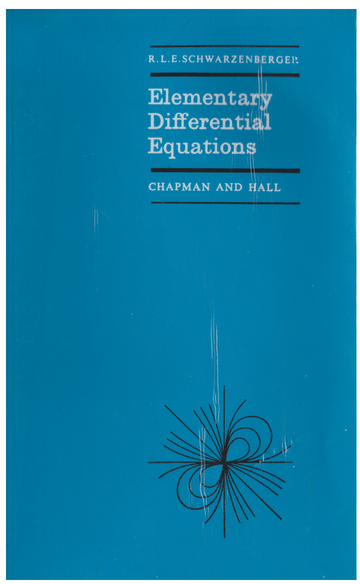
A Solution Manual For

Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)



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December 19, 2024 Compiled on December 19, 2024 at 6:33pm

Contents

CHAPTER 1

Lookup tables for all problems in current book

1.1 Chapter 3. Solutions of first-order equations. Exercises at page 47

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE
18164	1 (i)	$x' = 3t^2 + 4t$
18165	1 (ii)	$x' = b\,e^t$
18166	1 (iii)	$x' = \frac{1}{t^2+1}$
18167	1 (iv)	$x' = \frac{1}{\sqrt{t^2+1}}$
18168	1 (v)	$x' = \cos(t)$
18169	1 (vi)	$x' = \frac{\cos(t)}{\sin(t)}$
18170	2 (i)	$x' = x^2 - 3x + 2$
18171	2 (ii)	$x' = b\,e^x$
18172	2 (iii)	$x' = (x-1)^2$
18173	2 (iv)	$x' = \sqrt{x^2 - 1}$
18174	2 (v)	$x' = 2\sqrt{x}$
18175	2 (vi)	$x' = \tan(x)$
18176	3 (i)	$3t^2x - xt + (3t^3x^2 + t^3x^4)\,x' = 0$
18177	3 (ii)	$1 + 2x + (-t^2 + 4)\,x' = 0$
18178	3 (iii)	$x' = \cos\left(\frac{x}{t}\right)$
18179	3 (iv)	$(t^2 - x^2)\,x' = xt$
18180	3 (v)	$e^{3t}x' + 3x\,e^{3t} = 2t$
18181	3 (vi)	$2t + 3x + (3t - x)\,x' = t^2$
18182	4 (i)	$x' + 2x = e^t$
18183	4 (ii)	$x' + x\tan(t) = 0$
18184	4 (iii)	$x' - x\tan(t) = 4\sin(t)$

Continued on next page

Table 1.1 Lookup table
Continued from previous page

ID	problem	ODE
18185	4 (iv)	$t^3x' + (-3t^2 + 2)\, x = t^3$
18186	4 (v)	$x' + 2xt + tx^4 = 0$
18187	4 (vi)	$tx' + x \ln{(t)} = t^2$
18188	5	$tx' + xg(t) = h(t)$
18189	6	$t^2x'' - 6tx' + 12x = 0$

1.2 Chapter 4. Autonomous systems. Exercises at page 69

Table 1.2: Lookup table for all problems in current section

ID	problem	ODE
18190	1	$x' = -\lambda x$
18191	2	$[x'(t) = x(t)\,, y'(t) = x(t) + 2y(t)]$
18192	3	$t^2x'' - 2tx' + 2x = 0$
18193	5 (i)	$x'' - 5x' + 6x = 0$
18194	5 (ii)	$x'' - 4x' + 4x = 0$
18195	5 (iiI=i)	$x'' - 4x' + 5x = 0$
18196	5 (iv)	$x'' + 3x' = 0$
18197	6 (i)	$x'' - 3x' + 2x = 0$
18198	6 (ii)	$x'' + x = 0$
18199	6 (iii)	$x'' + 2x' + x = 0$
18200	6 (iv)	$x'' - 2x' + 2x = 0$

1.3 Chapter 5. Linear equations. Exercises at page 85

Table 1.3: Lookup table for all problems in current section

ID	problem	ODE
18201	7 (i)	$x'' - x = t^2$
18202	7 (ii)	$x'' - x = e^t$
18203	7 (iii)	$x'' + 2x' + 4x = e^t \cos(2t)$
18204	7 (iv)	$x'' - x' + x = \sin(2t)$
18205	7 (v)	$x'' + 4x' + 3x = t \sin(t)$
18206	7 (vi)	$x'' + x = \cos(t)$

CHAPTER 2

Book Solved Problems

2.1 Chapter 3. Solutions of first-order equations. Exercises at page 47

2.1.1 problem 1 (i)

Internal problem ID [18164]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 1 (i)
Date solved : `Thursday, December 19, 2024 at 01:51:42 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = 3t^2 + 4t$$

With initial conditions

$$x(1) = 0$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + q(t)x = p(t)$$

Where here

$$q(t) = 0$$
$$p(t) = 3t^2 + 4t$$

Hence the ode is

$$x' = 3t^2 + 4t$$

The domain of $q(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $p(t) = 3t^2 + 4t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

Solved as first order quadrature ode

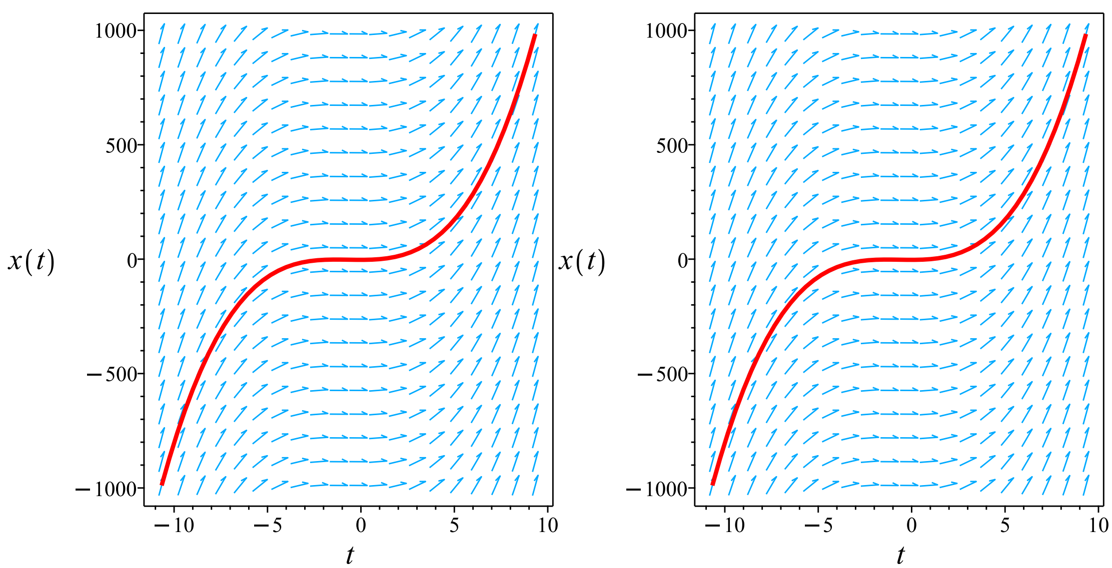
Time used: 0.094 (sec)

Since the ode has the form $x' = f(t)$, then we only need to integrate $f(t)$.

$$\int dx = \int 3t^2 + 4t \, dt$$
$$x = t^3 + 2t^2 + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

$$x = t^3 + 2t^2 - 3$$



(a) Solution plot
$x = t^3 + 2t^2 - 3$

(b) Slope field plot
$x' = 3t^2 + 4t$

Summary of solutions found

$$x = t^3 + 2t^2 - 3$$

Solved as first order Exact ode

Time used: 0.075 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x &= \left(3t^2 + 4t\right)\mathrm{d}t\\ \left(-3t^2 - 4t\right)\mathrm{d}t + \mathrm{d}x &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= -3t^2 - 4t\\ N(t,x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-3t^2 - 4t) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -3t^2 - 4t \, \mathrm{d}t \\ \phi &= -t^3 - 2t^2 + f(x) \qquad (3)\end{aligned}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 1$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (1)\,\mathrm{d}x$$

$$f(x) = x + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -t^3 - 2t^2 + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

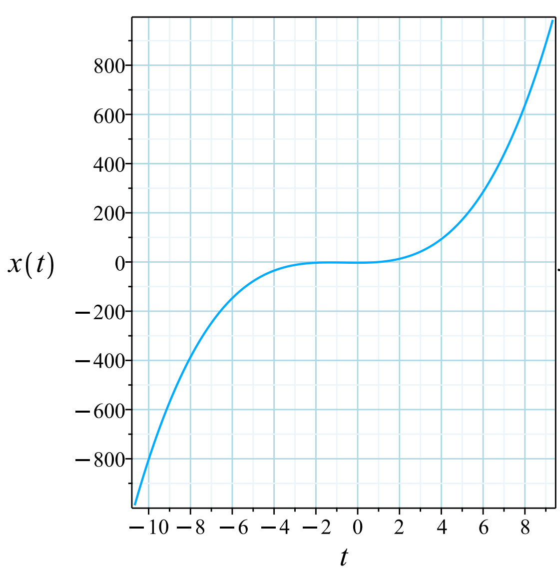
$$c_1 = -t^3 - 2t^2 + x$$

Solving for the constant of integration from initial conditions, the solution becomes

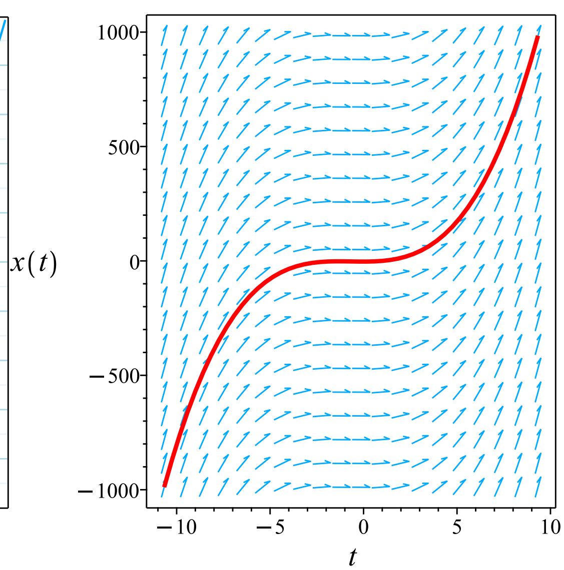
$$-t^3 - 2t^2 + x = -3$$

Solving for x gives

$$x = t^3 + 2t^2 - 3$$



(a) Solution plot
$x = t^3 + 2t^2 - 3$



(b) Slope field plot
$x' = 3t^2 + 4t$

Summary of solutions found

$$x = t^3 + 2t^2 - 3$$

Maple step by step solution

Let's solve
$[x' = 3t^2 + 4t, x(1) = 0]$

- Highest derivative means the order of the ODE is 1
  x'
- Integrate both sides with respect to t
  $\int x' dt = \int (3t^2 + 4t)\, dt + C1$
- Evaluate integral
  $x = t^3 + 2t^2 + C1$
- Solve for x
  $x = t^3 + 2t^2 + C1$
- Use initial condition $x(1) = 0$
  $0 = C1 + 3$
- Solve for _$C1$
  $C1 = -3$
- Substitute _$C1 = -3$ into general solution and simplify
  $x = t^3 + 2t^2 - 3$
- Solution to the IVP
  $x = t^3 + 2t^2 - 3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 14

```
dsolve([diff(x(t),t) = 3*t^2+4*t,
       op([x(1) = 0])],x(t),singsol=all)
```

$$x = t^3 + 2t^2 - 3$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
DSolve[{D[x[t],t]==3*t^2+4*t,{x[1]==0}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to t^3 + 2t^2 - 3$$

2.1.2 problem 1 (ii)

Internal problem ID [18165]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 1 (ii)
Date solved : `Thursday, December 19, 2024 at 01:51:43 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = b\,\mathrm{e}^{t}$$

With initial conditions

$$x(1) = 0$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + q(t)x = p(t)$$

Where here

$$q(t) = 0$$
$$p(t) = b\,\mathrm{e}^{t}$$

Hence the ode is

$$x' = b\,\mathrm{e}^{t}$$

The domain of $q(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $p(t) = b\,\mathrm{e}^t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

Solved as first order quadrature ode

Time used: 0.109 (sec)

Since the ode has the form $x' = f(t)$, then we only need to integrate $f(t)$.

$$\int dx = \int b\,\mathrm{e}^t\,dt$$
$$x = b\,\mathrm{e}^t + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

$$x = b\,\mathrm{e}^t - b\,\mathrm{e}$$

Summary of solutions found

$$x = b\,\mathrm{e}^t - b\,\mathrm{e}$$

Solved as first order Exact ode

Time used: 0.073 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x &= \left(b\,\mathrm{e}^t\right)\mathrm{d}t\\ \left(-b\,\mathrm{e}^t\right)\mathrm{d}t + \mathrm{d}x &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -b\,\mathrm{e}^t\\ N(t, x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-b\,\mathrm{e}^t\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -b\,\mathrm{e}^t\,\mathrm{d}t$$
$$\phi = -b\,\mathrm{e}^t + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 1$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (1)\,\mathrm{d}x$$
$$f(x) = x + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -b\,\mathrm{e}^t + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -b\,\mathrm{e}^t + x$$

Solving for the constant of integration from initial conditions, the solution becomes

$$-b\,\mathrm{e}^{t}+x=-b\,\mathrm{e}$$

Solving for x gives

$$x=b\,\mathrm{e}^{t}-b\,\mathrm{e}$$

Summary of solutions found

$$x=b\,\mathrm{e}^{t}-b\,\mathrm{e}$$

Maple step by step solution

Let's solve
$[x'=b\,\mathrm{e}^{t}, x(1)=0]$

- Highest derivative means the order of the ODE is 1
  x'
- Integrate both sides with respect to t
  $\int x'dt=\int b\,\mathrm{e}^{t}dt+\mathit{C1}$
- Evaluate integral
  $x=b\,\mathrm{e}^{t}+\mathit{C1}$
- Solve for x
  $x=b\,\mathrm{e}^{t}+\mathit{C1}$
- Use initial condition $x(1)=0$
  $0=b\,\mathrm{e}+\mathit{C1}$
- Solve for $\_\mathit{C1}$
  $\mathit{C1}=-b\,\mathrm{e}$
- Substitute $\_\mathit{C1}=-b\mathrm{e}$ into general solution and simplify
  $x=b(\mathrm{e}^{t}-\mathrm{e})$
- Solution to the IVP
  $x=b(\mathrm{e}^{t}-\mathrm{e})$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 14

```
dsolve([diff(x(t),t) = b*exp(t),
       op([x(1) = 0])],x(t),singsol=all)
```

$$x = -b(\mathrm{e} - \mathrm{e}^{t})$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
DSolve[{D[x[t],t]==b*Exp[t],{x[1]==0}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to b(e^{t} - e)$$

2.1.3 problem 1 (iii)

Internal problem ID [18166]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 1 (iii)
Date solved : `Thursday, December 19, 2024 at 01:51:43 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = \frac{1}{t^2+1}$$

With initial conditions

$$x(1) = 0$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + q(t)x = p(t)$$

Where here

$$q(t) = 0$$
$$p(t) = \frac{1}{t^2+1}$$

Hence the ode is

$$x' = \frac{1}{t^2+1}$$

The domain of $q(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $p(t) = \frac{1}{t^2+1}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

Solved as first order quadrature ode

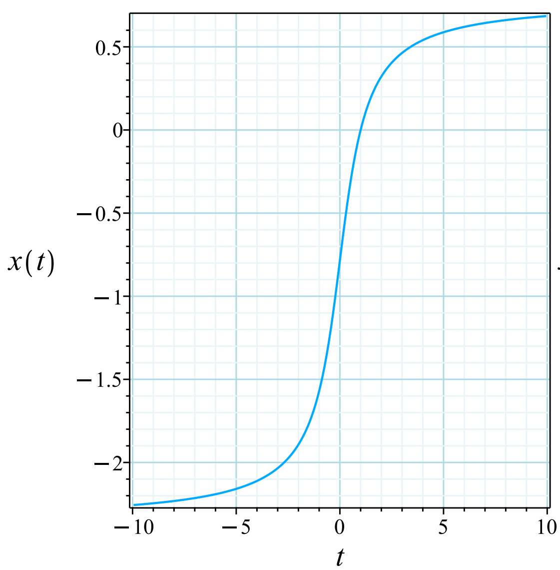
Time used: 0.105 (sec)

Since the ode has the form $x' = f(t)$, then we only need to integrate $f(t)$.

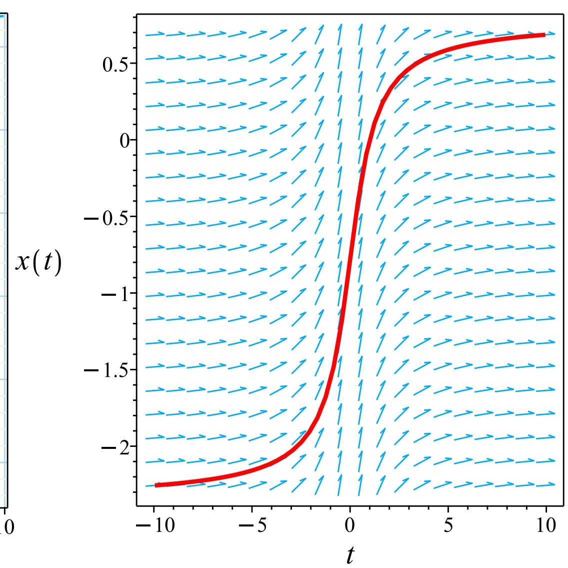
$$\int dx = \int \frac{1}{t^2+1}\, dt$$
$$x = \arctan(t) + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

$$x = \arctan(t) - \frac{\pi}{4}$$



(a) Solution plot
$x = \arctan(t) - \frac{\pi}{4}$



(b) Slope field plot
$x' = \frac{1}{t^2+1}$

Summary of solutions found

$$x = \arctan(t) - \frac{\pi}{4}$$

Solved as first order Exact ode

Time used: 0.325 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\mathrm{d}x=\left(\frac{1}{t^2+1}\right)\mathrm{d}t$$

$$\left(-\frac{1}{t^2+1}\right)\mathrm{d}t+\mathrm{d}x=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t,x) &= -\frac{1}{t^2+1} \\ N(t,x) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{t^2+1}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{t^2+1}\,\mathrm{d}t \end{aligned}$$

$$\phi = -\arctan(t) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial x} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 1$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (1)\,\mathrm{d}x$$

$$f(x) = x + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\arctan(t) + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

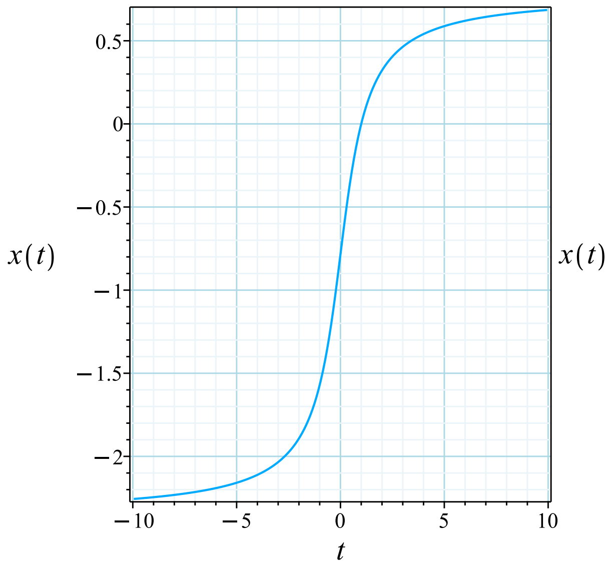
$$c_1 = -\arctan(t) + x$$

Solving for the constant of integration from initial conditions, the solution becomes

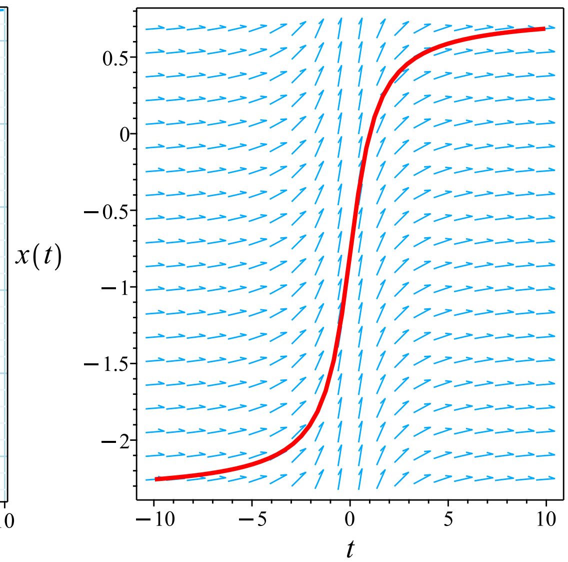
$$-\arctan(t) + x = -\frac{\pi}{4}$$

Solving for x gives

$$x = \arctan(t) - \frac{\pi}{4}$$



(a) Solution plot
$x = \arctan(t) - \frac{\pi}{4}$



(b) Slope field plot
$x' = \frac{1}{t^2+1}$

Summary of solutions found

$$x = \arctan(t) - \frac{\pi}{4}$$

Maple step by step solution

Let's solve
$\left[x' = \frac{1}{t^2+1}, x(1) = 0\right]$

- Highest derivative means the order of the ODE is 1
  x'
- Integrate both sides with respect to t
  $\int x' dt = \int \frac{1}{t^2+1} dt + C1$
- Evaluate integral
  $x = \arctan(t) + C1$
- Solve for x
  $x = \arctan(t) + C1$
- Use initial condition $x(1) = 0$
  $0 = \frac{\pi}{4} + C1$
- Solve for $\_C1$
  $C1 = -\frac{\pi}{4}$
- Substitute $\_C1 = -\frac{\pi}{4}$ into general solution and simplify

