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \frac{3\ln(x)}{2}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-i\sqrt{3}\,x^4 + 12i\sqrt{3}\,y^2x + i\sqrt{3}\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 6912}\right.}{}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{3y}{2x^{5/2}} \\
R_y &= \frac{1}{x^{3/2}} \\
S_x &= \frac{3}{2x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{18x^{3/2}\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\right.}{\left(-i\sqrt{3}\,x + x\right)\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\sqrt{-2\left(x^3 - \frac{27y^2}{2}\right)\left(x^3 - 16y^2\right)^2 y}\right)^{2/3} + \left(-2x^3 + 18y\right.} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{18R\left(\left(48R^3 - 3R\right)\sqrt{3}\sqrt{27R^2 - 2} + 432R^4 - 45}{\left(i\sqrt{3} - 1\right)\left(\left(48R^3 - 3R\right)\sqrt{3}\sqrt{27R^2 - 2} + 432R^4 - 45R^2 + 1\right)^{2/3} + \left(-18R^2 + 2\right)\left(\left(48R^3 - 3R\right)\sqrt{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{}{i\left(\left(48R^3 - 3R\right)\sqrt{3}\sqrt{27R^2 - 2} + 432R^4 - 45R^2 + 1\right)^{2/3}\sqrt{3} + 12i\sqrt{3}\,R^2 - 18\left(\left(48R^3 - 3R\right)\sqrt{3}}$$

$$S(R) = \int \frac{}{i\left(\left(48R^3 - 3R\right)\sqrt{3}\sqrt{27R^2 - 2} + 432R^4 - 45R^2 + 1\right)^{2/3}\sqrt{3} + 12i\sqrt{3}\,R^2 - 18\left(\left(48R^3 - 3R\right)\sqrt{3}}$$

$$S(R) = \int \frac{}{i\left(\left(48R^3 - 3R\right)\sqrt{3}\sqrt{27R^2 - 2} + 432R^4 - 45R^2 + 1\right)^{2/3}\sqrt{3} + 12i\sqrt{3}\,R^2 - 18\left(\left(48R^3 - 3R\right)\sqrt{3}}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{3\ln(x)}{2} = \int^{\frac{y}{x^{3/2}}} \frac{}{i\left(\left(48\_a^3 - 3\_a\right)\sqrt{3}\sqrt{27\_a^2 - 2} + 432\_a^4 - 45\_a^2 + 1\right)^{2/3}\sqrt{3} + 12i\sqrt{3}\,\_a^2 - 18\left(\left(48\_\right.\right.}$$

We now need to find the singular solutions, these are found by finding for what values

$$\frac{x^2}{6y} + \frac{i\sqrt{3}\left(\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1/3}}{6y} + \frac{x\left(-x^3+12y^2\right)}{6y\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1/3}}\right)}{2}$$

$$(-\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1/3}}{12y} + \frac{x\left(-x^3+12y^2\right)}{12y\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\right.}$$

is zero. These give

$$\begin{aligned} y = \operatorname{RootOf}\Big(&-i\sqrt{3}\,x^4 + 12i\sqrt{3}\,\_Z^2x + i\sqrt{3}\,\Big(x^6 - 45x^3\_Z^2 + 432\_Z^4 \\ &+ 3\sqrt{3}\,\sqrt{-2x^9 + 91\_Z^2x^6 - 1376\_Z^4x^3 + 6912\_Z^6}\,\_Z\Big)^{2/3} - x^4 + 12\_Z^2x \\ &+2x^2\left(x^6-45x^3\_Z^2+432\_Z^4+3\sqrt{3}\,\sqrt{-2x^9 + 91\_Z^2x^6 - 1376\_Z^4x^3 + 6912\_Z^6}\,\_Z\right)^{1/3} \\ &-\left(x^6-45x^3\_Z^2+432\_Z^4+3\sqrt{3}\,\sqrt{-2x^9 + 91\_Z^2x^6 - 1376\_Z^4x^3 + 6912\_Z^6}\,\_Z\right)^{2/3}\Big) \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution RootOf $\left(-i\sqrt{3}\,x^4+12i\sqrt{3}\,\_Z^2x+i\sqrt{3}\left(x^6-45x^3\_Z^2+432\_Z^4+3\sqrt{3}\,\sqrt{-2x^9+91\_Z^2x\right.\right.$
will not be used

Solving Eq. (3)

Writing the ode as

$$y'=-\frac{-i\sqrt{3}\,x^4+12i\sqrt{3}\,y^2x+i\sqrt{3}\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right.}{}$$

$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\text{Expression too large to display} \tag{5E}$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-\left(2x^3-27y^2\right)\left(x^3-16y^2\right)^2}, \left(x^6-45x^3y^2+432y^4 +3\sqrt{3}\,\sqrt{-\left(2x^3-27y^2\right)\left(x^3-16y^2\right)^2}\,y\right)^{1/3}, \left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-\left(2x^3-27y^2\right)\left(x^3-16y^2\right)^2}\,y\right)$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-\left(2x^3-27y^2\right)\left(x^3-16y^2\right)^2} = v_3, \left(x^6-45x^3y^2+432y^4 +3\sqrt{3}\,\sqrt{-\left(2x^3-27y^2\right)\left(x^3-16y^2\right)^2}\,y\right)^{1/3} = v_4, \left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-\left(2x^3-27y^2\right)\left(x^3-16y^2\right)^2}\right.$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

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3888b_2 &= 0\\
62208b_2 &= 0\\
-995328\sqrt{3}\, a_1 &= 0\\
-13104\sqrt{3}\, a_1 &= 0\\
288\sqrt{3}\, a_1 &= 0\\
198144\sqrt{3}\, a_1 &= 0\\
-446976\sqrt{3}\, a_3 &= 0\\
-3564\sqrt{3}\, a_3 &= 0\\
72\sqrt{3}\, a_3 &= 0\\
62640\sqrt{3}\, a_3 &= 0\\
995328\sqrt{3}\, a_3 &= 0\\
-99072\sqrt{3}\, b_1 &= 0\\
-144\sqrt{3}\, b_1 &= 0\\
6552\sqrt{3}\, b_1 &= 0\\
497664\sqrt{3}\, b_1 &= 0\\
-96768\sqrt{3}\, b_2 &= 0\\
-59760\sqrt{3}\, b_2 &= 0\\
-144\sqrt{3}\, b_2 &= 0\\
5688\sqrt{3}\, b_2 &= 0\\
2985984\sqrt{3}\, b_2 &= 0\\
-31104a_2 + 20736b_3 &= 0\\
-72a_2 + 48b_3 &= 0\\
3240a_2 - 2160b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{2b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{2x}{3} \\
\eta &= y
\end{aligned}
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The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{\frac{2x}{3}} \\
&= \frac{3y}{2x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^{3/2}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^{3/2}}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{\frac{2x}{3}} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \frac{3\ln(x)}{2} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-i\sqrt{3}\,x^4 + 12i\sqrt{3}\,y^2x + i\sqrt{3}\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-2x^9 + 91y^2x^6 - 1376y^4x^3 + 691}\right.}{}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{3y}{2x^{5/2}} \\ R_y &= \frac{1}{x^{3/2}} \\ S_x &= \frac{3}{2x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{18x^{3/2}\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-}\right.}{x\left(-i\sqrt{3}-1\right)\left(x^6 - 45x^3y^2 + 432y^4 + 3\sqrt{3}\,\sqrt{-2\left(x^3 - \frac{27y^2}{2}\right)\left(x^3-16y^2\right)^2 y}\right)^{2/3} + 2\left(x^3 - 9y^2\right)\left(\right.} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{18R\left((48R^3 - 3R)\sqrt{3}\sqrt{27R^2-2} + 432R^4 - 4\right.}{(1+i\sqrt{3})\left((48R^3-3R)\sqrt{3}\sqrt{27R^2-2}+432R^4-45R^2+1\right)^{2/3} + (18R^2-2)\left((48R^3-3R)\sqrt{3}\right.}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -\frac{}{i\left((48R^3-3R)\sqrt{3}\sqrt{27R^2-2}+432R^4-45R^2+1\right)^{2/3}\sqrt{3}+12i\sqrt{3}\,R^2+18\left((48R^3-3R)\sqrt{\right.}}$$

$$S(R) = \int -\frac{}{i\left((48R^3-3R)\sqrt{3}\sqrt{27R^2-2}+432R^4-45R^2+1\right)^{2/3}\sqrt{3}+12i\sqrt{3}\,R^2+18\left((48R^3-3R)\sqrt{\right.}}$$

$$S(R) = \int -\frac{}{i\left((48R^3-3R)\sqrt{3}\sqrt{27R^2-2}+432R^4-45R^2+1\right)^{2/3}\sqrt{3}+12i\sqrt{3}\,R^2+18\left((48R^3-3R)\sqrt{\right.}}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{3\ln(x)}{2} = \int^{\frac{y}{x^{3/2}}} -\frac{}{i\left((48\_a^3-3\_a)\sqrt{3}\sqrt{27\_a^2-2}+432\_a^4-45\_a^2+1\right)^{2/3}\sqrt{3}+12i\sqrt{3}\_a^2+18\left((48\right.}$$

We now need to find the singular solutions, these are found by finding for what values

$$(-\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1/3}}{12y}+\frac{x(-x^3+12y^2)}{12y\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\right.}$$

$$\frac{x^2}{6y}-\frac{i\sqrt{3}\left(\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1/3}}{6y}+\frac{x(-x^3+12y^2)}{6y\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\sqrt{-2x^9+91y^2x^6-1376y^4x^3+6912y^6}\,y\right)^{1/3}}\right)}{2}$$

is zero. These give

$$\begin{aligned} y = \operatorname{RootOf}\Big(&-i\sqrt{3}\,x^4+12i\sqrt{3}\_Z^2x+i\sqrt{3}\left(x^6-45x^3\_Z^2+432\_Z^4\right.\\ &\left.+3\sqrt{3}\sqrt{-2x^9+91\_Z^2x^6-1376\_Z^4x^3+6912\_Z^6}\,\_Z\right)^{2/3}+x^4\\ &-2x^2\left(x^6-45x^3\_Z^2+432\_Z^4+3\sqrt{3}\sqrt{-2x^9+91\_Z^2x^6-1376\_Z^4x^3+6912\_Z^6}\,\_Z\right)^{1/3}\\ &-12\_Z^2x\\ &+\left(x^6-45x^3\_Z^2+432\_Z^4+3\sqrt{3}\sqrt{-2x^9+91\_Z^2x^6-1376\_Z^4x^3+6912\_Z^6}\,\_Z\right)^{2/3}\Big)\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $\mathrm{RootOf}\left(-i\sqrt{3}\,x^4+12i\sqrt{3}\,\_Z^2x+i\sqrt{3}\left(x^6-45x^3\_Z^2+432\_Z^4+3\sqrt{3}\,\sqrt{-2x^9+91\_Z^2x\right.\right.$ will not be used

Maple step by step solution

Let's solve

$4yy'^3-2y'^2x^2+4xyy'+x^3=16y^2$

- Highest derivative means the order of the ODE is 1

  y'

- Solve for the highest derivative

  $\left[y'=\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376x^3y^4+6912y^6}\,y\right)^{1/3}}{6y}-\frac{x(-x^3+12y^2)}{6y\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-137\right.}\right.$

- Solve the equation $y'=\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376x^3y^4+6912y^6}\,y\right)^{1/3}}{6y}-\frac{}{6y\left(x^6-45x^3y^2+432y^4+3\right.}$

- Solve the equation $y'=-\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376x^3y^4+6912y^6}\,y\right)^{1/3}}{12y}+\frac{}{12y\left(x^6-45x^3y^2+432y^4\right.}$

- Solve the equation $y'=-\frac{\left(x^6-45x^3y^2+432y^4+3\sqrt{3}\,\sqrt{-2x^9+91y^2x^6-1376x^3y^4+6912y^6}\,y\right)^{1/3}}{12y}+\frac{}{12y\left(x^6-45x^3y^2+432y^4\right.}$

- Set of solutions

  $\{workingODE, workingODE, workingODE\}$

Maple dsolve solution

Solving time : `299.037` (sec)
Leaf size : maple_leaf_size

```
dsolve(4*y(x)*diff(y(x),x)^3-2*x^2*diff(y(x),x)^2+4*x*diff(y(x),x)*y(x)+x^3 = 16*y(x)^
```

No solution found

Mathematica DSolve solution

Solving time : 50.04 (sec)
Leaf size : 49162

```
DSolve[{4*y[x]*D[y[x],x]^3-2*x^2*D[y[x],x]^2+4*x*y[x]*D[y[x],x]+x^3==16*y[x]^2,{}},y[x],x,In
```

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2.8 Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163

2.8.1 Problem 1 (eq 100)

Internal problem ID [18580]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 1 (eq 100)
Date solved : `Tuesday, January 28, 2025 at 12:02:49 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$\theta'' - p^2\theta = 0$$

Solved as second order linear constant coeff ode

Time used: 0.076 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\theta''(x) + B\theta'(x) + C\theta(x) = 0$$

Where in the above $A = 1, B = 0, C = -p^2$. Let the solution be $\theta = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - p^2 \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - p^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -p^2$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-p^2)}$$
$$= \pm\sqrt{p^2}$$

Hence

$$\lambda_1 = +\sqrt{p^2}$$
$$\lambda_2 = -\sqrt{p^2}$$

Which simplifies to

$$\lambda_1 = p$$
$$\lambda_2 = -p$$

Since roots are real and distinct, then the solution is

$$\theta = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$\theta = c_1 e^{(p)x} + c_2 e^{(-p)x}$$

Or

$$\theta = c_1 \, \mathrm{e}^{xp} + c_2 \, \mathrm{e}^{-xp}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\theta = c_1 \, \mathrm{e}^{xp} + c_2 \, \mathrm{e}^{-xp}$$

Solved as second order can be made integrable

Time used: 2.370 (sec)

Multiplying the ode by θ' gives

$$\theta'\theta'' - p^2\theta'\theta = 0$$

Integrating the above w.r.t x gives

$$\int \left(\theta'\theta'' - p^2\theta'\theta\right) dx = 0$$
$$\frac{\theta'^2}{2} - \frac{p^2\theta^2}{2} = c_1$$

Which is now solved for θ. Solving for the derivative gives these ODE's to solve

$$\theta' = \sqrt{p^2\theta^2 + 2c_1} \tag{1}$$
$$\theta' = -\sqrt{p^2\theta^2 + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{p^2\theta^2 + 2c_1}} d\theta = dx$$
$$\frac{\ln\left(\frac{p^2\theta}{\sqrt{p^2}} + \sqrt{p^2\theta^2 + 2c_1}\right)}{\sqrt{p^2}} = x + c_2$$

Singular solutions are found by solving

$$\sqrt{p^2\theta^2 + 2c_1} = 0$$

for θ. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$\theta = \frac{\sqrt{-2c_1}}{p}$$
$$\theta = -\frac{\sqrt{-2c_1}}{p}$$

Solving for θ gives

$$\theta = \frac{\sqrt{-2c_1}}{p}$$
$$\theta = -\frac{\sqrt{-2c_1}}{p}$$
$$\theta = -\frac{\sqrt{p^2}\left(-e^{2c_2\sqrt{p^2}+2x\sqrt{p^2}} + 2c_1\right)e^{-c_2\sqrt{p^2}-x\sqrt{p^2}}}{2p^2}$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{p^2\theta^2+2c_1}}d\theta = dx$$
$$-\frac{\ln\left(\frac{p^2\theta}{\sqrt{p^2}}+\sqrt{p^2\theta^2+2c_1}\right)}{\sqrt{p^2}} = x + c_3$$

Singular solutions are found by solving

$$-\sqrt{p^2\theta^2+2c_1} = 0$$

for θ. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$\theta = \frac{\sqrt{-2c_1}}{p}$$
$$\theta = -\frac{\sqrt{-2c_1}}{p}$$

Solving for θ gives

$$\theta = \frac{\sqrt{-2c_1}}{p}$$
$$\theta = -\frac{\sqrt{-2c_1}}{p}$$
$$\theta = -\frac{\sqrt{p^2}\left(-\mathrm{e}^{-2c_3\sqrt{p^2}-2x\sqrt{p^2}}+2c_1\right)\mathrm{e}^{c_3\sqrt{p^2}+x\sqrt{p^2}}}{2p^2}$$

Will add steps showing solving for IC soon.

The solution

$$\theta = \frac{\sqrt{-2c_1}}{p}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$\theta = -\frac{\sqrt{-2c_1}}{p}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\theta = -\frac{\sqrt{p^2}\left(-\mathrm{e}^{2c_2\sqrt{p^2}+2x\sqrt{p^2}} + 2c_1\right)\mathrm{e}^{-c_2\sqrt{p^2}-x\sqrt{p^2}}}{2p^2}$$

$$\theta = -\frac{\sqrt{p^2}\left(-\mathrm{e}^{-2c_3\sqrt{p^2}-2x\sqrt{p^2}} + 2c_1\right)\mathrm{e}^{c_3\sqrt{p^2}+x\sqrt{p^2}}}{2p^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.056 (sec)

Writing the ode as

$$\theta'' - p^2\theta = 0 \tag{1}$$

$$A\theta'' + B\theta' + C\theta = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -p^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \theta e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{p^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = p^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (p^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then θ is found using the inverse transformation

$$\theta = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.47: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = p^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{x\sqrt{p^2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in θ is found from

$$\theta_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}\theta_1 &= z_1 \\ &= e^{x\sqrt{p^2}}\end{aligned}$$

Which simplifies to

$$\theta_1 = e^{x\sqrt{p^2}}$$

The second solution θ_2 to the original ode is found using reduction of order

$$\theta_2 = \theta_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{\theta_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}\theta_2 &= \theta_1 \int \frac{1}{\theta_1^2}\,dx \\ &= e^{x\sqrt{p^2}} \int \frac{1}{e^{2x\sqrt{p^2}}}\,dx \\ &= e^{x\sqrt{p^2}} \left(-\frac{\sqrt{p^2}\,e^{-2x\sqrt{p^2}}}{2p^2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}\theta &= c_1\theta_1 + c_2\theta_2 \\ &= c_1\left(e^{x\sqrt{p^2}}\right) + c_2\left(e^{x\sqrt{p^2}}\left(-\frac{\sqrt{p^2}\,e^{-2x\sqrt{p^2}}}{2p^2}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\theta = c_1 \, \mathrm{e}^{x\sqrt{p^2}} - \frac{c_2 \, \mathrm{csgn}\left(p\right) \mathrm{e}^{-x \, \mathrm{csgn}(p) p}}{2p}$$

Solved as second order ode adjoint method

Time used: 0.453 (sec)

In normal form the ode

$$\theta'' - p^2\theta = 0 \tag{1}$$

Becomes

$$\theta'' + p(x)\,\theta' + q(x)\,\theta = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -p^2 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + \left(-p^2\xi(x)\right) &= 0 \\ \xi''(x) - p^2\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = -p^2$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - p^2 \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - p^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -p^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-p^2)}\\ &= \pm\sqrt{p^2}\end{aligned}$$

Hence

$$\lambda_1 = +\sqrt{p^2}$$
$$\lambda_2 = -\sqrt{p^2}$$

Which simplifies to

$$\lambda_1 = p$$
$$\lambda_2 = -p$$

Since roots are real and distinct, then the solution is

$$\xi = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$\xi = c_1 e^{(p)x} + c_2 e^{(-p)x}$$

Or

$$\xi = c_1\,\mathrm{e}^{xp} + c_2\,\mathrm{e}^{-xp}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\,\theta' - \theta\xi'(x) + \xi(x)\,p(x)\,\theta &= \int \xi(x)\,r(x)\,dx\\ \theta' + \theta\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}\end{aligned}$$

Or

$$\theta' - \frac{\theta(c_1 p\,\mathrm{e}^{xp} - c_2 p\,\mathrm{e}^{-xp})}{c_1\,\mathrm{e}^{xp} + c_2\,\mathrm{e}^{-xp}} = 0$$

Which is now a first order ode. This is now solved for θ. In canonical form a linear first order is

$$\theta' + q(x)\theta = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{p(c_1\,\mathrm{e}^{2xp} - c_2)}{c_1\,\mathrm{e}^{2xp} + c_2} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\frac{p\left(c_1\,\mathrm{e}^{2xp}-c_2\right)}{c_1\,\mathrm{e}^{2xp}+c_2}dx} \\ &= \frac{\sqrt{\mathrm{e}^{2xp}}}{c_1\,\mathrm{e}^{2xp} + c_2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu\theta &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\theta\sqrt{\mathrm{e}^{2xp}}}{c_1\,\mathrm{e}^{2xp} + c_2}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{\theta\sqrt{\mathrm{e}^{2xp}}}{c_1\,\mathrm{e}^{2xp} + c_2} &= \int 0\,dx + c_3 \\ &= c_3 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\mathrm{e}^{2xp}}}{c_1\,\mathrm{e}^{2xp}+c_2}$ gives the final solution

$$\theta = \frac{(c_1\,\mathrm{e}^{2xp} + c_2)\,c_3}{\sqrt{\mathrm{e}^{2xp}}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$\theta = \frac{(c_1\,\mathrm{e}^{2xp} + c_2)\,c_3}{\sqrt{\mathrm{e}^{2xp}}}$$

The constants can be merged to give

$$\theta = \frac{c_1\,\mathrm{e}^{2xp} + c_2}{\sqrt{\mathrm{e}^{2xp}}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\theta = \frac{c_1 \,\mathrm{e}^{2xp} + c_2}{\sqrt{\mathrm{e}^{2xp}}}$$

Maple step by step solution

Let's solve
$\theta'' - p^2\theta = 0$

- Highest derivative means the order of the ODE is 2
  θ''
- Characteristic polynomial of ODE
  $-p^2 + r^2 = 0$
- Factor the characteristic polynomial
  $-(p-r)\,(p+r) = 0$
- Roots of the characteristic polynomial
  $r = (p, -p)$
- 1st solution of the ODE
  $\theta_1(x) = \mathrm{e}^{xp}$
- 2nd solution of the ODE
  $\theta_2(x) = \mathrm{e}^{-xp}$
- General solution of the ODE
  $\theta = C1\theta_1(x) + C2\theta_2(x)$
- Substitute in solutions
  $\theta = C1\,\mathrm{e}^{xp} + C2\,\mathrm{e}^{-xp}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 18

```
dsolve(diff(diff(theta(x),x),x)-p^2*theta(x) = 0,theta(x),singsol=all)
```

$$\theta(x) = c_1 \mathrm{e}^{-px} + c_2 \mathrm{e}^{px}$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 23

```
DSolve[{D[theta[x],{x,2}]-p^2*theta[x]==0,{}},theta[x],x,IncludeSingularSolutions->True]
```

$$\theta(x) \to c_1 e^{px} + c_2 e^{-px}$$

2.8.2 Problem 2

Internal problem ID [18581]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 2
Date solved : `Tuesday, January 28, 2025 at 12:02:52 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y = 0$$

Solved as second order linear constant coeff ode

Time used: 0.064 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos(x) + c_2 \sin(x)$$

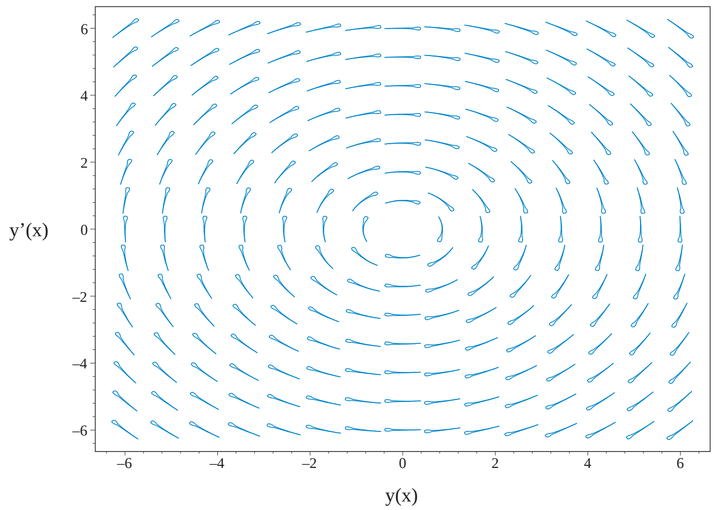

Figure 2.81: Slope field plot
$y'' + y = 0$

Solved as second order can be made integrable

Time used: 0.776 (sec)

Multiplying the ode by y' gives

$$y'y'' + y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + y'y)\, dx = 0$$
$$\frac{y'^2}{2} + \frac{y^2}{2} = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{-y^2 + 2c_1}} dy = dx$$
$$\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = x + c_2$$

Singular solutions are found by solving

$$\sqrt{-y^2+2c_1}=0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y=\sqrt{2}\,\sqrt{c_1}$$
$$y=-\sqrt{2}\,\sqrt{c_1}$$

Solving for y gives

$$y=\sqrt{2}\,\sqrt{c_1}$$
$$y=\tan(x+c_2)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(x+c_2)^2+1}}$$
$$y=-\sqrt{2}\,\sqrt{c_1}$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{-y^2+2c_1}}dy=dx$$
$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right)=x+c_3$$

Singular solutions are found by solving

$$-\sqrt{-y^2+2c_1}=0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y=\sqrt{2}\,\sqrt{c_1}$$
$$y=-\sqrt{2}\,\sqrt{c_1}$$

Solving for y gives

$$y=\sqrt{2}\,\sqrt{c_1}$$
$$y=-\sqrt{2}\,\sqrt{c_1}$$
$$y=-\tan(x+c_3)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(x+c_3)^2+1}}$$

Will add steps showing solving for IC soon.

The solution

$$y = \sqrt{2}\,\sqrt{c_1}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\sqrt{2}\,\sqrt{c_1}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = \tan(x + c_2)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(x + c_2)^2 + 1}}$$

$$y = -\tan(x + c_3)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(x + c_3)^2 + 1}}$$

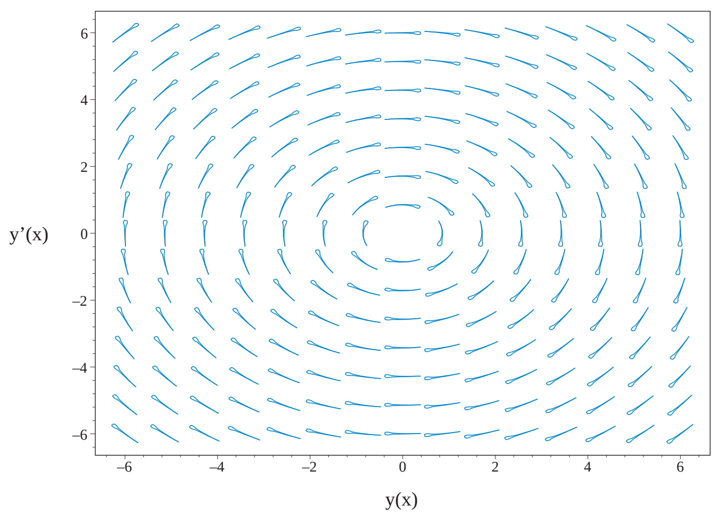

Figure 2.82: Slope field plot
$y'' + y = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.086 (sec)

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.49: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos(x) + c_2 \sin(x)$$

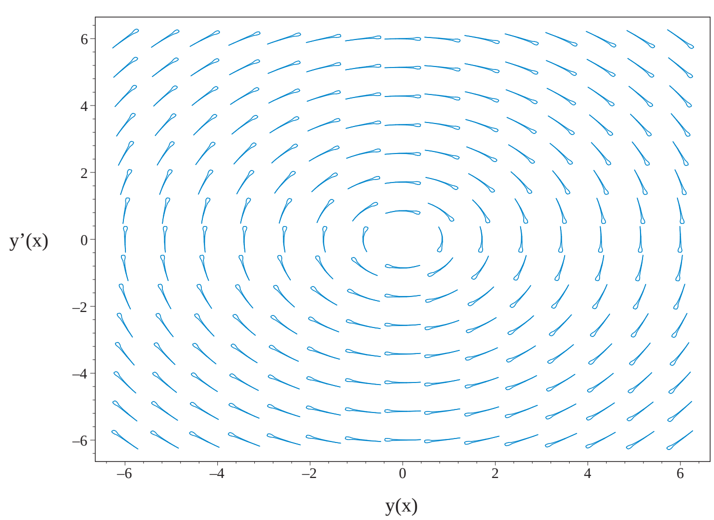

Figure 2.83: Slope field plot
$y'' + y = 0$

Solved as second order ode adjoint method

Time used: 0.569 (sec)

In normal form the ode

$$y'' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y = \int \xi(x)\, r(x)\, dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)}$$
$$p(x) = 0$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= e^{\int -\frac{-c_1 \sin(x)+c_2 \cos(x)}{c_1 \cos(x)+c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = 0$$

Integrating gives

$$\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x)+c_2 \sin(x)}$ gives the final solution

$$y = (c_1 \cos(x) + c_2 \sin(x))\, c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = (c_1 \cos(x) + c_2 \sin(x))\, c_3$$

The constants can be merged to give

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos(x) + c_2 \sin(x)$$

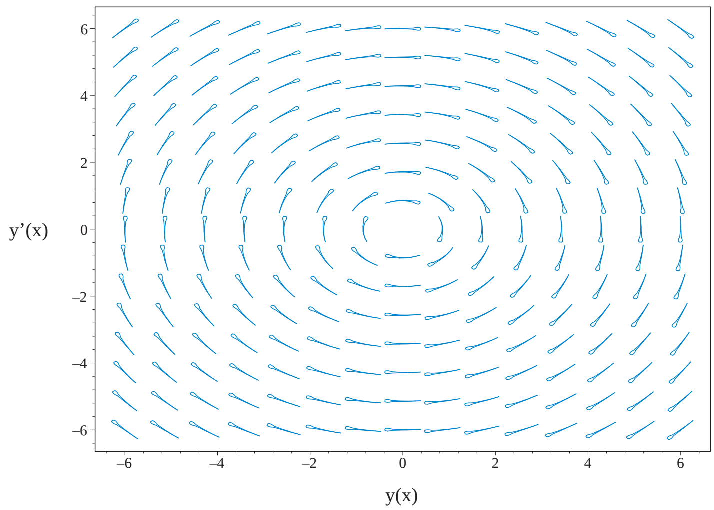

Figure 2.84: Slope field plot
$y'' + y = 0$

Maple step by step solution

Let's solve
$y'' + y = 0$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the ODE
  $y_1(x) = \cos(x)$
- 2nd solution of the ODE
  $y_2(x) = \sin(x)$
- General solution of the ODE
  $y = C1 y_1(x) + C2 y_2(x)$
- Substitute in solutions
  $y = C1 \cos(x) + C2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 13

```
dsolve(diff(diff(y(x),x),x)+y(x) = 0,y(x),singsol=all)
```

$$y(x) = c_1 \sin(x) + c_2 \cos(x)$$

Mathematica DSolve solution

Solving time : 0.01 (sec)
Leaf size : 16

```
DSolve[{D[y[x],{x,2}]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \cos(x) + c_2 \sin(x)$$

2.8.3 Problem 3

Internal problem ID [18582]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 3
Date solved : `Tuesday, January 28, 2025 at 12:02:55 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + 12y = 7y'$$

Solved as second order linear constant coeff ode

Time used: 0.037 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -7, C = 12$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - 7\lambda\, \mathrm{e}^{x\lambda} + 12\, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 7\lambda + 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -7, C = 12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{7}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-7^2 - (4)(1)(12)} \\ &= \frac{7}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{7}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{7}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4 \\ \lambda_2 &= 3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(4)x} + c_2 e^{(3)x}\end{aligned}$$

Or

$$y = c_1 \,\mathrm{e}^{4x} + c_2 \,\mathrm{e}^{3x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{4x} + c_2 \,\mathrm{e}^{3x}$$

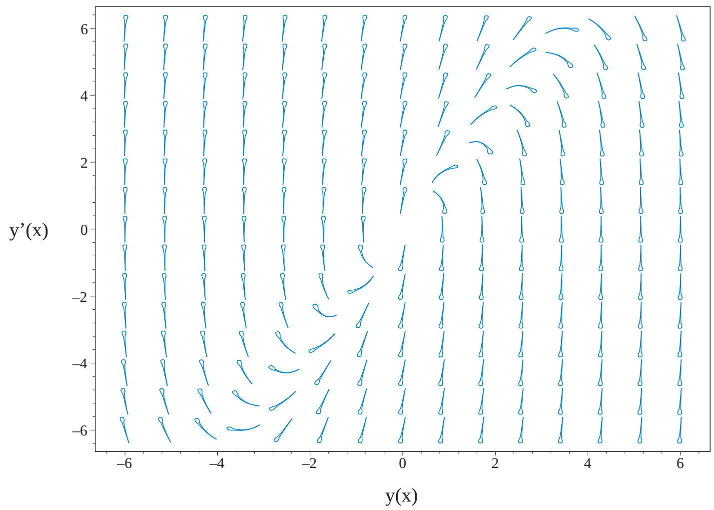

Figure 2.85: Slope field plot
$y'' + 12y = 7y'$

Solved as second order ode using Kovacic algorithm

Time used: 0.051 (sec)

Writing the ode as

$$y'' + 12y - 7y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -7 \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.51: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-7}{1}\,dx} \\ &= z_1 e^{\frac{7x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{7x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-7}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{7x}}{(y_1)^2}\,dx \\ &= y_1 \left(\mathrm{e}^{7x}\mathrm{e}^{-6x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{3x}\right) + c_2\left(\mathrm{e}^{3x}\left(\mathrm{e}^{7x}\mathrm{e}^{-6x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{3x} + c_2 \,\mathrm{e}^{4x}$$

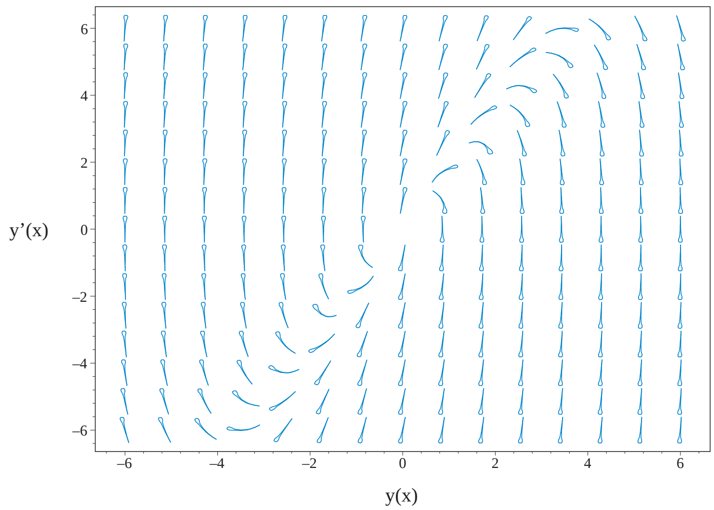

Figure 2.86: Slope field plot
$y'' + 12y = 7y'$

Solved as second order ode adjoint method

Time used: 0.439 (sec)

In normal form the ode

$$y'' + 12y = 7y' \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -7 \\ q(x) &= 12 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (-7\xi(x))' + (12\xi(x)) &= 0 \\ \xi''(x) + 7\xi'(x) + 12\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 7, C = 12$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + 7\lambda\, \mathrm{e}^{x\lambda} + 12\, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 7\lambda + 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 7, C = 12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-7}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{7^2 - (4)(1)(12)} \\ &= -\frac{7}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{7}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{7}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 \\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ \xi &= c_1 e^{(-3)x} + c_2 e^{(-4)x}\end{aligned}$$

Or

$$\xi = c_1\, \mathrm{e}^{-3x} + c_2\, \mathrm{e}^{-4x}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(-7 - \frac{-3c_1\,\mathrm{e}^{-3x} - 4c_2\,\mathrm{e}^{-4x}}{c_1\,\mathrm{e}^{-3x} + c_2\,\mathrm{e}^{-4x}}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{3\,\mathrm{e}^{-x}c_2 + 4c_1}{\mathrm{e}^{-x}c_2 + c_1} \\
p(x) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{3\,\mathrm{e}^{-x}c_2 + 4c_1}{\mathrm{e}^{-x}c_2 + c_1}dx} \\
&= \frac{\mathrm{e}^{-4x}}{\mathrm{e}^{-x}c_2 + c_1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,\mathrm{e}^{-4x}}{\mathrm{e}^{-x}c_2 + c_1}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\,\mathrm{e}^{-4x}}{\mathrm{e}^{-x}c_2 + c_1} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{e^{-4x}}{e^{-x}c_2+c_1}$ gives the final solution

$$y = (c_1 e^x + c_2) e^{3x} c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = (c_1 e^x + c_2) e^{3x} c_3$$

The constants can be merged to give

$$y = (c_1 e^x + c_2) e^{3x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = (c_1 e^x + c_2) e^{3x}$$

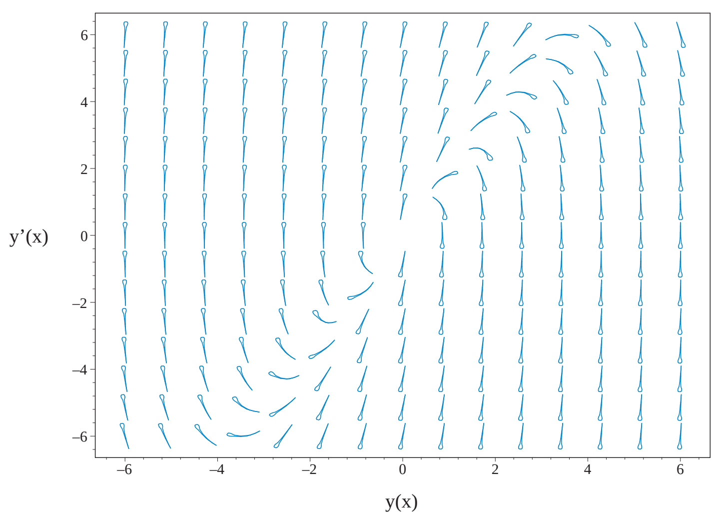

Figure 2.87: Slope field plot
$y'' + 12y = 7y'$

Maple step by step solution

Let's solve
$y'' + 12y = 7y'$

- Highest derivative means the order of the ODE is 2
  y''

- Isolate 2nd derivative
  $y'' = -12y + 7y'$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y'' + 12y - 7y' = 0$
- Characteristic polynomial of ODE
  $r^2 - 7r + 12 = 0$
- Factor the characteristic polynomial
  $(r-3)(r-4) = 0$
- Roots of the characteristic polynomial
  $r = (3, 4)$
- 1st solution of the ODE
  $y_1(x) = e^{3x}$
- 2nd solution of the ODE
  $y_2(x) = e^{4x}$
- General solution of the ODE
  $y = C1y_1(x) + C2y_2(x)$
- Substitute in solutions
  $y = C1\, e^{3x} + C2\, e^{4x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve(diff(diff(y(x),x),x)+12*y(x) = 7*diff(y(x),x),y(x),singsol=all)
```

$$y(x) = c_1 e^{4x} + e^{3x} c_2$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+12*y[x]==7*D[y[x],x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{3x}(c_2e^x + c_1)$$

2.8.4 Problem 4

Internal problem ID [18583]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 4
Date solved : `Tuesday, January 28, 2025 at 12:02:56 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$r'' - a^2 r = 0$$

Solved as second order linear constant coeff ode

Time used: 0.074 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ar''(\phi) + Br'(\phi) + Cr(\phi) = 0$$

Where in the above $A = 1, B = 0, C = -a^2$. Let the solution be $r = e^{\lambda\phi}$. Substituting this into the ODE gives

$$\lambda^2 e^{\phi\lambda} - a^2 e^{\phi\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\phi}$ gives

$$-a^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -a^2$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-a^2)} = \pm\sqrt{a^2}$$

Hence

$$\lambda_1 = +\sqrt{a^2}$$
$$\lambda_2 = -\sqrt{a^2}$$

Which simplifies to

$$\lambda_1 = a$$
$$\lambda_2 = -a$$

Since roots are real and distinct, then the solution is

$$r = c_1 e^{\lambda_1 \phi} + c_2 e^{\lambda_2 \phi}$$
$$r = c_1 e^{(a)\phi} + c_2 e^{(-a)\phi}$$

Or

$$r = c_1 \,\mathrm{e}^{\phi a} + c_2 \,\mathrm{e}^{-\phi a}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$r = c_1 \,\mathrm{e}^{\phi a} + c_2 \,\mathrm{e}^{-\phi a}$$

Solved as second order can be made integrable

Time used: 2.390 (sec)

Multiplying the ode by r' gives

$$r'r'' - a^2 r'r = 0$$

Integrating the above w.r.t ϕ gives

$$\int (r'r'' - a^2 r'r)\, d\phi = 0$$
$$\frac{r'^2}{2} - \frac{a^2 r^2}{2} = c_1$$

Which is now solved for r. Solving for the derivative gives these ODE's to solve

$$r' = \sqrt{a^2r^2 + 2c_1} \tag{1}$$
$$r' = -\sqrt{a^2r^2 + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{a^2r^2 + 2c_1}} dr = d\phi$$
$$\frac{\ln\left(\frac{a^2 r}{\sqrt{a^2}} + \sqrt{a^2r^2 + 2c_1}\right)}{\sqrt{a^2}} = \phi + c_2$$

Singular solutions are found by solving

$$\sqrt{a^2r^2 + 2c_1} = 0$$

for r. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$r = \frac{\sqrt{-2c_1}}{a}$$
$$r = -\frac{\sqrt{-2c_1}}{a}$$

Solving for r gives

$$r = \frac{\sqrt{-2c_1}}{a}$$
$$r = -\frac{\sqrt{-2c_1}}{a}$$
$$r = -\frac{\sqrt{a^2}\left(-e^{2c_2\sqrt{a^2}+2\phi\sqrt{a^2}} + 2c_1\right) e^{-c_2\sqrt{a^2}-\phi\sqrt{a^2}}}{2a^2}$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{a^2r^2+2c_1}}dr = d\phi$$
$$-\frac{\ln\left(\frac{a^2r}{\sqrt{a^2}}+\sqrt{a^2r^2+2c_1}\right)}{\sqrt{a^2}} = \phi + c_3$$

Singular solutions are found by solving

$$-\sqrt{a^2r^2+2c_1} = 0$$

for r. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$r = \frac{\sqrt{-2c_1}}{a}$$
$$r = -\frac{\sqrt{-2c_1}}{a}$$

Solving for r gives

$$r = \frac{\sqrt{-2c_1}}{a}$$
$$r = -\frac{\sqrt{-2c_1}}{a}$$
$$r = -\frac{\sqrt{a^2}\left(-\mathrm{e}^{-2c_3\sqrt{a^2}-2\phi\sqrt{a^2}}+2c_1\right)\mathrm{e}^{c_3\sqrt{a^2}+\phi\sqrt{a^2}}}{2a^2}$$

Will add steps showing solving for IC soon.