$x = \arctan(t) - \frac{\pi}{4}$

- Solution to the IVP
  $x = \arctan(t) - \frac{\pi}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 10

```
dsolve([diff(x(t),t) = 1/(t^2+1),
       op([x(1) = 0])],x(t),singsol=all)
```

$$x = \arctan(t) - \frac{\pi}{4}$$

Mathematica DSolve solution

Solving time : 0.005 (sec)
Leaf size : 13

```
DSolve[{D[x[t],t]==1/(1+t^2),{x[1]==0}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \arctan(t) - \frac{\pi}{4}$$

2.1.4 problem 1 (iv)

Internal problem ID [18167]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 1 (iv)
Date solved : `Thursday, December 19, 2024 at 01:51:44 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = \frac{1}{\sqrt{t^2+1}}$$

With initial conditions

$$x(1) = 0$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + q(t)x = p(t)$$

Where here

$$q(t) = 0$$
$$p(t) = \frac{1}{\sqrt{t^2+1}}$$

Hence the ode is

$$x' = \frac{1}{\sqrt{t^2+1}}$$

The domain of $q(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $p(t) = \frac{1}{\sqrt{t^2+1}}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

Solved as first order quadrature ode

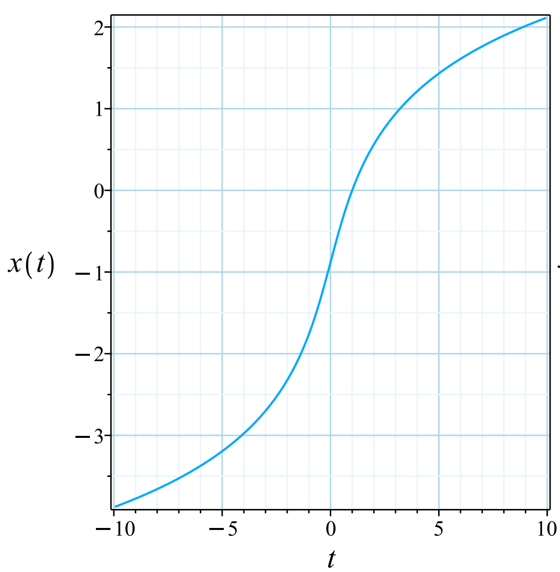
Time used: 0.108 (sec)

Since the ode has the form $x' = f(t)$, then we only need to integrate $f(t)$.

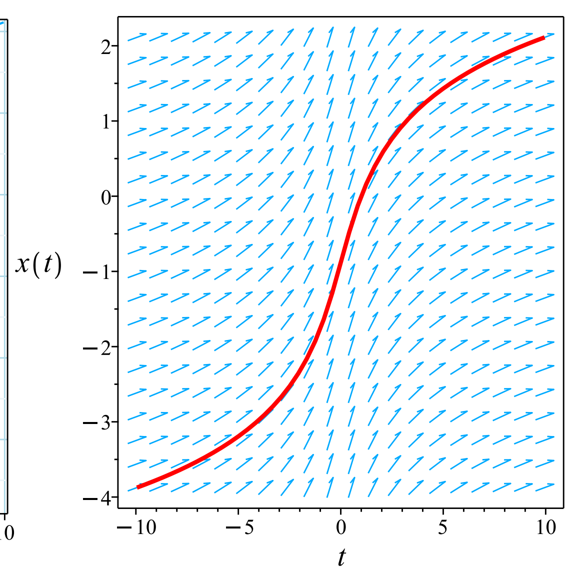
$$\int dx = \int \frac{1}{\sqrt{t^2+1}}\, dt$$
$$x = \operatorname{arcsinh}(t) + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

$$x = \operatorname{arcsinh}(t) - \ln\left(1+\sqrt{2}\right)$$



(a) Solution plot
$x = \operatorname{arcsinh}(t) - \ln\left(1+\sqrt{2}\right)$



(b) Slope field plot
$x' = \frac{1}{\sqrt{t^2+1}}$

Summary of solutions found

$$x = \operatorname{arcsinh}(t) - \ln\left(1+\sqrt{2}\right)$$

Solved as first order Exact ode

Time used: 0.064 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = \left(\frac{1}{\sqrt{t^2+1}}\right)\mathrm{d}t$$

$$\left(-\frac{1}{\sqrt{t^2+1}}\right)\mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,x) &= -\frac{1}{\sqrt{t^2+1}}\\
N(t,x) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{\sqrt{t^2+1}}\right)\\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{\sqrt{t^2+1}}\,\mathrm{d}t\\
\phi &= -\operatorname{arcsinh}(t) + f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 1$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (1)\,\mathrm{d}x$$
$$f(x) = x + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\operatorname{arcsinh}(t) + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

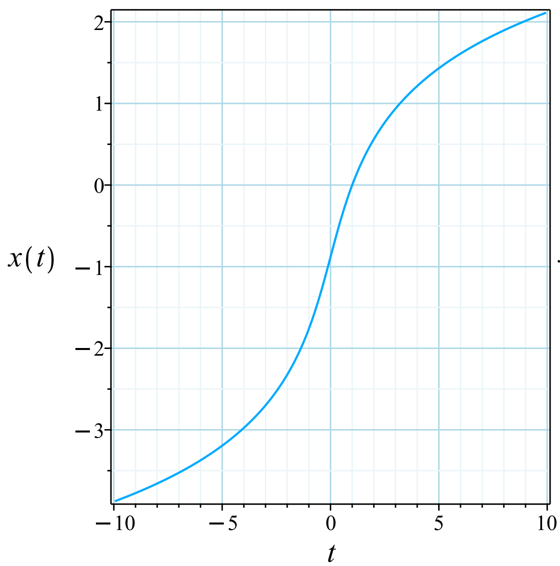
$$c_1 = -\operatorname{arcsinh}(t) + x$$

Solving for the constant of integration from initial conditions, the solution becomes

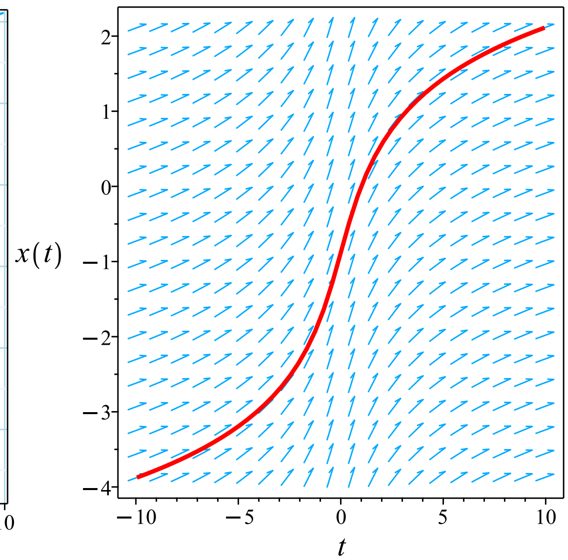
$$-\operatorname{arcsinh}(t) + x = -\ln\left(1 + \sqrt{2}\right)$$

Solving for x gives

$$x = \operatorname{arcsinh}(t) - \ln\left(1 + \sqrt{2}\right)$$



(a) Solution plot
$x = \operatorname{arcsinh}(t) - \ln\left(1+\sqrt{2}\right)$



(b) Slope field plot
$x' = \frac{1}{\sqrt{t^2+1}}$

Summary of solutions found

$$x = \operatorname{arcsinh}(t) - \ln\left(1+\sqrt{2}\right)$$

Maple step by step solution

Let's solve

$\left[x' = \frac{1}{\sqrt{t^2+1}}, x(1) = 0\right]$

- Highest derivative means the order of the ODE is 1
  x'
- Integrate both sides with respect to t
  $\int x' dt = \int \frac{1}{\sqrt{t^2+1}} dt + C1$
- Evaluate integral
  $x = \operatorname{arcsinh}(t) + C1$
- Solve for x
  $x = \operatorname{arcsinh}(t) + C1$
- Use initial condition $x(1) = 0$
  $0 = \ln\left(1+\sqrt{2}\right) + C1$
- Solve for $\_C1$

$$C1 = -\ln\left(1+\sqrt{2}\right)$$

- Substitute $\_C1 = -\ln\left(1+\sqrt{2}\right)$ into general solution and simplify

  $$x = \operatorname{arcsinh}(t) - \ln\left(1+\sqrt{2}\right)$$

- Solution to the IVP

  $$x = \operatorname{arcsinh}(t) - \ln\left(1+\sqrt{2}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 15

```
dsolve([diff(x(t),t) = 1/(t^2+1)^(1/2),
       op([x(1) = 0])],x(t),singsol=all)
```

$$x = \operatorname{arcsinh}(t) - \ln\left(1+\sqrt{2}\right)$$

Mathematica DSolve solution

Solving time : `0.022` (sec)
Leaf size : 26

```
DSolve[{D[x[t],t]==1/Sqrt[1+t^2],{x[1]==0}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \operatorname{arctanh}\left(\frac{t}{\sqrt{t^2+1}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{2}}\right)$$

2.1.5 problem 1 (v)

Internal problem ID [18168]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 1 (v)
Date solved : `Thursday, December 19, 2024 at 01:51:44 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = \cos(t)$$

With initial conditions

$$x(1) = 0$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + q(t)x = p(t)$$

Where here

$$q(t) = 0$$
$$p(t) = \cos(t)$$

Hence the ode is

$$x' = \cos(t)$$

The domain of $q(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $p(t) = \cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

Solved as first order quadrature ode

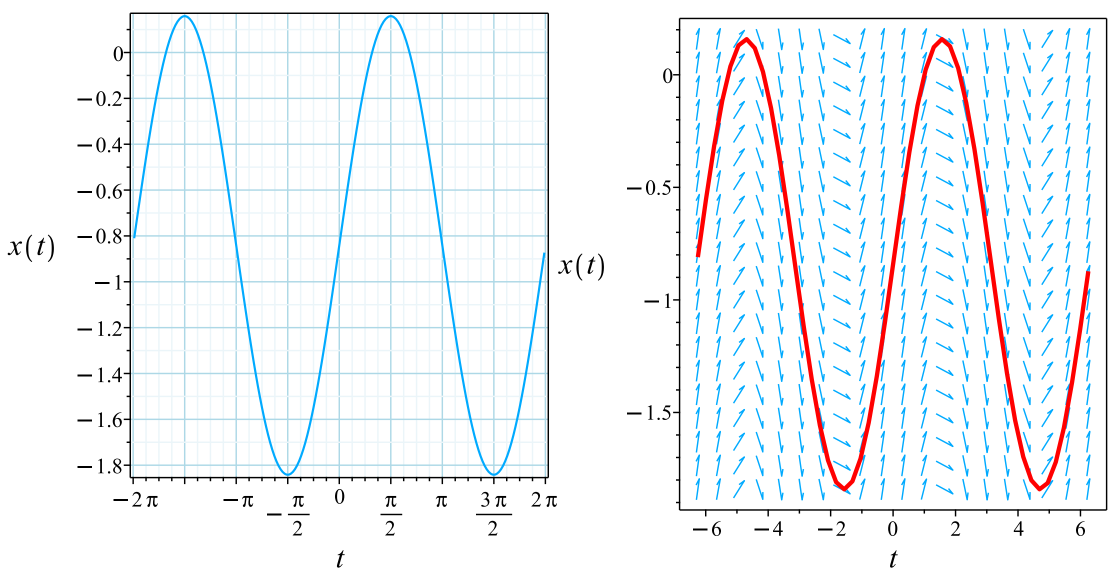
Time used: 0.109 (sec)

Since the ode has the form $x' = f(t)$, then we only need to integrate $f(t)$.

$$\int dx = \int \cos(t)\, dt$$
$$x = \sin(t) + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

$$x = \sin(t) - \sin(1)$$



(a) Solution plot $x = \sin(t) - \sin(1)$

(b) Slope field plot $x' = \cos(t)$

Summary of solutions found

$$x = \sin(t) - \sin(1)$$

Solved as first order Exact ode

Time used: 0.321 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x&=(\cos(t))\,\mathrm{d}t\\ (-\cos(t))\,\mathrm{d}t+\mathrm{d}x&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-\cos(t)\\ N(t,x)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-\cos(t)) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\cos(t) \, \mathrm{d}t \\ \phi &= -\sin(t) + f(x) \end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 1$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (1)\,\mathrm{d}x$$

$$f(x) = x + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\sin(t) + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

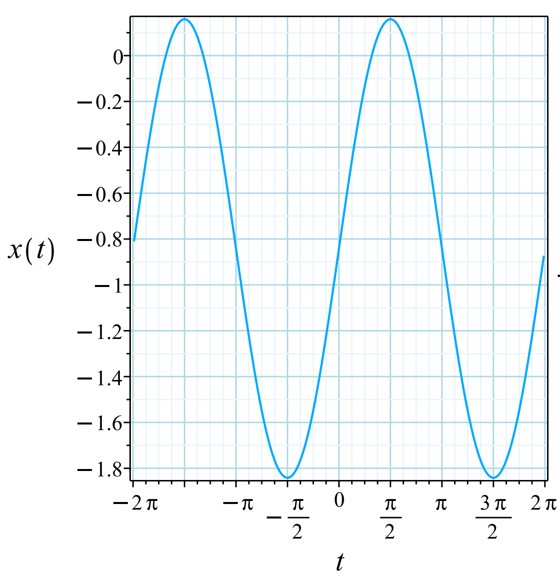
$$c_1 = -\sin(t) + x$$

Solving for the constant of integration from initial conditions, the solution becomes

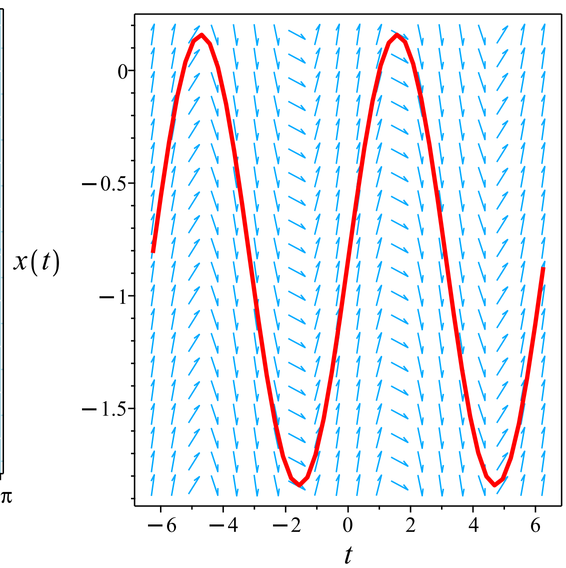
$$-\sin(t) + x = -\sin(1)$$

Solving for x gives

$$x = \sin(t) - \sin(1)$$



(a) Solution plot
$x = \sin(t) - \sin(1)$



(b) Slope field plot
$x' = \cos(t)$

Summary of solutions found

$$x = \sin(t) - \sin(1)$$

Maple step by step solution

Let's solve
$[x' = \cos(t), x(1) = 0]$

- Highest derivative means the order of the ODE is 1
  x'
- Integrate both sides with respect to t
  $\int x' dt = \int \cos(t) dt + C1$
- Evaluate integral
  $x = \sin(t) + C1$
- Solve for x
  $x = \sin(t) + C1$
- Use initial condition $x(1) = 0$
  $0 = \sin(1) + C1$
- Solve for _$C1$
  $C1 = -\sin(1)$
- Substitute _$C1 = -\sin(1)$ into general solution and simplify
  $x = \sin(t) - \sin(1)$
- Solution to the IVP
  $x = \sin(t) - \sin(1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 11

```
dsolve([diff(x(t),t) = cos(t),
       op([x(1) = 0])],x(t),singsol=all)
```

$$x = \sin(t) - \sin(1)$$

Mathematica DSolve solution

Solving time : 0.009 (sec)
Leaf size : 12

```
DSolve[{D[x[t],t]==Cos[t],{x[1]==0}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \sin(t) - \sin(1)$$

2.1.6 problem 1 (vi)

Internal problem ID [18169]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 1 (vi)
Date solved : `Thursday, December 19, 2024 at 01:51:45 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = \frac{\cos(t)}{\sin(t)}$$

With initial conditions

$$x(1) = 0$$

Solved as first order quadrature ode

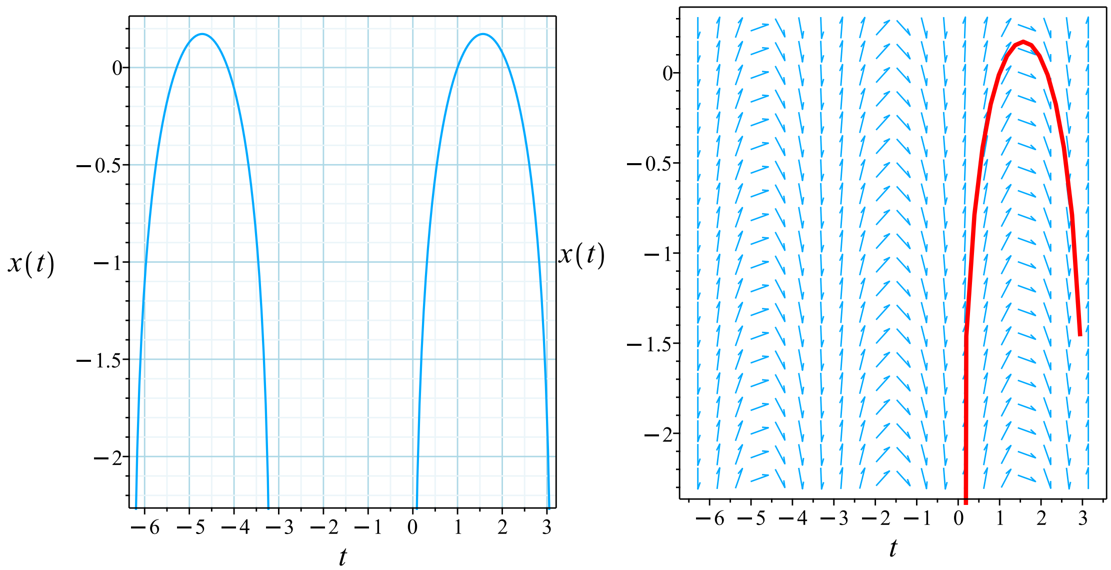
Time used: 0.146 (sec)

Since the ode has the form $x' = f(t)$, then we only need to integrate $f(t)$.

$$\int dx = \int \frac{\cos(t)}{\sin(t)}\, dt$$
$$x = \ln(\sin(t)) + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

$$x = \ln(\sin(t)) - \ln(\sin(1))$$



(a) Solution plot
$x = \ln(\sin(t)) - \ln(\sin(1))$

(b) Slope field plot
$x' = \frac{\cos(t)}{\sin(t)}$

<u>Summary of solutions found</u>

$$x = \ln(\sin(t)) - \ln(\sin(1))$$

Solved as first order Exact ode

Time used: 0.072 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = \left(\frac{\cos(t)}{\sin(t)}\right) \mathrm{d}t$$

$$\left(-\frac{\cos(t)}{\sin(t)}\right) \mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, x) &= -\frac{\cos(t)}{\sin(t)} \\ N(t, x) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\cos(t)}{\sin(t)}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int M \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int -\frac{\cos(t)}{\sin(t)} \,\mathrm{d}t$$

$$\phi = -\ln(\sin(t)) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 1$$

Integrating the above w.r.t x gives

$$\int f'(x) \,\mathrm{d}x = \int (1) \,\mathrm{d}x$$

$$f(x) = x + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(t)) + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

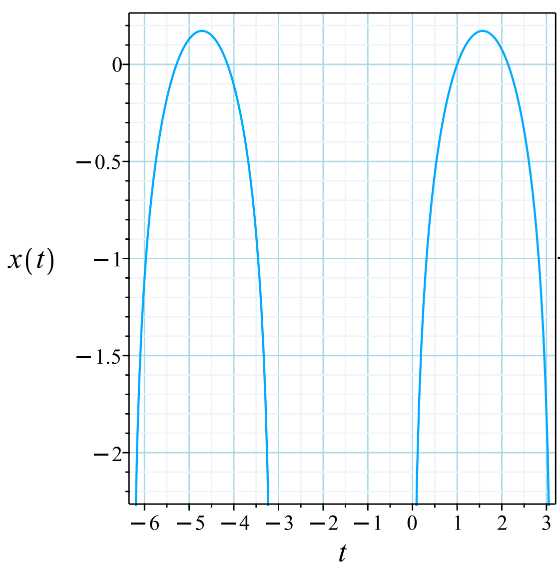
$$c_1 = -\ln(\sin(t)) + x$$

Solving for the constant of integration from initial conditions, the solution becomes

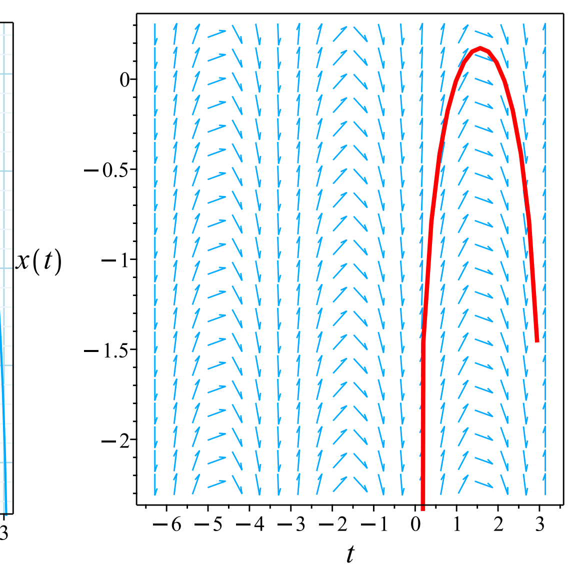
$$-\ln(\sin(t)) + x = -\ln(\sin(1))$$

Solving for x gives

$$x = \ln(\sin(t)) - \ln(\sin(1))$$



(a) Solution plot
$x = \ln(\sin(t)) - \ln(\sin(1))$



(b) Slope field plot
$x' = \frac{\cos(t)}{\sin(t)}$

<u>Summary of solutions found</u>

$$x = \ln(\sin(t)) - \ln(\sin(1))$$

Maple step by step solution

Let's solve

$\left[x' = \frac{\cos(t)}{\sin(t)}, x(1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int \frac{\cos(t)}{\sin(t)} dt + C1$

- Evaluate integral

  $x = \ln(\sin(t)) + C1$

- Solve for x

  $x = \ln(\sin(t)) + C1$

- Use initial condition $x(1) = 0$

  $0 = \ln(\sin(1)) + C1$

- Solve for $\_C1$

  $C1 = -\ln(\sin(1))$

- Substitute $\_C1 = -\ln(\sin(1))$ into general solution and simplify

  $x = \ln(\sin(t)) - \ln(\sin(1))$

- Solution to the IVP

  $x = \ln(\sin(t)) - \ln(\sin(1))$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 13

```
dsolve([diff(x(t),t) = cos(t)/sin(t),
       op([x(1) = 0])],x(t),singsol=all)
```

$$x = \ln(\sin(t)) - \ln(\sin(1))$$

Mathematica DSolve solution

Solving time : 0.009 (sec)
Leaf size : 11

```
DSolve[{D[x[t],t]==Cos[t]/Sin[t],{x[1]==0}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \log(\csc(1)\sin(t))$$

2.1.7 problem 2 (i)

Internal problem ID [18170]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 2 (i)
Date solved : `Thursday, December 19, 2024 at 01:51:47 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = x^2 - 3x + 2$$

With initial conditions

$$x(0) = 1$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= x^2 - 3x + 2 \end{aligned}$$

The x domain of $f(t,x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 - 3x + 2\right) \\ &= 2x - 3 \end{aligned}$$

The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

Solved as first order autonomous ode

Time used: 0.401 (sec)

Since the ode has the form $x' = f(x)$ and initial conditions (t_0, x_0) are given such that they satisfy the ode itself, then we can write

$$
\begin{aligned}
0 &= f(x)|_{x=x_0} \\
0 &= 0
\end{aligned}
$$

And the solution is immediately written as

$$
\begin{aligned}
x &= x_0 \\
x &= 1
\end{aligned}
$$

Singular solutions are found by solving

$$
x^2 - 3x + 2 = 0
$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
x = 1
$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

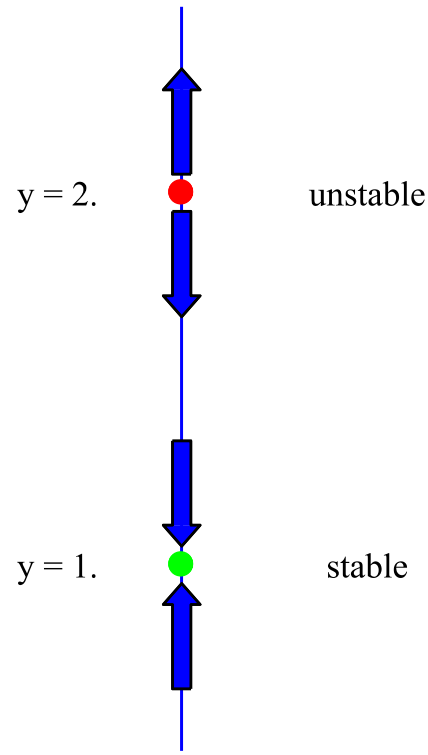
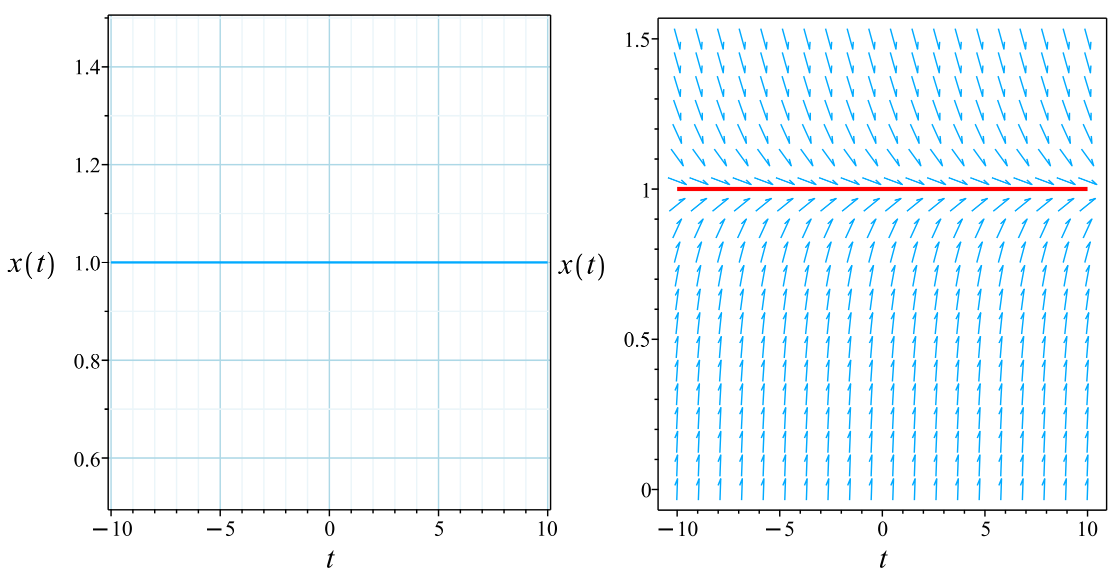