The solution

$$r = \frac{\sqrt{-2c_1}}{a}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$r = -\frac{\sqrt{-2c_1}}{a}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$r = -\frac{\sqrt{a^2}\left(-\mathrm{e}^{2c_2\sqrt{a^2}+2\phi\sqrt{a^2}}+2c_1\right)\mathrm{e}^{-c_2\sqrt{a^2}-\phi\sqrt{a^2}}}{2a^2}$$
$$r = -\frac{\sqrt{a^2}\left(-\mathrm{e}^{-2c_3\sqrt{a^2}-2\phi\sqrt{a^2}}+2c_1\right)\mathrm{e}^{c_3\sqrt{a^2}+\phi\sqrt{a^2}}}{2a^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.052 (sec)

Writing the ode as

$$r'' - a^2 r = 0 \tag{1}$$
$$Ar'' + Br' + Cr = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(\phi) = re^{\int \frac{B}{2A}\,d\phi}$$

Then (2) becomes

$$z''(\phi) = rz(\phi) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(\phi) = (a^2)\, z(\phi) \tag{7}$$

Equation (7) is now solved. After finding $z(\phi)$ then r is found using the inverse transformation

$$r = z(\phi)\, e^{-\int \frac{B}{2A}\, d\phi}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.53: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = a^2$ is not a function of ϕ, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(\phi) = \mathrm{e}^{\phi\sqrt{a^2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in r is found from

$$r_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,d\phi}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} r_1 &= z_1 \\ &= \mathrm{e}^{\phi\sqrt{a^2}} \end{aligned}$$

Which simplifies to

$$r_1 = \mathrm{e}^{\phi\sqrt{a^2}}$$

The second solution r_2 to the original ode is found using reduction of order

$$r_2 = r_1 \int \frac{e^{\int -\frac{B}{A}\,d\phi}}{r_1^2}\,d\phi$$

Since $B = 0$ then the above becomes

$$\begin{aligned} r_2 &= r_1 \int \frac{1}{r_1^2}\,d\phi \\ &= \mathrm{e}^{\phi\sqrt{a^2}} \int \frac{1}{\mathrm{e}^{2\phi\sqrt{a^2}}}\,d\phi \\ &= \mathrm{e}^{\phi\sqrt{a^2}} \left(-\frac{\sqrt{a^2}\,\mathrm{e}^{-2\phi\sqrt{a^2}}}{2a^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} r &= c_1 r_1 + c_2 r_2 \\ &= c_1\left(\mathrm{e}^{\phi\sqrt{a^2}}\right) + c_2\left(\mathrm{e}^{\phi\sqrt{a^2}}\left(-\frac{\sqrt{a^2}\,\mathrm{e}^{-2\phi\sqrt{a^2}}}{2a^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$r = c_1 \,\mathrm{e}^{\phi \sqrt{a^2}} - \frac{c_2 \,\mathrm{csgn}\left(a\right) \mathrm{e}^{-\phi \,\mathrm{csgn}(a) a}}{2a}$$

Solved as second order ode adjoint method

Time used: 0.457 (sec)

In normal form the ode

$$r'' - a^2 r = 0 \tag{1}$$

Becomes

$$r'' + p(\phi)\, r' + q(\phi)\, r = r(\phi) \tag{2}$$

Where

$$\begin{aligned} p(\phi) &= 0 \\ q(\phi) &= -a^2 \\ r(\phi) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + \left(-a^2 \xi(\phi)\right) &= 0 \\ \xi''(\phi) - a^2 \xi(\phi) &= 0 \end{aligned}$$

Which is solved for $\xi(\phi)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(\phi) + B\xi'(\phi) + C\xi(\phi) = 0$$

Where in the above $A = 1, B = 0, C = -a^2$. Let the solution be $\xi = e^{\lambda \phi}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\phi \lambda} - a^2 \mathrm{e}^{\phi \lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \phi}$ gives

$$-a^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-a^2)}\\ &= \pm\sqrt{a^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{a^2}\\ \lambda_2 &= -\sqrt{a^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= a\\ \lambda_2 &= -a\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}\xi &= c_1 e^{\lambda_1 \phi} + c_2 e^{\lambda_2 \phi}\\ \xi &= c_1 e^{(a)\phi} + c_2 e^{(-a)\phi}\end{aligned}$$

Or

$$\xi = c_1 \, e^{\phi a} + c_2 \, e^{-\phi a}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(\phi)\, r' - r\xi'(\phi) + \xi(\phi)\, p(\phi)\, r &= \int \xi(\phi)\, r(\phi)\, d\phi\\ r' + r\left(p(\phi) - \frac{\xi'(\phi)}{\xi(\phi)}\right) &= \frac{\int \xi(\phi)\, r(\phi)\, d\phi}{\xi(\phi)}\end{aligned}$$

Or

$$r' - \frac{r\left(c_1 a \, e^{\phi a} - c_2 a \, e^{-\phi a}\right)}{c_1 \, e^{\phi a} + c_2 \, e^{-\phi a}} = 0$$

Which is now a first order ode. This is now solved for r. In canonical form a linear first order is

$$r' + q(\phi)r = p(\phi)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(\phi) &= -\frac{a\left(c_1\,\mathrm{e}^{2\phi a} - c_2\right)}{c_1\,\mathrm{e}^{2\phi a} + c_2} \\ p(\phi) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,d\phi} \\ &= \mathrm{e}^{\int -\frac{a\left(c_1\,\mathrm{e}^{2\phi a} - c_2\right)}{c_1\,\mathrm{e}^{2\phi a} + c_2}d\phi} \\ &= \frac{\sqrt{\mathrm{e}^{2\phi a}}}{c_1\,\mathrm{e}^{2\phi a} + c_2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\phi}\mu r &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{r\sqrt{\mathrm{e}^{2\phi a}}}{c_1\,\mathrm{e}^{2\phi a} + c_2}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{r\sqrt{\mathrm{e}^{2\phi a}}}{c_1\,\mathrm{e}^{2\phi a} + c_2} &= \int 0\,d\phi + c_3 \\ &= c_3 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\mathrm{e}^{2\phi a}}}{c_1\,\mathrm{e}^{2\phi a} + c_2}$ gives the final solution

$$r = \frac{\left(c_1\,\mathrm{e}^{2\phi a} + c_2\right)c_3}{\sqrt{\mathrm{e}^{2\phi a}}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$r = \frac{\left(c_1\,\mathrm{e}^{2\phi a} + c_2\right)c_3}{\sqrt{\mathrm{e}^{2\phi a}}}$$

The constants can be merged to give

$$r = \frac{c_1\,\mathrm{e}^{2\phi a} + c_2}{\sqrt{\mathrm{e}^{2\phi a}}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$r = \frac{c_1 \,\mathrm{e}^{2\phi a} + c_2}{\sqrt{\mathrm{e}^{2\phi a}}}$$

Maple step by step solution

Let's solve
$r'' - a^2 r = 0$

- Highest derivative means the order of the ODE is 2
  r''
- Characteristic polynomial of ODE
  $-a^2 + s^2 = 0$
- Factor the characteristic polynomial
  $-(a - s)(a + s) = 0$
- Roots of the characteristic polynomial
  $s = (a, -a)$
- 1st solution of the ODE
  $r_1(\phi) = \mathrm{e}^{\phi a}$
- 2nd solution of the ODE
  $r_2(\phi) = \mathrm{e}^{-\phi a}$
- General solution of the ODE
  $r = C1 r_1(\phi) + C2 r_2(\phi)$
- Substitute in solutions
  $r = C1 \,\mathrm{e}^{\phi a} + C2 \,\mathrm{e}^{-\phi a}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 18

```
dsolve(diff(diff(r(phi),phi),phi)-a^2*r(phi) = 0,r(phi),singsol=all)
```

$$r = c_1 \mathrm{e}^{-a\phi} + c_2 \mathrm{e}^{a\phi}$$

Mathematica DSolve solution

Solving time : `0.013` (sec)
Leaf size : 23

```
DSolve[{D[r[phi],{phi,2}]-a^2*r[phi]==0,{}},r[phi],phi,IncludeSingularSolutions->True]
```

$$r(\phi) \to c_1 e^{a\phi} + c_2 e^{-a\phi}$$

2.8.5 Problem 5

Internal problem ID [18584]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 5
Date solved : `Tuesday, January 28, 2025 at 12:03:00 PM`
CAS classification : `[[_high_order, _missing_x]]`

Solve

$$y'''' - a^4 y = 0$$

Solved as higher order constant coeff ode

Time used: 0.037 (sec)

The characteristic equation is

$$-a^4 + \lambda^4 = 0$$

The roots of the above equation are

$$\lambda_1 = a$$
$$\lambda_2 = -a$$
$$\lambda_3 = ia$$
$$\lambda_4 = -ia$$

Therefore the homogeneous solution is

$$y_h(x) = e^{ax}c_1 + e^{-ax}c_2 + e^{iax}c_3 + e^{-iax}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{ax} \\
y_2 &= e^{-ax} \\
y_3 &= e^{iax} \\
y_4 &= e^{-iax}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 30

```
dsolve(diff(diff(diff(diff(y(x),x),x),x),x)-a^4*y(x) = 0,y(x),singsol=all)
```

$$
y(x) = c_1 e^{-ax} + c_2 e^{ax} + c_3 \sin(ax) + c_4 \cos(ax)
$$

Mathematica DSolve solution

Solving time : `0.003` (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,4}]-a^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to c_2 e^{-\sqrt{a}x} + c_4 e^{\sqrt{a}x} + c_1 \cos\left(\sqrt{a}x\right) + c_3 \sin\left(\sqrt{a}x\right)
$$

2.8.6 Problem 6

Internal problem ID [18585]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 6
Date solved : `Tuesday, January 28, 2025 at 12:03:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$v'' - 6v' + 13v = \mathrm{e}^{-2u}$$

Solved as second order linear constant coeff ode

Time used: 0.105 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(u) + Bv'(u) + Cv(u) = f(u)$$

Where $A = 1, B = -6, C = 13, f(u) = \mathrm{e}^{-2u}$. Let the solution be

$$v = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(u)+Bv'(u)+Cv(u) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(u)+Bv'(u)+Cv(u) = f(u)$. v_h is the solution to

$$v'' - 6v' + 13v = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(u) + Bv'(u) + Cv(u) = 0$$

Where in the above $A = 1, B = -6, C = 13$. Let the solution be $v = e^{\lambda u}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{u\lambda} - 6\lambda \,\mathrm{e}^{u\lambda} + 13\,\mathrm{e}^{u\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda u}$ gives

$$\lambda^2 - 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-6^2 - (4)\,(1)\,(13)}\\ &= 3 \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = 3 + 2i$$
$$\lambda_2 = 3 - 2i$$

Which simplifies to

$$\lambda_1 = 3 + 2i$$
$$\lambda_2 = 3 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$v = e^{\alpha u}(c_1 \cos(\beta u) + c_2 \sin(\beta u))$$

Which becomes

$$v = e^{3u}(c_1 \cos(2u) + c_2 \sin(2u))$$

Therefore the homogeneous solution v_h is

$$v_h = \mathrm{e}^{3u}(c_1 \cos(2u) + c_2 \sin(2u))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2u}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2u}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3u}\cos(2u), e^{3u}\sin(2u)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$v_p = A_1 e^{-2u}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$29A_1 e^{-2u} = e^{-2u}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{29}\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = \frac{e^{-2u}}{29}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(e^{3u}(c_1\cos(2u) + c_2\sin(2u))\right) + \left(\frac{e^{-2u}}{29}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$v = \frac{e^{-2u}}{29} + e^{3u}(c_1 \cos(2u) + c_2 \sin(2u))$$

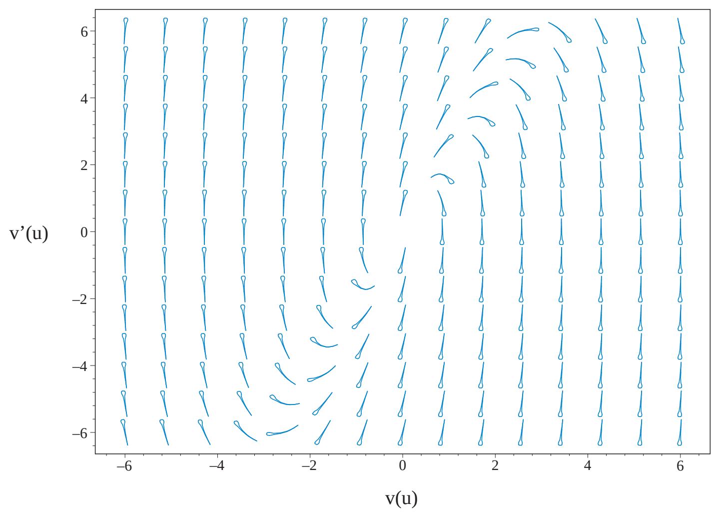

Figure 2.88: Slope field plot
$v'' - 6v' + 13v = e^{-2u}$

Solved as second order ode using Kovacic algorithm

Time used: 0.165 (sec)

Writing the ode as

$$v'' - 6v' + 13v = 0 \tag{1}$$

$$Av'' + Bv' + Cv = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(u) = ve^{\int \frac{B}{2A} du}$$

Then (2) becomes

$$z''(u) = rz(u) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(u) = -4z(u) \tag{7}$$

Equation (7) is now solved. After finding $z(u)$ then v is found using the inverse transformation

$$v = z(u)\, e^{-\int \frac{B}{2A}\, du}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.55: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of u, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(u) = \cos(2u)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in v is found from

$$\begin{aligned} v_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, du} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\, du} \\ &= z_1 e^{3u} \\ &= z_1 \left(\mathrm{e}^{3u}\right) \end{aligned}$$

Which simplifies to

$$v_1 = \mathrm{e}^{3u}\cos(2u)$$

The second solution v_2 to the original ode is found using reduction of order

$$v_2 = v_1 \int \frac{e^{\int -\frac{B}{A}\, du}}{v_1^2}\, du$$

Substituting gives

$$\begin{aligned} v_2 &= v_1 \int \frac{e^{\int -\frac{-6}{1}\, du}}{(v_1)^2}\, du \\ &= v_1 \int \frac{e^{6u}}{(v_1)^2}\, du \\ &= v_1 \left(\frac{\tan(2u)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
v &= c_1 v_1 + c_2 v_2 \\
&= c_1 \left(e^{3u} \cos(2u)\right) + c_2 \left(e^{3u} \cos(2u) \left(\frac{\tan(2u)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
v = v_h + v_p
$$

Where v_h is the solution to the homogeneous ODE $Av''(u)+Bv'(u)+Cv(u) = 0$, and v_p is a particular solution to the nonhomogeneous ODE $Av''(u) + Bv'(u) + Cv(u) = f(u)$. v_h is the solution to

$$
v'' - 6v' + 13v = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
v_h = c_1 e^{3u} \cos(2u) + \frac{c_2 e^{3u} \sin(2u)}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^{-2u}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{-2u}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{e^{3u} \cos(2u), \frac{e^{3u} \sin(2u)}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
v_p = A_1 e^{-2u}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$29A_1 e^{-2u} = e^{-2u}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{29}\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = \frac{e^{-2u}}{29}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(c_1 e^{3u} \cos(2u) + \frac{c_2 e^{3u} \sin(2u)}{2}\right) + \left(\frac{e^{-2u}}{29}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$v = c_1 e^{3u} \cos(2u) + \frac{c_2 e^{3u} \sin(2u)}{2} + \frac{e^{-2u}}{29}$$

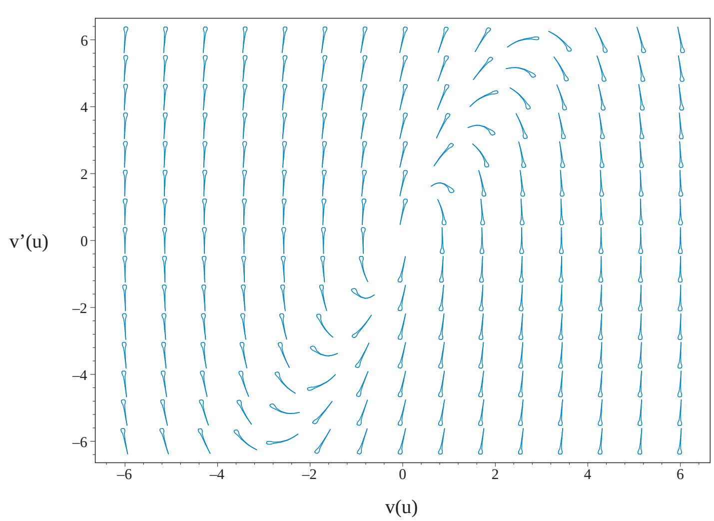

Figure 2.89: Slope field plot
$v'' - 6v' + 13v = e^{-2u}$

Solved as second order ode adjoint method

Time used: 11.036 (sec)

In normal form the ode

$$v'' - 6v' + 13v = e^{-2u} \tag{1}$$

Becomes

$$v'' + p(u)\, v' + q(u)\, v = r(u) \tag{2}$$

Where

$$\begin{aligned} p(u) &= -6 \\ q(u) &= 13 \\ r(u) &= e^{-2u} \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi^{''} - (\xi\, p)' + \xi q &= 0 \\ \xi^{''} - (-6\xi(u))' + (13\xi(u)) &= 0 \\ \xi''(u) + 6\xi'(u) + 13\xi(u) &= 0 \end{aligned}$$

Which is solved for $\xi(u)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(u) + B\xi'(u) + C\xi(u) = 0$$

Where in the above $A = 1, B = 6, C = 13$. Let the solution be $\xi = e^{\lambda u}$. Substituting this into the ODE gives

$$\lambda^2 e^{u\lambda} + 6\lambda\, e^{u\lambda} + 13\, e^{u\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda u}$ gives

$$\lambda^2 + 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 13$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{6^2 - (4)\,(1)\,(13)} \\ &= -3 \pm 2i \end{aligned}$$

Hence

$$\lambda_1 = -3 + 2i$$
$$\lambda_2 = -3 - 2i$$

Which simplifies to

$$\lambda_1 = -3 + 2i$$
$$\lambda_2 = -3 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha u}(c_1 \cos(\beta u) + c_2 \sin(\beta u))$$

Which becomes

$$\xi = e^{-3u}(c_1 \cos(2u) + c_2 \sin(2u))$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(u)\, v' - v\xi'(u) + \xi(u)\, p(u)\, v = \int \xi(u)\, r(u)\, du$$
$$v' + v\left(p(u) - \frac{\xi'(u)}{\xi(u)}\right) = \frac{\int \xi(u)\, r(u)\, du}{\xi(u)}$$

Or

$$v' + v\left(-6 - \frac{(-3\,\mathrm{e}^{-3u}(c_1\cos(2u) + c_2\sin(2u)) + \mathrm{e}^{-3u}(-2c_1\sin(2u) + 2c_2\cos(2u)))\,\mathrm{e}^{3u}}{c_1\cos(2u) + c_2\sin(2u)}\right) = \frac{\mathrm{e}^{3u}\left(2c_1\left(\frac{(-5}{}\right.\right.}{}$$

Which is now a first order ode. This is now solved for v. In canonical form a linear first order is

$$v' + q(u)v = p(u)$$

Comparing the above to the given ode shows that

$$q(u) = -\frac{(3c_1 + 2c_2)\cos(2u) - 2\left(c_1 - \frac{3c_2}{2}\right)\sin(2u)}{c_1\cos(2u) + c_2\sin(2u)}$$

$$p(u) = -\frac{5\left(\left(c_1 + \frac{2c_2}{5}\right)\cos(2u) - \frac{2\left(c_1 - \frac{5c_2}{2}\right)\sin(2u)}{5}\right)e^{-2u}}{29c_1\cos(2u) + 29c_2\sin(2u)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,du}\\ &= e^{\int -\frac{(3c_1+2c_2)\cos(2u)-2\left(c_1-\frac{3c_2}{2}\right)\sin(2u)}{c_1\cos(2u)+c_2\sin(2u)}du}\\ &= e^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\tan(2u))-3u}\end{aligned}$$

The ode becomes

$$\frac{d}{du}(\mu v) = \mu p$$

$$\frac{d}{du}(\mu v) = (\mu)\left(-\frac{5\left(\left(c_1 + \frac{2c_2}{5}\right)\cos(2u) - \frac{2\left(c_1 - \frac{5c_2}{2}\right)\sin(2u)}{5}\right)e^{-2u}}{29c_1\cos(2u) + 29c_2\sin(2u)}\right)$$

$$\begin{aligned}&\frac{d}{du}\left(v\,e^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\tan(2u))-3u}\right)\\ &= \left(e^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\tan(2u))-3u}\right)\left(-\frac{5\left(\left(c_1 + \frac{2c_2}{5}\right)\cos(2u) - \frac{2\left(c_1 - \frac{5c_2}{2}\right)\sin(2u)}{5}\right)e^{-2u}}{29c_1\cos(2u) + 29c_2\sin(2u)}\right)\end{aligned}$$

$$\begin{aligned}&d\left(v\,e^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\tan(2u))-3u}\right)\\ &= \left(-\frac{5\left(\left(c_1 + \frac{2c_2}{5}\right)\cos(2u) - \frac{2\left(c_1 - \frac{5c_2}{2}\right)\sin(2u)}{5}\right)e^{-2u}e^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\tan(2u))-3u}}{29c_1\cos(2u) + 29c_2\sin(2u)}\right)du\end{aligned}$$

Integrating gives

$$
\begin{aligned}
v\,\mathrm{e}^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\tan(2u))-3u} &= \int -\frac{5\left(\left(c_1+\frac{2c_2}{5}\right)\cos\left(2u\right)-\frac{2\left(c_1-\frac{5c_2}{2}\right)\sin(2u)}{5}\right)\mathrm{e}^{-2u}\mathrm{e}^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\,\mathrm{ta}}}{29c_1\cos\left(2u\right)+29c_2\sin\left(2u\right)} \\
&= \int -\frac{5\left(\left(c_1+\frac{2c_2}{5}\right)\cos\left(2u\right)-\frac{2\left(c_1-\frac{5c_2}{2}\right)\sin(2u)}{5}\right)\mathrm{e}^{-2u}\mathrm{e}^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\,\mathrm{ta}}}{29c_1\cos\left(2u\right)+29c_2\sin\left(2u\right)}
\end{aligned}
$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\tan(2u))-3u}$ gives the final solution

$$
v=\left(c_1+c_2\tan\left(2u\right)\right)\mathrm{e}^{\ln\left(\frac{1}{\sqrt{\tan(2u)^2+1}}\right)+3u}\left(\int -\frac{5\left(\left(c_1+\frac{2c_2}{5}\right)\cos\left(2u\right)-\frac{2\left(c_1-\frac{5c_2}{2}\right)\sin(2u)}{5}\right)\mathrm{e}^{-2u}\mathrm{e}^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\mathrm{l}}}{29c_1\cos\left(2u\right)+29c_2\sin\left(2u\right)}\right.
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
v=\left(c_1+c_2\tan\left(2u\right)\right)\mathrm{e}^{\ln\left(\frac{1}{\sqrt{\tan(2u)^2+1}}\right)+3u}\left(\int -\frac{5\left(\left(c_1+\frac{2c_2}{5}\right)\cos\left(2u\right)-\frac{2\left(c_1-\frac{5c_2}{2}\right)\sin(2u)}{5}\right)\mathrm{e}^{-2u}\mathrm{e}^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\tan(2u))-3u}}{29c_1\cos\left(2u\right)+29c_2\sin\left(2u\right)}du+c_3\right)
$$

The constants can be merged to give

$$
v=\left(c_1+c_2\tan\left(2u\right)\right)\mathrm{e}^{\ln\left(\frac{1}{\sqrt{\tan(2u)^2+1}}\right)+3u}\left(\int -\frac{5\left(\left(c_1+\frac{2c_2}{5}\right)\cos\left(2u\right)-\frac{2\left(c_1-\frac{5c_2}{2}\right)\sin(2u)}{5}\right)\mathrm{e}^{-2u}\mathrm{e}^{\frac{\ln\left(\tan(2u)^2+1\right)}{2}-\ln(c_1+c_2\tan(2u))-3u}}{29c_1\cos\left(2u\right)+29c_2\sin\left(2u\right)}du+1\right)
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$v = (c_1 + c_2 \tan(2u))\, e^{\ln\left(\frac{1}{\sqrt{\tan(2u)^2+1}}\right)+3u} \left(\int -\frac{5\left(\left(c_1 + \frac{2c_2}{5}\right)\cos(2u) - \frac{2\left(c_1 - \frac{5c_2}{2}\right)\sin(2u)}{5}\right) e^{-2u} e^{\frac{\ln\left(\tan(2u)^2+1\right)}{2} - \ln(c_1+c_2\tan(2u)) - 3u}}{29c_1\cos(2u) + 29c_2\sin(2u)} du + 1\right)$$

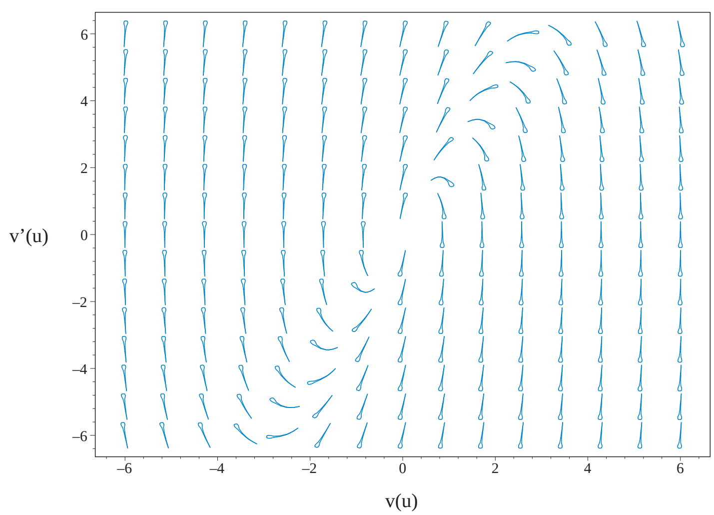

Figure 2.90: Slope field plot
$v'' - 6v' + 13v = e^{-2u}$

Maple step by step solution

Let's solve
$v'' - 6v' + 13v = e^{-2u}$

- Highest derivative means the order of the ODE is 2
  v''
- Characteristic polynomial of homogeneous ODE

$r^2 - 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{6 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (3 - 2\,\mathrm{I}, 3 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $v_1(u) = \mathrm{e}^{3u} \cos(2u)$

- 2nd solution of the homogeneous ODE

  $v_2(u) = \mathrm{e}^{3u} \sin(2u)$

- General solution of the ODE

  $v = C1 v_1(u) + C2 v_2(u) + v_p(u)$

- Substitute in solutions of the homogeneous ODE

  $v = C1\,\mathrm{e}^{3u} \cos(2u) + C2\,\mathrm{e}^{3u} \sin(2u) + v_p(u)$

☐ Find a particular solution $v_p(u)$ of the ODE

  ◦ Use variation of parameters to find v_p here $f(u)$ is the forcing function

  $\left[v_p(u) = -v_1(u) \left(\int \frac{v_2(u) f(u)}{W(v_1(u), v_2(u))} du\right) + v_2(u) \left(\int \frac{v_1(u) f(u)}{W(v_1(u), v_2(u))} du\right), f(u) = \mathrm{e}^{-2u}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(v_1(u), v_2(u)) = \begin{bmatrix} \mathrm{e}^{3u} \cos(2u) & \mathrm{e}^{3u} \sin(2u) \\ 3\,\mathrm{e}^{3u} \cos(2u) - 2\,\mathrm{e}^{3u} \sin(2u) & 3\,\mathrm{e}^{3u} \sin(2u) + 2\,\mathrm{e}^{3u} \cos(2u) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(v_1(u), v_2(u)) = 2\,\mathrm{e}^{6u}$

  ◦ Substitute functions into equation for $v_p(u)$

  $v_p(u) = \frac{\mathrm{e}^{3u} \left(-\cos(2u) \left(\int \sin(2u)\mathrm{e}^{-5u} du\right) + \sin(2u) \left(\int \cos(2u)\mathrm{e}^{-5u} du\right)\right)}{2}$

  ◦ Compute integrals

  $v_p(u) = \frac{\mathrm{e}^{-2u}}{29}$

- Substitute particular solution into general solution to ODE

  $v = C2\,\mathrm{e}^{3u} \sin(2u) + C1\,\mathrm{e}^{3u} \cos(2u) + \frac{\mathrm{e}^{-2u}}{29}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 33

```
dsolve(diff(diff(v(u),u),u)-6*diff(v(u),u)+13*v(u) = exp(-2*u),v(u),singsol=all)
```

$$v = (c_1 \cos(2u) + c_2 \sin(2u)) \mathrm{e}^{-2u} \mathrm{e}^{5u} + \frac{\mathrm{e}^{-2u}}{29}$$

Mathematica DSolve solution

Solving time : 0.109 (sec)
Leaf size : 39

```
DSolve[{D[v[u],{u,2}]-6*D[v[u],u]+13*v[u]==Exp[-2*u],{}},v[u],u,IncludeSingularSolutions->Tr
```

$$v(u) \to \frac{e^{-2u}}{29} + c_2 e^{3u} \cos(2u) + c_1 e^{3u} \sin(2u)$$

2.8.7 Problem 7

Internal problem ID [18586]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 7
Date solved : `Tuesday, January 28, 2025 at 12:03:12 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + 4y' - y = \sin(t)$$

Solved as second order linear constant coeff ode

Time used: 0.131 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = -1, f(t) = \sin(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = -1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} + 4\lambda\, \mathrm{e}^{t\lambda} - \mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(-1)}\\ &= -2 \pm \sqrt{5}\end{aligned}$$

Hence

$$\lambda_1 = -2 + \sqrt{5}$$
$$\lambda_2 = -2 - \sqrt{5}$$

Which simplifies to

$$\lambda_1 = -2 + \sqrt{5}$$
$$\lambda_2 = -2 - \sqrt{5}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{\left(-2+\sqrt{5}\right)t} + c_2 e^{\left(-2-\sqrt{5}\right)t}$$

Or

$$y = c_1\, \mathrm{e}^{t\left(-2+\sqrt{5}\right)} + c_2\, \mathrm{e}^{t\left(-2-\sqrt{5}\right)}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\, \mathrm{e}^{t\left(-2+\sqrt{5}\right)} + c_2\, \mathrm{e}^{t\left(-2-\sqrt{5}\right)}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{t\left(-2-\sqrt{5}\right)}, e^{t\left(-2+\sqrt{5}\right)}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \cos(t) - 2A_2 \sin(t) - 4A_1 \sin(t) + 4A_2 \cos(t) = \sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = -\frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(t)}{5} - \frac{\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{t\left(-2+\sqrt{5}\right)} + c_2 e^{t\left(-2-\sqrt{5}\right)}\right) + \left(-\frac{\cos(t)}{5} - \frac{\sin(t)}{10}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(t)}{5} - \frac{\sin(t)}{10} + c_1 \mathrm{e}^{t\left(-2+\sqrt{5}\right)} + c_2 \mathrm{e}^{t\left(-2-\sqrt{5}\right)}$$

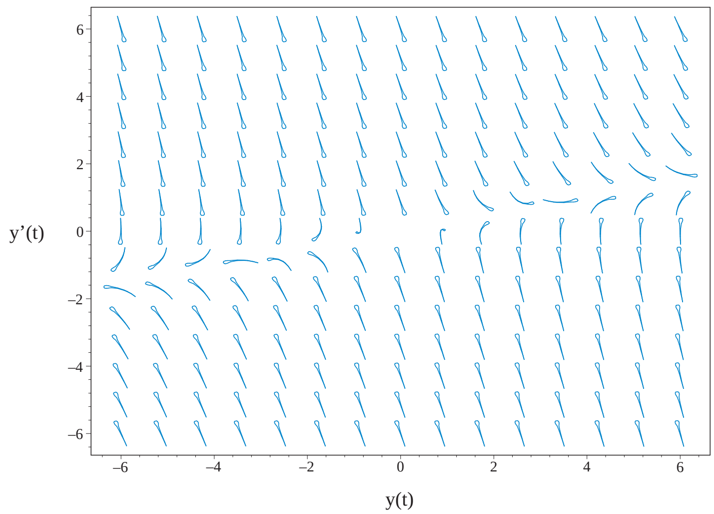

Figure 2.91: Slope field plot
$y'' + 4y' - y = \sin(t)$

Solved as second order ode using Kovacic algorithm

Time used: 0.135 (sec)

Writing the ode as

$$y'' + 4y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 5z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.57: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 5$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\sqrt{5}\,t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\ &= z_1 e^{-2t} \\ &= z_1\left(\mathrm{e}^{-2t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t\left(2+\sqrt{5}\right)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{\left(y_1\right)^2}\,dt \\ &= y_1 \int \frac{e^{-4t}}{\left(y_1\right)^2}\,dt \\ &= y_1 \left(\frac{\sqrt{5}\,\mathrm{e}^{-4t}\mathrm{e}^{2t\left(2+\sqrt{5}\right)}}{10}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t\left(2+\sqrt{5}\right)}\right) + c_2 \left(\mathrm{e}^{-t\left(2+\sqrt{5}\right)} \left(\frac{\sqrt{5}\,\mathrm{e}^{-4t} \mathrm{e}^{2t\left(2+\sqrt{5}\right)}}{10}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \,\mathrm{e}^{-t\left(2+\sqrt{5}\right)} + \frac{c_2 \sqrt{5}\,\mathrm{e}^{t\left(-2+\sqrt{5}\right)}}{10}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(t), \sin(t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\sqrt{5}\,\mathrm{e}^{t\left(-2+\sqrt{5}\right)}}{10}, \mathrm{e}^{-t\left(2+\sqrt{5}\right)}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(t) + A_2 \sin(t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \cos(t) - 2A_2 \sin(t) - 4A_1 \sin(t) + 4A_2 \cos(t) = \sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = -\frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(t)}{5} - \frac{\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \,\mathrm{e}^{-t\left(2+\sqrt{5}\right)} + \frac{c_2\sqrt{5}\,\mathrm{e}^{t\left(-2+\sqrt{5}\right)}}{10}\right) + \left(-\frac{\cos(t)}{5} - \frac{\sin(t)}{10}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{-t\left(2+\sqrt{5}\right)} + \frac{c_2\sqrt{5}\,\mathrm{e}^{t\left(-2+\sqrt{5}\right)}}{10} - \frac{\cos(t)}{5} - \frac{\sin(t)}{10}$$

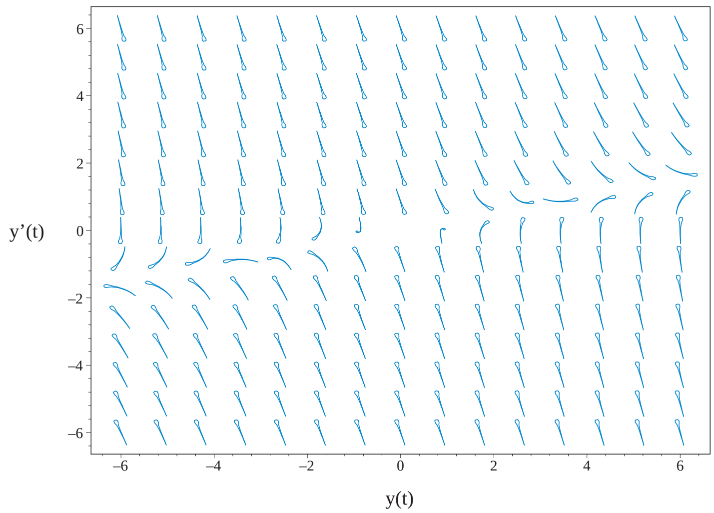

Figure 2.92: Slope field plot
$y'' + 4y' - y = \sin(t)$

Solved as second order ode adjoint method

Time used: 1.406 (sec)

In normal form the ode

$$y'' + 4y' - y = \sin(t) \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= 4 \\ q(t) &= -1 \\ r(t) &= \sin(t) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (4\xi(t))' + (-\xi(t)) &= 0 \\ \xi''(t) - 4\xi'(t) - \xi(t) &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = -4, C = -1$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} - 4\lambda\, \mathrm{e}^{t\lambda} - \mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4\lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(-1)}\\ &= 2 \pm \sqrt{5}\end{aligned}$$

Hence

$$\lambda_1 = 2 + \sqrt{5}$$
$$\lambda_2 = 2 - \sqrt{5}$$

Which simplifies to

$$\lambda_1 = 2 + \sqrt{5}$$
$$\lambda_2 = 2 - \sqrt{5}$$

Since roots are real and distinct, then the solution is

$$\xi = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$\xi = c_1 e^{\left(2+\sqrt{5}\right)t} + c_2 e^{\left(2-\sqrt{5}\right)t}$$

Or

$$\xi = c_1\, \mathrm{e}^{t\left(2+\sqrt{5}\right)} + c_2\, \mathrm{e}^{t\left(2-\sqrt{5}\right)}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(t)\,y' - y\xi'(t) + \xi(t)\,p(t)\,y = \int \xi(t)\,r(t)\,dt$$
$$y' + y\left(p(t) - \frac{\xi'(t)}{\xi(t)}\right) = \frac{\int \xi(t)\,r(t)\,dt}{\xi(t)}$$

Or

$$y' + y\left(4 - \frac{c_1(2+\sqrt{5})\,\mathrm{e}^{t(2+\sqrt{5})} + c_2(2-\sqrt{5})\,\mathrm{e}^{t(2-\sqrt{5})}}{c_1\,\mathrm{e}^{t(2+\sqrt{5})} + c_2\,\mathrm{e}^{t(2-\sqrt{5})}}\right) = \frac{c_1\left(-\frac{\mathrm{e}^{t(2+\sqrt{5})}\cos(t)}{(2+\sqrt{5})^2+1} + \frac{(2+\sqrt{5})\mathrm{e}^{t(2+\sqrt{5})}\sin(t)}{(2+\sqrt{5})^2+1}\right) + c}{c_1\,\mathrm{e}^{t(2+\sqrt{5})} + c}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(t)y = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = -\frac{-c_2(2+\sqrt{5})\,\mathrm{e}^{-t(-2+\sqrt{5})} + c_1\,\mathrm{e}^{t(2+\sqrt{5})}(-2+\sqrt{5})}{c_1\,\mathrm{e}^{t(2+\sqrt{5})} + c_2\,\mathrm{e}^{-t(-2+\sqrt{5})}}$$
$$p(t) = \frac{-2\left(\left(\cos(t) + \frac{\sin(t)}{2}\right)\sqrt{5} + \frac{5\cos(t)}{2}\right)c_2\,\mathrm{e}^{-t(-2+\sqrt{5})} + 2c_1\left(\left(\cos(t) + \frac{\sin(t)}{2}\right)\sqrt{5} - \frac{5\cos(t)}{2}\right)\mathrm{e}^{t(2+\sqrt{5})}}{10c_1\,\mathrm{e}^{t(2+\sqrt{5})} + 10c_2\,\mathrm{e}^{-t(-2+\sqrt{5})}}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt}\\ &= \mathrm{e}^{\int -\frac{-c_2(2+\sqrt{5})\mathrm{e}^{-t(-2+\sqrt{5})} + c_1\,\mathrm{e}^{t(2+\sqrt{5})}(-2+\sqrt{5})}{c_1\,\mathrm{e}^{t(2+\sqrt{5})} + c_2\,\mathrm{e}^{-t(-2+\sqrt{5})}}dt}\\ &= \frac{\mathrm{e}^{t(2+\sqrt{5})}}{\mathrm{e}^{2\sqrt{5}\,t}c_1 + c_2}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu y)$$
$$= (\mu)\left(\frac{-2\left(\left(\cos(t) + \frac{\sin(t)}{2}\right)\sqrt{5} + \frac{5\cos(t)}{2}\right)c_2\,\mathrm{e}^{-t(-2+\sqrt{5})} + 2c_1\left(\left(\cos(t) + \frac{\sin(t)}{2}\right)\sqrt{5} - \frac{5\cos(t)}{2}\right)\mathrm{e}^{t(2+\sqrt{5})}}{10c_1\,\mathrm{e}^{t(2+\sqrt{5})} + 10c_2\,\mathrm{e}^{-t(-2+\sqrt{5})}}\right)$$

$$\frac{d}{dt}\left(\frac{y\,e^{t\left(2+\sqrt{5}\right)}}{e^{2\sqrt{5}\,t}c_1+c_2}\right)$$
$$=\left(\frac{e^{t\left(2+\sqrt{5}\right)}}{e^{2\sqrt{5}\,t}c_1+c_2}\right)\left(\frac{-2\left(\left(\cos(t)+\frac{\sin(t)}{2}\right)\sqrt{5}+\frac{5\cos(t)}{2}\right)c_2\,e^{-t\left(-2+\sqrt{5}\right)}+2c_1\left(\left(\cos(t)+\frac{\sin(t)}{2}\right)\sqrt{5}-\frac{5\cos(t}{2}\right.}{10c_1\,e^{t\left(2+\sqrt{5}\right)}+10c_2\,e^{-t\left(-2+\sqrt{5}\right)}}\right.$$

$$d\left(\frac{y\,e^{t\left(2+\sqrt{5}\right)}}{e^{2\sqrt{5}\,t}c_1+c_2}\right)$$
$$=\left(\frac{\left(-2\left(\left(\cos(t)+\frac{\sin(t)}{2}\right)\sqrt{5}+\frac{5\cos(t)}{2}\right)c_2\,e^{-t\left(-2+\sqrt{5}\right)}+2c_1\left(\left(\cos(t)+\frac{\sin(t)}{2}\right)\sqrt{5}-\frac{5\cos(t)}{2}\right)e^{t\left(2+\sqrt{5}\right)}\right)e^{t(}}{\left(10c_1\,e^{t\left(2+\sqrt{5}\right)}+10c_2\,e^{-t\left(-2+\sqrt{5}\right)}\right)\left(e^{2\sqrt{5}\,t}c_1+c_2\right)}\right.$$

Integrating gives

$$\frac{y\,e^{t\left(2+\sqrt{5}\right)}}{e^{2\sqrt{5}\,t}c_1+c_2}=\int\frac{\left(-2\left(\left(\cos(t)+\frac{\sin(t)}{2}\right)\sqrt{5}+\frac{5\cos(t)}{2}\right)c_2\,e^{-t\left(-2+\sqrt{5}\right)}+2c_1\left(\left(\cos(t)+\frac{\sin(t)}{2}\right)\sqrt{5}-\frac{5\cos(t)}{2}\right)\right.}{\left(10c_1\,e^{t\left(2+\sqrt{5}\right)}+10c_2\,e^{-t\left(-2+\sqrt{5}\right)}\right)\left(e^{2\sqrt{5}\,t}c_1+c_2\right)}$$
$$=\frac{\left(i\sqrt{5}-2\sqrt{5}\right)\sqrt{5}\,e^{t\left(\sqrt{5}+2+i\right)}}{100\,e^{2\sqrt{5}\,t}c_1+100c_2}-\frac{\left(i\sqrt{5}+2\sqrt{5}\right)\sqrt{5}\,e^{t\left(\sqrt{5}+2-i\right)}}{100\left(e^{2\sqrt{5}\,t}c_1+c_2\right)}+c_3$$

Dividing throughout by the integrating factor $\frac{e^{t\left(2+\sqrt{5}\right)}}{e^{2\sqrt{5}\,t}c_1+c_2}$ gives the final solution

$$y=-\frac{\left((2+i)\,e^{t\left(\sqrt{5}+2-i\right)}+(2-i)\,e^{t\left(\sqrt{5}+2+i\right)}-20\,e^{2\sqrt{5}\,t}c_1c_3-20c_2c_3\right)e^{-t\left(2+\sqrt{5}\right)}}{20}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y=-\frac{\left((2+i)\,e^{t\left(\sqrt{5}+2-i\right)}+(2-i)\,e^{t\left(\sqrt{5}+2+i\right)}-20\,e^{2\sqrt{5}\,t}c_1c_3-20c_2c_3\right)e^{-t\left(2+\sqrt{5}\right)}}{20}$$

The constants can be merged to give

$$y=-\frac{\left((2+i)\,e^{t\left(\sqrt{5}+2-i\right)}+(2-i)\,e^{t\left(\sqrt{5}+2+i\right)}-20\,e^{2\sqrt{5}\,t}c_1-20c_2\right)e^{-t\left(2+\sqrt{5}\right)}}{20}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\left((2+i)\,e^{t\left(\sqrt{5}+2-i\right)} + (2-i)\,e^{t\left(\sqrt{5}+2+i\right)} - 20\,e^{2\sqrt{5}\,t}c_1 - 20c_2\right)e^{-t\left(2+\sqrt{5}\right)}}{20}$$

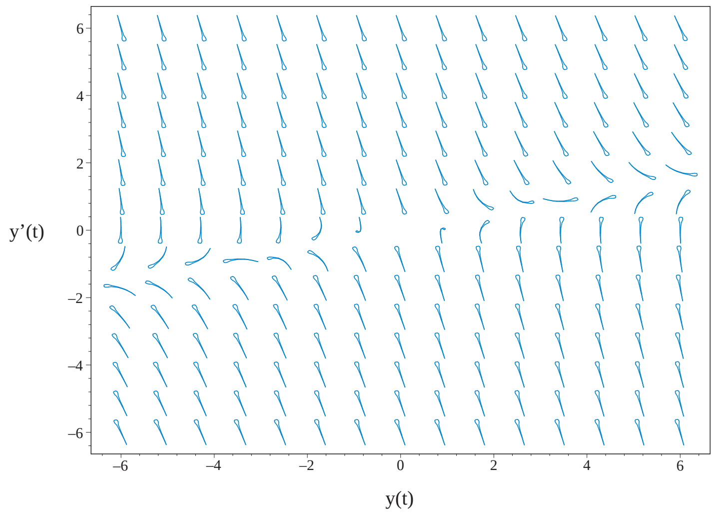

Figure 2.93: Slope field plot
$y'' + 4y' - y = \sin(t)$

Maple step by step solution

Let's solve
$y'' + 4y' - y = \sin(t)$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 4r - 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{(-4)\pm\left(\sqrt{20}\right)}{2}$
- Roots of the characteristic polynomial
  $r = \left(-2-\sqrt{5}, -2+\sqrt{5}\right)$
- 1st solution of the homogeneous ODE

$y_1(t) = e^{t\left(-2-\sqrt{5}\right)}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{t\left(-2+\sqrt{5}\right)}$