Figure 2.11: Phase line diagram



(a) Solution plot
$x = 1$

(b) Slope field plot
$x' = x^2 - 3x + 2$

Summary of solutions found

$$x = 1$$

Maple step by step solution

Let's solve
$[x' = x^2 - 3x + 2, x(0) = 1]$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = x^2 - 3x + 2$
- Separate variables
  $\frac{x'}{x^2-3x+2} = 1$
- Integrate both sides with respect to t
  $\int \frac{x'}{x^2-3x+2}dt = \int 1dt + C1$
- Evaluate integral
  $\ln(x-2) - \ln(x-1) = t + C1$
- Solve for x
  $x = \frac{-2+e^{t+C1}}{e^{t+C1}-1}$
- Use initial condition $x(0) = 1$
  $1 = \frac{-2+e^{C1}}{e^{C1}-1}$
- Solve for $\_C1$
  $C1 = ()$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 5

```
dsolve([diff(x(t),t) = x(t)^2-3*x(t)+2,
       op([x(0) = 1])],x(t),singsol=all)
```

$$x = 1$$

Mathematica DSolve solution

Solving time : 0.001 (sec)
Leaf size : 6

```
DSolve[{D[x[t],t]==x[t]^2-3*x[t]+2,{x[0]==1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to 1$$

2.1.8 problem 2 (ii)

Internal problem ID [18171]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 2 (ii)
Date solved : `Thursday, December 19, 2024 at 01:51:51 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = b\,e^{x}$$

With initial conditions

$$x(0) = 1$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t,x) \\ &= b\,e^{x} \end{aligned}$$

The x domain of $f(t,x)$ when $t=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(b\,e^{x}) \\ &= b\,e^{x} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

Solved as first order autonomous ode

Time used: 0.110 (sec)

Integrating gives

$$\int \frac{e^{-x}}{b} dx = dt$$
$$-\frac{e^{-x}}{b} = t + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

$$-\frac{e^{-x}}{b} = t - \frac{e^{-1}}{b}$$

Solving for x gives

$$x = -\ln\left(-(tb\,e - 1)\,e^{-1}\right)$$

Summary of solutions found

$$x = -\ln\left(-(tb\,e - 1)\,e^{-1}\right)$$

Solved as first order Exact ode

Time used: 0.375 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x &= (b\,\mathrm{e}^{x})\,\mathrm{d}t\\ (-b\,\mathrm{e}^{x})\,\mathrm{d}t + \mathrm{d}x &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -b\,\mathrm{e}^{x}\\ N(t, x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-b\,\mathrm{e}^{x})\\ &= -b\,\mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1((-b\,\mathrm{e}^{x}) - (0)) \\
&= -b\,\mathrm{e}^{x}
\end{aligned}
$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right) \\
&= -\frac{\mathrm{e}^{-x}}{b}((0) - (-b\,\mathrm{e}^{x})) \\
&= -1
\end{aligned}
$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}x} \\
&= e^{\int -1 \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-b\,\mathrm{e}^{x}) \\
&= -b
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\\ (-b) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -b\,\mathrm{d}t\end{aligned}$$

$$\phi = -tb + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \mathrm{e}^{-x}$$

Integrating the above w.r.t x gives

$$\begin{aligned}\int f'(x)\,\mathrm{d}x &= \int \left(\mathrm{e}^{-x}\right)\mathrm{d}x\\ f(x) &= -\mathrm{e}^{-x} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -tb - \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -tb - \mathrm{e}^{-x}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$-tb - \mathrm{e}^{-x} = -\mathrm{e}^{-1}$$

Solving for x gives

$$x = -\ln\left(-(tb\,\mathrm{e} - 1)\,\mathrm{e}^{-1}\right)$$

Summary of solutions found

$$x = -\ln\left(-(tb\,\mathrm{e} - 1)\,\mathrm{e}^{-1}\right)$$

Solved using Lie symmetry for first order ode

Time used: 0.569 (sec)

Writing the ode as

$$x' = b\,\mathrm{e}^{x}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = t a_2 + x a_3 + a_1 \tag{1E}$$
$$\eta = t b_2 + x b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + b\,\mathrm{e}^x(b_3 - a_2) - b^2\mathrm{e}^{2x}a_3 - b\,\mathrm{e}^x(tb_2 + xb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-b^2\mathrm{e}^{2x}a_3 - \mathrm{e}^x btb_2 - \mathrm{e}^x bxb_3 - b\,\mathrm{e}^x a_2 - \mathrm{e}^x bb_1 + b\,\mathrm{e}^x b_3 + b_2 = 0$$

Setting the numerator to zero gives

$$-b^2\mathrm{e}^{2x}a_3 - \mathrm{e}^x btb_2 - \mathrm{e}^x bxb_3 - b\,\mathrm{e}^x a_2 - \mathrm{e}^x bb_1 + b\,\mathrm{e}^x b_3 + b_2 = 0 \tag{6E}$$

Simplifying the above gives

$$-b^2\mathrm{e}^{2x}a_3 - \mathrm{e}^x btb_2 - \mathrm{e}^x bxb_3 - b\,\mathrm{e}^x a_2 - \mathrm{e}^x bb_1 + b\,\mathrm{e}^x b_3 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x, \mathrm{e}^x, \mathrm{e}^{2x}\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2, \mathrm{e}^x = v_3, \mathrm{e}^{2x} = v_4\}$$

The above PDE (6E) now becomes

$$-b^2v_4a_3 - v_3bv_1b_2 - v_3bv_2b_3 - bv_3a_2 - v_3bb_1 + bv_3b_3 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-v_3bv_1b_2 - v_3bv_2b_3 + (-ba_2 - bb_1 + bb_3)\,v_3 - b^2v_4a_3 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-bb_2 &= 0 \\
-bb_3 &= 0 \\
-b^2 a_3 &= 0 \\
-ba_2 - bb_1 + bb_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= -b_1 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t, x)\,\xi \\
&= 0 - (b\,\mathrm{e}^{x})\,(1) \\
&= -b\,\mathrm{e}^{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-b\,\mathrm{e}^{x}} dy \end{aligned}$$

Which results in

$$S = \frac{\mathrm{e}^{-x}}{b}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = b\,\mathrm{e}^{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 0 \\ S_x &= -\frac{\mathrm{e}^{-x}}{b} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -1\, dR$$
$$S(R) = -R + c_2$$

To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$\frac{e^{-x}}{b} = -t + c_2$$

Which gives

$$x = -\ln\left(c_2 b - tb\right)$$

Solving for the constant of integration from initial conditions, the solution becomes

$$x = -\ln\left(-tb + e^{-1}\right)$$

<u>Summary of solutions found</u>

$$x = -\ln\left(-tb + e^{-1}\right)$$

Solved as first order ode of type ID 1

Time used: 0.092 (sec)

Writing the ode as

$$x' = b\,e^{x} \tag{1}$$

And using the substitution $u = e^{-x}$ then

$$u' = -x' e^{-x}$$

The above shows that

$$\begin{aligned} x' &= -u'(t)\, e^{x} \\ &= -\frac{u'(t)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(t)}{u} = \frac{b}{u}$$

The above simplifies to

$$u'(t) = -b \tag{2}$$

Now ode (2) is solved for $u(t)$.

Since the ode has the form $u'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\int du = \int -b\,dt$$
$$u(t) = -tb + c_1$$

Substituting the solution found for $u(t)$ in $u = e^{-x}$ gives

$$\begin{aligned} x &= -\ln(u(t)) \\ &= -\ln(-\ln(-tb + c_1)) \\ &= -\ln(-tb + c_1) \end{aligned}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$x = -\ln\left(-tb + e^{-1}\right)$$

Summary of solutions found

$$x = -\ln\left(-tb + e^{-1}\right)$$

Maple step by step solution

Let's solve
$[x' = b\,e^{x}, x(0) = 1]$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = b\,e^{x}$
- Separate variables
  $\frac{x'}{e^{x}} = b$
- Integrate both sides with respect to t
  $\int \frac{x'}{e^{x}} dt = \int b dt + C1$
- Evaluate integral

$-\frac{1}{e^x} = tb + C1$

- Solve for x

  $x = \ln\left(-\frac{1}{tb+C1}\right)$
- Use initial condition $x(0) = 1$

  $1 = \ln\left(-\frac{1}{C1}\right)$
- Solve for $\_C1$

  $C1 = -\frac{1}{e}$
- Substitute $\_C1 = -\frac{1}{e}$ into general solution and simplify

  $x = \ln\left(\frac{1}{-tb+e^{-1}}\right)$
- Solution to the IVP

  $x = \ln\left(\frac{1}{-tb+e^{-1}}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 14

```
dsolve([diff(x(t),t) = b*exp(x(t)),
       op([x(0) = 1])],x(t),singsol=all)
```

$$x = -\ln\left(-tb + e^{-1}\right)$$

Mathematica DSolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
DSolve[{D[x[t],t]==b*Exp[x[t]],{x[0]==1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to 1 - \log(1 - ebt)$$

2.1.9 problem 2 (iii)

Internal problem ID [18172]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 2 (iii)
Date solved : `Thursday, December 19, 2024 at 01:51:52 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = (x-1)^2$$

With initial conditions

$$x(0) = 1$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= (x-1)^2 \end{aligned}$$

The x domain of $f(t, x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left((x-1)^2\right) \\ &= 2x - 2 \end{aligned}$$

The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

Solved as first order autonomous ode

Time used: 0.083 (sec)

Since the ode has the form $x' = f(x)$ and initial conditions (t_0, x_0) are given such that they satisfy the ode itself, then we can write

$$\begin{aligned} 0 &= f(x)|_{x=x_0} \\ 0 &= 0 \end{aligned}$$

And the solution is immediately written as

$$\begin{aligned} x &= x_0 \\ x &= 1 \end{aligned}$$

Singular solutions are found by solving

$$(x-1)^2 = 0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x = 1$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

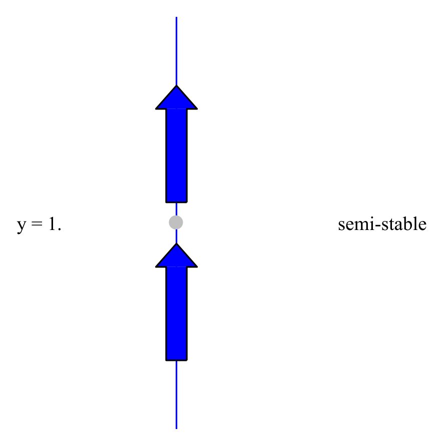
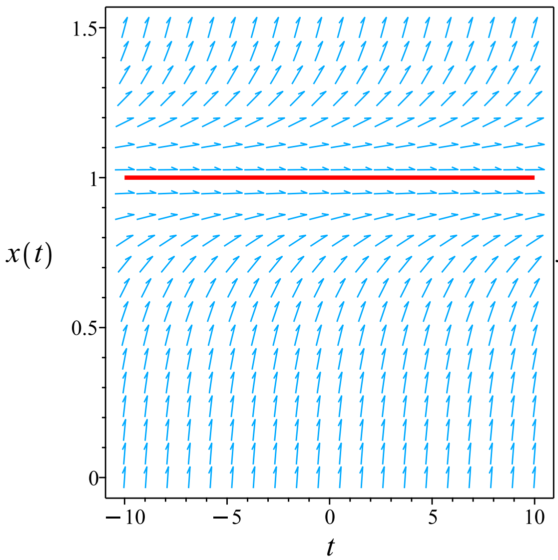
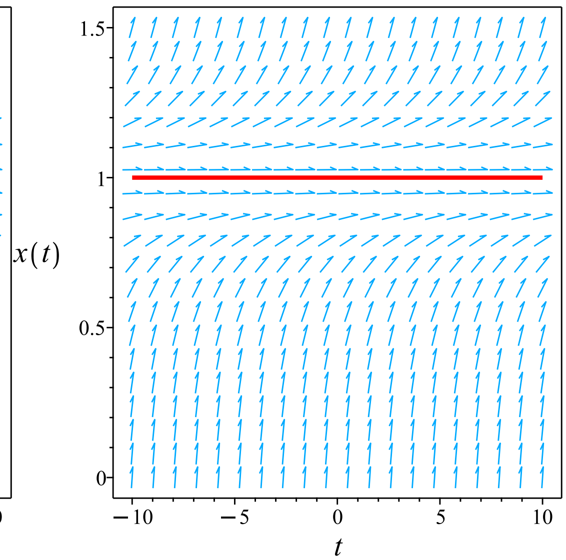


Figure 2.13: Phase line diagram



(a) Solution plot
$x = 1$



(b) Slope field plot
$x' = (x-1)^2$

Summary of solutions found

$$x = 1$$

Solved as first order Exact ode

Time used: 0.205 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}x &= \left((x-1)^2\right)\mathrm{d}t \\
\left(-(x-1)^2\right)\mathrm{d}t + \mathrm{d}x &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(t, x) &= -(x-1)^2 \\
N(t, x) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-(x-1)^2\right) \\
&= -2x + 2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1((-2x + 2) - (0)) \\
&= -2x + 2
\end{aligned}$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right) \\ &= -\frac{1}{(x-1)^2}((0) - (-2x+2)) \\ &= -\frac{2}{x-1} \end{aligned}$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x-1} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(x-1)} \\ &= \frac{1}{(x-1)^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{(x-1)^2}\left(-(x-1)^2\right) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{(x-1)^2}(1) \\ &= \frac{1}{(x-1)^2} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ (-1) + \left(\frac{1}{(x-1)^2}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial\phi}{\partial t}=\overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial x}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t=\int\overline{M}\,\mathrm{d}t$$

$$\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t=\int -1\,\mathrm{d}t$$

$$\phi=-t+f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x}=0+f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial x}=\frac{1}{(x-1)^2}$. Therefore equation (4) becomes

$$\frac{1}{(x-1)^2}=0+f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x)=\frac{1}{(x-1)^2}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x=\int\left(\frac{1}{(x-1)^2}\right)\mathrm{d}x$$

$$f(x)=-\frac{1}{x-1}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

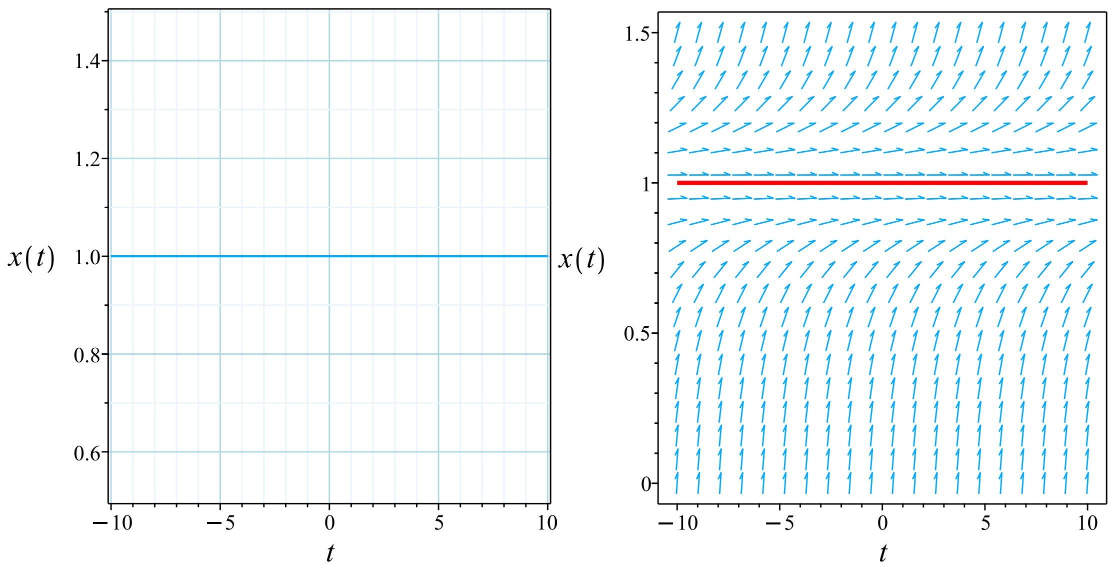
$$\phi=-t-\frac{1}{x-1}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -t - \frac{1}{x-1}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$x = 1$$



(a) Solution plot
$x = 1$

(b) Slope field plot
$x' = (x-1)^2$

Summary of solutions found

$$x = 1$$

Solved using Lie symmetry for first order ode

Time used: 0.560 (sec)

Writing the ode as

$$x' = (x-1)^2$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$

$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (x-1)^2 (b_3 - a_2) - (x-1)^4 a_3 - (2x-2)(tb_2 + xb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\begin{gathered}-x^4 a_3 + 4x^3 a_3 - 2txb_2 - x^2 a_2 - 6x^2 a_3 - x^2 b_3 + 2tb_2 \\ + 2xa_2 + 4xa_3 - 2xb_1 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0\end{gathered}$$

Setting the numerator to zero gives

$$\begin{gathered}-x^4 a_3 + 4x^3 a_3 - 2txb_2 - x^2 a_2 - 6x^2 a_3 - x^2 b_3 + 2tb_2 \\ + 2xa_2 + 4xa_3 - 2xb_1 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0\end{gathered} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$\begin{gathered}-a_3 v_2^4 + 4a_3 v_2^3 - a_2 v_2^2 - 6a_3 v_2^2 - 2b_2 v_1 v_2 - b_3 v_2^2 + 2a_2 v_2 \\ + 4a_3 v_2 - 2b_1 v_2 + 2b_2 v_1 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0\end{gathered} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-2b_2v_1v_2 + 2b_2v_1 - a_3v_2^4 + 4a_3v_2^3 + (-a_2 - 6a_3 - b_3)\,v_2^2 + (2a_2 + 4a_3 - 2b_1)\,v_2 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_3 &= 0\\
4a_3 &= 0\\
-2b_2 &= 0\\
2b_2 &= 0\\
-a_2 - 6a_3 - b_3 &= 0\\
2a_2 + 4a_3 - 2b_1 &= 0\\
-a_2 - a_3 + 2b_1 + b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= a_1\\
a_2 &= -b_3\\
a_3 &= 0\\
b_1 &= -b_3\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 1\\
\eta &= 0
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(t, x)\,\xi\\
&= 0 - \left((x-1)^2\right)(1)\\
&= -x^2 + 2x - 1\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-x^2 + 2x - 1} dy \end{aligned}$$

Which results in

$$S = \frac{1}{x - 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = (x - 1)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 0 \\ S_x &= -\frac{1}{(x - 1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}\int dS &= \int -1\, dR \\ S(R) &= -R + c_2\end{aligned}$$

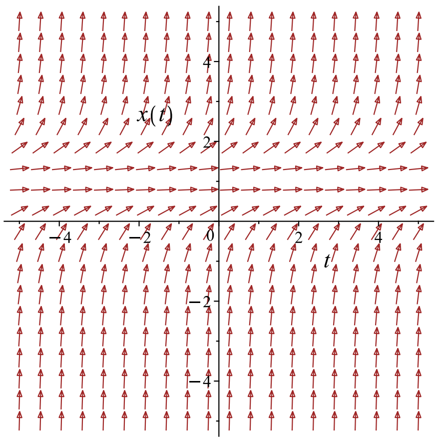
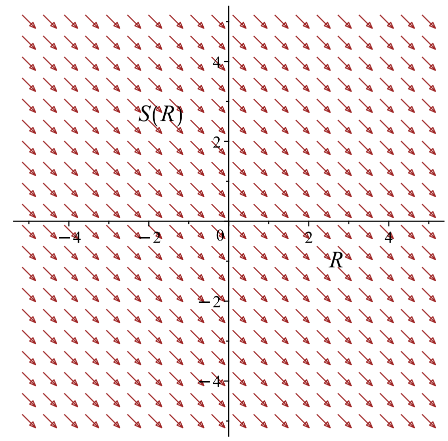
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$\frac{1}{x-1} = -t + c_2$$

Which gives

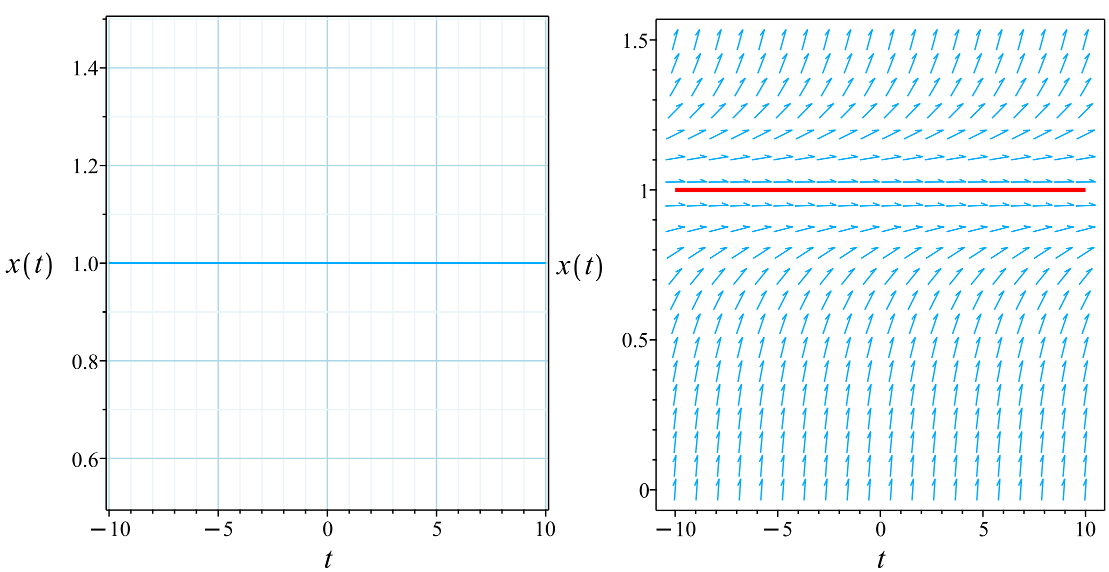
$$x = \frac{c_2 - t + 1}{-t + c_2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = (x-1)^2$ 	$R = t$ $S = \frac{1}{x-1}$	$\frac{dS}{dR} = -1$ 

Solving for the constant of integration from initial conditions, the solution becomes

$$x = 1$$



(a) Solution plot
$x = 1$

(b) Slope field plot
$x' = (x-1)^2$

Summary of solutions found

$$x = 1$$

Maple step by step solution

Let's solve
$\left[x' = (x-1)^2, x(0) = 1\right]$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = (x-1)^2$
- Separate variables
  $\frac{x'}{(x-1)^2} = 1$
- Integrate both sides with respect to t
  $\int \frac{x'}{(x-1)^2} dt = \int 1 dt + C1$
- Evaluate integral
  $-\frac{1}{x-1} = t + C1$
- Solve for x
  $x = \frac{C1+t-1}{t+C1}$
- Use initial condition $x(0) = 1$
  $1 = \frac{C1-1}{C1}$
- Solve for $\_C1$
  $C1 = ()$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 5

```
dsolve([diff(x(t),t) = (x(t)-1)^2,
       op([x(0) = 1])],x(t),singsol=all)
```

$$x = 1$$

Mathematica DSolve solution

Solving time : 0.001 (sec)
Leaf size : 6

```
DSolve[{D[x[t],t]==(x[t]-1)^2,{x[0]==1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to 1$$

2.1.10 problem 2 (iv)

Internal problem ID [18173]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 2 (iv)
Date solved : `Thursday, December 19, 2024 at 01:51:54 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = \sqrt{x^2 - 1}$$

With initial conditions

$$x(0) = 1$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= \sqrt{x^2 - 1} \end{aligned}$$

The x domain of $f(t, x)$ when $t = 0$ is

$$\{1 \leq x \leq \infty, -\infty \leq x \leq -1\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{x^2 - 1}\right) \\ &= \frac{x}{\sqrt{x^2 - 1}} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty \leq x < -1, -1 < x < 1, 1 < x \leq \infty\}$$

But the point $x_0 = 1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

Solved as first order autonomous ode

Time used: 0.126 (sec)

Since the ode has the form $x' = f(x)$ and initial conditions (t_0, x_0) are given such that they satisfy the ode itself, then we can write

$$\begin{aligned} 0 &= f(x)|_{x=x_0} \\ 0 &= 0 \end{aligned}$$

And the solution is immediately written as

$$\begin{aligned} x &= x_0 \\ x &= 1 \end{aligned}$$

Singular solutions are found by solving

$$\sqrt{x^2 - 1} = 0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x = 1$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

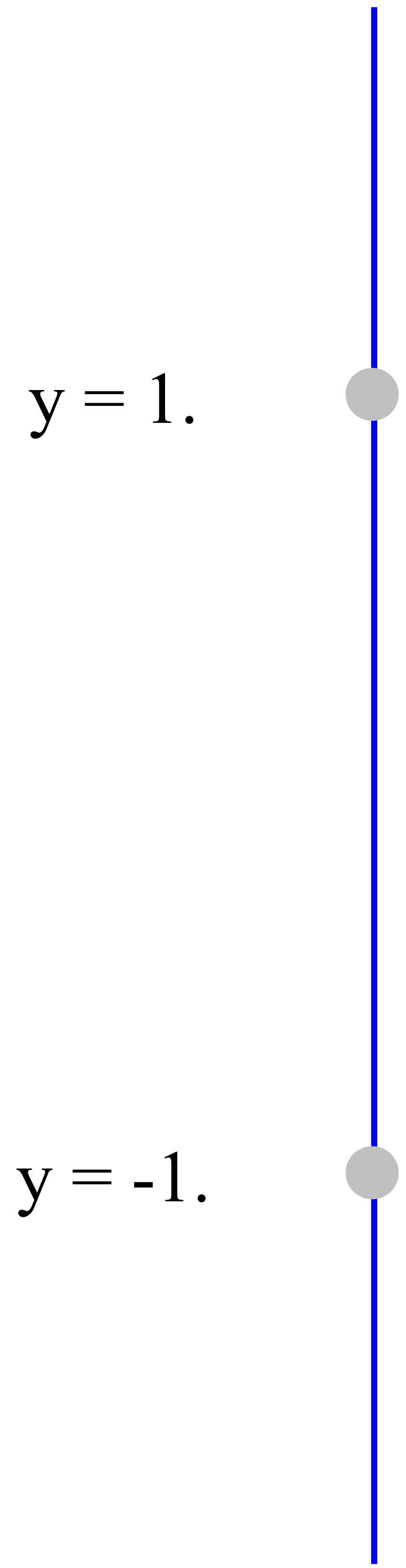
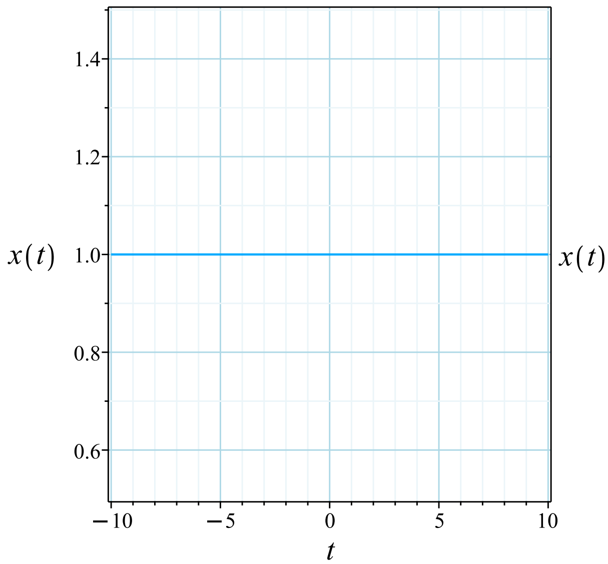
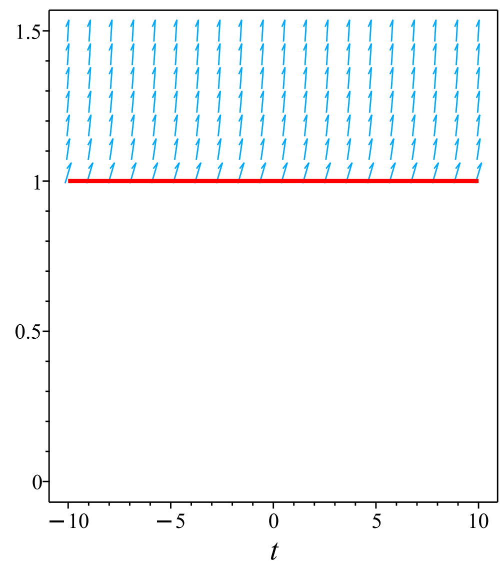