- General solution of the ODE

  $y = C1 y_1(t) + C2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = C1\, e^{t\left(-2-\sqrt{5}\right)} + e^{t\left(-2+\sqrt{5}\right)} C2 + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \sin(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{t\left(-2-\sqrt{5}\right)} & e^{t\left(-2+\sqrt{5}\right)} \\ \left(-2-\sqrt{5}\right) e^{t\left(-2-\sqrt{5}\right)} & \left(-2+\sqrt{5}\right) e^{t\left(-2+\sqrt{5}\right)} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\sqrt{5}\, e^{-4t}$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\sqrt{5}\left(e^{-t\left(2+\sqrt{5}\right)}\left(\int \sin(t) e^{t\left(2+\sqrt{5}\right)} dt\right) - e^{t\left(-2+\sqrt{5}\right)}\left(\int \sin(t) e^{-t\left(-2+\sqrt{5}\right)} dt\right)\right)}{10}$

  ◦ Compute integrals

  $y_p(t) = -\frac{\cos(t)}{5} - \frac{\sin(t)}{10}$

- Substitute particular solution into general solution to ODE

  $y = C1\, e^{t\left(-2-\sqrt{5}\right)} + e^{t\left(-2+\sqrt{5}\right)} C2 - \frac{\cos(t)}{5} - \frac{\sin(t)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
```

```
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 34

```
dsolve(diff(diff(y(t),t),t)+4*diff(y(t),t)-y(t) = sin(t),y(t),singsol=all)
```

$$y = \mathrm{e}^{\left(-2+\sqrt{5}\right)t} c_2 + \mathrm{e}^{-\left(2+\sqrt{5}\right)t} c_1 - \frac{\cos(t)}{5} - \frac{\sin(t)}{10}$$

Mathematica DSolve solution

Solving time : 0.25 (sec)
Leaf size : 47

```
DSolve[{D[y[t],{t,2}]+4*D[y[t],t]-y[t]==Sin[t],{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to -\frac{\sin(t)}{10} - \frac{\cos(t)}{5} + e^{-\left(\left(2+\sqrt{5}\right)t\right)} \left(c_2 e^{2\sqrt{5}t} + c_1\right)$$

2.8.8 Problem 8

Internal problem ID [18587]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 8
Date solved : `Tuesday, January 28, 2025 at 12:03:15 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + 3y = \sin(x) + \frac{\sin(3x)}{3}$$

Solved as second order linear constant coeff ode

Time used: 0.177 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 3, f(x) = \sin(x) + \frac{\sin(3x)}{3}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + 3\,e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(3)}\\ &= \pm i\sqrt{3}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{3}\\ \lambda_2 &= -i\sqrt{3}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{3}\\ \lambda_2 &= -i\sqrt{3}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{3}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)\right)$$

Or

$$y = c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x) + \frac{\sin(3x)}{3}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{3}\, x\right), \sin\left(\sqrt{3}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(3x) + A_4 \sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) + 2A_2 \sin(x) - 6A_3 \cos(3x) - 6A_4 \sin(3x) = \sin(x) + \frac{\sin(3x)}{3}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}, A_3 = 0, A_4 = -\frac{1}{18}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)}{2} - \frac{\sin(3x)}{18}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)\right) + \left(\frac{\sin(x)}{2} - \frac{\sin(3x)}{18}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{\sin(x)}{2} - \frac{\sin(3x)}{18} + c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)
$$

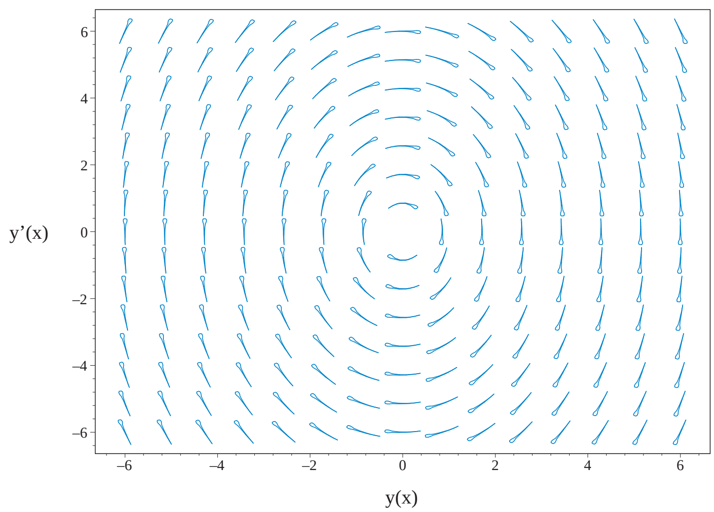

Figure 2.94: Slope field plot
$y'' + 3y = \sin(x) + \frac{\sin(3x)}{3}$

Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$
y'' + 3y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.59: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{3}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos\left(\sqrt{3}\, x\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{3}\, x\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos\left(\sqrt{3}\, x\right) \int \frac{1}{\cos\left(\sqrt{3}\, x\right)^2}\,dx \\ &= \cos\left(\sqrt{3}\, x\right)\left(\frac{\sqrt{3}\, \tan\left(\sqrt{3}\, x\right)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos\left(\sqrt{3}\, x\right)\right) + c_2\left(\cos\left(\sqrt{3}\, x\right)\left(\frac{\sqrt{3}\, \tan\left(\sqrt{3}\, x\right)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(\sqrt{3}\, x\right) + \frac{c_2\sqrt{3}\, \sin\left(\sqrt{3}\, x\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x) + \frac{\sin(3x)}{3}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{3}\, \sin\left(\sqrt{3}\, x\right)}{3}, \cos\left(\sqrt{3}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(3x) + A_4 \sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) + 2A_2 \sin(x) - 6A_3 \cos(3x) - 6A_4 \sin(3x) = \sin(x) + \frac{\sin(3x)}{3}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}, A_3 = 0, A_4 = -\frac{1}{18}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)}{2} - \frac{\sin(3x)}{18}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{3}\, x\right) + \frac{c_2\sqrt{3}\, \sin\left(\sqrt{3}\, x\right)}{3}\right) + \left(\frac{\sin(x)}{2} - \frac{\sin(3x)}{18}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(\sqrt{3}\, x\right) + \frac{c_2\sqrt{3}\, \sin\left(\sqrt{3}\, x\right)}{3} + \frac{\sin(x)}{2} - \frac{\sin(3x)}{18}$$

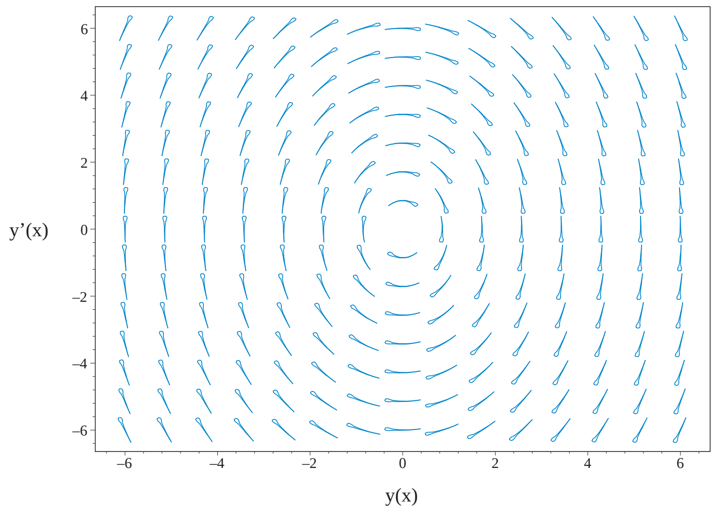

Figure 2.95: Slope field plot
$y'' + 3y = \sin(x) + \frac{\sin(3x)}{3}$

Solved as second order ode adjoint method

Time used: 3.651 (sec)

In normal form the ode

$$y'' + 3y = \sin(x) + \frac{\sin(3x)}{3} \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= 0 \\
q(x) &= 3 \\
r(x) &= \sin(x) + \frac{\sin(3x)}{3}
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (0)' + (3\xi(x)) &= 0 \\
\xi''(x) + 3\xi(x) &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 3$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + 3\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(3)}\\ &= \pm i\sqrt{3}\end{aligned}$$

Hence

$$\lambda_1 = +i\sqrt{3}$$
$$\lambda_2 = -i\sqrt{3}$$

Which simplifies to

$$\lambda_1 = i\sqrt{3}$$
$$\lambda_2 = -i\sqrt{3}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{3}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0\left(c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)\right)$$

Or

$$\xi = c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' - \frac{y\left(-c_1\sqrt{3}\,\sin\left(\sqrt{3}\,x\right) + c_2\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)\right)}{c_1\cos\left(\sqrt{3}\,x\right) + c_2\sin\left(\sqrt{3}\,x\right)} = \frac{\frac{c_1\cos\left(x\left(-3+\sqrt{3}\right)\right)}{-18+6\sqrt{3}} - \frac{c_1\cos\left(x\left(1+\sqrt{3}\right)\right)}{2\left(1+\sqrt{3}\right)} - \frac{c_1\cos\left(x\left(3+\sqrt{3}\right)\right)}{6\left(3+\sqrt{3}\right)} + \frac{c_1 \cos}{}}{c_1 \cos}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{\sqrt{3}\left(c_1\sin\left(\sqrt{3}\,x\right) - c_2\cos\left(\sqrt{3}\,x\right)\right)}{c_1\cos\left(\sqrt{3}\,x\right) + c_2\sin\left(\sqrt{3}\,x\right)}$$
$$p(x) = \frac{\left(2\left(\cos(x)^2 - \frac{5}{2}\right)\sin(x)\,c_2\sqrt{3} - 6c_1\cos(x)^3 + 9c_1\cos(x)\right)\cos\left(\sqrt{3}\,x\right) - 2\sin\left(\sqrt{3}\,x\right)\left(c_1\sin(x)\left(\cos\right.\right.}{9c_2\sin\left(\sqrt{3}\,x\right) + 9c_1\cos\left(\sqrt{3}\,x\right)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= \mathrm{e}^{\int \frac{\sqrt{3}\left(c_1\sin\left(\sqrt{3}\,x\right) - c_2\cos\left(\sqrt{3}\,x\right)\right)}{c_1\cos\left(\sqrt{3}\,x\right) + c_2\sin\left(\sqrt{3}\,x\right)}dx}\\ &= \frac{1}{c_1\cos\left(\sqrt{3}\,x\right) + c_2\sin\left(\sqrt{3}\,x\right)}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)$$
$$= (\mu)\left(\frac{\left(2\left(\cos(x)^2 - \frac{5}{2}\right)\sin(x)\,c_2\sqrt{3} - 6c_1\cos(x)^3 + 9c_1\cos(x)\right)\cos\left(\sqrt{3}\,x\right) - 2\sin\left(\sqrt{3}\,x\right)\left(c_1\sin(x)\left(\cos\right.\right.}{9c_2\sin\left(\sqrt{3}\,x\right) + 9c_1\cos\left(\sqrt{3}\,x\right)}\right.$$

$$\frac{d}{dx}\left(\frac{y}{c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)}\right)$$
$$=\left(\frac{1}{c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)}\right)\left(\frac{\left(2\left(\cos(x)^2-\frac{5}{2}\right)\sin(x)\,c_2\sqrt{3}-6c_1\cos(x)^3+9c_1\cos(x)\right)\cos\left(\sqrt{3}\,x\right)}{9c_2\sin\left(\sqrt{3}\,x\right)}\right.$$

$$d\left(\frac{y}{c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)}\right)$$
$$=\left(\frac{\left(2\left(\cos(x)^2-\frac{5}{2}\right)\sin(x)\,c_2\sqrt{3}-6c_1\cos(x)^3+9c_1\cos(x)\right)\cos\left(\sqrt{3}\,x\right)-2\sin\left(\sqrt{3}\,x\right)\left(c_1\sin(x)\left(\cos(x\right.\right.}{\left(9c_2\sin\left(\sqrt{3}\,x\right)+9c_1\cos\left(\sqrt{3}\,x\right)\right)\left(c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)\right)}\right.$$

Integrating gives

$$\frac{y}{c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)}=\int\frac{\left(2\left(\cos(x)^2-\frac{5}{2}\right)\sin(x)\,c_2\sqrt{3}-6c_1\cos(x)^3+9c_1\cos(x)\right)\cos\left(\sqrt{3}\,x\right)-}{\left(9c_2\sin\left(\sqrt{3}\,x\right)+9c_1\cos\left(\sqrt{3}\,x\right)\right)}$$
$$=\frac{i\left(e^{ix\left(3+\sqrt{3}\right)}-9\,e^{ix\left(1+\sqrt{3}\right)}+9\,e^{ix\left(\sqrt{3}-1\right)}-e^{ix\left(-3+\sqrt{3}\right)}\right)}{-18ic_2\,e^{2i\sqrt{3}\,x}+18c_1\,e^{2i\sqrt{3}\,x}+18ic_2+18c_1}+c_3$$

Dividing throughout by the integrating factor $\frac{1}{c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)}$ gives the final solution

$$y=\frac{\left(18ie^{2i\sqrt{3}\,x}c_1c_3+18ic_1c_3+18\,e^{2i\sqrt{3}\,x}c_2c_3-18c_2c_3-e^{ix\left(3+\sqrt{3}\right)}+9\,e^{ix\left(1+\sqrt{3}\right)}-9\,e^{ix\left(\sqrt{3}-1\right)}+e^{ix\left(-3+\sqrt{3}\right.}\right.}{\left(18ic_1+18c_2\right)e^{2i\sqrt{3}\,x}+18ic_1-18c_2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y$$
$$=\frac{\left(18ie^{2i\sqrt{3}\,x}c_1c_3+18ic_1c_3+18\,e^{2i\sqrt{3}\,x}c_2c_3-18c_2c_3-e^{ix\left(3+\sqrt{3}\right)}+9\,e^{ix\left(1+\sqrt{3}\right)}-9\,e^{ix\left(\sqrt{3}-1\right)}+e^{ix\left(-3+\sqrt{3}\right)}\right.}{\left(18ic_1+18c_2\right)e^{2i\sqrt{3}\,x}+18ic_1-18c_2}$$

The constants can be merged to give

$$y$$
$$=\frac{\left(18ic_1\,e^{2i\sqrt{3}\,x}+18ic_1+18c_2\,e^{2i\sqrt{3}\,x}-18c_2-e^{ix\left(3+\sqrt{3}\right)}+9\,e^{ix\left(1+\sqrt{3}\right)}-9\,e^{ix\left(\sqrt{3}-1\right)}+e^{ix\left(-3+\sqrt{3}\right)}\right)\left(c_1\cos\right.}{\left(18ic_1+18c_2\right)e^{2i\sqrt{3}\,x}+18ic_1-18c_2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(18ic_1\,\mathrm{e}^{2i\sqrt{3}\,x} + 18ic_1 + 18c_2\,\mathrm{e}^{2i\sqrt{3}\,x} - 18c_2 - \mathrm{e}^{ix\left(3+\sqrt{3}\right)} + 9\,\mathrm{e}^{ix\left(1+\sqrt{3}\right)} - 9\,\mathrm{e}^{ix\left(\sqrt{3}-1\right)} + \mathrm{e}^{ix\left(-3+\sqrt{3}\right)}\right)\left(c_1 \cos\right.}{\left(18ic_1 + 18c_2\right)\mathrm{e}^{2i\sqrt{3}\,x} + 18ic_1 - 18c_2}$$

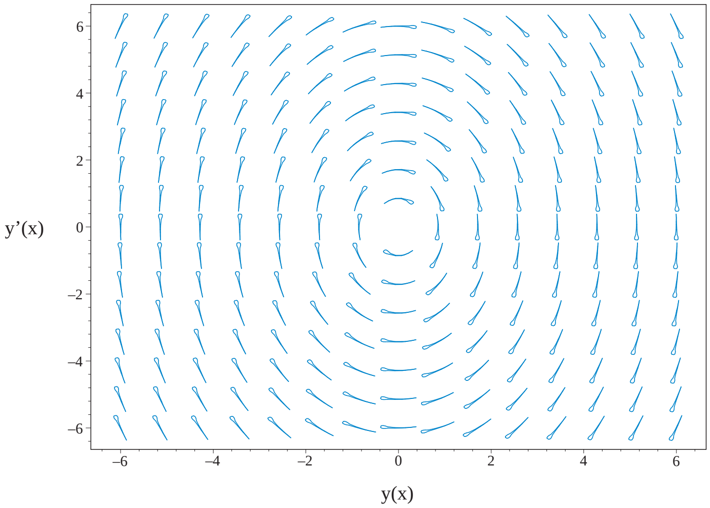

Figure 2.96: Slope field plot
$y'' + 3y = \sin(x) + \frac{\sin(3x)}{3}$

Maple step by step solution

Let's solve
$y'' + 3y = \sin(x) + \frac{\sin(3x)}{3}$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 3 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-12})}{2}$
- Roots of the characteristic polynomial
  $r = \left(-\mathrm{I}\sqrt{3}, \mathrm{I}\sqrt{3}\right)$
- 1st solution of the homogeneous ODE
  $y_1(x) = \cos\left(\sqrt{3}\,x\right)$

- 2nd solution of the homogeneous ODE
  $y_2(x) = \sin\left(\sqrt{3}\, x\right)$
- General solution of the ODE
  $y = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y = C1 \cos\left(\sqrt{3}\, x\right) + C2 \sin\left(\sqrt{3}\, x\right) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x) + \frac{\sin(3x)}{3}\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos\left(\sqrt{3}\, x\right) & \sin\left(\sqrt{3}\, x\right) \\ -\sqrt{3}\sin\left(\sqrt{3}\, x\right) & \sqrt{3}\cos\left(\sqrt{3}\, x\right) \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = \sqrt{3}$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -\frac{\sqrt{3}\left(\cos\left(\sqrt{3}\, x\right)\left(\int \sin\left(\sqrt{3}\, x\right)(\sin(3x)+3\sin(x))dx\right) - \sin\left(\sqrt{3}\, x\right)\left(\int \cos\left(\sqrt{3}\, x\right)(\sin(3x)+3\sin(x))dx\right)\right)}{9}$
  - Compute integrals
    $y_p(x) = \frac{\sin(x)}{2} - \frac{\sin(3x)}{18}$
- Substitute particular solution into general solution to ODE
  $y = \frac{\sin(x)}{2} - \frac{\sin(3x)}{18} + C1 \cos\left(\sqrt{3}\, x\right) + C2 \sin\left(\sqrt{3}\, x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 31

```
dsolve(diff(diff(y(x),x),x)+3*y(x) = sin(x)+1/3*sin(3*x),y(x),singsol=all)
```

$$y(x) = \sin\left(\sqrt{3}\,x\right) c_2 + \cos\left(\sqrt{3}\,x\right) c_1 + \frac{\sin(x)}{2} - \frac{\sin(3x)}{18}$$

Mathematica DSolve solution

Solving time : `0.559` (sec)
Leaf size : 42

```
DSolve[{D[y[x],{x,2}]+3*y[x]==Sin[x]+1/3*Sin[3*x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sin(x)}{2} - \frac{1}{18}\sin(3x) + c_1 \cos\left(\sqrt{3}x\right) + c_2 \sin\left(\sqrt{3}x\right)$$

2.8.9 Problem 10

Internal problem ID [18588]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 10
Date solved : `Tuesday, January 28, 2025 at 12:03:20 PM`
CAS classification : `[[_linear, 'class A']]`

Solve

$$5x' + x = \sin(3t)$$

Solved as first order linear ode

Time used: 0.108 (sec)

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = \frac{1}{5}$$
$$p(t) = \frac{\sin(3t)}{5}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt}\\ &= e^{\int \frac{1}{5}dt}\\ &= e^{\frac{t}{5}}\end{aligned}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = \mu p$$
$$\frac{d}{dt}(\mu x) = (\mu)\left(\frac{\sin(3t)}{5}\right)$$
$$\frac{d}{dt}\left(x\,e^{\frac{t}{5}}\right) = \left(e^{\frac{t}{5}}\right)\left(\frac{\sin(3t)}{5}\right)$$
$$d\left(x\,e^{\frac{t}{5}}\right) = \left(\frac{\sin(3t)\,e^{\frac{t}{5}}}{5}\right) dt$$

Integrating gives

$$x\,e^{\frac{t}{5}} = \int \frac{\sin(3t)\,e^{\frac{t}{5}}}{5}\,dt$$
$$= -\frac{15\cos(3t)\,e^{\frac{t}{5}}}{226} + \frac{\sin(3t)\,e^{\frac{t}{5}}}{226} + c_1$$

Dividing throughout by the integrating factor $e^{\frac{t}{5}}$ gives the final solution

$$x = \frac{\sin(3t)}{226} - \frac{15\cos(3t)}{226} + c_1\,e^{-\frac{t}{5}}$$

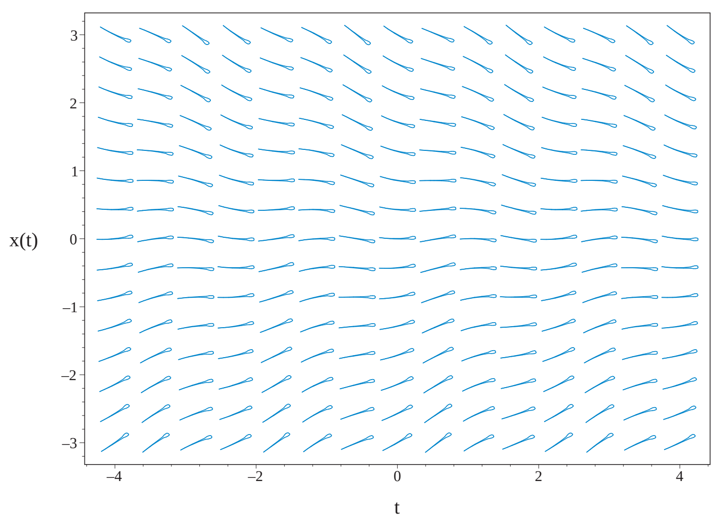

Figure 2.97: Slope field plot
$5x' + x = \sin(3t)$

Summary of solutions found

$$x = \frac{\sin(3t)}{226} - \frac{15\cos(3t)}{226} + c_1\,e^{-\frac{t}{5}}$$

Solved as first order Exact ode

Time used: 0.105 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}(5)\,\mathrm{d}x&=(-x+\sin(3t))\,\mathrm{d}t\\ (x-\sin(3t))\,\mathrm{d}t+(5)\,\mathrm{d}x&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=x-\sin(3t)\\ N(t,x)&=5\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(x - \sin(3t)) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(5) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\ &= \frac{1}{5}((1) - (0)) \\ &= \frac{1}{5}\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \frac{1}{5} \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{t}{5}} \\ &= \mathrm{e}^{\frac{t}{5}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{\frac{t}{5}}(x - \sin(3t)) \\ &= (x - \sin(3t))\,\mathrm{e}^{\frac{t}{5}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{\frac{t}{5}}(5) \\ &= 5\,\mathrm{e}^{\frac{t}{5}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left((x-\sin(3t))\,\mathrm{e}^{\frac{t}{5}}\right) + \left(5\,\mathrm{e}^{\frac{t}{5}}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{N}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 5\,\mathrm{e}^{\frac{t}{5}}\,\mathrm{d}x \\ \phi &= 5x\,\mathrm{e}^{\frac{t}{5}} + f(t)\end{aligned} \tag{3}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$\frac{\partial \phi}{\partial t} = x\,\mathrm{e}^{\frac{t}{5}} + f'(t) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial t} = (x-\sin(3t))\,\mathrm{e}^{\frac{t}{5}}$. Therefore equation (4) becomes

$$(x-\sin(3t))\,\mathrm{e}^{\frac{t}{5}} = x\,\mathrm{e}^{\frac{t}{5}} + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$f'(t) = -\sin(3t)\,\mathrm{e}^{\frac{t}{5}}$$