Figure 2.17: Phase line diagram



(a) Solution plot
$x = 1$



(b) Slope field plot
$x' = \sqrt{x^2 - 1}$

Summary of solutions found

$$x = 1$$

Solved as first order Exact ode

Time used: 8.006 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = \left(\sqrt{x^2-1}\right)\mathrm{d}t$$
$$\left(-\sqrt{x^2-1}\right)\mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,x) = -\sqrt{x^2-1}$$
$$N(t,x) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\sqrt{x^2-1}\right)\\ &= -\frac{x}{\sqrt{x^2-1}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(-\frac{x}{\sqrt{x^2-1}}\right) - (0)\right) \\
&= -\frac{x}{\sqrt{x^2-1}}
\end{aligned}
$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right) \\
&= -\frac{1}{\sqrt{x^2-1}}\left((0) - \left(-\frac{x}{\sqrt{x^2-1}}\right)\right) \\
&= -\frac{x}{x^2-1}
\end{aligned}
$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}x} \\
&= e^{\int -\frac{x}{x^2-1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\
&= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\left(-\sqrt{x^2-1}\right) \\
&= -\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}
\end{aligned}
$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}(1) \\ &= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left(-\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\right) + \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\,\mathrm{d}t \\ \phi &= -\frac{\sqrt{x^2-1}\,t}{\sqrt{x-1}\,\sqrt{x+1}} + f(x)\end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -\frac{tx}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{x+1}} + \frac{\sqrt{x^2-1}\,t}{2\,(x-1)^{3/2}\,\sqrt{x+1}} + \frac{\sqrt{x^2-1}\,t}{2\sqrt{x-1}\,(x+1)^{3/2}} + f'(x) \tag{4}$$

$$= 0 + f'(x)$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{\sqrt{x-1}\sqrt{x+1}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{x-1}\,\sqrt{x+1}} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$$

Integrating the above w.r.t x gives

$$\int f'(x)\, dx = \int \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right) dx$$

$$f(x) = \frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{\sqrt{x^2-1}\,t}{\sqrt{x-1}\,\sqrt{x+1}} + \frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{\sqrt{x^2-1}\,t}{\sqrt{x-1}\,\sqrt{x+1}} + \frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$-\frac{\sqrt{x^2-1}\,t}{\sqrt{x-1}\,\sqrt{x+1}} + \frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}} = 0$$

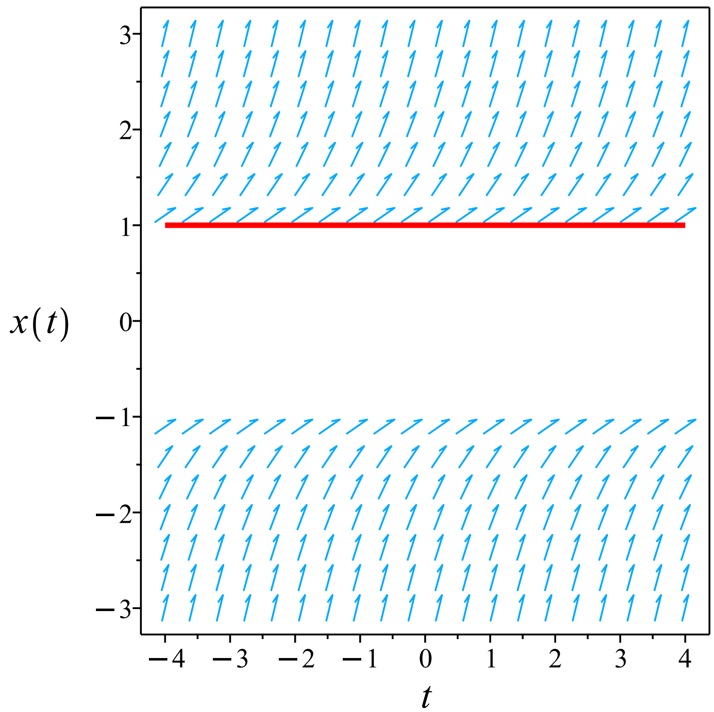


Figure 2.19: Slope field plot
$x' = \sqrt{x^2 - 1}$

Summary of solutions found

$$-\frac{\sqrt{x^2-1}\,t}{\sqrt{x-1}\,\sqrt{x+1}} + \frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}} = 0$$

Solved using Lie symmetry for first order ode

Time used: 1.941 (sec)

Writing the ode as

$$x' = \sqrt{x^2-1}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = t a_2 + x a_3 + a_1 \tag{1E}$$
$$\eta = t b_2 + x b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \sqrt{x^2-1}\,(b_3 - a_2) - \left(x^2-1\right) a_3 - \frac{x(tb_2 + xb_3 + b_1)}{\sqrt{x^2-1}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\sqrt{x^2-1}\,x^2a_3 + txb_2 + x^2a_2 - a_3\sqrt{x^2-1} - b_2\sqrt{x^2-1} + xb_1 - a_2 + b_3}{\sqrt{x^2-1}} = 0$$

Setting the numerator to zero gives

$$-\sqrt{x^2-1}\,x^2a_3 - txb_2 - x^2a_2 + a_3\sqrt{x^2-1} + b_2\sqrt{x^2-1} - xb_1 + a_2 - b_3 = 0 \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-\sqrt{x^2-1}\,x^2a_3 - \left(x^2-1\right) a_2 + \left(x^2-1\right) b_3 - txb_2 \\ &- x^2b_3 + a_3\sqrt{x^2-1} + b_2\sqrt{x^2-1} - xb_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-\sqrt{x^2-1}\,x^2a_3 - txb_2 - x^2a_2 + a_3\sqrt{x^2-1} + b_2\sqrt{x^2-1} - xb_1 + a_2 - b_3 = 0$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\left\{t, x, \sqrt{x^2-1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\left\{t = v_1, x = v_2, \sqrt{x^2-1} = v_3\right\}$$

The above PDE (6E) now becomes

$$-v_3v_2^2a_3 - v_2^2a_2 - v_1v_2b_2 + a_3v_3 - v_2b_1 + b_2v_3 + a_2 - b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-v_1 v_2 b_2 - v_3 v_2^2 a_3 - v_2^2 a_2 - v_2 b_1 + (a_3 + b_2) v_3 - b_3 + a_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_2 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ a_3 + b_2 &= 0 \\ -b_3 + a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 0 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(t, x)\, \xi \\ &= 0 - \left(\sqrt{x^2 - 1}\right)(1) \\ &= -\sqrt{x^2 - 1} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\sqrt{x^2 - 1}} dy \end{aligned}$$

Which results in

$$S = -\ln\left(x + \sqrt{x^2 - 1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = \sqrt{x^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 0 \\ S_x &= -\frac{1}{\sqrt{x^2 - 1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}\int dS &= \int -1\, dR \\ S(R) &= -R + c_2\end{aligned}$$

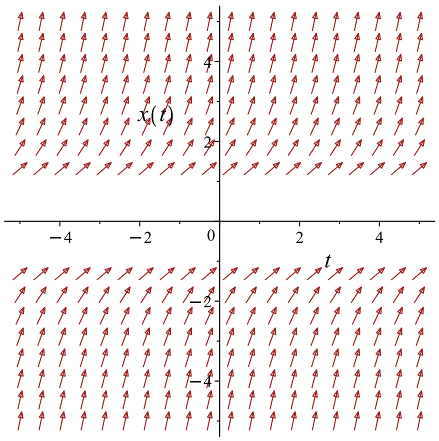
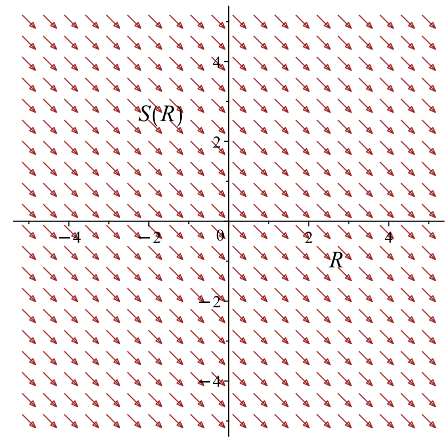
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$-\ln\left(x + \sqrt{x^2 - 1}\right) = -t + c_2$$

Which gives

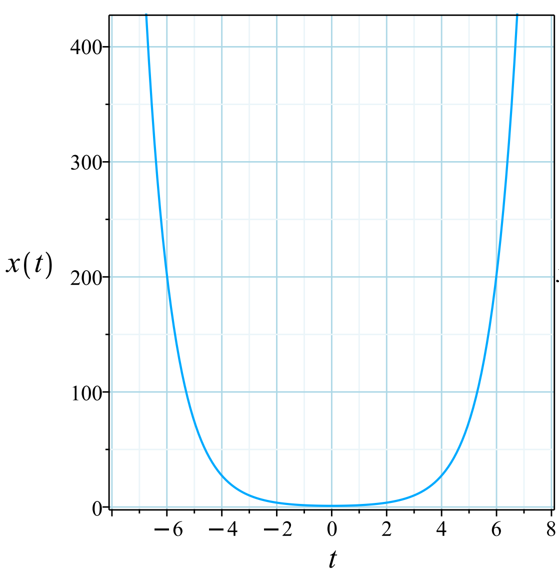
$$x = \frac{(\mathrm{e}^{2t-2c_2} + 1)\,\mathrm{e}^{-t+c_2}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

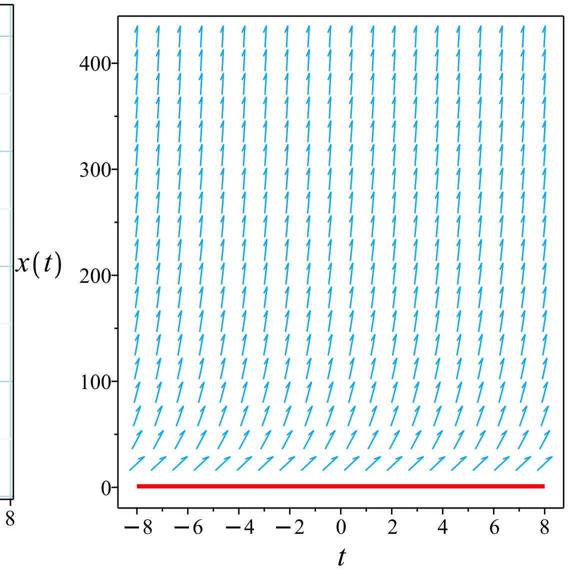
Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \sqrt{x^2 - 1}$ 	$R = t$ $S = -\ln\left(x + \sqrt{x^2 - 1}\right)$	$\frac{dS}{dR} = -1$ 

Solving for the constant of integration from initial conditions, the solution becomes

$$x = \frac{(e^{2t} + 1)\,e^{-t}}{2}$$



(a) Solution plot
$x = \frac{(e^{2t}+1)e^{-t}}{2}$



(b) Slope field plot
$x' = \sqrt{x^2 - 1}$

Summary of solutions found

$$x = \frac{(e^{2t} + 1)\, e^{-t}}{2}$$

Maple step by step solution

Let's solve

$\left[x' = \sqrt{x^2 - 1}, x(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  x'
- Solve for the highest derivative

  $x' = \sqrt{x^2 - 1}$
- Separate variables

  $\frac{x'}{\sqrt{x^2-1}} = 1$
- Integrate both sides with respect to t

  $\int \frac{x'}{\sqrt{x^2-1}} dt = \int 1 dt + C1$
- Evaluate integral

  $\ln\left(x + \sqrt{x^2 - 1}\right) = t + C1$
- Solve for x

  $x = \frac{\left(e^{t+C1}\right)^2 + 1}{2\, e^{t+C1}}$
- Use initial condition $x(0) = 1$

  $1 = \frac{\left(e^{C1}\right)^2 + 1}{2\, e^{C1}}$
- Solve for $\_C1$

  $C1 = 0$
- Substitute $\_C1 = 0$ into general solution and simplify

  $x = \frac{e^t}{2} + \frac{e^{-t}}{2}$
- Solution to the IVP

  $x = \frac{e^t}{2} + \frac{e^{-t}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 5

```
dsolve([diff(x(t),t) = (x(t)^2-1)^(1/2),
       op([x(0) = 1])],x(t),singsol=all)
```

$$x = 1$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[x[t],t]==Sqrt[x[t]^2-1],{x[0]==1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$\{\}$$

2.1.11 problem 2 (v)

Internal problem ID [18174]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 2 (v)
Date solved : `Thursday, December 19, 2024 at 01:52:05 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = 2\sqrt{x}$$

With initial conditions

$$x(0) = 1$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= 2\sqrt{x} \end{aligned}$$

The x domain of $f(t, x)$ when $t = 0$ is

$$\{0 \leq x\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(2\sqrt{x}) \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

Solved as first order autonomous ode

Time used: 0.207 (sec)

Integrating gives

$$\int \frac{1}{2\sqrt{x}} dx = dt$$
$$\sqrt{x} = t + c_1$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

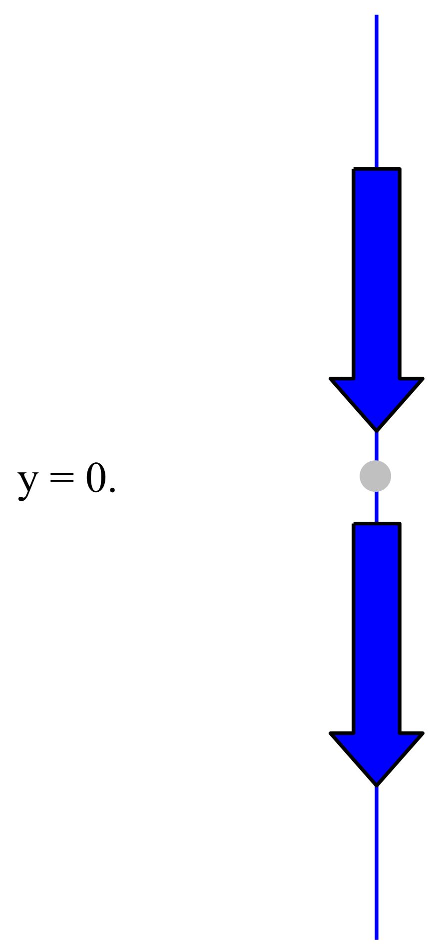


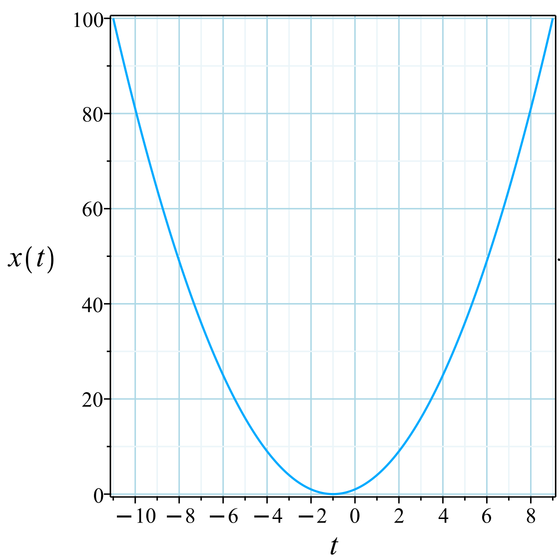
Figure 2.21: Phase line diagram

Solving for the constant of integration from initial conditions, the solution becomes

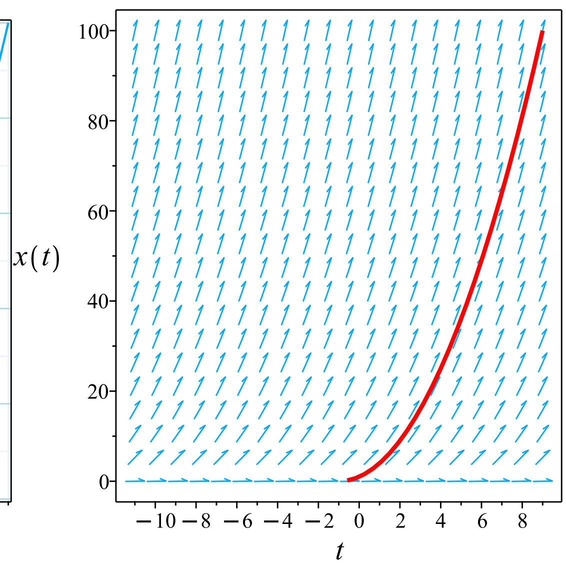
$$\sqrt{x} = t + 1$$

Solving for x gives

$$x = t^2 + 2t + 1$$



(a) Solution plot
$x = t^2 + 2t + 1$



(b) Slope field plot
$x' = 2\sqrt{x}$

Summary of solutions found

$$x = t^2 + 2t + 1$$

Solved as first order Bernoulli ode

Time used: 0.074 (sec)

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= 2\sqrt{x} \end{aligned}$$

This is a Bernoulli ODE.

$$x' = (2)\sqrt{x} \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(t)x + f_1(t)x^n \tag{2}$$

Comparing this to (1) shows that

$$
\begin{aligned}
f_0 &= 0\\
f_1 &= 2
\end{aligned}
$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_1(t) \tag{3}$$

The next step is use the substitution $v = x^{1-n}$ in equation (3) which generates a new ODE in $v(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(t) &= 0\\
f_1(t) &= 2\\
n &= \frac{1}{2}
\end{aligned}
$$

Dividing both sides of ODE (1) by $x^n = \sqrt{x}$ gives

$$x'\frac{1}{\sqrt{x}} = 0 + 2 \tag{4}$$

Let

$$
\begin{aligned}
v &= x^{1-n}\\
&= \sqrt{x} \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t t gives

$$v' = \frac{1}{2\sqrt{x}}x' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
2v'(t) &= 2\\
v' &= 1 \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $v(t)$ which is now solved.

Since the ode has the form $v'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\int dv = \int 1\, dt$$
$$v(t) = t + c_1$$

The substitution $v = x^{1-n}$ is now used to convert the above solution back to x which results in

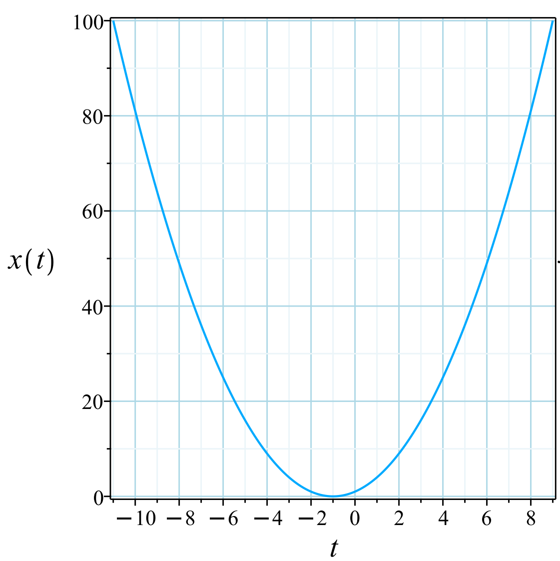
$$\sqrt{x} = t + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

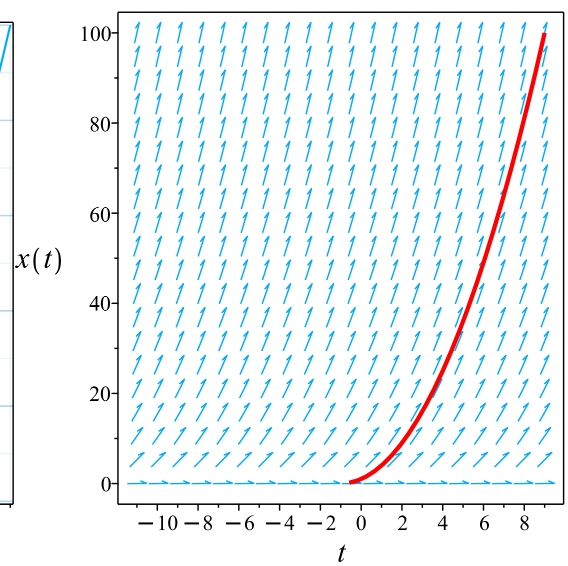
$$\sqrt{x} = t + 1$$

Solving for x gives

$$x = t^2 + 2t + 1$$



(a) Solution plot
$x = t^2 + 2t + 1$



(b) Slope field plot
$x' = 2\sqrt{x}$

Summary of solutions found

$$x = t^2 + 2t + 1$$

Solved as first order Exact ode

Time used: 0.109 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x&=\left(2\sqrt{x}\right)\mathrm{d}t\\ \left(-2\sqrt{x}\right)\mathrm{d}t+\mathrm{d}x&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-2\sqrt{x}\\ N(t,x)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-2\sqrt{x}) \\ &= -\frac{1}{\sqrt{x}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\ &= 1\left(\left(-\frac{1}{\sqrt{x}}\right) - (0)\right) \\ &= -\frac{1}{\sqrt{x}}\end{aligned}$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right) \\ &= -\frac{1}{2\sqrt{x}}\left((0) - \left(-\frac{1}{\sqrt{x}}\right)\right) \\ &= -\frac{1}{2x}\end{aligned}$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}x} \\ &= e^{\int -\frac{1}{2x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln(x)}{2}} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{x}}(-2\sqrt{x}) \\
&= -2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{x}}(1) \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
(-2) + \left(\frac{1}{\sqrt{x}}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -2\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -2t + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{\sqrt{x}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{x}} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \frac{1}{\sqrt{x}}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(\frac{1}{\sqrt{x}}\right) \mathrm{d}x$$
$$f(x) = 2\sqrt{x} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -2t + 2\sqrt{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

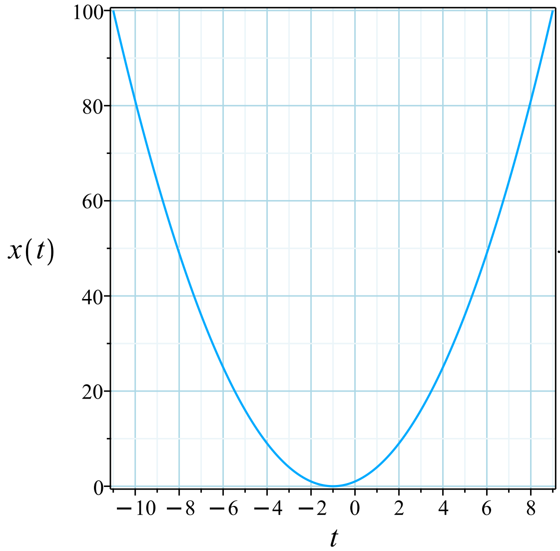
$$c_1 = -2t + 2\sqrt{x}$$

Solving for the constant of integration from initial conditions, the solution becomes

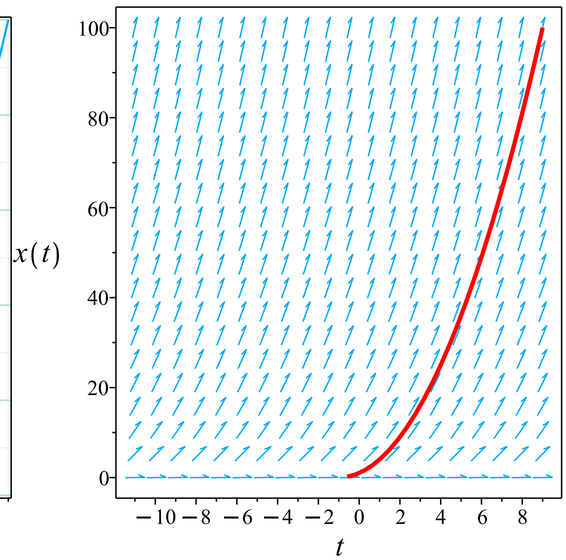
$$-2t + 2\sqrt{x} = 2$$

Solving for x gives

$$x = t^2 + 2t + 1$$



(a) Solution plot
$x = t^2 + 2t + 1$



(b) Slope field plot
$x' = 2\sqrt{x}$

Summary of solutions found

$$x = t^2 + 2t + 1$$

Solved using Lie symmetry for first order ode

Time used: 0.672 (sec)

Writing the ode as

$$x' = 2\sqrt{x}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$
$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + 2\sqrt{x}\,(b_3 - a_2) - 4xa_3 - \frac{tb_2 + xb_3 + b_1}{\sqrt{x}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2xa_2 - xb_3 + 4x^{3/2}a_3 - b_2\sqrt{x} + tb_2 + b_1}{\sqrt{x}} = 0$$

Setting the numerator to zero gives

$$-4x^{3/2}a_3 + b_2\sqrt{x} - tb_2 - 2xa_2 + xb_3 - b_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x, \sqrt{x}, x^{3/2}\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2, \sqrt{x} = v_3, x^{3/2} = v_4\}$$

The above PDE (6E) now becomes

$$-2v_2a_2 - 4v_4a_3 - v_1b_2 + b_2v_3 + v_2b_3 - b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-v_1b_2 + (-2a_2 + b_3)\,v_2 + b_2v_3 - 4v_4a_3 - b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -4a_3 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ -2a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t, x)\,\xi \\
&= 0 - \left(2\sqrt{x}\right)(1) \\
&= -2\sqrt{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-2\sqrt{x}} dy
\end{aligned}
$$

Which results in

$$S = -\sqrt{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = 2\sqrt{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= 0 \\
S_x &= -\frac{1}{2\sqrt{x}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int -1\, dR \\
S(R) &= -R + c_2
\end{aligned}$$

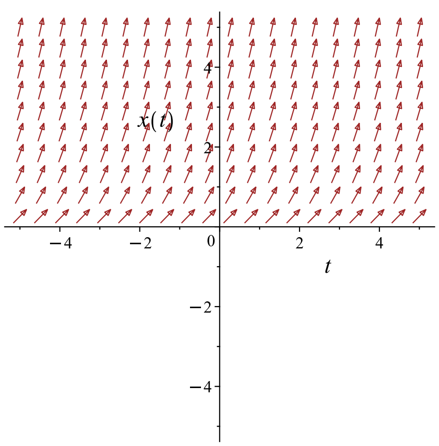
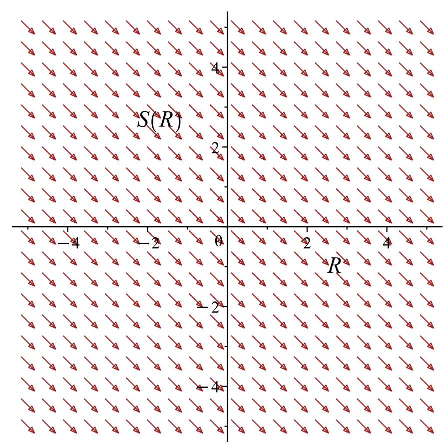
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$-\sqrt{x} = -t + c_2$$

Which gives

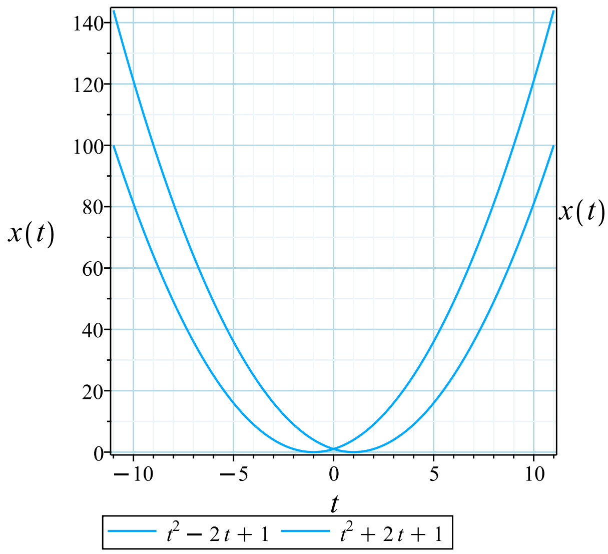
$$x = c_2^2 - 2c_2 t + t^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

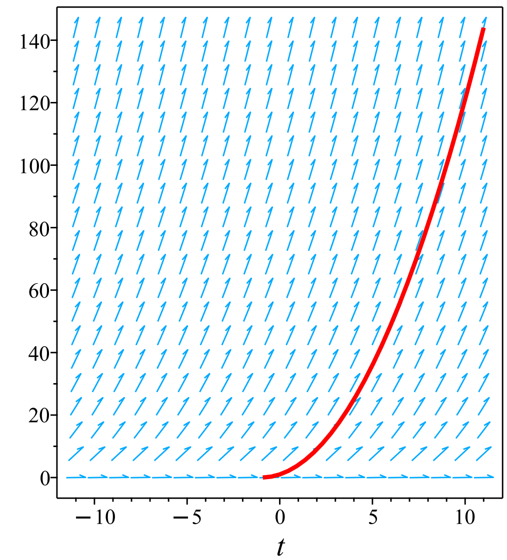
<table>
<tr><td>Original ode in t, x coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dx}{dt} = 2\sqrt{x}$

</td><td>$R = t$
$S = -\sqrt{x}$</td><td>$\frac{dS}{dR} = -1$

</td></tr>
</table>

Solving for the constant of integration from initial conditions gives

$$x = t^2 - 2t + 1$$
$$x = t^2 + 2t + 1$$



(a) Solutions plot



(b) Slope field plot
$x' = 2\sqrt{x}$

Summary of solutions found

$$x = t^2 - 2t + 1$$

$$x = t^2 + 2t + 1$$

Maple step by step solution

Let's solve
$\left[x' = 2\sqrt{x}, x(0) = 1\right]$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = 2\sqrt{x}$
- Separate variables
  $\frac{x'}{\sqrt{x}} = 2$
- Integrate both sides with respect to t
  $\int \frac{x'}{\sqrt{x}}dt = \int 2dt + C1$
- Evaluate integral
  $2\sqrt{x} = 2t + C1$
- Solve for x
  $x = t^2 + tC1 + \frac{1}{4}C1^2$

- Use initial condition $x(0) = 1$
  $1 = \frac{C1^2}{4}$
- Solve for $\_C1$
  $C1 = (-2, 2)$
- Substitute $\_C1 = (-2, 2)$ into general solution and simplify
  $x = (t-1)^2$
- Solution to the IVP
  $x = (t-1)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.031 (sec)
Leaf size : 9

```
dsolve([diff(x(t),t) = 2*x(t)^(1/2),
       op([x(0) = 1])],x(t),singsol=all)
```

$$x = (t+1)^2$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 10

```
DSolve[{D[x[t],t]==2*Sqrt[x[t]],{x[0]==1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to (t+1)^2$$

2.1.12 problem 2 (vi)

Internal problem ID [18175]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 2 (vi)
Date solved : `Thursday, December 19, 2024 at 01:52:07 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = \tan(x)$$

With initial conditions

$$x(0) = 1$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t,x) \\ &= \tan(x) \end{aligned}$$

The x domain of $f(t,x)$ when $t=0$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z10 \vee \frac{1}{2}\pi + \pi\_Z10 < x\right\}$$

And the point $x_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(\tan(x)) \\ &= 1 + \tan(x)^2 \end{aligned}$$

The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z10 \vee \frac{1}{2}\pi + \pi\_Z10 < x\right\}$$

And the point $x_0 = 1$ is inside this domain. Therefore solution exists and is unique.

Solved as first order autonomous ode

Time used: 0.165 (sec)

Integrating gives

$$\int \frac{1}{\tan(x)} dx = dt$$
$$\ln(\sin(x)) = t + c_1$$

Applying the exponential to both sides gives

$$e^{\ln(\sin(x))} = e^{t+c_1}$$
$$\sin(x) = \mathrm{e}^t c_1$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

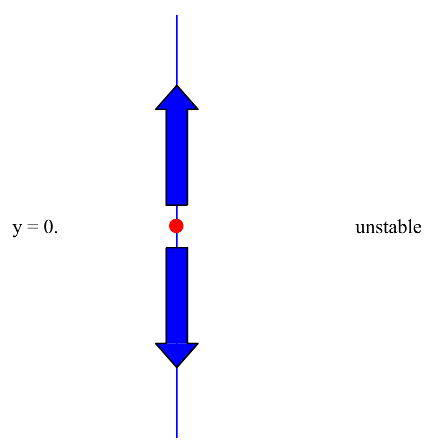


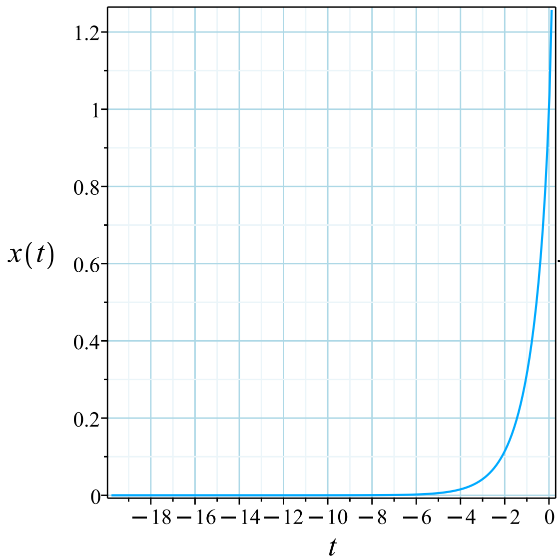
Figure 2.26: Phase line diagram

Solving for the constant of integration from initial conditions, the solution becomes

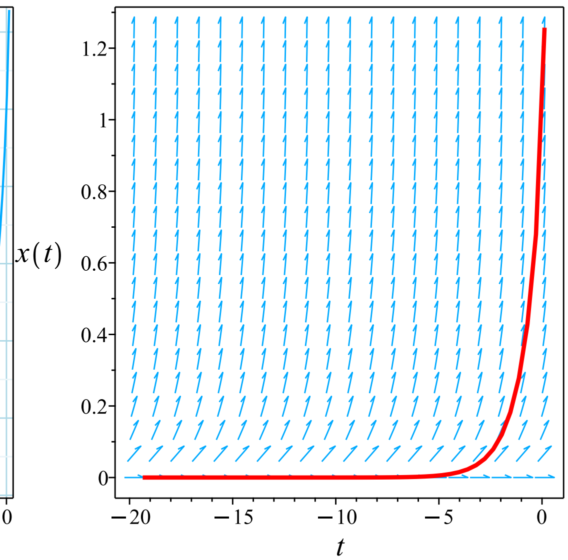
$$\sin(x) = \mathrm{e}^t \sin(1)$$

Solving for x gives

$$x = \arcsin\left(\mathrm{e}^t \sin(1)\right)$$



(a) Solution plot
$x = \arcsin\left(\mathrm{e}^{t} \sin\left(1\right)\right)$



(b) Slope field plot
$x' = \tan\left(x\right)$

Summary of solutions found

$$x = \arcsin\left(\mathrm{e}^{t} \sin\left(1\right)\right)$$

Solved as first order Exact ode

Time used: 0.205 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x &= (\tan(x))\,\mathrm{d}t\\ (-\tan(x))\,\mathrm{d}t + \mathrm{d}x &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -\tan(x)\\ N(t, x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-\tan(x))\\ &= -\sec(x)^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(-1 - \tan(x)^2\right) - (0)\right)\\ &= -\sec(x)^2\end{aligned}$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right) \\ &= -\cot(x)\left((0) - \left(-1 - \tan(x)^2\right)\right) \\ &= -\cot(x) - \tan(x) \end{aligned}$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}x} \\ &= e^{\int -\cot(x)-\tan(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(\sin(x))+\ln(\cos(x))} \\ &= \frac{\cos(x)}{\sin(x)} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{\cos(x)}{\sin(x)}(-\tan(x)) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{\cos(x)}{\sin(x)}(1) \\ &= \cot(x) \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ (-1) + (\cot(x))\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int -1 \,\mathrm{d}t$$
$$\phi = -t + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \cot(x)$. Therefore equation (4) becomes

$$\cot(x) = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \cot(x)$$

Integrating the above w.r.t x gives

$$\int f'(x) \,\mathrm{d}x = \int (\cot(x)) \,\mathrm{d}x$$
$$f(x) = \ln(\sin(x)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -t + \ln(\sin(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

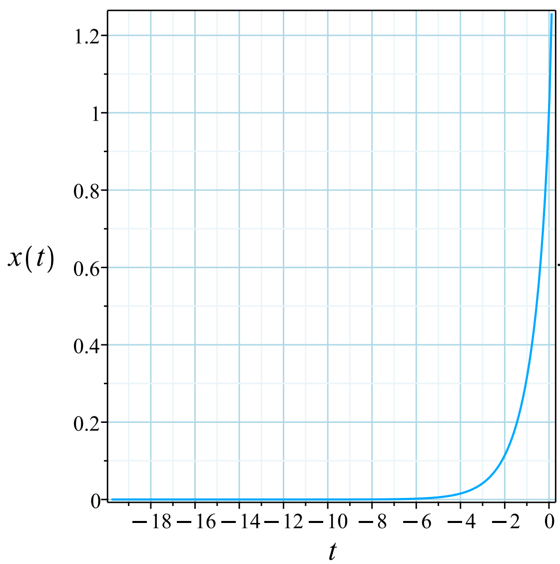
$$c_1 = -t + \ln(\sin(x))$$

Solving for the constant of integration from initial conditions, the solution becomes

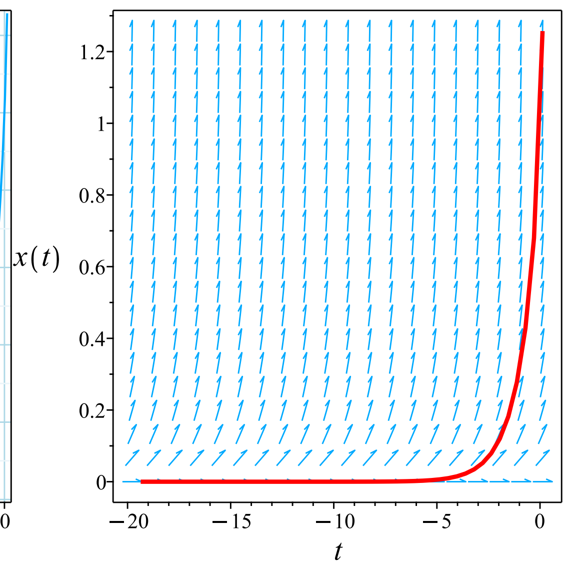
$$-t + \ln(\sin(x)) = \ln(\sin(1))$$

Solving for x gives

$$x = \arcsin\left(e^t \sin(1)\right)$$



(a) Solution plot
$x = \arcsin\left(e^t \sin(1)\right)$



(b) Slope field plot
$x' = \tan(x)$

Summary of solutions found

$$x = \arcsin\left(e^t \sin(1)\right)$$

Solved using Lie symmetry for first order ode

Time used: 0.626 (sec)

Writing the ode as

$$x' = \tan(x)$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = t a_2 + x a_3 + a_1 \tag{1E}$$
$$\eta = t b_2 + x b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \tan(x)(b_3 - a_2) - \tan(x)^2 a_3 - \left(1 + \tan(x)^2\right)(t b_2 + x b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\tan(x)^2 t b_2 - \tan(x)^2 x b_3 - \tan(x)^2 a_3 - \tan(x)^2 b_1 - \tan(x) a_2 + \tan(x) b_3 - t b_2 - x b_3 - b_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-\tan(x)^2 t b_2 - \tan(x)^2 x b_3 - \tan(x)^2 a_3 - \tan(x)^2 b_1 - \tan(x) a_2 + \tan(x) b_3 - t b_2 - x b_3 - b_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x, \tan(x)\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2, \tan(x) = v_3\}$$

The above PDE (6E) now becomes

$$-v_3^2 v_1 b_2 - v_3^2 v_2 b_3 - v_3^2 a_3 - v_3^2 b_1 - v_3 a_2 - v_1 b_2 - v_2 b_3 + v_3 b_3 - b_1 + b_2 = 0 \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-v_3^2 v_1 b_2 - v_1 b_2 - v_3^2 v_2 b_3 - v_2 b_3 + (-a_3 - b_1) v_3^2 + (b_3 - a_2) v_3 - b_1 + b_2 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -b_2 &= 0 \\ -b_3 &= 0 \\ -a_3 - b_1 &= 0 \\ -b_1 + b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 0 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t,x)\,\xi \\
&= 0 - (\tan(x))\,(1) \\
&= -\tan(x) \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\tan(x)} dy
\end{aligned}
$$

Which results in

$$S = -\ln(\sin(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \tan(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= 0 \\
S_x &= -\cot(x)
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int -1\, dR \\
S(R) &= -R + c_2
\end{aligned}$$

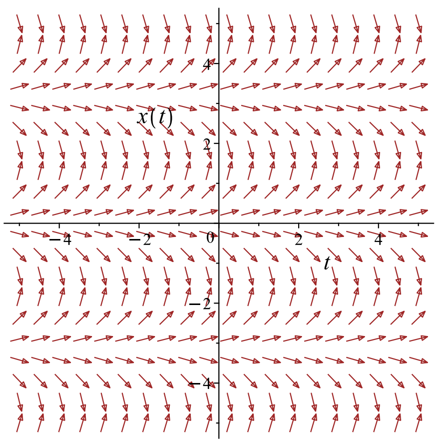
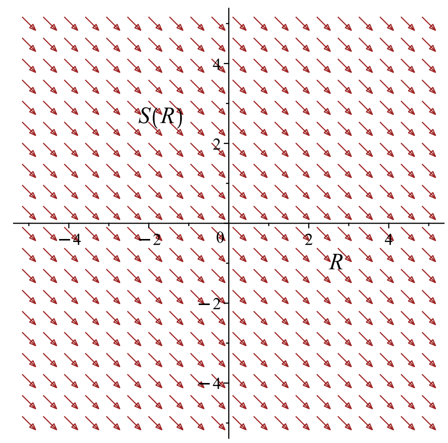
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$-\ln(\sin(x)) = -t + c_2$$

Which gives

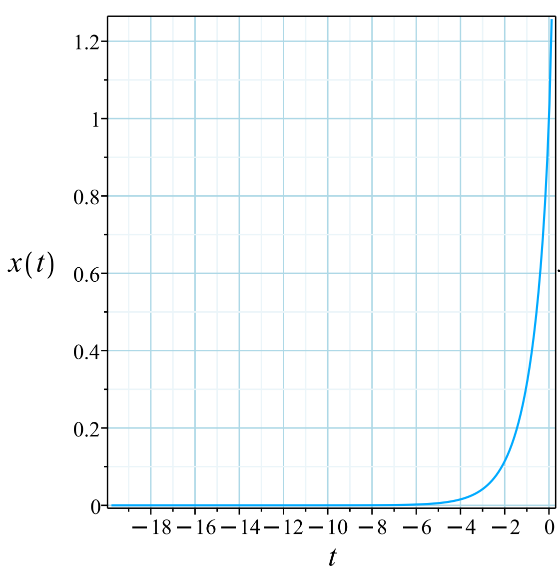
$$x = \arcsin\left(\mathrm{e}^{t-c_2}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

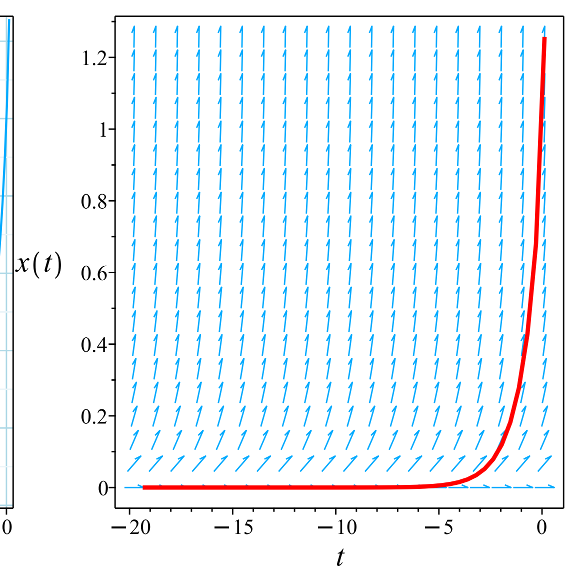
Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \tan(x)$ 	$R = t$ $S = -\ln(\sin(x))$	$\frac{dS}{dR} = -1$ 

Solving for the constant of integration from initial conditions, the solution becomes

$$x = \arcsin\left(e^{t+\ln(\sin(1))}\right)$$



(a) Solution plot
$x = \arcsin\left(e^{t+\ln(\sin(1))}\right)$



(b) Slope field plot
$x' = \tan(x)$

Summary of solutions found

$$x = \arcsin\left(e^{t+\ln(\sin(1))}\right)$$

Maple step by step solution

Let's solve
$[x' = \tan(x), x(0) = 1]$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = \tan(x)$
- Separate variables
  $\frac{x'}{\tan(x)} = 1$
- Integrate both sides with respect to t
  $\int \frac{x'}{\tan(x)} dt = \int 1 dt + C1$
- Evaluate integral
  $\ln(\sin(x)) = t + C1$
- Solve for x
  $x = \arcsin\left(e^{t+C1}\right)$
- Use initial condition $x(0) = 1$
  $1 = \arcsin\left(e^{C1}\right)$
- Solve for $\_C1$
  $C1 = \ln(\sin(1))$
- Substitute $\_C1 = \ln(\sin(1))$ into general solution and simplify
  $x = \arcsin(e^t \sin(1))$
- Solution to the IVP
  $x = \arcsin(e^t \sin(1))$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
```

```
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.093 (sec)
Leaf size : 10

```
dsolve([diff(x(t),t) = tan(x(t)),
       op([x(0) = 1])],x(t),singsol=all)
```

$$x = \arcsin\left(\mathrm{e}^{t} \sin\left(1\right)\right)$$

Mathematica DSolve solution

Solving time : 0.007 (sec)
Leaf size : 12

```
DSolve[{D[x[t],t]==Tan[x[t]],{x[0]==1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \arcsin\left(e^{t} \sin(1)\right)$$

2.1.13 problem 3 (i)

Internal problem ID [18176]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 3 (i)
Date solved : `Thursday, December 19, 2024 at 01:52:09 PM`
CAS classification : `[_separable]`

Solve

$$3t^2x - xt + \left(3t^3x^2 + t^3x^4\right)x' = 0$$

Factoring the ode gives these factors

$$x = 0 \tag{1}$$

$$x'x^3t^2 + 3x'xt^2 + 3t - 1 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for x from

$$x = 0$$

Solving gives $x = 0$

Solving equation (2)

Solved as first order separable ode

Time used: 0.903 (sec)

The ode $x' = -\frac{3t-1}{t^2 x(x^2+3)}$ is separable as it can be written as

$$\begin{aligned} x' &= -\frac{3t-1}{t^2 x\left(x^2+3\right)} \\ &= f(t)g(x) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= -\frac{3t-1}{t^2} \\ g(x) &= \frac{1}{x\left(x^2+3\right)} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(x)}\, dx &= \int f(t)\, dt \\ \int x\left(x^2+3\right)\, dx &= \int -\frac{3t-1}{t^2}\, dt \\ \frac{\left(x^2+3\right)^2}{4} &= -\frac{1}{t} + \ln\left(\frac{1}{t^3}\right) + c_1 \end{aligned}$$

Solving for x gives

$$x = \frac{\sqrt{t\left(-3t+2\sqrt{\ln\left(\frac{1}{t^3}\right) t^2 + c_1\, t^2 - t}\right)}}{t}$$

$$x = \frac{\sqrt{-t\left(3t+2\sqrt{\ln\left(\frac{1}{t^3}\right) t^2 + c_1\, t^2 - t}\right)}}{t}$$

$$x = -\frac{\sqrt{t\left(-3t+2\sqrt{\ln\left(\frac{1}{t^3}\right) t^2 + c_1\, t^2 - t}\right)}}{t}$$

$$x = -\frac{\sqrt{-t\left(3t+2\sqrt{\ln\left(\frac{1}{t^3}\right) t^2 + c_1\, t^2 - t}\right)}}{t}$$

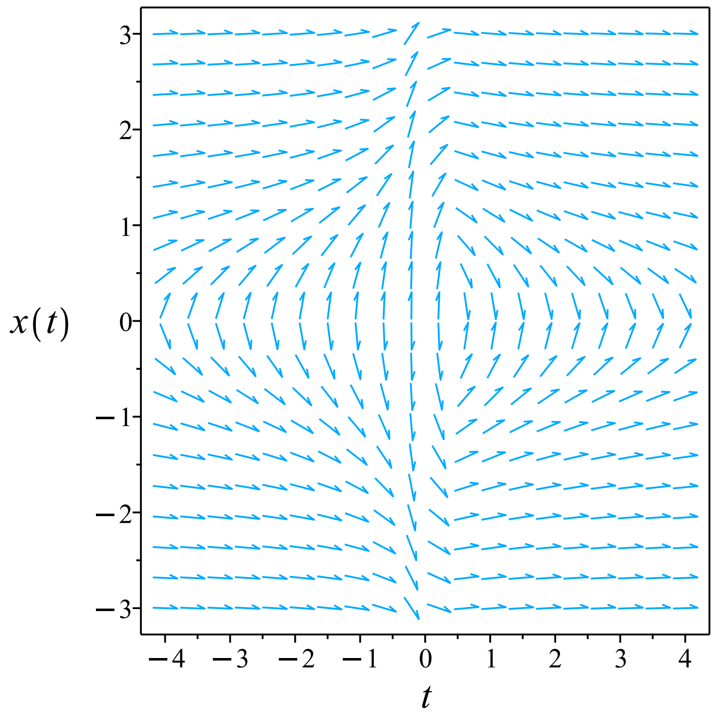


Figure 2.30: Slope field plot
$x' x^3 t^2 + 3x' x t^2 + 3t - 1 = 0$

Summary of solutions found

$$x = \frac{\sqrt{t\left(-3t + 2\sqrt{\ln\left(\frac{1}{t^3}\right) t^2 + c_1 t^2 - t}\right)}}{t}$$

$$x = \frac{\sqrt{-t\left(3t + 2\sqrt{\ln\left(\frac{1}{t^3}\right) t^2 + c_1 t^2 - t}\right)}}{t}$$

$$x = -\frac{\sqrt{t\left(-3t + 2\sqrt{\ln\left(\frac{1}{t^3}\right) t^2 + c_1 t^2 - t}\right)}}{t}$$

$$x = -\frac{\sqrt{-t\left(3t + 2\sqrt{\ln\left(\frac{1}{t^3}\right) t^2 + c_1 t^2 - t}\right)}}{t}$$