Integrating the above w.r.t t gives

$$\int f'(t)\,\mathrm{d}t = \int \left(-\sin(3t)\,\mathrm{e}^{\frac{t}{5}}\right)\mathrm{d}t$$

$$f(t) = \frac{75\cos(3t)\,\mathrm{e}^{\frac{t}{5}}}{226} - \frac{5\sin(3t)\,\mathrm{e}^{\frac{t}{5}}}{226} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(t)$ into equation (3) gives ϕ

$$\phi = 5x\,\mathrm{e}^{\frac{t}{5}} + \frac{75\cos(3t)\,\mathrm{e}^{\frac{t}{5}}}{226} - \frac{5\sin(3t)\,\mathrm{e}^{\frac{t}{5}}}{226} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = 5x\,\mathrm{e}^{\frac{t}{5}} + \frac{75\cos(3t)\,\mathrm{e}^{\frac{t}{5}}}{226} - \frac{5\sin(3t)\,\mathrm{e}^{\frac{t}{5}}}{226}$$

Solving for x gives

$$x = \frac{\left(5\sin(3t)\,\mathrm{e}^{\frac{t}{5}} - 75\cos(3t)\,\mathrm{e}^{\frac{t}{5}} + 226c_1\right)\mathrm{e}^{-\frac{t}{5}}}{1130}$$

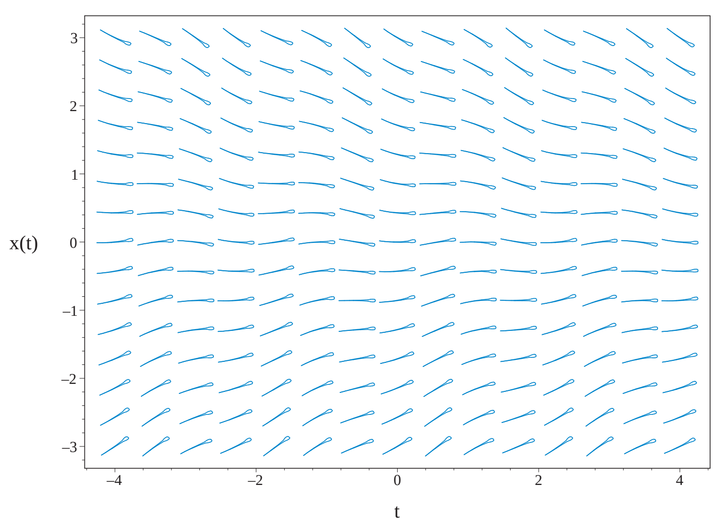

Figure 2.98: Slope field plot
$5x' + x = \sin(3t)$

Summary of solutions found

$$x = \frac{\left(5 \sin \left(3 t\right) {\mathrm e}^{\frac{t}{5}}-75 \cos \left(3 t\right) {\mathrm e}^{\frac{t}{5}}+226 c_{1}\right) {\mathrm e}^{-\frac{t}{5}}}{1130}$$

Maple step by step solution

Let's solve
$5x'+x=\sin(3t)$

- Highest derivative means the order of the ODE is 1
x'
- Solve for the highest derivative
$x'=-\frac{x}{5}+\frac{\sin(3t)}{5}$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE
$x'+\frac{x}{5}=\frac{\sin(3t)}{5}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
$\mu(t)\left(x'+\frac{x}{5}\right)=\frac{\mu(t)\sin(3t)}{5}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(x\mu(t))$
$\mu(t)\left(x'+\frac{x}{5}\right)=x'\mu(t)+x\mu'(t)$
- Isolate $\mu'(t)$
$\mu'(t)=\frac{\mu(t)}{5}$
- Solve to find the integrating factor
$\mu(t)=\mathrm{e}^{\frac{t}{5}}$
- Integrate both sides with respect to t
$\int\left(\frac{d}{dt}(x\mu(t))\right)dt=\int\frac{\mu(t)\sin(3t)}{5}dt+C1$
- Evaluate the integral on the lhs
$x\mu(t)=\int\frac{\mu(t)\sin(3t)}{5}dt+C1$
- Solve for x
$x=\frac{\int\frac{\mu(t)\sin(3t)}{5}dt+C1}{\mu(t)}$
- Substitute $\mu(t)=\mathrm{e}^{\frac{t}{5}}$
$x=\frac{\int\frac{\sin(3t)\mathrm{e}^{\frac{t}{5}}}{5}dt+C1}{\mathrm{e}^{\frac{t}{5}}}$
- Evaluate the integrals on the rhs

$$x = \frac{-\frac{15\cos(3t)\,\mathrm{e}^{\frac{t}{5}}}{226} + \frac{\sin(3t)\,\mathrm{e}^{\frac{t}{5}}}{226} + C1}{\mathrm{e}^{\frac{t}{5}}}$$

- Simplify

$$x = \frac{\sin(3t)}{226} - \frac{15\cos(3t)}{226} + C1\,\mathrm{e}^{-\frac{t}{5}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 23

```
dsolve(5*diff(x(t),t)+x(t) = sin(3*t),x(t),singsol=all)
```

$$x = -\frac{15\cos(3t)}{226} + \frac{\sin(3t)}{226} + \mathrm{e}^{-\frac{t}{5}} c_1$$

Mathematica DSolve solution

Solving time : `0.08` (sec)
Leaf size : 31

```
DSolve[{5*D[x[t],t]+x[t]==Sin[3*t],{}},x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{226}(\sin(3t) - 15\cos(3t)) + c_1 e^{-t/5}$$

2.8.10 Problem 11

Internal problem ID [18589]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 11
Date solved : `Tuesday, January 28, 2025 at 12:03:22 PM`
CAS classification : `[[_high_order, _missing_y]]`

Solve

$$x'''' - 6x''' + 11x'' - 6x' = \mathrm{e}^{-3t}$$

Solved as higher order constant coeff ode

Time used: 0.063 (sec)

The characteristic equation is

$$\lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$
$$\lambda_4 = 3$$

Therefore the homogeneous solution is

$$x_h(t) = c_1 + \mathrm{e}^t c_2 + \mathrm{e}^{2t} c_3 + \mathrm{e}^{3t} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = 1$$
$$x_2 = e^t$$
$$x_3 = e^{2t}$$
$$x_4 = e^{3t}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE And x_p is a particular solution to the nonhomogeneous ODE. x_h is the solution to

$$x'''' - 6x''' + 11x'' - 6x' = 0$$

Now the particular solution to the given ODE is found

$$x'''' - 6x''' + 11x'' - 6x' = e^{-3t}$$

The particular solution is now found using the method of undetermined coefficients.

Looking at the RHS of the ode, which is

$$e^{-3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^t, e^{2t}, e^{3t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^{-3t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$360A_1 e^{-3t} = e^{-3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{360}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{e^{-3t}}{360}$$

Therefore the general solution is

$$\begin{aligned}x &= x_h + x_p \\ &= \left(c_1 + e^t c_2 + e^{2t} c_3 + e^{3t} c_4\right) + \left(\frac{e^{-3t}}{360}\right)\end{aligned}$$

Maple step by step solution

Let's solve
$x'''' - 6x''' + 11x'' - 6x' = e^{-3t}$

- Highest derivative means the order of the ODE is 4
  x''''
- Characteristic polynomial of homogeneous ODE
  $r^4 - 6r^3 + 11r^2 - 6r = 0$
- Roots of the characteristic polynomial
  $r = [0, 1, 2, 3]$
- Homogeneous solution from $r = 0$
  $x_1(t) = 1$
- Homogeneous solution from $r = 1$
  $x_2(t) = e^t$
- Homogeneous solution from $r = 2$
  $x_3(t) = e^{2t}$
- Homogeneous solution from $r = 3$
  $x_4(t) = e^{3t}$
- General solution of the ODE
  $x = C1x_1(t) + C2x_2(t) + C3x_3(t) + C4x_4(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE
  $x = C1 + e^t C2 + e^{2t} C3 + e^{3t} C4 + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- Define the forcing function of the ODE

  $f(t) = e^{-3t}$
- Form of the particular solution to the ODE where the $u_i(t)$ are to be found

  $$x_p(t) = \sum_{i=1}^{4} u_i(t)\, x_i(t)$$
- Calculate the 1st derivative of $x_p(t)$

  $$x_p'(t) = \sum_{i=1}^{4} \left(u_i'(t)\, x_i(t) + u_i(t)\, x_i'(t)\right)$$
- Choose equation to add to a system of equations in $u_i'(t)$

  $$\sum_{i=1}^{4} u_i'(t)\, x_i(t) = 0$$
- Calculate the 2nd derivative of $x_p(t)$

  $$x_p''(t) = \sum_{i=1}^{4} \left(u_i'(t)\, x_i'(t) + u_i(t)\, x_i''(t)\right)$$
- Choose equation to add to a system of equations in $u_i'(t)$

  $$\sum_{i=1}^{4} u_i'(t)\, x_i'(t) = 0$$
- Calculate the 3rd derivative of $x_p(t)$

  $$x_p'''(t) = \sum_{i=1}^{4} \left(u_i'(t)\, x_i''(t) + u_i(t)\, x_i'''(t)\right)$$
- Choose equation to add to a system of equations in $u_i'(t)$

  $$\sum_{i=1}^{4} u_i'(t)\, x_i''(t) = 0$$
- The ODE is of the following form where the $P_i(t)$ in this situation are the coefficients of the d

  $$x'''' + \left(\sum_{i=0}^{3} P_i(t)\, x^{(i)}\right) = f(t)$$
- Substitute $x_p(t) = \sum_{i=1}^{4} u_i(t)\, x_i(t)$ into the ODE

  $$\left(\sum_{j=0}^{3} P_j(t) \left(\sum_{i=1}^{4} u_i(t)\, x_i^{(j)}(t)\right)\right) + \sum_{i=1}^{4} \left(u_i'(t)\, x_i'''(t) + u_i(t)\, x_i''''(t)\right) = f(t)$$
- Rearrange the ODE

  $$\sum_{i=1}^{4} \left(u_i(t) \cdot \left(\left(\sum_{j=0}^{3} P_j(t)\, x_i^{(j)}(t)\right) + x_i''''(t)\right) + u_i'(t)\, x_i'''(t)\right) = f(t)$$

- Notice that $x_i(t)$ are solutions to the homogeneous equation so the first term in the sum is 0
  $\sum_{i=1}^{4} u_i'(t)\, x_i'''(t) = f(t)$
- We have now made a system of 4 equations in 4 unknowns ($u_i'(t)$)
  $\left[\sum_{i=1}^{4} u_i'(t)\, x_i(t) = 0, \sum_{i=1}^{4} u_i'(t)\, x_i'(t) = 0, \sum_{i=1}^{4} u_i'(t)\, x_i''(t) = 0, \sum_{i=1}^{4} u_i'(t)\, x_i'''(t) = f(t)\right]$
- Convert the system to linear algebra format, notice that the matrix is the wronskian W
  $$\begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \\ x_1'(t) & x_2'(t) & x_3'(t) & x_4'(t) \\ x_1''(t) & x_2''(t) & x_3''(t) & x_4''(t) \\ x_1'''(t) & x_2'''(t) & x_3'''(t) & x_4'''(t) \end{bmatrix} \cdot \begin{bmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \\ u_4'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{bmatrix}$$
- Solve for the varied parameters
  $$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = \int \frac{1}{W} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{bmatrix} dt$$
- Substitute in the homogeneous solutions and forcing function and solve
  $$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{-3t}}{18} \\ -\frac{e^{-3t}}{8\, e^{t}} \\ \frac{e^{-3t}}{10\, e^{2t}} \\ -\frac{e^{-3t}}{36\, e^{3t}} \end{bmatrix}$$
  Find a particular solution $x_p(t)$ of the ODE
  $x_p(t) = \frac{e^{-3t}}{360}$
- Substitute particular solution into general solution to ODE
  $x = C1 + e^{t} C2 + e^{2t} C3 + e^{3t} C4 + \frac{e^{-3t}}{360}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = 6*(diff(diff(
```

```
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 39

```
dsolve(diff(diff(diff(diff(x(t),t),t),t),t)-6*diff(diff(diff(x(t),t),t),t)+11*diff(dif
```

$$x = \frac{\left(c_3\,\mathrm{e}^{6t} + 3c_1\mathrm{e}^{4t} + \frac{3\,\mathrm{e}^{5t}c_2}{2} + 3c_4\,\mathrm{e}^{3t} + \frac{1}{120}\right)\mathrm{e}^{-3t}}{3}$$

Mathematica DSolve solution

Solving time : 0.039 (sec)
Leaf size : 45

```
DSolve[{D[x[t],{t,4}]-6*D[x[t],{t,3}]+11*D[x[t],{t,2}]-6*D[x[t],t]==Exp[-3*t],{}},x[t],t,Inc
```

$$x(t) \to \frac{e^{-3t}}{360} + c_1e^t + \frac{1}{2}c_2e^{2t} + \frac{1}{3}c_3e^{3t} + c_4$$

2.8.11 Problem 14

Internal problem ID [18590]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 14
Date solved : `Tuesday, January 28, 2025 at 12:03:23 PM`
CAS classification : `[[_high_order, _missing_y]]`

Solve

$$x^4y'''' + x^3y''' - 20x^2y'' + 20xy' = 17x^6$$

Solved as higher order Euler type ode

Time used: 0.244 (sec)

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}
\end{aligned}$$

Substituting these back into

$$x^4y'''' + x^3y''' - 20x^2y'' + 20xy' = 17x^6$$

gives

$$\begin{aligned}
&20x\lambda\, x^{\lambda-1} - 20x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
&+ x^4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} = 0
\end{aligned}$$

Which simplifies to

$$20\lambda\, x^\lambda - 20\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$20\lambda - 20\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

Simplifying gives the characteristic equation as

$$\lambda^4 - 5\lambda^3 - 12\lambda^2 + 36\lambda = 0$$

Solving the above gives the following roots

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= 2\\ \lambda_3 &= 6\\ \lambda_4 &= -3\end{aligned}$$

This table summarises the result

root	multiplicity	type of root
0	1	real root
-3	1	real root
2	1	real root
6	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 + \frac{c_2}{x^3} + c_3\, x^2 + c_4\, x^6$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= 1\\ y_2 &= \frac{1}{x^3}\\ y_3 &= x^2\\ y_4 &= x^6\end{aligned}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^4 y'''' + x^3 y''' - 20x^2 y'' + 20xy' = 0$$

Now the particular solution to the given ODE is found

$$x^4 y'''' + x^3 y''' - 20x^2 y'' + 20xy' = 17x^6$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & \frac{1}{x^3} & x^2 & x^6 \\ 0 & -\frac{3}{x^4} & 2x & 6x^5 \\ 0 & \frac{12}{x^5} & 2 & 30x^4 \\ 0 & -\frac{60}{x^6} & 0 & 120x^3 \end{bmatrix}$$

$$|W| = -\frac{6480}{x}$$

The determinant simplifies to

$$|W| = -\frac{6480}{x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \frac{1}{x^3} & x^2 & x^6 \\ -\frac{3}{x^4} & 2x & 6x^5 \\ \frac{12}{x^5} & 2 & 30x^4 \end{bmatrix} \\ &= 180x^2 \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} 1 & x^2 & x^6 \\ 0 & 2x & 6x^5 \\ 0 & 2 & 30x^4 \end{bmatrix} \\ &= 48x^5 \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} 1 & \frac{1}{x^3} & x^6 \\ 0 & -\frac{3}{x^4} & 6x^5 \\ 0 & \frac{12}{x^5} & 30x^4 \end{bmatrix} \\ &= -162 \end{aligned}$$

$$\begin{aligned} W_4(x) &= \det \begin{bmatrix} 1 & \frac{1}{x^3} & x^2 \\ 0 & -\frac{3}{x^4} & 2x \\ 0 & \frac{12}{x^5} & 2 \end{bmatrix} \\ &= -\frac{30}{x^4} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3 \int \frac{\left(17x^6\right)\left(180x^2\right)}{\left(x^4\right)\left(-\frac{6480}{x}\right)}\,dx \\ &= -\int \frac{3060x^8}{-6480x^3}\,dx \\ &= -\int \left(-\frac{17x^5}{36}\right)dx \\ &= \frac{17x^6}{216} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(17x^6)(48x^5)}{(x^4)\left(-\frac{6480}{x}\right)}\,dx \\
&= \int \frac{816x^{11}}{-6480x^3}\,dx \\
&= \int \left(-\frac{17x^8}{135}\right)dx \\
&= -\frac{17x^9}{1215}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(17x^6)(-162)}{(x^4)\left(-\frac{6480}{x}\right)}\,dx \\
&= -\int \frac{-2754x^6}{-6480x^3}\,dx \\
&= -\int \left(\frac{17x^3}{40}\right)dx \\
&= -\frac{17x^4}{160}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(17x^6)\left(-\frac{30}{x^4}\right)}{(x^4)\left(-\frac{6480}{x}\right)}\,dx \\
&= \int \frac{-510x^2}{-6480x^3}\,dx \\
&= \int \left(\frac{17}{216x}\right)dx \\
&= \frac{17\ln(x)}{216}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4
$$

Hence

$$y_p = \left(\frac{17x^6}{216}\right) + \left(-\frac{17x^9}{1215}\right)\left(\frac{1}{x^3}\right) + \left(-\frac{17x^4}{160}\right)\left(x^2\right) + \left(\frac{17\ln(x)}{216}\right)\left(x^6\right)$$

Therefore the particular solution is

$$y_p = \frac{17x^6(-19 + 36\ln(x))}{7776}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2}{x^3} + c_3\,x^2 + c_4\,x^6\right) + \left(\frac{17x^6(-19 + 36\ln(x))}{7776}\right) \end{aligned}$$

Maple step by step solution

Let's solve
$x^4y'''' + x^3y''' - 20x^2y'' + 20xy' = 17x^6$

- Highest derivative means the order of the ODE is 4
y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = -(-17*_a^5+(c
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
```

```
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
   Euler equation successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 41

```
dsolve(x^4*diff(diff(diff(diff(y(x),x),x),x),x)+x^3*diff(diff(diff(y(x),x),x),x)-20*x^
```

$$y(x) = \frac{612 \ln(x)\, x^9 + (1296 c_3 - 323)\, x^9 + 3888 c_1 x^5 + 7776 c_4\, x^3 - 2592 c_2}{7776 x^3}$$

Mathematica DSolve solution

Solving time : 0.014 (sec)
Leaf size : 49