Solved as first order Exact ode

Time used: 0.313 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(t^2x^3+3t^2x\right)\mathrm{d}x&=(-3t+1)\,\mathrm{d}t\\ (3t-1)\,\mathrm{d}t+\left(t^2x^3+3t^2x\right)\mathrm{d}x&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=3t-1\\ N(t,x)&=t^2x^3+3t^2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(3t-1)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(t^2x^3+3t^2x\right)\\ &= 2t\,x^3+6tx\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x}-\frac{\partial N}{\partial t}\right)\\ &= \frac{1}{t^2x\,(x^2+3)}\left((0)-\left(2t\,x^3+6tx\right)\right)\\ &= -\frac{2}{t}\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -\frac{2}{t}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(t)}\\ &= \frac{1}{t^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{t^2}(3t-1)\\ &= \frac{3t-1}{t^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{t^2}\left(t^2x^3 + 3t^2x\right) \\ &= x\left(x^2+3\right)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left(\frac{3t-1}{t^2}\right) + \left(x\left(x^2+3\right)\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\ \int \frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int \frac{3t-1}{t^2}\,\mathrm{d}t \\ \phi &= \frac{1}{t} + 3\ln(t) + f(x) \qquad (3)\end{aligned}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial x} = x(x^2+3)$. Therefore equation (4) becomes

$$x\left(x^2+3\right) = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = x\left(x^2+3\right)$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(x\left(x^2+3\right)\right)\mathrm{d}x$$

$$f(x) = \frac{\left(x^2+3\right)^2}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{1}{t} + 3\ln(t) + \frac{\left(x^2+3\right)^2}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{1}{t} + 3\ln(t) + \frac{\left(x^2+3\right)^2}{4}$$

Solving for x gives

$$x = \frac{\sqrt{t\left(-3t+2\sqrt{-3\ln(t)\,t^2+c_1\,t^2-t}\right)}}{t}$$

$$x = \frac{\sqrt{-t\left(3t+2\sqrt{-3\ln(t)\,t^2+c_1\,t^2-t}\right)}}{t}$$

$$x = -\frac{\sqrt{t\left(-3t+2\sqrt{-3\ln(t)\,t^2+c_1\,t^2-t}\right)}}{t}$$

$$x = -\frac{\sqrt{-t\left(3t+2\sqrt{-3\ln(t)\,t^2+c_1\,t^2-t}\right)}}{t}$$

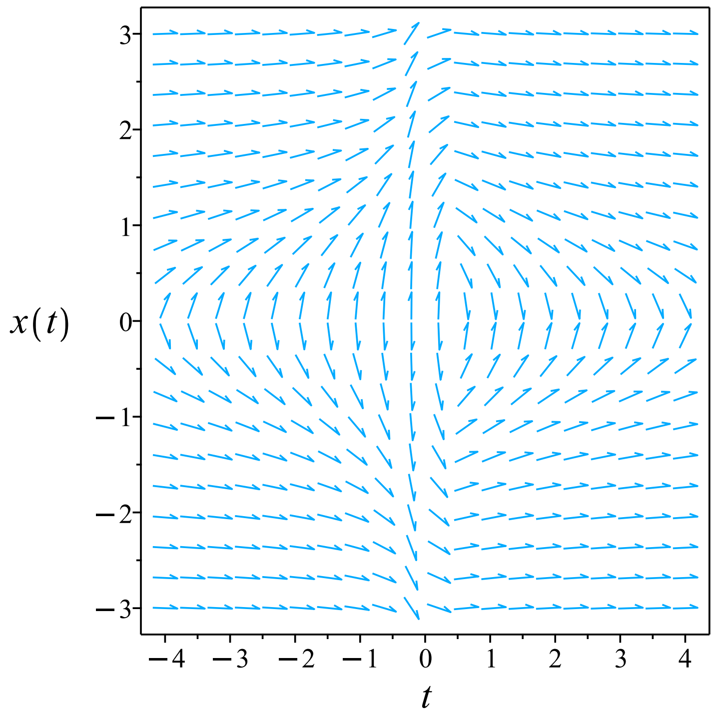


Figure 2.31: Slope field plot
$x'x^3t^2 + 3x'xt^2 + 3t - 1 = 0$

Summary of solutions found

$$x = \frac{\sqrt{t\left(-3t + 2\sqrt{-3\ln(t)\,t^2 + c_1\,t^2 - t}\right)}}{t}$$

$$x = \frac{\sqrt{-t\left(3t + 2\sqrt{-3\ln(t)\,t^2 + c_1\,t^2 - t}\right)}}{t}$$

$$x = -\frac{\sqrt{t\left(-3t + 2\sqrt{-3\ln(t)\,t^2 + c_1\,t^2 - t}\right)}}{t}$$

$$x = -\frac{\sqrt{-t\left(3t + 2\sqrt{-3\ln(t)\,t^2 + c_1\,t^2 - t}\right)}}{t}$$

Maple step by step solution

Let's solve

$3t^2x - xt + (3t^3x^2 + t^3x^4)\,x' = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Solve for the highest derivative

  $x' = \frac{-3t^2x+xt}{3t^3x^2+t^3x^4}$

- Separate variables

  $x'x(x^2+3) = -\frac{3t-1}{t^2}$

- Integrate both sides with respect to t

  $\int x'x(x^2+3)\,dt = \int -\frac{3t-1}{t^2}dt + C1$

- Evaluate integral

  $\frac{(x^2+3)^2}{4} = -\frac{1}{t} - 3\ln(t) + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 139

```
dsolve(3*t^2*x(t)-x(t)*t+(3*t^3*x(t)^2+t^3*x(t)^4)*diff(x(t),t) = 0,
       x(t),singsol=all)
```

$$x = 0$$
$$x = \frac{\sqrt{-3t^2 + 2t\sqrt{-t\left(1 + 3\ln\left(t\right)t + tc_1\right)}}}{t}$$
$$x = \frac{\sqrt{-3t^2 - 2t\sqrt{-t\left(1 + 3\ln\left(t\right)t + tc_1\right)}}}{t}$$
$$x = -\frac{\sqrt{-3t^2 + 2t\sqrt{-t\left(1 + 3\ln\left(t\right)t + tc_1\right)}}}{t}$$
$$x = -\frac{\sqrt{-3t^2 - 2t\sqrt{-t\left(1 + 3\ln\left(t\right)t + tc_1\right)}}}{t}$$

Mathematica DSolve solution

Solving time : 6.967 (sec)
Leaf size : 157

```
DSolve[{(3*t^2*x[t]-t*x[t])+(3*t^3*x[t]^2+t^3*x[t]^4)*D[x[t],t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to 0$$
$$x(t) \to -\sqrt{-3 - \frac{\sqrt{9t - 12t\log(t) + 4c_1t - 4}}{\sqrt{t}}}$$
$$x(t) \to \sqrt{-3 - \frac{\sqrt{9t - 12t\log(t) + 4c_1t - 4}}{\sqrt{t}}}$$
$$x(t) \to -\sqrt{-3 + \frac{\sqrt{9t - 12t\log(t) + 4c_1t - 4}}{\sqrt{t}}}$$
$$x(t) \to \sqrt{-3 + \frac{\sqrt{9t - 12t\log(t) + 4c_1t - 4}}{\sqrt{t}}}$$
$$x(t) \to 0$$

2.1.14 problem 3 (ii)

Internal problem ID [18177]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 3 (ii)
Date solved : `Thursday, December 19, 2024 at 01:52:12 PM`
CAS classification : `[_separable]`

Solve

$$1 + 2x + \left(-t^2 + 4\right) x' = 0$$

Solved as first order linear ode

Time used: 0.125 (sec)

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = -\frac{2}{t^2 - 4}$$
$$p(t) = \frac{1}{t^2 - 4}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dt} \\ &= e^{\int -\frac{2}{t^2-4} dt} \\ &= \frac{\sqrt{t+2}}{\sqrt{t-2}}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = (\mu)\left(\frac{1}{t^2-4}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x\sqrt{t+2}}{\sqrt{t-2}}\right) = \left(\frac{\sqrt{t+2}}{\sqrt{t-2}}\right)\left(\frac{1}{t^2-4}\right)$$

$$\mathrm{d}\left(\frac{x\sqrt{t+2}}{\sqrt{t-2}}\right) = \left(\frac{\sqrt{t+2}}{(t^2-4)\sqrt{t-2}}\right)\mathrm{d}t$$

Integrating gives

$$\begin{aligned}\frac{x\sqrt{t+2}}{\sqrt{t-2}} &= \int \frac{\sqrt{t+2}}{(t^2-4)\sqrt{t-2}}\, dt \\ &= -\frac{\sqrt{t-2}\,(t+2)^{3/2}}{2\,(t^2-4)} + c_1\end{aligned}$$

Dividing throughout by the integrating factor $\frac{\sqrt{t+2}}{\sqrt{t-2}}$ gives the final solution

$$x = \frac{\sqrt{t-2}\left(-\sqrt{t-2}\,(t+2)^{3/2} + 2c_1\,t^2 - 8c_1\right)}{\sqrt{t+2}\,(2t^2-8)}$$

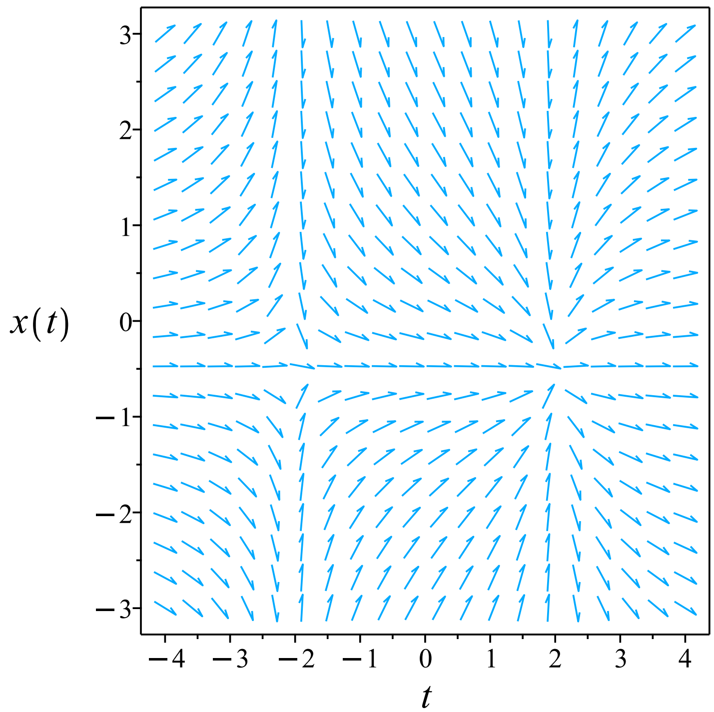


Figure 2.32: Slope field plot
$1+2x+\left(-t^{2}+4\right)x' = 0$

Summary of solutions found

$$x = \frac{\sqrt{t-2}\,\left(-\sqrt{t-2}\,(t+2)^{3/2}+2c_1\,t^{2}-8c_1\right)}{\sqrt{t+2}\,\left(2t^{2}-8\right)}$$

Solved as first order separable ode

Time used: 0.171 (sec)

The ode $x' = \frac{1+2x}{t^2-4}$ is separable as it can be written as

$$\begin{aligned} x' &= \frac{1+2x}{t^2-4} \\ &= f(t)g(x) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= \frac{1}{t^2-4} \\ g(x) &= 2x+1 \end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(x)}\,dx &= \int f(t)\,dt \\
\int \frac{1}{2x+1}\,dx &= \int \frac{1}{t^2-4}\,dt \\
\frac{\ln(1+2x)}{2} &= \ln\left(\frac{(t-2)^{1/4}}{(t+2)^{1/4}}\right) + c_1
\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(x)$ is zero, since we had to divide by this above. Solving $g(x) = 0$ or $2x + 1 = 0$ for x gives

$$x = -\frac{1}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}
\frac{\ln(1+2x)}{2} &= \ln\left(\frac{(t-2)^{1/4}}{(t+2)^{1/4}}\right) + c_1 \\
x &= -\frac{1}{2}
\end{aligned}$$

Solving for x gives

$$\begin{aligned}
x &= -\frac{1}{2} \\
x &= -\frac{-\mathrm{e}^{2c_1}\sqrt{t-2} + \sqrt{t+2}}{2\sqrt{t+2}}
\end{aligned}$$

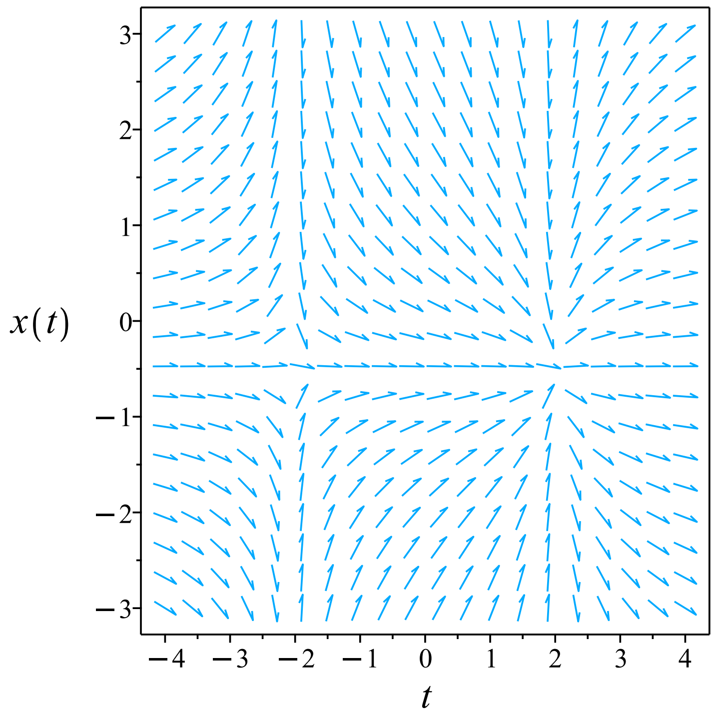


Figure 2.33: Slope field plot
$1 + 2x + (-t^2 + 4)\, x' = 0$

Summary of solutions found

$$x = -\frac{1}{2}$$

$$x = -\frac{-e^{2c_1}\sqrt{t-2} + \sqrt{t+2}}{2\sqrt{t+2}}$$

Solved as first order Exact ode

Time used: 0.486 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-t^2+4\right)\mathrm{d}x &= (-1-2x)\,\mathrm{d}t\\ (2x+1)\,\mathrm{d}t + \left(-t^2+4\right)\mathrm{d}x &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= 2x+1\\ N(t,x) &= -t^2+4\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(2x+1)\\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-t^2+4\right)\\ &= -2t\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= -\frac{1}{t^2-4}((2)-(-2t))\\ &= \frac{-2t-2}{t^2-4}\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int \frac{-2t-2}{t^2-4}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(t+2)}{2}-\frac{3\ln(t-2)}{2}}\\ &= \frac{1}{\sqrt{t+2}\,(t-2)^{3/2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\sqrt{t+2}\,(t-2)^{3/2}}(2x+1)\\ &= \frac{2x+1}{\sqrt{t+2}\,(t-2)^{3/2}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{\sqrt{t+2}\,(t-2)^{3/2}}\left(-t^2+4\right)\\ &= -\frac{\sqrt{t+2}}{\sqrt{t-2}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

$$\left(\frac{2x+1}{\sqrt{t+2}\,(t-2)^{3/2}}\right) + \left(-\frac{\sqrt{t+2}}{\sqrt{t-2}}\right)\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{N}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{\sqrt{t+2}}{\sqrt{t-2}}\,\mathrm{d}x$$

$$\phi = -\frac{x\sqrt{t+2}}{\sqrt{t-2}} + f(t) \tag{3}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$\frac{\partial \phi}{\partial t} = -\frac{x}{2\sqrt{t+2}\,\sqrt{t-2}} + \frac{x\sqrt{t+2}}{2\,(t-2)^{3/2}} + f'(t) \tag{4}$$

$$= \frac{2x}{\sqrt{t+2}\,(t-2)^{3/2}} + f'(t)$$

But equation (1) says that $\frac{\partial \phi}{\partial t} = \frac{2x+1}{\sqrt{t+2}\,(t-2)^{3/2}}$. Therefore equation (4) becomes

$$\frac{2x+1}{\sqrt{t+2}\,(t-2)^{3/2}} = \frac{2x}{\sqrt{t+2}\,(t-2)^{3/2}} + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$f'(t) = \frac{1}{\sqrt{t+2}\,(t-2)^{3/2}}$$

Integrating the above w.r.t t gives

$$\int f'(t)\,\mathrm{d}t = \int \left(\frac{1}{\sqrt{t+2}\,(t-2)^{3/2}}\right) \mathrm{d}t$$
$$f(t) = -\frac{\sqrt{t+2}}{2\sqrt{t-2}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(t)$ into equation (3) gives ϕ

$$\phi = -\frac{x\sqrt{t+2}}{\sqrt{t-2}} - \frac{\sqrt{t+2}}{2\sqrt{t-2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{x\sqrt{t+2}}{\sqrt{t-2}} - \frac{\sqrt{t+2}}{2\sqrt{t-2}}$$

Solving for x gives

$$x = -\frac{2c_1\sqrt{t-2} + \sqrt{t+2}}{2\sqrt{t+2}}$$

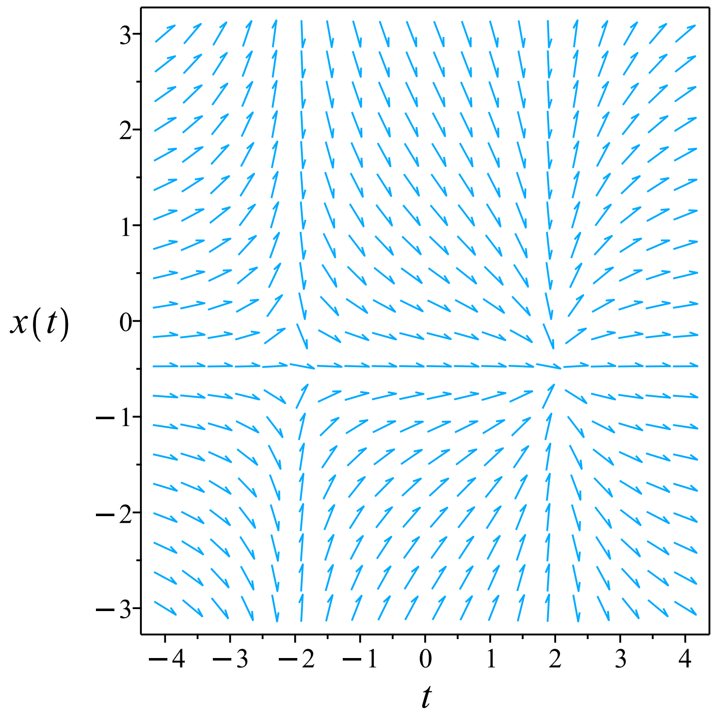


Figure 2.34: Slope field plot
$1 + 2x + (-t^2 + 4)\,x' = 0$

Summary of solutions found

$$x = -\frac{2c_1\sqrt{t-2} + \sqrt{t+2}}{2\sqrt{t+2}}$$

Solved using Lie symmetry for first order ode

Time used: 0.819 (sec)

Writing the ode as

$$x' = \frac{2x+1}{t^2-4}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$

$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &+ \frac{(2x+1)(b_3-a_2)}{t^2-4} - \frac{(2x+1)^2 a_3}{(t^2-4)^2} \\ &+ \frac{2(2x+1)\,t(ta_2+xa_3+a_1)}{(t^2-4)^2} - \frac{2(tb_2+xb_3+b_1)}{t^2-4} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{t^4b_2 - 2t^3b_2 + 2t^2xa_2 + 4t\,x^2a_3 + t^2a_2 - 2t^2b_1 - 8t^2b_2 + t^2b_3 + 4txa_1 + 2txa_3 - 4x^2a_3 + 2ta_1 + 8tb_2 + 8x}{(t^2-4)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} t^4b_2 - 2t^3b_2 + 2t^2xa_2 + 4t\,x^2a_3 + t^2a_2 - 2t^2b_1 - 8t^2b_2 + t^2b_3 + 4txa_1 + 2txa_3 \\ - 4x^2a_3 + 2ta_1 + 8tb_2 + 8xa_2 - 4xa_3 + 4a_2 - a_3 + 8b_1 + 16b_2 - 4b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} b_2v_1^4 + 2a_2v_1^2v_2 + 4a_3v_1v_2^2 - 2b_2v_1^3 + 4a_1v_1v_2 + a_2v_1^2 + 2a_3v_1v_2 - 4a_3v_2^2 - 2b_1v_1^2 \\ - 8b_2v_1^2 + b_3v_1^2 + 2a_1v_1 + 8a_2v_2 - 4a_3v_2 + 8b_2v_1 + 4a_2 - a_3 + 8b_1 + 16b_2 - 4b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &b_2v_1^4 - 2b_2v_1^3 + 2a_2v_1^2v_2 + (a_2 - 2b_1 - 8b_2 + b_3)\, v_1^2 + 4a_3v_1v_2^2 + (4a_1 + 2a_3)\, v_1v_2 \\ &+ (2a_1 + 8b_2)\, v_1 - 4a_3v_2^2 + (8a_2 - 4a_3)\, v_2 + 4a_2 - a_3 + 8b_1 + 16b_2 - 4b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ 2a_2 &= 0 \\ -4a_3 &= 0 \\ 4a_3 &= 0 \\ -2b_2 &= 0 \\ 2a_1 + 8b_2 &= 0 \\ 4a_1 + 2a_3 &= 0 \\ 8a_2 - 4a_3 &= 0 \\ a_2 - 2b_1 - 8b_2 + b_3 &= 0 \\ 4a_2 - a_3 + 8b_1 + 16b_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 2b_1 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= 2x + 1 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{2x + 1} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(2x + 1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = \frac{2x + 1}{t^2 - 4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 0 \\ S_x &= \frac{1}{2x + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{t^2 - 4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 - 4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}\int dS &= \int \frac{1}{R^2 - 4}\, dR \\ S(R) &= -\frac{\ln(R+2)}{4} + \frac{\ln(R-2)}{4} + c_2\end{aligned}$$

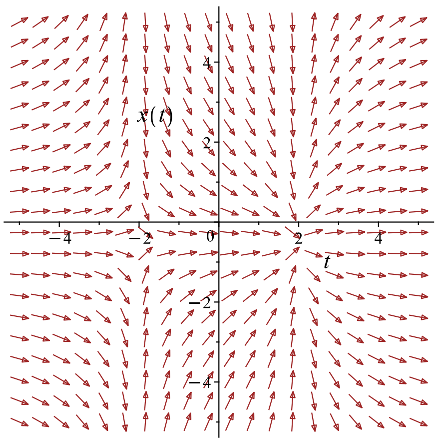
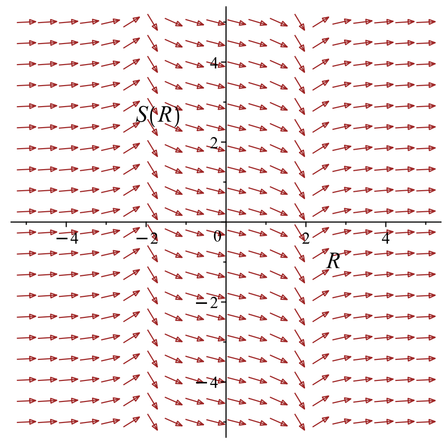
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$\frac{\ln(1+2x)}{2} = -\frac{\ln(t+2)}{4} + \frac{\ln(t-2)}{4} + c_2$$

Which gives

$$x = \frac{e^{-\frac{\ln(t+2)}{2} + \frac{\ln(t-2)}{2} + 2c_2}}{2} - \frac{1}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{2x+1}{t^2-4}$ 	$R = t$ $S = \frac{\ln(2x+1)}{2}$	$\frac{dS}{dR} = \frac{1}{R^2-4}$ 

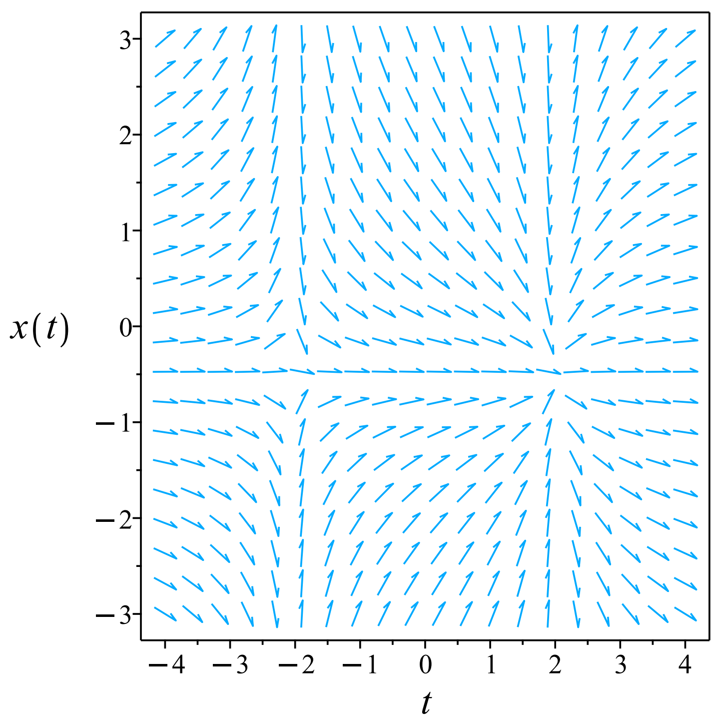


Figure 2.35: Slope field plot
$1 + 2x + (-t^2 + 4)\,x' = 0$

Summary of solutions found

$$x = \frac{e^{-\frac{\ln(t+2)}{2}+\frac{\ln(t-2)}{2}+2c_2}}{2} - \frac{1}{2}$$

Maple step by step solution

Let's solve

$1 + 2x + (-t^2 + 4)\, x' = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Solve for the highest derivative

  $x' = \frac{-1-2x}{-t^2+4}$

- Separate variables

  $\frac{x'}{-1-2x} = \frac{1}{-t^2+4}$

- Integrate both sides with respect to t

  $\int \frac{x'}{-1-2x} dt = \int \frac{1}{-t^2+4} dt + C1$

- Evaluate integral

  $-\frac{\ln(-1-2x)}{2} = \frac{\ln(t+2)}{4} - \frac{\ln(t-2)}{4} + C1$

- Solve for x

  $\left\{x = -\frac{e^{4C1} t + \sqrt{e^{4C1} t^2 - 4\,e^{4C1}} + 2\,e^{4C1}}{2\,e^{4C1} (t+2)}, x = -\frac{e^{4C1} t + 2\,e^{4C1} - \sqrt{e^{4C1} t^2 - 4\,e^{4C1}}}{2\,e^{4C1} (t+2)}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 18

```
dsolve(1+2*x(t)+(-t^2+4)*diff(x(t),t) = 0,
       x(t),singsol=all)
```

$$x = -\frac{1}{2} + \frac{\sqrt{t-2}\, c_1}{\sqrt{t+2}}$$

Mathematica DSolve solution

Solving time : 0.078 (sec)
Leaf size : 87

```
DSolve[{(1+2*x[t])+(4-t^2)*D[x[t],t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{\sqrt{2-t}\left(\sqrt{4-t^2} - 2\sqrt{t+2} + 2c_1\left(t + 2\sqrt{2-t} - 2\right)\right)}{2\sqrt{t+2}\left(t + 2\sqrt{2-t} - 2\right)}$$
$$x(t) \to -\frac{1}{2}$$

2.1.15 problem 3 (iii)

Internal problem ID [18178]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 3 (iii)
Date solved : `Thursday, December 19, 2024 at 01:52:17 PM`
CAS classification : `[[_homogeneous, 'class A'], _dAlembert]`

Solve

$$x' = \cos\left(\frac{x}{t}\right)$$

Solved as first order homogeneous class A ode

Time used: 0.369 (sec)

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= \cos\left(\frac{x}{t}\right) \end{aligned} \tag{1}$$

An ode of the form $x' = \frac{M(t,x)}{N(t,x)}$ is called homogeneous if the functions $M(t, x)$ and $N(t, x)$ are both homogeneous functions and of the same order. Recall that a function $f(t, x)$ is homogeneous of order n if

$$f(t^n t, t^n x) = t^n f(t, x)$$

In this case, it can be seen that both $M = \cos\left(\frac{x}{t}\right)$ and $N = 1$ are both homogeneous and of the same order $n = 0$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{x}{t}$, or $x = ut$. Hence

$$\frac{dx}{dt} = \frac{du}{dt}t + u$$

Applying the transformation $x = ut$ to the above ODE in (1) gives

$$\frac{du}{dt}t + u = \cos(u)$$
$$\frac{du}{dt} = \frac{\cos(u(t)) - u(t)}{t}$$

Or

$$u'(t) - \frac{\cos(u(t)) - u(t)}{t} = 0$$

Or

$$u'(t)\,t - \cos(u(t)) + u(t) = 0$$

Which is now solved as separable in $u(t)$.

The ode $u'(t) = \frac{\cos(u(t)) - u(t)}{t}$ is separable as it can be written as

$$u'(t) = \frac{\cos(u(t)) - u(t)}{t}$$
$$= f(t)g(u)$$

Where

$$f(t) = \frac{1}{t}$$
$$g(u) = \cos(u) - u$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(t)\,dt$$
$$\int \frac{1}{\cos(u) - u}\,du = \int \frac{1}{t}\,dt$$
$$\int^{u(t)} \frac{1}{\cos(\tau) - \tau}d\tau = \ln(t) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\cos(u) - u = 0$ for $u(t)$ gives

$$u(t) = \mathrm{RootOf}(-\cos(\_Z) + \_Z)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution RootOf $(-\cos(\_Z)+\_Z)$ will not be used

Converting $\int^{u(t)} \frac{1}{\cos(\tau)-\tau} d\tau = \ln(t) + c_1$ back to x gives

$$\int^{\frac{x}{t}} \frac{1}{\cos(\tau)-\tau} d\tau = \ln(t) + c_1$$

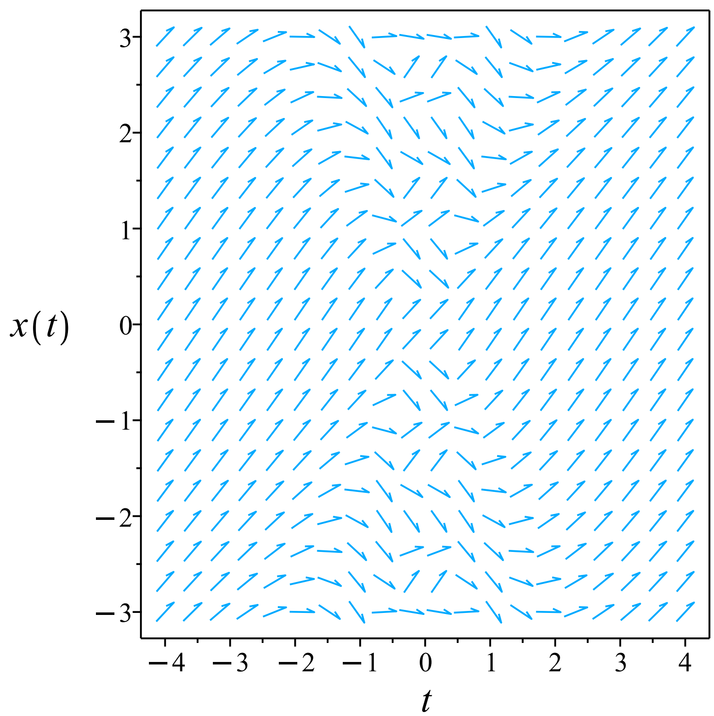


Figure 2.36: Slope field plot
$x' = \cos\left(\frac{x}{t}\right)$

Summary of solutions found

$$\int^{\frac{x}{t}} \frac{1}{\cos(\tau)-\tau} d\tau = \ln(t) + c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.151 (sec)

Applying change of variables $x = u(t)\,t$, then the ode becomes

$$u'(t)\,t + u(t) = \cos(u(t))$$

Which is now solved The ode $u'(t) = -\frac{u(t)-\cos(u(t))}{t}$ is separable as it can be written as

$$\begin{aligned} u'(t) &= -\frac{u(t) - \cos(u(t))}{t} \\ &= f(t)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= \frac{1}{t} \\ g(u) &= -u + \cos(u) \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(t)\,dt \\ \int \frac{1}{-u + \cos(u)}\,du &= \int \frac{1}{t}\,dt \\ \int^{u(t)} \frac{1}{-\tau + \cos(\tau)}d\tau &= \ln(t) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $-u + \cos(u) = 0$ for $u(t)$ gives

$$u(t) = \text{RootOf}(-\cos(\_Z) + \_Z)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $\text{RootOf}(-\cos(\_Z) + \_Z)$ will not be used

Converting $\int^{u(t)} \frac{1}{-\tau+\cos(\tau)}d\tau = \ln(t) + c_1$ back to x gives

$$\int^{\frac{x}{t}} \frac{1}{-\tau + \cos(\tau)}d\tau = \ln(t) + c_1$$

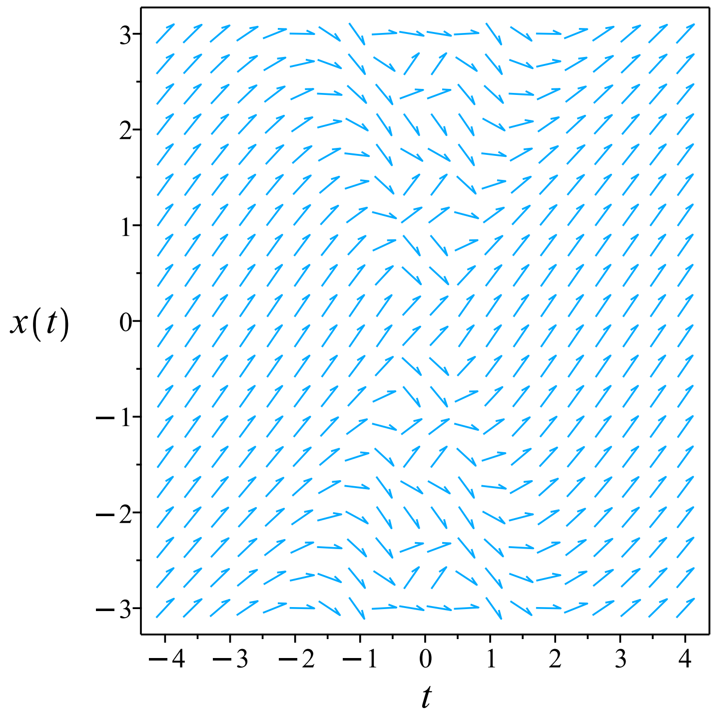


Figure 2.37: Slope field plot
$x' = \cos\left(\frac{x}{t}\right)$

Summary of solutions found

$$\int^{\frac{x}{t}} \frac{1}{-\tau + \cos(\tau)} d\tau = \ln(t) + c_1$$

Solved as first order isobaric ode

Time used: 0.585 (sec)

Solving for x' gives

$$x' = \cos\left(\frac{x}{t}\right) \tag{1}$$

Each of the above ode's is now solved An ode $x' = f(t, x)$ is isobaric if

$$f(tt, t^m x) = t^{m-1} f(t, x) \tag{1}$$

Where here

$$f(t, x) = \cos\left(\frac{x}{t}\right) \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} x &= ut^m \\ &= ut \end{aligned}$$

Converts the ODE to a separable in $u(t)$. Performing this substitution gives

$$u(t) + tu'(t) = \cos(u(t))$$

The ode $u'(t) = -\frac{u(t)-\cos(u(t))}{t}$ is separable as it can be written as

$$\begin{aligned} u'(t) &= -\frac{u(t) - \cos(u(t))}{t} \\ &= f(t)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= \frac{1}{t} \\ g(u) &= -u + \cos(u) \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(t)\, dt \\ \int \frac{1}{-u + \cos(u)}\, du &= \int \frac{1}{t}\, dt \\ \int^{u(t)} \frac{1}{-\tau + \cos(\tau)} d\tau &= \ln(t) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $-u + \cos(u) = 0$ for $u(t)$ gives

$$u(t) = \operatorname{RootOf}(-\cos(\_Z) + \_Z)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $\operatorname{RootOf}(-\cos(\_Z) + \_Z)$ will not be used

Converting $\int^{u(t)} \frac{1}{-\tau+\cos(\tau)} d\tau = \ln(t) + c_1$ back to x gives

$$\int^{\frac{x}{t}} \frac{1}{-\tau + \cos(\tau)} d\tau = \ln(t) + c_1$$

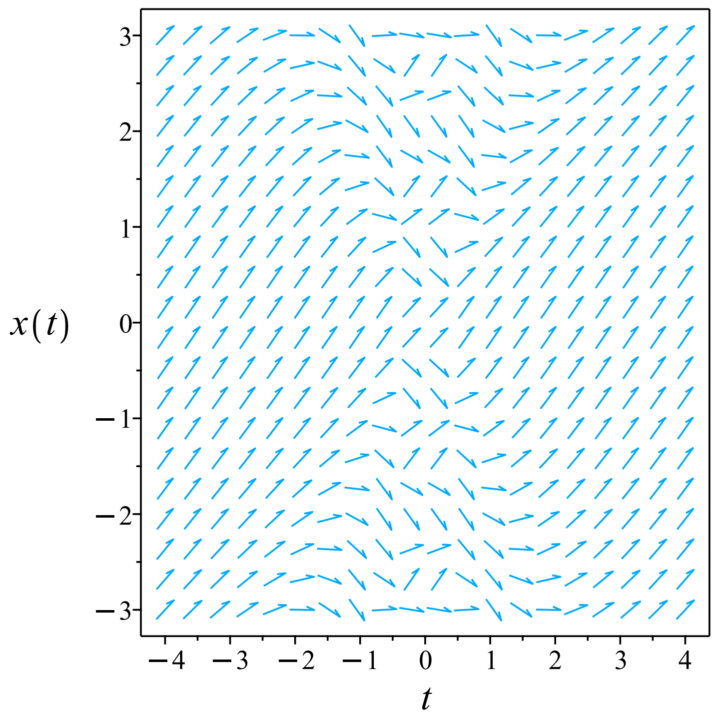


Figure 2.38: Slope field plot
$x' = \cos\left(\frac{x}{t}\right)$

Summary of solutions found

$$\int^{\frac{x}{t}} \frac{1}{-\tau + \cos(\tau)} d\tau = \ln(t) + c_1$$

Solved using Lie symmetry for first order ode

Time used: 1.046 (sec)

Writing the ode as

$$x' = \cos\left(\frac{x}{t}\right)$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$

$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \cos\left(\frac{x}{t}\right)(b_3 - a_2) - \cos\left(\frac{x}{t}\right)^2 a_3 - \frac{x\sin\left(\frac{x}{t}\right)(ta_2 + xa_3 + a_1)}{t^2} + \frac{\sin\left(\frac{x}{t}\right)(tb_2 + xb_3 + b_1)}{t} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\cos\left(\frac{x}{t}\right)^2 a_3t^2 + \cos\left(\frac{x}{t}\right)t^2a_2 - \cos\left(\frac{x}{t}\right)t^2b_3 - \sin\left(\frac{x}{t}\right)t^2b_2 + \sin\left(\frac{x}{t}\right)txa_2 - \sin\left(\frac{x}{t}\right)txb_3 + \sin\left(\frac{x}{t}\right)x^2a_3 - \mathrm{si}}{t^2} = 0$$

Setting the numerator to zero gives

$$-\cos\left(\frac{x}{t}\right)^2 a_3t^2 - \cos\left(\frac{x}{t}\right)t^2a_2 + \cos\left(\frac{x}{t}\right)t^2b_3 + \sin\left(\frac{x}{t}\right)t^2b_2 - \sin\left(\frac{x}{t}\right)txa_2 + \sin\left(\frac{x}{t}\right)txb_3 - \sin\left(\frac{x}{t}\right)x^2a_3 + \sin\left(\frac{x}{t}\right)tb_1 - \sin\left(\frac{x}{t}\right)xa_1 + b_2t^2 = 0 \tag{6E}$$

Simplifying the above gives

$$b_2t^2 - \frac{a_3t^2}{2} - \frac{a_3t^2\cos\left(\frac{2x}{t}\right)}{2} - \cos\left(\frac{x}{t}\right)t^2a_2 + \cos\left(\frac{x}{t}\right)t^2b_3 + \sin\left(\frac{x}{t}\right)t^2b_2 - \sin\left(\frac{x}{t}\right)txa_2 + \sin\left(\frac{x}{t}\right)txb_3 - \sin\left(\frac{x}{t}\right)x^2a_3 + \sin\left(\frac{x}{t}\right)tb_1 - \sin\left(\frac{x}{t}\right)xa_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\left\{t, x, \cos\left(\frac{x}{t}\right), \cos\left(\frac{2x}{t}\right), \sin\left(\frac{x}{t}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\left\{t = v_1, x = v_2, \cos\left(\frac{x}{t}\right) = v_3, \cos\left(\frac{2x}{t}\right) = v_4, \sin\left(\frac{x}{t}\right) = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & b_2 v_1^2 - \frac{1}{2} a_3 v_1^2 - \frac{1}{2} a_3 v_1^2 v_4 - v_3 v_1^2 a_2 + v_3 v_1^2 b_3 + v_5 v_1^2 b_2 \\ & - v_5 v_1 v_2 a_2 + v_5 v_1 v_2 b_3 - v_5 v_2^2 a_3 + v_5 v_1 b_1 - v_5 v_2 a_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned} & (b_3 - a_2) v_2 v_1 v_5 + \left(b_2 - \frac{a_3}{2}\right) v_1^2 - \frac{a_3 v_1^2 v_4}{2} + (b_3 - a_2) v_1^2 v_3 \\ & + v_5 v_1^2 b_2 - v_5 v_2^2 a_3 + v_5 v_1 b_1 - v_5 v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ b_2 &= 0 \\ -a_1 &= 0 \\ -a_3 &= 0 \\ -\frac{a_3}{2} &= 0 \\ b_2 - \frac{a_3}{2} &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= t \\
\eta &= x
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{\eta}{\xi} \\
&= \frac{x}{t} \\
&= \frac{x}{t}
\end{aligned}
$$

This is easily solved to give

$$
x = tc_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{x}{t}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dt}{\xi} \\ &= \frac{dt}{t} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dt}{T} \\ &= \ln(t) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \cos\left(\frac{x}{t}\right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= -\frac{x}{t^2} \\ R_x &= \frac{1}{t} \\ S_t &= \frac{1}{t} \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{t}{\cos\left(\frac{x}{t}\right) t - x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\cos(R) - R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

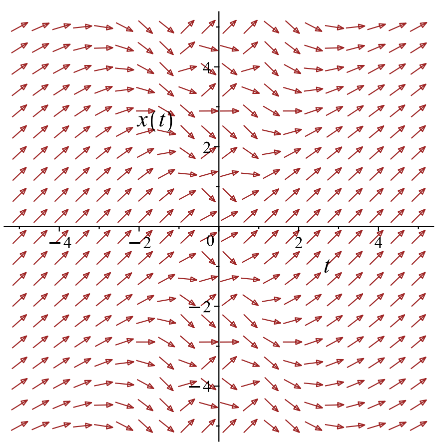
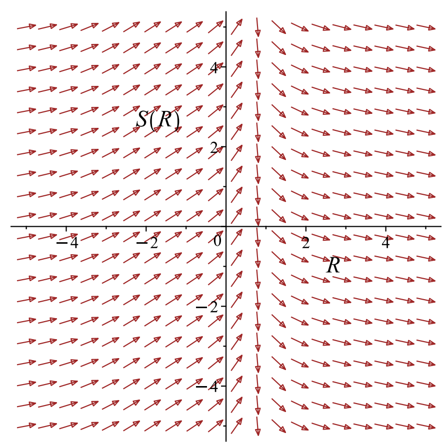
$$\int dS = \int \frac{1}{\cos(R) - R} dR$$
$$S(R) = \int \frac{1}{\cos(R) - R} dR + c_2$$

$$S(R) = \int \frac{1}{\cos(R) - R} dR + c_2$$

To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$\ln(t) = \int^{\frac{x}{t}} \frac{1}{\cos(\_a) - \_a} d\_a + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \cos\left(\frac{x}{t}\right)$ 	$R = \frac{x}{t}$ $S = \ln(t)$	$\frac{dS}{dR} = \frac{1}{\cos(R)-R}$ 

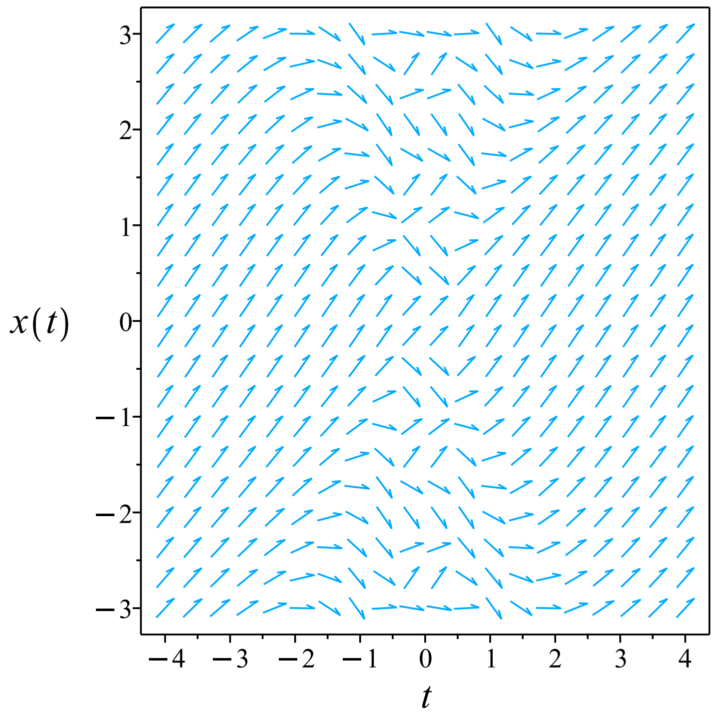


Figure 2.39: Slope field plot
$x' = \cos\left(\frac{x}{t}\right)$

Summary of solutions found

$$\ln(t) = \int^{\frac{x}{t}} \frac{1}{\cos(\_a) - \_a} d\_a + c_2$$

Solved as first order ode of type dAlembert

Time used: 1.377 (sec)

Let $p = x'$ the ode becomes

$$p = \cos\left(\frac{x}{t}\right)$$

Solving for x from the above results in

$$x = \arccos(p)\, t \tag{1}$$

This has the form

$$x = tf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = x'(t)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. t gives

$$
\begin{aligned}
p &= f + (tf' + g')\frac{dp}{dt} \\
p - f &= (tf' + g')\frac{dp}{dt}
\end{aligned} \tag{2}
$$

Comparing the form $x = tf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \arccos(p) \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \arccos(p) = -\frac{tp'(t)}{\sqrt{-p^2 + 1}} \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dt} = 0$ in the above which gives

$$
p - \arccos(p) = 0
$$

No singular solution are found.

The general solution is found when $\frac{\mathrm{dp}}{\mathrm{dt}} \neq 0$. From eq. (2A). This results in

$$
p'(t) = -\frac{(p(t) - \arccos(p(t)))\sqrt{-p(t)^2 + 1}}{t} \tag{3}
$$

This ODE is now solved for $p(t)$. No inversion is needed. The ode $p'(t) = -\frac{(p(t)-\arccos(p(t)))\sqrt{-p(t)^2+1}}{t}$ is separable as it can be written as

$$
\begin{aligned}
p'(t) &= -\frac{(p(t) - \arccos(p(t)))\sqrt{-p(t)^2 + 1}}{t} \\
&= f(t)g(p)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(t) &= -\frac{1}{t} \\
g(p) &= (p - \arccos(p))\sqrt{-p^2 + 1}
\end{aligned}
$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(t)\,dt$$
$$\int \frac{1}{(p-\arccos(p))\sqrt{-p^2+1}}\,dp = \int -\frac{1}{t}\,dt$$
$$\int^{p(t)} \frac{1}{(\tau-\arccos(\tau))\sqrt{-\tau^2+1}}d\tau = \ln\left(\frac{1}{t}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or $(p-\arccos(p))\sqrt{-p^2+1}=0$ for $p(t)$ gives

$$p(t) = -1$$
$$p(t) = 1$$
$$p(t) = \operatorname{RootOf}(-\cos(\_Z)+\_Z)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $\operatorname{RootOf}(-\cos(\_Z)+\_Z)$ will not be used

Therefore the solutions found are

$$\int^{p(t)} \frac{1}{(\tau-\arccos(\tau))\sqrt{-\tau^2+1}}d\tau = \ln\left(\frac{1}{t}\right) + c_1$$
$$p(t) = -1$$
$$p(t) = 1$$

Substituing the above solution for p in (2A) gives

$$x = \arccos\left(\operatorname{RootOf}\left(-\int^{\_Z} \frac{1}{(\tau-\arccos(\tau))\sqrt{-\tau^2+1}}d\tau + \ln\left(\frac{1}{t}\right) + c_1\right)\right)t$$
$$x = \pi t$$
$$x = 0$$

The solution

$$x = 0$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$x = \pi t$$

was found not to satisfy the ode or the IC. Hence it is removed.