```
DSolve[{x^4*D[y[x],{x,4}]+x^3*D[y[x],{x,3}]-20*x^2*D[y[x],{x,2}]+20*x*D[y[x],x]==17*x^6,{}},
```

$$y(x) \to \frac{17}{216} x^6 \log(x) + \left(-\frac{323}{7776} + \frac{c_3}{6}\right) x^6 - \frac{c_1}{3x^3} + \frac{c_2 x^2}{2} + c_4$$

2.8.12 Problem 15

Internal problem ID [18591]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 56. Problems at page 163
Problem number : 15
Date solved : `Tuesday, January 28, 2025 at 12:03:23 PM`
CAS classification : `[[_high_order, _exact, _linear, _nonhomogeneous]]`

Solve

$$t^4x'''' - 2t^3x''' - 20t^2x'' + 12tx' + 16x = \cos(3\ln(t))$$

Solved as higher order Euler type ode

Time used: 0.617 (sec)

This is Euler ODE of higher order. Let $x = t^\lambda$. Hence

$$\begin{aligned} x' &= \lambda\, t^{\lambda-1} \\ x'' &= \lambda(\lambda-1)\, t^{\lambda-2} \\ x''' &= \lambda(\lambda-1)(\lambda-2)\, t^{\lambda-3} \\ x'''' &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)\, t^{\lambda-4} \end{aligned}$$

Substituting these back into

$$t^4x'''' - 2t^3x''' - 20t^2x'' + 12tx' + 16x = \cos(3\ln(t))$$

gives

$$\begin{aligned} &12t\lambda\, t^{\lambda-1} - 20t^2\lambda(\lambda-1)\, t^{\lambda-2} - 2t^3\lambda(\lambda-1)(\lambda-2)\, t^{\lambda-3} \\ &+ t^4\lambda(\lambda-1)(\lambda-2)(\lambda-3)\, t^{\lambda-4} + 16t^\lambda = 0 \end{aligned}$$

Which simplifies to

$$12\lambda\, t^\lambda - 20\lambda(\lambda-1)\, t^\lambda - 2\lambda(\lambda-1)(\lambda-2)\, t^\lambda + \lambda(\lambda-1)(\lambda-2)(\lambda-3)\, t^\lambda + 16t^\lambda = 0$$

And since $t^{\lambda} \neq 0$ then dividing through by t^{λ}, the above becomes

$$12\lambda - 20\lambda(\lambda - 1) - 2\lambda(\lambda - 1)(\lambda - 2) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + 16 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda - 2)(\lambda - 8)(\lambda + 1)^2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 2$$
$$\lambda_2 = 8$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	2	real root
2	1	real root
8	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 t^{\lambda}$ basis solution. Each real root of multiplicty two, generates $c_1 t^{\lambda}$ and $c_2 t^{\lambda} \ln(t)$ basis solutions. Each real root of multiplicty three, generates $c_1 t^{\lambda}$ and $c_2 t^{\lambda} \ln(t)$ and $c_3 t^{\lambda} \ln(t)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $t^{\alpha}(c_1 \cos(\beta \ln(t)) + c_2 \sin(\beta \ln(t)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(t)\, t^{\alpha}(c_1 \cos(\beta \ln(t)) + c_2 \sin(\beta \ln(t)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(t)^2\, t^{\alpha}(c_1 \cos(\beta \ln(t)) + c_2 \sin(\beta \ln$ basis solutions. And so on. Using the above show that the solution is

$$x = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t} + c_3 t^2 + c_4 t^8$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = \frac{1}{t}$$
$$x_2 = \frac{\ln(t)}{t}$$
$$x_3 = t^2$$
$$x_4 = t^8$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous Euler ODE And x_p is a particular solution to the nonhomogeneous Euler ODE. x_h is the solution to

$$t^4 x'''' - 2t^3 x''' - 20t^2 x'' + 12tx' + 16x = 0$$

Now the particular solution to the given ODE is found

$$t^4 x'''' - 2t^3 x''' - 20t^2 x'' + 12tx' + 16x = \cos(3\ln(t))$$

Let the particular solution be

$$x_p = U_1 x_1 + U_2 x_2 + U_3 x_3 + U_4 x_4$$

Where x_i are the basis solutions found above for the homogeneous solution x_h and $U_i(t)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(t) W_i(t)}{a W(t)} \, dt$$

Where $W(t)$ is the Wronskian and $W_i(t)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(t)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(t)$. This is given by

$$W(t) = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1' & x_2' & x_3' & x_4' \\ x_1'' & x_2'' & x_3'' & x_4'' \\ x_1''' & x_2''' & x_3''' & x_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions x_i found above in the Wronskian gives

$$W = \begin{bmatrix} \frac{1}{t} & \frac{\ln(t)}{t} & t^2 & t^8 \\ -\frac{1}{t^2} & \frac{1-\ln(t)}{t^2} & 2t & 8t^7 \\ \frac{2}{t^3} & \frac{-3+2\ln(t)}{t^3} & 2 & 56t^6 \\ -\frac{6}{t^4} & \frac{11-6\ln(t)}{t^4} & 0 & 336t^5 \end{bmatrix}$$

$$|W| = 4374t^2$$

The determinant simplifies to

$$|W| = 4374t^2$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(t) &= \det \begin{bmatrix} \frac{\ln(t)}{t} & t^2 & t^8 \\ \frac{1-\ln(t)}{t^2} & 2t & 8t^7 \\ \frac{-3+2\ln(t)}{t^3} & 2 & 56t^6 \end{bmatrix} \\ &= 18t^6(-4 + 9\ln(t)) \end{aligned}$$

$$\begin{aligned} W_2(t) &= \det \begin{bmatrix} \frac{1}{t} & t^2 & t^8 \\ -\frac{1}{t^2} & 2t & 8t^7 \\ \frac{2}{t^3} & 2 & 56t^6 \end{bmatrix} \\ &= 162t^6 \end{aligned}$$

$$\begin{aligned} W_3(t) &= \det \begin{bmatrix} \frac{1}{t} & \frac{\ln(t)}{t} & t^8 \\ -\frac{1}{t^2} & \frac{1-\ln(t)}{t^2} & 8t^7 \\ \frac{2}{t^3} & \frac{-3+2\ln(t)}{t^3} & 56t^6 \end{bmatrix} \\ &= 81t^3 \end{aligned}$$

$$\begin{aligned} W_4(t) &= \det \begin{bmatrix} \frac{1}{t} & \frac{\ln(t)}{t} & t^2 \\ -\frac{1}{t^2} & \frac{1-\ln(t)}{t^2} & 2t \\ \frac{2}{t^3} & \frac{-3+2\ln(t)}{t^3} & 2 \end{bmatrix} \\ &= \frac{9}{t^3} \end{aligned}$$

Now we are ready to evaluate each $U_i(t)$.

$$
\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(t)W_1(t)}{aW(t)}\, dt \\
&= (-1)^{3} \int \frac{(\cos(3\ln(t)))\,(18t^6(-4+9\ln(t)))}{(t^4)\,(4374t^2)}\, dt \\
&= -\int \frac{18\cos(3\ln(t))\, t^6(-4+9\ln(t))}{4374t^6}\, dt \\
&= -\int \left(\frac{\cos(3\ln(t))\,(-4+9\ln(t))}{243}\right) dt \\
&= -\frac{\left(\frac{8}{25}+\frac{9\ln(t)}{10}\right) t\cos(3\ln(t))}{243} + \frac{\left(\frac{87}{50}-\frac{27\ln(t)}{10}\right) t\sin(3\ln(t))}{243}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(t)W_2(t)}{aW(t)}\, dt \\
&= (-1)^{2} \int \frac{(\cos(3\ln(t)))\,(162t^6)}{(t^4)\,(4374t^2)}\, dt \\
&= \int \frac{162\cos(3\ln(t))\, t^6}{4374t^6}\, dt \\
&= \int \left(\frac{\cos(3\ln(t))}{27}\right) dt \\
&= \frac{\cos(3\ln(t))\, t}{270} + \frac{t\sin(3\ln(t))}{90}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(t)W_3(t)}{aW(t)}\, dt \\
&= (-1)^{1} \int \frac{(\cos(3\ln(t)))\,(81t^3)}{(t^4)\,(4374t^2)}\, dt \\
&= -\int \frac{81\cos(3\ln(t))\, t^3}{4374t^6}\, dt \\
&= -\int \left(\frac{\cos(3\ln(t))}{54t^3}\right) dt \\
&= -\frac{-\frac{1}{351}+\frac{\tan\left(\frac{3\ln(t)}{2}\right)^2}{351}+\frac{\tan\left(\frac{3\ln(t)}{2}\right)}{117}}{\left(1+\tan\left(\frac{3\ln(t)}{2}\right)^2\right) t^2}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(t) W_4(t)}{a W(t)} \, dt \\
&= (-1)^0 \int \frac{(\cos(3\ln(t)))\left(\frac{9}{t^3}\right)}{(t^4)(4374 t^2)} \, dt \\
&= \int \frac{\frac{9\cos(3\ln(t))}{t^3}}{4374 t^6} \, dt \\
&= \int \left(\frac{\cos(3\ln(t))}{486 t^9}\right) dt \\
&= \frac{\left(-\frac{2}{17739} - \frac{i}{23652}\right) t^{3i}}{t^8} + \frac{\left(-\frac{2}{17739} + \frac{i}{23652}\right) t^{-3i}}{t^8}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
x_p = U_1 x_1 + U_2 x_2 + U_3 x_3 + U_4 x_4
$$

Hence

$$
\begin{aligned}
x_p = {} & \left(-\frac{\left(\frac{8}{25} + \frac{9\ln(t)}{10}\right) t \cos(3\ln(t))}{243} + \frac{\left(\frac{87}{50} - \frac{27\ln(t)}{10}\right) t \sin(3\ln(t))}{243}\right)\left(\frac{1}{t}\right) \\
& + \left(\frac{\cos(3\ln(t))\, t}{270} + \frac{t \sin(3\ln(t))}{90}\right)\left(\frac{\ln(t)}{t}\right) \\
& + \left(-\frac{-\frac{1}{351} + \frac{\tan\left(\frac{3\ln(t)}{2}\right)^2}{351} + \frac{\tan\left(\frac{3\ln(t)}{2}\right)}{117}}{\left(1 + \tan\left(\frac{3\ln(t)}{2}\right)^2\right) t^2}\right)(t^2) \\
& + \left(\frac{\left(-\frac{2}{17739} - \frac{i}{23652}\right) t^{3i}}{t^8} + \frac{\left(-\frac{2}{17739} + \frac{i}{23652}\right) t^{-3i}}{t^8}\right)(t^8)
\end{aligned}
$$

Therefore the particular solution is

$$
x_p = \left(\frac{31}{47450} - \frac{141i}{94900}\right) t^{-3i} t^{6i} + \left(\frac{31}{47450} + \frac{141i}{94900}\right) t^{-3i}
$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(\frac{c_1}{t} + \frac{c_2 \ln(t)}{t} + c_3 t^2 + c_4 t^8\right) + \left(\left(\frac{31}{47450} - \frac{141i}{94900}\right) t^{-3i} t^{6i} + \left(\frac{31}{47450} + \frac{141i}{94900}\right) t^{-3i}\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$t^4x'''' - 2t^3x''' - 20t^2x'' + 12x't + 16x = \cos(3\ln(t))$

- Highest derivative means the order of the ODE is 4

  x''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = (c__1-16*_b(_
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   -> Calling odsolve with the ODE`, diff(diff(_g(_f), _f), _f) = c__2+8*_g(_f)/_f^2+6
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying high order exact linear fully integrable
      trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
      trying a double symmetry of the form [xi=0, eta=F(x)]
      -> Try solving first the homogeneous part of the ODE
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         <- LODE of Euler type successful
      <- solving first the homogeneous part of the ODE successful
   <- high order exact_linear_nonhomogeneous successful
<- high order exact_linear_nonhomogeneous successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 43

```
dsolve(t^4*diff(diff(diff(diff(x(t),t),t),t),t)-2*t^3*diff(diff(diff(x(t),t),t),t)-20*
```

$$x = \frac{(15066 + 34263i)\, t^{1-3i} + (15066 - 34263i)\, t^{1+3i} + 23060700t^9 c_3 - 1281150 c_2 t^3 + 854100 c_1 \ln(t) + 9490}{23060700t}$$

Mathematica DSolve solution

Solving time : `0.076` (sec)
Leaf size : 48

```
DSolve[{t^4*D[x[t],{t,4}]-2*t^3*D[x[t],{t,3}]-20*t^2*D[x[t],{t,2}]+12*t*D[x[t],t]+16*x[t]==0
```

$$x(t) \to \frac{c_4 t^9 + c_3 t^3 + c_2 \log(t) + c_1}{t} + \frac{141 \sin(3 \log(t))}{47450} + \frac{31 \cos(3 \log(t))}{23725}$$

2.9 Chapter VII. Linear equations of order higher than the first. section 63. Problems at page 196

2.9.1 Problem 1

Internal problem ID [18592]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 63. Problems at page 196
Problem number : 1
Date solved : `Tuesday, January 28, 2025 at 12:03:24 PM`
CAS classification : `[[_3rd_order, _missing_x]]`

Solve

$$y''' - y'' - y' + y = 0$$

Solved as higher order constant coeff ode

Time used: 0.023 (sec)

The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-x} c_1 + \mathrm{e}^{x} c_2 + x\,\mathrm{e}^{x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{x}$$
$$y_3 = x\,\mathrm{e}^{x}$$

Maple step by step solution

Let's solve
$y''' - y'' - y' + y = 0$

- Highest derivative means the order of the ODE is 3
  y'''
- Characteristic polynomial of ODE
  $r^3 - r^2 - r + 1 = 0$
- Roots of the characteristic polynomial and corresponding multiplicities
  $r = [[-1, 1], [1, 2]]$
- Solution from $r = -1$
  $y_1(x) = e^{-x}$
- 1st solution from $r = 1$
  $y_2(x) = e^{x}$
- 2nd solution from $r = 1$
  $y_3(x) = x\,e^{x}$
- General solution of the ODE
  $y = C1y_1(x) + C2y_2(x) + C3y_3(x)$
- Substitute in solutions and simplify
  $y = e^{-x}C1 + e^{x}(C3x + C2)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 19

```
dsolve(diff(diff(diff(y(x),x),x),x)-diff(diff(y(x),x),x)-diff(y(x),x)+y(x) = 0,y(x),si
```

$$y(x) = e^{-x}c_1 + e^{x}(c_3x + c_2)$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 25

```
DSolve[{D[y[x],{x,3}]-D[y[x],{x,2}]-D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to c_1 e^{-x} + e^x (c_3 x + c_2)$$

2.9.2 Problem 2

Internal problem ID [18593]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 63. Problems at page 196
Problem number : 2
Date solved : `Tuesday, January 28, 2025 at 12:03:25 PM`
CAS classification : `[[_high_order, _missing_y]]`

Solve

$$y'''' - 3y''' + 3y'' - y' = e^{2x}$$

Solved as higher order constant coeff ode

Time used: 0.059 (sec)

The characteristic equation is

$$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^x c_2 + x\, e^x c_3 + x^2 e^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = x\,e^x$$
$$y_4 = x^2 e^x$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 3y''' + 3y'' - y' = 0$$

Now the particular solution to the given ODE is found

$$y'''' - 3y''' + 3y'' - y' = e^{2x}$$

The particular solution is now found using the method of undetermined coefficients.

Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\,e^x, x^2 e^x, e^x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + e^x c_2 + x\,e^x c_3 + x^2 e^x c_4\right) + \left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Maple step by step solution

Let's solve
$y'''' - 3y''' + 3y'' - y' = e^{2x}$

- Highest derivative means the order of the ODE is 4
  y''''
- Characteristic polynomial of homogeneous ODE
  $r^4 - 3r^3 + 3r^2 - r = 0$
- Roots of the characteristic polynomial and corresponding multiplicities
  $r = [[0,1],[1,3]]$
- Homogeneous solution from $r = 0$
  $y_1(x) = 1$
- 1st homogeneous solution from $r = 1$
  $y_2(x) = e^x$
- 2nd homogeneous solution from $r = 1$
  $y_3(x) = x\,e^x$
- 3rd homogeneous solution from $r = 1$
  $y_4(x) = x^2 e^x$
- General solution of the ODE
  $y = C1 y_1(x) + C2 y_2(x) + C3 y_3(x) + C4 y_4(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y = C1 + e^x C2 + x\,e^x C3 + x^2 e^x C4 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Define the forcing function of the ODE
  $f(x) = e^{2x}$
- Form of the particular solution to the ODE where the $u_i(x)$ are to be found
  $$y_p(x) = \sum_{i=1}^{4} u_i(x)\, y_i(x)$$
- Calculate the 1st derivative of $y_p(x)$
  $$y_p'(x) = \sum_{i=1}^{4} \left(u_i'(x)\, y_i(x) + u_i(x)\, y_i'(x)\right)$$
- Choose equation to add to a system of equations in $u_i'(x)$
  $$\sum_{i=1}^{4} u_i'(x)\, y_i(x) = 0$$
- Calculate the 2nd derivative of $y_p(x)$
  $$y_p''(x) = \sum_{i=1}^{4} \left(u_i'(x)\, y_i'(x) + u_i(x)\, y_i''(x)\right)$$
- Choose equation to add to a system of equations in $u_i'(x)$
  $$\sum_{i=1}^{4} u_i'(x)\, y_i'(x) = 0$$
- Calculate the 3rd derivative of $y_p(x)$
  $$y_p'''(x) = \sum_{i=1}^{4} \left(u_i'(x)\, y_i''(x) + u_i(x)\, y_i'''(x)\right)$$
- Choose equation to add to a system of equations in $u_i'(x)$
  $$\sum_{i=1}^{4} u_i'(x)\, y_i''(x) = 0$$
- The ODE is of the following form where the $P_i(x)$ in this situation are the coefficients of the c
  $$y'''' + \left(\sum_{i=0}^{3} P_i(x)\, y^{(i)}\right) = f(x)$$
- Substitute $y_p(x) = \sum_{i=1}^{4} u_i(x)\, y_i(x)$ into the ODE
  $$\left(\sum_{j=0}^{3} P_j(x) \left(\sum_{i=1}^{4} u_i(x)\, y_i^{(j)}(x)\right)\right) + \sum_{i=1}^{4} \left(u_i'(x)\, y_i'''(x) + u_i(x)\, y_i''''(x)\right) = f(x)$$
- Rearrange the ODE
  $$\sum_{i=1}^{4} \left(u_i(x) \cdot \left(\left(\sum_{j=0}^{3} P_j(x)\, y_i^{(j)}(x)\right) + y_i''''(x)\right) + u_i'(x)\, y_i'''(x)\right) = f(x)$$

- Notice that $y_i(x)$ are solutions to the homogeneous equation so the first term in the sum is 0

  $\sum_{i=1}^{4} u_i'(x)\, y_i'''(x) = f(x)$
- We have now made a system of 4 equations in 4 unknowns ($u_i'(x)$)

  $\left[\sum_{i=1}^{4} u_i'(x)\, y_i(x) = 0, \sum_{i=1}^{4} u_i'(x)\, y_i'(x) = 0, \sum_{i=1}^{4} u_i'(x)\, y_i''(x) = 0, \sum_{i=1}^{4} u_i'(x)\, y_i'''(x) = f(x)\right]$
- Convert the system to linear algebra format, notice that the matrix is the wronskian W

$$\begin{bmatrix} y_1(x) & y_2(x) & y_3(x) & y_4(x) \\ y_1'(x) & y_2'(x) & y_3'(x) & y_4'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) & y_4''(x) \\ y_1'''(x) & y_2'''(x) & y_3'''(x) & y_4'''(x) \end{bmatrix} \cdot \begin{bmatrix} u_1'(x) \\ u_2'(x) \\ u_3'(x) \\ u_4'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(x) \end{bmatrix}$$
- Solve for the varied parameters

$$\begin{bmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ u_4(x) \end{bmatrix} = \int \frac{1}{W} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(x) \end{bmatrix} dx$$
- Substitute in the homogeneous solutions and forcing function and solve

$$\begin{bmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ u_4(x) \end{bmatrix} = \begin{bmatrix} -\frac{e^{2x}}{2} \\ \frac{(x^2+2)e^{2x}}{2\,e^x} \\ -\frac{x\,e^{2x}}{e^x} \\ \frac{e^{2x}}{2\,e^x} \end{bmatrix}$$

  Find a particular solution $y_p(x)$ of the ODE

  $y_p(x) = \frac{e^{2x}}{2}$
- Substitute particular solution into general solution to ODE

  $y = C1 + e^x C2 + x\,e^x C3 + x^2 e^x C4 + \frac{e^{2x}}{2}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = 3*(diff(diff(
```

```
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 33

```
dsolve(diff(diff(diff(diff(y(x),x),x),x),x)-3*diff(diff(diff(y(x),x),x),x)+3*diff(dif
```

$$y(x) = \frac{e^{2x}}{2} + \left(\left(x^2 - 2x + 2\right) c_2 + c_3 x + c_1 - c_3\right) e^x + c_4$$

Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 41

```
DSolve[{D[y[x],{x,4}]-3*D[y[x],{x,3}]+3*D[y[x],{x,2}]-D[y[x],x]==Exp[2*x],{}},y[x],x,Include
```

$$y(x) \to \frac{1}{2} e^x \left(e^x + 2\left(c_3\left(x^2 - 2x + 2\right) + c_2(x - 1) + c_1\right)\right) + c_4$$

2.9.3 Problem 3

Internal problem ID [18594]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 63. Problems at page 196
Problem number : 3
Date solved : `Tuesday, January 28, 2025 at 12:03:25 PM`
CAS classification : `[[_3rd_order, _linear, _nonhomogeneous]]`

Solve

$$y''' - y'' + y' - y = \cos(x)$$

Solved as higher order constant coeff ode

Time used: 0.463 (sec)

The characteristic equation is

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{-ix} c_2 + e^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{-ix}$$
$$y_3 = e^{ix}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' + y' - y = 0$$

Now the particular solution to the given ODE is found

$$y''' - y'' + y' - y = \cos(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{x} & e^{-ix} & e^{ix} \\ e^{x} & -i e^{-ix} & i e^{ix} \\ e^{x} & -e^{-ix} & -e^{ix} \end{bmatrix}$$

$$|W| = 4i e^{x} e^{-ix} e^{ix}$$

The determinant simplifies to

$$|W| = 4i e^{x}$$

Now we determine W_i for each U_i.

$$
\begin{aligned}
W_1(x) &= \det \begin{bmatrix} \mathrm{e}^{-ix} & \mathrm{e}^{ix} \\ -i\mathrm{e}^{-ix} & i\mathrm{e}^{ix} \end{bmatrix} \\
&= 2i
\end{aligned}
$$

$$
\begin{aligned}
W_2(x) &= \det \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{ix} \\ \mathrm{e}^{x} & i\mathrm{e}^{ix} \end{bmatrix} \\
&= (-1+i)\,\mathrm{e}^{(1+i)x}
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= \det \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{-ix} \\ \mathrm{e}^{x} & -i\mathrm{e}^{-ix} \end{bmatrix} \\
&= (-1-i)\,\mathrm{e}^{(1-i)x}
\end{aligned}
$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(\cos(x))\,(2i)}{(1)\,(4i\mathrm{e}^{x})}\,dx \\
&= \int \frac{2i\cos(x)}{4i\mathrm{e}^{x}}\,dx \\
&= \int \left(\frac{\cos(x)\,\mathrm{e}^{-x}}{2}\right) dx \\
&= -\frac{\cos(x)\,\mathrm{e}^{-x}}{4} + \frac{\mathrm{e}^{-x}\sin(x)}{4}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(\cos(x))\,\left((-1+i)\,\mathrm{e}^{(1+i)x}\right)}{(1)\,(4i\mathrm{e}^{x})}\,dx \\
&= -\int \frac{(-1+i)\cos(x)\,\mathrm{e}^{(1+i)x}}{4i\mathrm{e}^{x}}\,dx \\
&= -\int \left(\left(\frac{1}{4}+\frac{i}{4}\right)\cos(x)\,\mathrm{e}^{ix}\right) dx \\
&= -\frac{x}{8} - \frac{ix}{8} - \frac{\mathrm{e}^{2ix}}{16} + \frac{i\mathrm{e}^{2ix}}{16} \\
&= -\frac{x}{8} - \frac{ix}{8} - \frac{\mathrm{e}^{2ix}}{16} + \frac{i\mathrm{e}^{2ix}}{16}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(\cos(x))\left((-1-i)\,\mathrm{e}^{(1-i)x}\right)}{(1)\,(4i\mathrm{e}^x)}\,dx \\
&= \int \frac{(-1-i)\cos(x)\,\mathrm{e}^{(1-i)x}}{4i\mathrm{e}^x}\,dx \\
&= \int \left(\left(-\frac{1}{4}+\frac{i}{4}\right)\cos(x)\,\mathrm{e}^{-ix}\right)\,dx \\
&= \int \left(-\frac{1}{4}+\frac{i}{4}\right)\cos(x)\,\mathrm{e}^{-ix}dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{\cos(x)\,\mathrm{e}^{-x}}{4}+\frac{\mathrm{e}^{-x}\sin(x)}{4}\right)(\mathrm{e}^x) \\
&+ \left(-\frac{x}{8}-\frac{ix}{8}-\frac{\mathrm{e}^{2ix}}{16}+\frac{i\mathrm{e}^{2ix}}{16}\right)(\mathrm{e}^{-ix}) \\
&+ \left(\int \left(-\frac{1}{4}+\frac{i}{4}\right)\cos(x)\,\mathrm{e}^{-ix}dx\right)(\mathrm{e}^{ix})
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{(-5+i-4x)\cos(x)}{16} + \frac{(1+i-4x)\sin(x)}{16}
$$

Which simplifies to

$$
y_p = \frac{(-5+i-4x)\cos(x)}{16} + \frac{(1+i-4x)\sin(x)}{16}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^x c_1 + \mathrm{e}^{-ix} c_2 + \mathrm{e}^{ix} c_3\right) + \left(\frac{(-5+i-4x)\cos(x)}{16} + \frac{(1+i-4x)\sin(x)}{16}\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$y''' - y'' + y' - y = \cos(x)$

- Highest derivative means the order of the ODE is 3

  y'''

- Characteristic polynomial of homogeneous ODE

  $r^3 - r^2 + r - 1 = 0$

- Roots of the characteristic polynomial

  $r = [1, \mathrm{I}, -\mathrm{I}]$

- Homogeneous solution from $r = 1$

  $y_1(x) = \mathrm{e}^x$

- Homogeneous solutions from $r = \mathrm{I}$ and $r = -\mathrm{I}$

  $[y_2(x) = \sin(x), y_3(x) = \cos(x)]$

- General solution of the ODE

  $y = C1 y_1(x) + C2 y_2(x) + C3 y_3(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = C1\,\mathrm{e}^x + C2 \sin(x) + C3 \cos(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Define the forcing function of the ODE

  $f(x) = \cos(x)$

- ○ Form of the particular solution to the ODE where the $u_i(x)$ are to be found

  $y_p(x) = \sum_{i=1}^{3} u_i(x)\, y_i(x)$

- ○ Calculate the 1st derivative of $y_p(x)$

  $y_p'(x) = \sum_{i=1}^{3} \left(u_i'(x)\, y_i(x) + u_i(x)\, y_i'(x)\right)$

- ○ Choose equation to add to a system of equations in $u_i'(x)$

  $\sum_{i=1}^{3} u_i'(x)\, y_i(x) = 0$

- ○ Calculate the 2nd derivative of $y_p(x)$

  $y_p''(x) = \sum_{i=1}^{3} \left(u_i'(x)\, y_i'(x) + u_i(x)\, y_i''(x)\right)$

- ○ Choose equation to add to a system of equations in $u_i'(x)$

  $\sum_{i=1}^{3} u_i'(x)\, y_i'(x) = 0$

- The ODE is of the following form where the $P_i(x)$ in this situation are the coefficients of the

  $y''' + \left(\sum_{i=0}^{2} P_i(x)\, y^{(i)}\right) = f(x)$

- Substitute $y_p(x) = \sum_{i=1}^{3} u_i(x)\, y_i(x)$ into the ODE

  $\left(\sum_{j=0}^{2} P_j(x) \left(\sum_{i=1}^{3} u_i(x)\, y_i^{(j)}(x)\right)\right) + \sum_{i=1}^{3} \left(u_i'(x)\, y_i''(x) + u_i(x)\, y_i'''(x)\right) = f(x)$

- Rearrange the ODE

  $\sum_{i=1}^{3} \left(u_i(x) \cdot \left(\left(\sum_{j=0}^{2} P_j(x)\, y_i^{(j)}(x)\right) + y_i'''(x)\right) + u_i'(x)\, y_i''(x)\right) = f(x)$

- Notice that $y_i(x)$ are solutions to the homogeneous equation so the first term in the sum is 0

  $\sum_{i=1}^{3} u_i'(x)\, y_i''(x) = f(x)$

- We have now made a system of 3 equations in 3 unknowns ($u_i'(x)$)

  $\left[\sum_{i=1}^{3} u_i'(x)\, y_i(x) = 0, \sum_{i=1}^{3} u_i'(x)\, y_i'(x) = 0, \sum_{i=1}^{3} u_i'(x)\, y_i''(x) = f(x)\right]$

- Convert the system to linear algebra format, notice that the matrix is the wronskian W

  $\begin{bmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{bmatrix} \cdot \begin{bmatrix} u_1'(x) \\ u_2'(x) \\ u_3'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f(x) \end{bmatrix}$

- Solve for the varied parameters

  $\begin{bmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{bmatrix} = \int \frac{1}{W} \cdot \begin{bmatrix} 0 \\ 0 \\ f(x) \end{bmatrix} dx$

- Substitute in the homogeneous solutions and forcing function and solve

  $\begin{bmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{bmatrix} = \begin{bmatrix} -\frac{\cos(x) e^{-x}}{4} + \frac{e^{-x} \sin(x)}{4} \\ -\frac{x}{4} - \frac{\sin(2x)}{8} + \frac{\cos(2x)}{8} \\ -\frac{x}{4} - \frac{\sin(2x)}{8} - \frac{\cos(2x)}{8} \end{bmatrix}$

  Find a particular solution $y_p(x)$ of the ODE

  $y_p(x) = \frac{(-2x-3)\cos(x)}{8} + \frac{(-2x+1)\sin(x)}{8}$

- Substitute particular solution into general solution to ODE

  $y = C1\, e^{x} + C2 \sin(x) + C3 \cos(x) + \frac{(-2x-3)\cos(x)}{8} + \frac{(-2x+1)\sin(x)}{8}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 33

```
dsolve(diff(diff(diff(y(x),x),x),x)-diff(diff(y(x),x),x)+diff(y(x),x)-y(x) = cos(x),y(
```

$$y(x) = \frac{(-x + 4c_1 - 2)\cos(x)}{4} + \frac{(-x + 4c_3 + 1)\sin(x)}{4} + c_2 e^x$$

Mathematica DSolve solution

Solving time : 0.038 (sec)
Leaf size : 40

```
DSolve[{D[y[x],{x,3}]-D[y[x],{x,2}]+D[y[x],x]-y[x]==Cos[x],{}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{1}{4}\left(4c_3 e^x - (x + 2 - 4c_1)\cos(x) + (-x + 1 + 4c_2)\sin(x)\right)$$

2.9.4 Problem 8

Internal problem ID [18595]
Book : Elementary Differential Equations. By Thornton C. Fry. D Van Nostrand. NY. First Edition (1929)
Section : Chapter VII. Linear equations of order higher than the first. section 63. Problems at page 196
Problem number : 8
Date solved : `Tuesday, January 28, 2025 at 12:03:26 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _nonhomogeneous]]`

Solve

$$x^2y'' + 3xy' + y = \frac{1}{x}$$

Solved as second order Euler type ode

Time used: 0.124 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''+3xy'+y=0$$

This is Euler second order ODE. Let the solution be $y=x^r$, then $y'=rx^{r-1}$ and $y''=r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2}+3xrx^{r-1}+x^r=0$$

Simplifying gives

$$r(r-1)\,x^r+3r\,x^r+x^r=0$$

Since $x^r\neq 0$ then dividing throughout by x^r gives

$$r(r-1)+3r+1=0$$

Or

$$r^2+2r+1=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -1 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y=c_1y_1+c_2y_2$$

Where $y_1=x^r$ and $y_2=x^r\ln(x)$. Hence

$$y=\frac{c_1}{x}+\frac{c_2\ln(x)}{x}$$

Next, we find the particular solution to the ODE

$$x^2y''+3xy'+y=\frac{1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & -\frac{\ln(x)}{x^2} + \frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x^2} + \frac{1}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)}{x^2}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x}dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x}dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{\ln(x)^2 + 2c_2\ln(x) + 2c_1}{2x} \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\ln(x)^2 + 2c_2\ln(x) + 2c_1}{2x}$$

Solved as second order linear exact ode

Time used: 0.097 (sec)

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= 3x \\ r(x) &= 1 \\ s(x) &= \frac{1}{x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 3 \end{aligned}$$

Therefore (1) becomes

$$2 - (3) + (1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$x^2 y' + yx = \int \frac{1}{x}\, dx$$

We now have a first order ode to solve which is

$$x^2 y' + yx = \ln(x) + c_1$$

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{1}{x}$$
$$p(x) = \frac{\ln(x) + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\ln(x) + c_1}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(yx) &= (x)\left(\frac{\ln(x) + c_1}{x^2}\right) \\ \mathrm{d}(yx) &= \left(\frac{\ln(x) + c_1}{x}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}yx &= \int \frac{\ln(x) + c_1}{x}\,dx \\ &= \frac{\ln(x)^2}{2} + c_1\ln(x) + c_2\end{aligned}$$

Dividing throughout by the integrating factor x gives the final solution

$$y = \frac{\frac{\ln(x)^2}{2} + c_1\ln(x) + c_2}{x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\frac{\ln(x)^2}{2} + c_1\ln(x) + c_2}{x}$$

Solved as second order integrable as is ode

Time used: 0.075 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2 y'' + 3xy' + y\right) dx = \int \frac{1}{x} dx$$
$$x^2 y' + yx = \ln(x) + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{1}{x}$$
$$p(x) = \frac{\ln(x) + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= e^{\int \frac{1}{x} dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\ln(x) + c_1}{x^2}\right) \\ \frac{d}{dx}(yx) &= (x)\left(\frac{\ln(x) + c_1}{x^2}\right) \\ d(yx) &= \left(\frac{\ln(x) + c_1}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}yx &= \int \frac{\ln(x) + c_1}{x}\, dx \\ &= \frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2\end{aligned}$$

Dividing throughout by the integrating factor x gives the final solution

$$y = \frac{\frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2}{x}$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$y = \frac{\frac{\ln(x)^2}{2} + c_1 \ln(x) + c_2}{x}$$

Solved as second order integrable as is ode (ABC method)

Time used: 0.118 (sec)

Writing the ode as

$$x^2 y'' + 3xy' + y = \frac{1}{x}$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int (x^2 y'' + 3xy' + y)\, dx &= \int \frac{1}{x} dx \\ x^2 y' + yx &= \ln(x) + c_1\end{aligned}$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= \frac{1}{x} \\ p(x) &= \frac{\ln(x) + c_1}{x^2}\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= e^{\int \frac{1}{x} dx} \\ &= x\end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\ln(x)+c_1}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(yx) &= (x)\left(\frac{\ln(x)+c_1}{x^2}\right) \\
\mathrm{d}(yx) &= \left(\frac{\ln(x)+c_1}{x}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
yx &= \int \frac{\ln(x)+c_1}{x}\,dx \\
&= \frac{\ln(x)^2}{2} + c_1\ln(x) + c_2
\end{aligned}
$$

Dividing throughout by the integrating factor x gives the final solution

$$
y = \frac{\frac{\ln(x)^2}{2} + c_1\ln(x) + c_2}{x}
$$

Will add steps showing solving for IC soon.

Solved as second order ode using change of variable on x method 2

Time used: 0.598 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
x^2y''+3xy'+y=0
$$

In normal form the ode

$$
x^2y''+3xy'+y=0 \tag{1}
$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\int p(x)dx} dx \\ &= \int \mathrm{e}^{-\int \frac{3}{x}dx} dx \\ &= \int e^{-3\ln(x)}\, dx \\ &= \int \frac{1}{x^3} dx \\ &= -\frac{1}{2x^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{x^2}}{\frac{1}{x^6}} \\ &= x^4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + x^4 y(\tau) &= 0
\end{aligned}
$$

But in terms of τ

$$
x^4 = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. Writing the ode as

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0 \tag{1}
$$

$$
A\frac{d^2}{d\tau^2}y(\tau) + B\frac{d}{d\tau}y(\tau) + Cy(\tau) = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= \frac{1}{4\tau^2}
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(\tau) = y(\tau)\, e^{\int \frac{B}{2A}\, d\tau}
$$

Then (2) becomes

$$
z''(\tau) = rz(\tau) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4\tau^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4\tau^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(\tau) = \left(-\frac{1}{4\tau^2}\right) z(\tau) \tag{7}$$

Equation (7) is now solved. After finding $z(\tau)$ then $y(\tau)$ is found using the inverse transformation

$$y(\tau) = z(\tau)\, e^{-\int \frac{B}{2A}\, d\tau}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.68: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4\tau^2$. There is a pole at $\tau = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4\tau^2}$$

For the pole at $\tau = 0$ let b be the coefficient of $\frac{1}{\tau^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{\tau^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4\tau^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4\tau^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{\tau - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{\tau - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2\tau} + (-)(0) \\ &= \frac{1}{2\tau} \\ &= \frac{1}{2\tau} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(\tau)$ of degree $d = 0$ to solve the ode. The polynomial $p(\tau)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(\tau) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2\tau}\right)(0) + \left(\left(-\frac{1}{2\tau^2}\right) + \left(\frac{1}{2\tau}\right)^2 - \left(-\frac{1}{4\tau^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(\tau) &= p e^{\int \omega \, d\tau} \\ &= \mathrm{e}^{\int \frac{1}{2\tau} d\tau} \\ &= \sqrt{\tau} \end{aligned}$$

The first solution to the original ode in $y(\tau)$ is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, d\tau}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \sqrt{\tau} \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{\tau}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, d\tau}}{y_1^2} \, d\tau$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, d\tau \\ &= \sqrt{\tau} \int \frac{1}{\tau} \, d\tau \\ &= \sqrt{\tau}(\ln(\tau)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y(\tau) &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{\tau}\right) + c_2\left(\sqrt{\tau}(\ln(\tau))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$
y = c_1\sqrt{-\frac{1}{2x^2}} + c_2\sqrt{-\frac{1}{2x^2}}\,\ln\left(-\frac{1}{2x^2}\right)
$$

Therefore the homogeneous solution y_h is

$$
y_h = c_1\sqrt{-\frac{1}{2x^2}} + c_2\sqrt{-\frac{1}{2x^2}}\,\ln\left(-\frac{1}{2x^2}\right)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{-\frac{1}{2x^2}} \\
y_2 &= -\sqrt{-\frac{1}{2x^2}}\,\ln(2) + \sqrt{-\frac{1}{2x^2}}\,\ln\left(-\frac{1}{x^2}\right)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{2x^2}} & -\sqrt{-\frac{1}{2x^2}}\ln(2) + \sqrt{-\frac{1}{2x^2}}\ln\left(-\frac{1}{x^2}\right) \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{2x^2}}\right) & \frac{d}{dx}\left(-\sqrt{-\frac{1}{2x^2}}\ln(2) + \sqrt{-\frac{1}{2x^2}}\ln\left(-\frac{1}{x^2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{2x^2}} & -\sqrt{-\frac{1}{2x^2}}\ln(2) + \sqrt{-\frac{1}{2x^2}}\ln\left(-\frac{1}{x^2}\right) \\ \frac{1}{2\sqrt{-\frac{1}{2x^2}}\,x^3} & -\frac{\ln(2)}{2\sqrt{-\frac{1}{2x^2}}\,x^3} + \frac{\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{2x^2}}\,x^3} - \frac{2\sqrt{-\frac{1}{2x^2}}}{x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{2x^2}}\right)\left(-\frac{\ln(2)}{2\sqrt{-\frac{1}{2x^2}}\,x^3} + \frac{\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{2x^2}}\,x^3} - \frac{2\sqrt{-\frac{1}{2x^2}}}{x}\right) - \left(-\sqrt{-\frac{1}{2x^2}}\ln(2) + \sqrt{-\frac{1}{2x^2}}\ln\left(-\frac{1}{x^2}\right)\right)\left(\frac{1}{2\sqrt{-\frac{1}{2x^2}}\,x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{-\sqrt{-\frac{1}{2x^2}}\ln(2)+\sqrt{-\frac{1}{2x^2}}\ln\left(-\frac{1}{x^2}\right)}{x}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\left(\ln(2) - \ln\left(-\frac{1}{x^2}\right)\right)}{2}dx$$

Hence

$$u_1 = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,x\ln(2)\ln(x)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,x\ln\left(-\frac{1}{x^2}\right)^2}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sqrt{-\frac{1}{2x^2}}}{x}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}}{2}dx$$

Hence

$$u_2 = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,x\ln(2)\ln(x)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,x\ln\left(-\frac{1}{x^2}\right)^2}{8}\right)\sqrt{-\frac{1}{2x^2}} + \frac{\left(-\sqrt{-\frac{1}{2x^2}}\,\ln(2) + \sqrt{-\frac{1}{2x^2}}\,\ln\left(-\frac{1}{x^2}\right)\right)\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\ln\left(-\frac{1}{x^2}\right)\left(\ln\left(-\frac{1}{x^2}\right) + 4\ln(x)\right)}{8x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\sqrt{-\frac{1}{2x^2}} + c_2\sqrt{-\frac{1}{2x^2}}\,\ln\left(-\frac{1}{2x^2}\right)\right) + \left(-\frac{\ln\left(-\frac{1}{x^2}\right)\left(\ln\left(-\frac{1}{x^2}\right) + 4\ln(x)\right)}{8x}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\ln\left(-\frac{1}{x^2}\right)\left(\ln\left(-\frac{1}{x^2}\right) + 4\ln(x)\right)}{8x} + c_1\sqrt{-\frac{1}{2x^2}} + c_2\sqrt{-\frac{1}{2x^2}}\ln\left(-\frac{1}{2x^2}\right)$$

Solved as second order ode using change of variable on y method 2

Time used: 0.148 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 3xy' + y = 0$$

In normal form the ode

$$x^2y'' + 3xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{1}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu u = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(ux) = 0$$

Integrating gives

$$ux = \int 0\,dx + c_3$$
$$= c_3$$

Dividing throughout by the integrating factor x gives the final solution

$$u(x) = \frac{c_3}{x}$$

Now that $u(x)$ is known, then

$$v'(x) = u(x)$$
$$v(x) = \int u(x)\,dx + c_4$$
$$= c_3 \ln(x) + c_4$$

Hence

$$y = v(x)\,x^n$$
$$= \frac{c_3 \ln(x) + c_4}{x}$$
$$= \frac{c_3 \ln(x) + c_4}{x}$$

Now the particular solution to this ODE is found

$$x^2 y'' + 3xy' + y = \frac{1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & -\frac{\ln(x)}{x^2} + \frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x^2} + \frac{1}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)}{x^2}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x}dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x}dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_3 \ln(x) + c_4}{x}\right) + \left(\frac{\ln(x)^2}{2x}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_3 \ln(x) + c_4}{x} + \frac{\ln(x)^2}{2x}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.194 (sec)

Writing the ode as

$$x^2y'' + 3xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.69: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1\left(\frac{1}{x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(\ln(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 3xy' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & -\frac{\ln(x)}{x^2} + \frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x^2} + \frac{1}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)}{x^2}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 \ln(x)}{x}\right) + \left(\frac{\ln(x)^2}{2x}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x} + \frac{\ln(x)^2}{2x}$$

Solved as second order ode adjoint method

Time used: 0.179 (sec)

In normal form the ode

$$x^2y'' + 3xy' + y = \frac{1}{x} \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \\ r(x) &= \frac{1}{x^3} \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(\frac{3\xi(x)}{x}\right)' + \left(\frac{\xi(x)}{x^2}\right) &= 0 \\ \xi''(x) + \frac{4\xi(x)}{x^2} - \frac{3\xi'(x)}{x} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is Euler second order ODE. Let the solution be $\xi = x^r$, then $\xi' = rx^{r-1}$ and $\xi'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are equal, then the general solution is

$$\xi = c_1\xi_1 + c_2\xi_2$$

Where $\xi_1 = x^r$ and $\xi_2 = x^r \ln(x)$. Hence

$$\xi = c_1\,x^2 + c_2\,x^2 \ln(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)} \end{aligned}$$

Or

$$y' + y\left(\frac{3}{x} - \frac{2c_1x + 2c_2x\ln(x) + c_2x}{c_1\,x^2 + c_2\,x^2\ln(x)}\right) = \frac{\frac{c_2\ln(x)^2}{2} + c_1\ln(x)}{c_1\,x^2 + c_2\,x^2\ln(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_2 \ln (x) - c_1 + c_2}{x (c_2 \ln (x) + c_1)} \\
p(x) &= \frac{\ln (x) (c_2 \ln (x) + 2c_1)}{2x^2 (c_2 \ln (x) + c_1)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= \mathrm{e}^{\int -\frac{-c_2 \ln(x) - c_1 + c_2}{x(c_2 \ln(x)+c_1)} dx} \\
&= \frac{x}{c_2 \ln (x) + c_1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(\frac{\ln (x) (c_2 \ln (x) + 2c_1)}{2x^2 (c_2 \ln (x) + c_1)} \right) \\
\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{yx}{c_2 \ln (x) + c_1} \right) &= \left(\frac{x}{c_2 \ln (x) + c_1} \right) \left(\frac{\ln (x) (c_2 \ln (x) + 2c_1)}{2x^2 (c_2 \ln (x) + c_1)} \right) \\
\mathrm{d} \left(\frac{yx}{c_2 \ln (x) + c_1} \right) &= \left(\frac{\ln (x) (c_2 \ln (x) + 2c_1)}{2x (c_2 \ln (x) + c_1)^2} \right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{yx}{c_2 \ln (x) + c_1} &= \int \frac{\ln (x) (c_2 \ln (x) + 2c_1)}{2x (c_2 \ln (x) + c_1)^2} \, dx \\
&= \frac{\ln (x)}{2c_2} + \frac{c_1^2}{2c_2^2 (c_2 \ln (x) + c_1)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{x}{c_2 \ln(x)+c_1}$ gives the final solution

$$
y = \frac{\ln (x)^2 c_2^2 + c_2(2c_3 c_2^2 + c_1) \ln (x) + c_1(2c_3 c_2^2 + c_1)}{2x c_2^2}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{\ln (x)^2 c_2^2 + c_2(2c_3 c_2^2 + c_1) \ln (x) + c_1(2c_3 c_2^2 + c_1)}{2x c_2^2}
$$

The constants can be merged to give

$$y = \frac{\ln(x)^2 c_2^2 + c_2(2c_2^2 + c_1)\ln(x) + c_1(2c_2^2 + c_1)}{2x\, c_2^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\ln(x)^2 c_2^2 + c_2(2c_2^2 + c_1)\ln(x) + c_1(2c_2^2 + c_1)}{2x\, c_2^2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 20

```
dsolve(x^2*diff(diff(y(x),x),x)+3*x*diff(y(x),x)+y(x) = 1/x,y(x),singsol=all)
```

$$y(x) = \frac{c_2 + c_1 \ln(x) + \frac{\ln(x)^2}{2}}{x}$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 27

```
DSolve[{x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+y[x]==1/x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\log^2(x) + 2c_2 \log(x) + 2c_1}{2x}$$