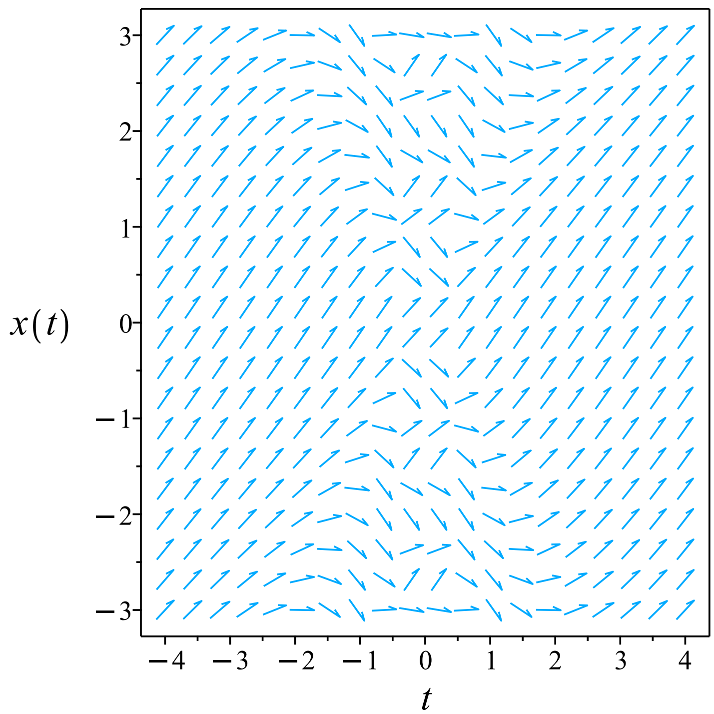


Figure 2.40: Slope field plot
$x' = \cos\left(\frac{x}{t}\right)$

Summary of solutions found

$$x = \arccos\left(\operatorname{RootOf}\left(-\int^{\_Z} \frac{1}{(\tau - \arccos(\tau))\sqrt{-\tau^2 + 1}} d\tau + \ln\left(\frac{1}{t}\right) + c_1\right)\right) t$$

Maple step by step solution

Let's solve
$x' = \cos\left(\frac{x}{t}\right)$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = \cos\left(\frac{x}{t}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 27

```
dsolve(diff(x(t),t) = cos(x(t)/t),
       x(t),singsol=all)
```

$$x = \mathrm{RootOf}\left(-\int^{\_Z} -\frac{1}{-\cos\left(\_a\right)+\_a}d\_a+\ln\left(t\right)+c_1\right)t$$

Mathematica DSolve solution

Solving time : 0.351 (sec)
Leaf size : 33

```
DSolve[{D[x[t],t]==Cos[x[t]/t],{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$\mathrm{Solve}\left[\int_1^{\frac{x(t)}{t}}\frac{1}{K[1]-\cos(K[1])}dK[1]=-\log(t)+c_1,x(t)\right]$$

2.1.16 problem 3 (iv)

Internal problem ID [18179]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 3 (iv)
Date solved : `Thursday, December 19, 2024 at 01:52:21 PM`
CAS classification : `[[_homogeneous, ‘class A‘], _rational, _dAlembert]`

Solve

$$(t^2 - x^2)\, x' = xt$$

Solved as first order homogeneous class A ode

Time used: 0.650 (sec)

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= -\frac{xt}{-t^2 + x^2} \end{aligned} \tag{1}$$

An ode of the form $x' = \frac{M(t,x)}{N(t,x)}$ is called homogeneous if the functions $M(t,x)$ and $N(t,x)$ are both homogeneous functions and of the same order. Recall that a function $f(t,x)$ is homogeneous of order n if

$$f(t^n t, t^n x) = t^n f(t,x)$$

In this case, it can be seen that both $M = tx$ and $N = t^2 - x^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{x}{t}$, or $x = ut$. Hence

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t}t + u$$

Applying the transformation $x = ut$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t}t + u &= -\frac{u}{u^2 - 1}\\ \frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{-\frac{u(t)}{u(t)^2 - 1} - u(t)}{t}\end{aligned}$$

Or

$$u'(t) - \frac{-\frac{u(t)}{u(t)^2 - 1} - u(t)}{t} = 0$$

Or

$$u'(t)\,u(t)^2\,t + u(t)^3 - u'(t)\,t = 0$$

Or

$$t\left(u(t)^2 - 1\right)u'(t) + u(t)^3 = 0$$

Which is now solved as separable in $u(t)$.

The ode $u'(t) = -\frac{u(t)^3}{t\left(u(t)^2 - 1\right)}$ is separable as it can be written as

$$\begin{aligned}u'(t) &= -\frac{u(t)^3}{t\left(u\left(t\right)^2 - 1\right)}\\ &= f(t)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(t) &= -\frac{1}{t}\\ g(u) &= \frac{u^3}{u^2 - 1}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(t)\,dt\\ \int \frac{u^2 - 1}{u^3}\,du &= \int -\frac{1}{t}\,dt\\ \ln\left(u(t)\right) + \frac{1}{2u\left(t\right)^2} &= \ln\left(\frac{1}{t}\right) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^3}{u^2-1} = 0$ for $u(t)$ gives

$$u(t) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(t)) + \frac{1}{2u(t)^2} = \ln\left(\frac{1}{t}\right) + c_1$$
$$u(t) = 0$$

Solving for $u(t)$ gives

$$u(t) = 0$$
$$u(t) = \sqrt{-\frac{1}{\operatorname{LambertW}(-\mathrm{e}^{-2c_1}t^2)}}$$

Converting $u(t) = 0$ back to x gives

$$x = 0$$

Converting $u(t) = \sqrt{-\frac{1}{\operatorname{LambertW}(-\mathrm{e}^{-2c_1}t^2)}}$ back to x gives

$$x = t\sqrt{-\frac{1}{\operatorname{LambertW}(-\mathrm{e}^{-2c_1}t^2)}}$$

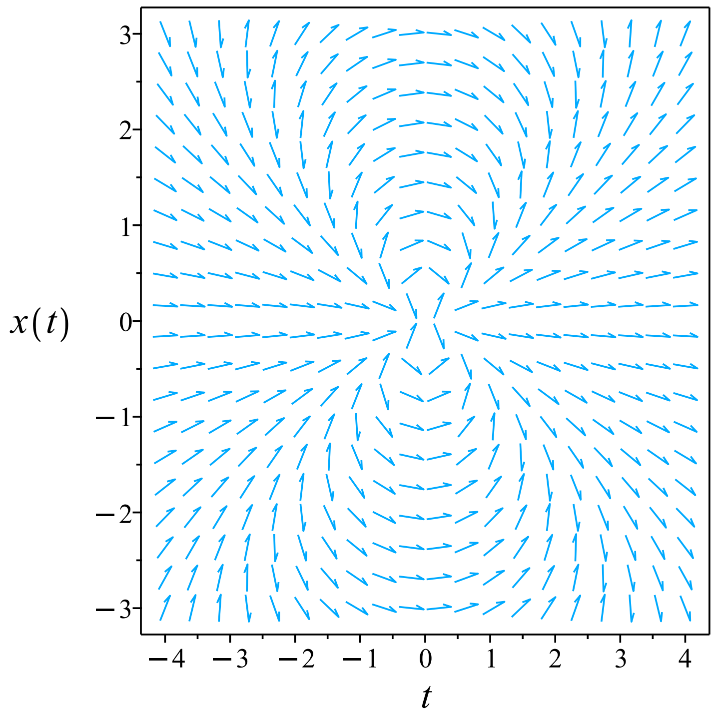


Figure 2.41: Slope field plot
$\left(t^2 - x^2\right) x' = xt$

Summary of solutions found

$$x = 0$$

$$x = t\sqrt{-\frac{1}{\operatorname{LambertW}\left(-\mathrm{e}^{-2c_1} t^2\right)}}$$

Solved as first order homogeneous class D2 ode

Time used: 0.530 (sec)

Applying change of variables $x = u(t)\, t$, then the ode becomes

$$\left(t^2 - u(t)^2 t^2\right) \left(u'(t)\, t + u(t)\right) = u(t)\, t^2$$

Which is now solved The ode $u'(t) = -\frac{u(t)^3}{\left(u(t)^2 - 1\right) t}$ is separable as it can be written as

$$\begin{aligned} u'(t) &= -\frac{u(t)^3}{\left(u\left(t\right)^2 - 1\right) t} \\ &= f(t) g(u) \end{aligned}$$

Where

$$f(t) = -\frac{1}{t}$$
$$g(u) = \frac{u^3}{u^2 - 1}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(t)\, dt$$
$$\int \frac{u^2-1}{u^3}\, du = \int -\frac{1}{t}\, dt$$
$$\ln(u(t)) + \frac{1}{2u(t)^2} = \ln\left(\frac{1}{t}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^3}{u^2-1} = 0$ for $u(t)$ gives

$$u(t) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(t)) + \frac{1}{2u(t)^2} = \ln\left(\frac{1}{t}\right) + c_1$$
$$u(t) = 0$$

Solving for $u(t)$ gives

$$u(t) = 0$$
$$u(t) = \sqrt{-\frac{1}{\operatorname{LambertW}\left(-\mathrm{e}^{-2c_1}t^2\right)}}$$

Converting $u(t) = 0$ back to x gives

$$x = 0$$

Converting $u(t) = \sqrt{-\frac{1}{\operatorname{LambertW}(-\mathrm{e}^{-2c_1}t^2)}}$ back to x gives

$$x = t\sqrt{-\frac{1}{\operatorname{LambertW}\left(-\mathrm{e}^{-2c_1}t^2\right)}}$$

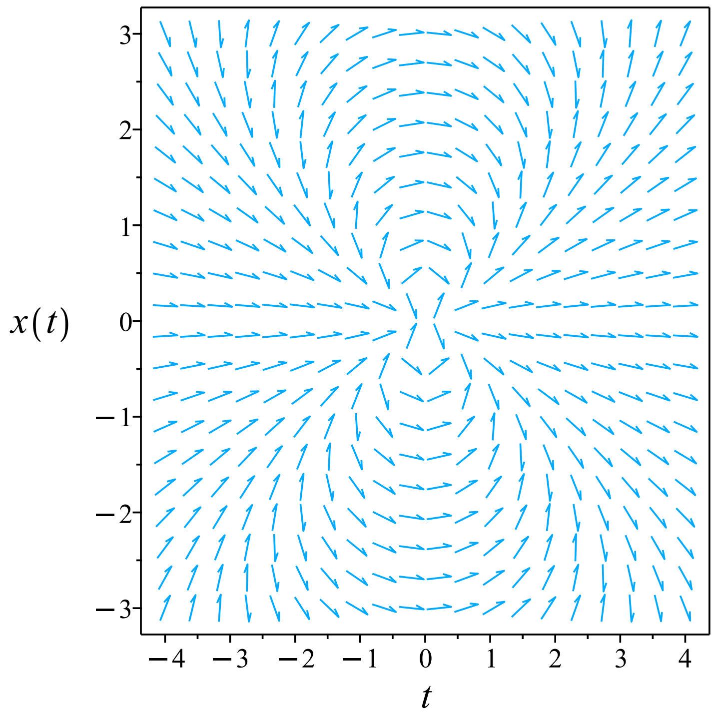


Figure 2.42: Slope field plot
$(t^2 - x^2) x' = xt$

Summary of solutions found

$$x = 0$$

$$x = t\sqrt{-\frac{1}{\text{LambertW}\left(-\mathrm{e}^{-2c_1} t^2\right)}}$$

Summary of solutions found

$$x = 0$$

$$x = t\sqrt{-\frac{1}{\text{LambertW}\left(-\mathrm{e}^{-2c_1} t^2\right)}}$$

Solved as first order homogeneous class Maple C ode

Time used: 1.020 (sec)

Let $Y = x - y_0$ and $X = t - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{(Y(X)+y_0)(x_0+X)}{(Y(X)+y_0)^2-(x_0+X)^2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{Y(X)X}{-X^2+Y(X)^2}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{YX}{-X^2+Y^2} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^nf(X,Y)$$

In this case, it can be seen that both $M = YX$ and $N = X^2-Y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -\frac{u}{u^2-1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-\frac{u(X)}{u(X)^2-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{-\frac{u(X)}{u(X)^2-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) u(X)^2 X + u(X)^3 - \left(\frac{d}{dX}u(X)\right) X = 0$$

Or

$$X\left(u(X)^2 - 1\right)\left(\frac{d}{dX}u(X)\right) + u(X)^3 = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = -\frac{u(X)^3}{X\left(u(X)^2-1\right)}$ is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{u(X)^3}{X\left(u\left(X\right)^2 - 1\right)}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= -\frac{1}{X}\\ g(u) &= \frac{u^3}{u^2-1}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(X)\,dX\\ \int \frac{u^2-1}{u^3}\,du &= \int -\frac{1}{X}\,dX\\ \ln\left(u(X)\right) + \frac{1}{2u\left(X\right)^2} &= \ln\left(\frac{1}{X}\right) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^3}{u^2-1} = 0$ for $u(X)$ gives

$$u(X) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(X)) + \frac{1}{2u(X)^2} = \ln\left(\frac{1}{X}\right) + c_1$$
$$u(X) = 0$$

Solving for $u(X)$ gives

$$u(X) = 0$$
$$u(X) = \sqrt{-\frac{1}{\operatorname{LambertW}(-\mathrm{e}^{-2c_1}X^2)}}$$

Converting $u(X) = 0$ back to $Y(X)$ gives

$$Y(X) = 0$$

Converting $u(X) = \sqrt{-\frac{1}{\operatorname{LambertW}(-\mathrm{e}^{-2c_1}X^2)}}$ back to $Y(X)$ gives

$$Y(X) = X\sqrt{-\frac{1}{\operatorname{LambertW}(-\mathrm{e}^{-2c_1}X^2)}}$$

Using the solution for $Y(X)$

$$Y(X) = 0 \tag{A}$$

And replacing back terms in the above solution using

$$Y = x + y_0$$
$$X = t + x_0$$

Or

$$Y = x$$
$$X = t$$

Then the solution in x becomes using EQ (A)

$$x = 0$$

Using the solution for $Y(X)$

$$Y(X) = X\sqrt{-\frac{1}{\operatorname{LambertW}(-\mathrm{e}^{-2c_1}X^2)}} \tag{A}$$

And replacing back terms in the above solution using

$$Y = x + y_0$$
$$X = t + x_0$$

Or

$$Y = x$$
$$X = t$$

Then the solution in x becomes using EQ (A)

$$x = t\sqrt{-\frac{1}{\mathrm{LambertW}\left(-\mathrm{e}^{-2c_1}t^2\right)}}$$

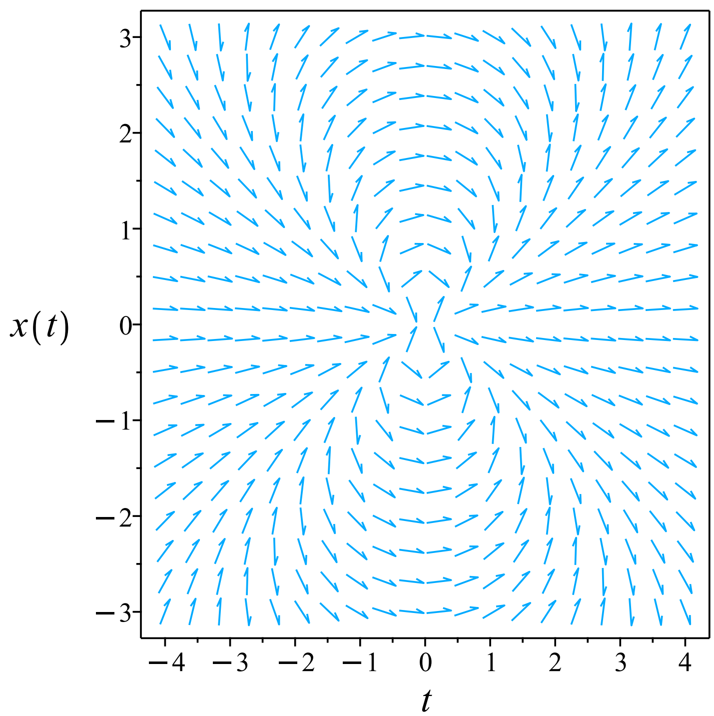


Figure 2.43: Slope field plot
$(t^2 - x^2)\, x' = xt$

Solved as first order Exact ode

Time used: 0.237 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(t^2-x^2\right)\mathrm{d}x&=(tx)\,\mathrm{d}t\\ (-tx)\,\mathrm{d}t+\left(t^2-x^2\right)\mathrm{d}x&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-tx\\ N(t,x)&=t^2-x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-tx) \\ &= -t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t^2 - x^2) \\ &= 2t\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\ &= \frac{1}{t^2 - x^2}((-t) - (2t)) \\ &= -\frac{3t}{t^2 - x^2}\end{aligned}$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right) \\ &= -\frac{1}{tx}((2t) - (-t)) \\ &= -\frac{3}{x}\end{aligned}$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}x} \\ &= e^{\int -\frac{3}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)} \\ &= \frac{1}{x^3}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^3}(-tx)\\ &= -\frac{t}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^3}\left(t^2 - x^2\right)\\ &= \frac{t^2 - x^2}{x^3}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\\ \left(-\frac{t}{x^2}\right) + \left(\frac{t^2 - x^2}{x^3}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{t}{x^2}\,\mathrm{d}t\end{aligned}$$

$$\phi = -\frac{t^2}{2x^2} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \frac{t^2}{x^3} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial x} = \frac{t^2-x^2}{x^3}$. Therefore equation (4) becomes

$$\frac{t^2 - x^2}{x^3} = \frac{t^2}{x^3} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{1}{x}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{1}{x}\right) \mathrm{d}x$$
$$f(x) = -\ln(x) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2x^2} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2x^2} - \ln(x)$$

Solving for x gives

$$x = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-t^2 \mathrm{e}^{2c_1}\right)}{2} - c_1}$$

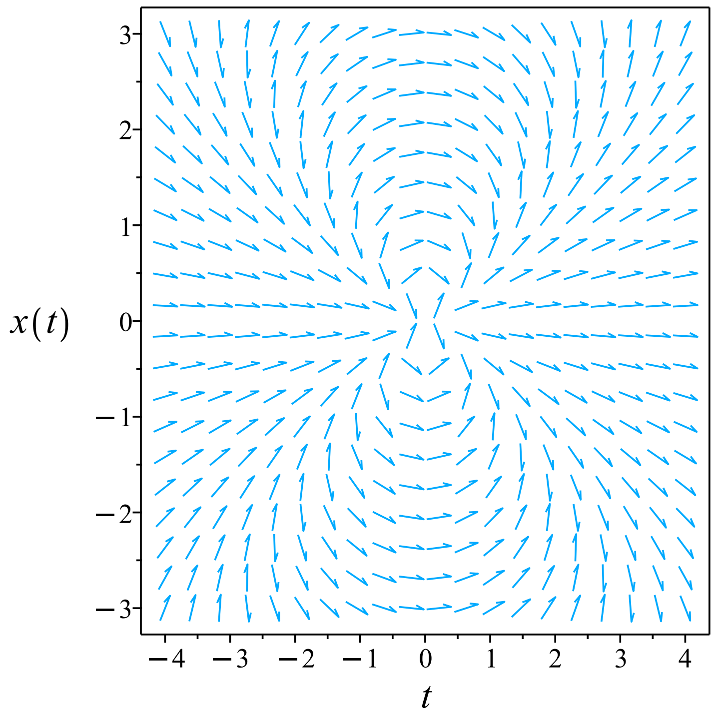


Figure 2.44: Slope field plot
$(t^2 - x^2)\, x' = xt$

Summary of solutions found

$$x = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-t^2 \mathrm{e}^{2c_1}\right)}{2} - c_1}$$

Solved as first order isobaric ode

Time used: 0.615 (sec)

Solving for x' gives

$$x' = -\frac{xt}{x^2 - t^2} \tag{1}$$

Each of the above ode's is now solved An ode $x' = f(t, x)$ is isobaric if

$$f(tt, t^m x) = t^{m-1} f(t, x) \tag{1}$$

Where here

$$f(t, x) = -\frac{xt}{x^2 - t^2} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} x &= ut^m \\ &= ut \end{aligned}$$

Converts the ODE to a separable in $u(t)$. Performing this substitution gives

$$u(t) + tu'(t) = -\frac{t^2 u(t)}{t^2 u(t)^2 - t^2}$$

The ode $u'(t) = -\frac{u(t)^3}{\left(u(t)^2-1\right)t}$ is separable as it can be written as

$$\begin{aligned} u'(t) &= -\frac{u(t)^3}{\left(u(t)^2 - 1\right)t} \\ &= f(t)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= -\frac{1}{t} \\ g(u) &= \frac{u^3}{u^2 - 1} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)} du &= \int f(t)\, dt \\ \int \frac{u^2 - 1}{u^3} du &= \int -\frac{1}{t} dt \\ \ln(u(t)) + \frac{1}{2u(t)^2} &= \ln\left(\frac{1}{t}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^3}{u^2-1} = 0$ for $u(t)$ gives

$$u(t) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(t)) + \frac{1}{2u(t)^2} &= \ln\left(\frac{1}{t}\right) + c_1 \\ u(t) &= 0 \end{aligned}$$

Solving for $u(t)$ gives

$$u(t) = 0$$

$$u(t) = \sqrt{-\frac{1}{\operatorname{LambertW}\left(-\mathrm{e}^{-2c_1} t^2\right)}}$$

Converting $u(t) = 0$ back to x gives

$$\frac{x}{t} = 0$$

Converting $u(t) = \sqrt{-\frac{1}{\operatorname{LambertW}\left(-\mathrm{e}^{-2c_1} t^2\right)}}$ back to x gives

$$\frac{x}{t} = \sqrt{-\frac{1}{\operatorname{LambertW}\left(-\mathrm{e}^{-2c_1} t^2\right)}}$$

Solving for x gives

$$x = 0$$

$$x = t\sqrt{-\frac{1}{\operatorname{LambertW}\left(-\mathrm{e}^{-2c_1} t^2\right)}}$$

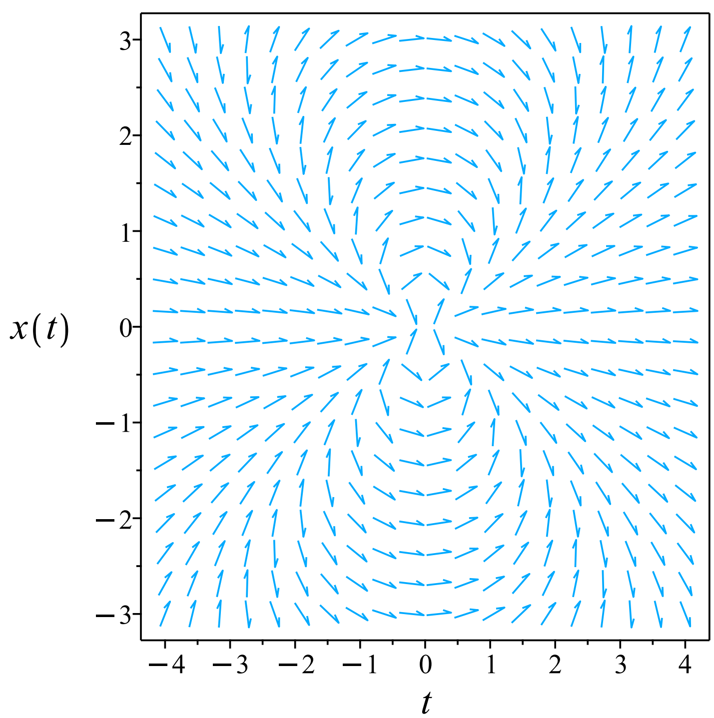


Figure 2.45: Slope field plot
$(t^2 - x^2)\, x' = xt$

Summary of solutions found

$$x = 0$$
$$x = t\sqrt{-\frac{1}{\operatorname{LambertW}\left(-\mathrm{e}^{-2c_1}t^2\right)}}$$

Solved using Lie symmetry for first order ode

Time used: 0.829 (sec)

Writing the ode as

$$x' = -\frac{xt}{-t^2+x^2}$$
$$x' = \omega(t,x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$
$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{xt(b_3-a_2)}{-t^2+x^2} - \frac{x^2t^2a_3}{(-t^2+x^2)^2} - \left(-\frac{x}{-t^2+x^2} - \frac{2x\,t^2}{(-t^2+x^2)^2}\right)(ta_2+xa_3+a_1) - \left(-\frac{t}{-t^2+x^2} + \frac{2x^2t}{(-t^2+x^2)^2}\right)(tb_2+xb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3t^2x^2b_2 - 2t\,x^3a_2 + 2t\,x^3b_3 - x^4a_3 - x^4b_2 + t^3b_1 - t^2xa_1 + t\,x^2b_1 - x^3a_1}{(t^2-x^2)^2} = 0$$

Setting the numerator to zero gives

$$-3t^2x^2b_2 + 2t\,x^3a_2 - 2t\,x^3b_3 + x^4a_3 + x^4b_2 - t^3b_1 + t^2xa_1 - t\,x^2b_1 + x^3a_1 = 0 \quad \text{(6E)}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1v_2^3 + a_3v_2^4 - 3b_2v_1^2v_2^2 + b_2v_2^4 - 2b_3v_1v_2^3 + a_1v_1^2v_2 + a_1v_2^3 - b_1v_1^3 - b_1v_1v_2^2 = 0 \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_1v_1^3 - 3b_2v_1^2v_2^2 + a_1v_1^2v_2 + (2a_2 - 2b_3)\,v_1v_2^3 - b_1v_1v_2^2 + (a_3 + b_2)\,v_2^4 + a_1v_2^3 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -b_1 &= 0 \\ -3b_2 &= 0 \\ 2a_2 - 2b_3 &= 0 \\ a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = t$$
$$\eta = x$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(t,x)\,\xi \\ &= x - \left(-\frac{xt}{-t^2+x^2}\right)(t) \\ &= -\frac{x^3}{t^2-x^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right)S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{x^3}{t^2-x^2}} dy\end{aligned}$$

Which results in

$$S = \ln(x) + \frac{t^2}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = -\frac{xt}{-t^2 + x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= \frac{t}{x^2} \\ S_x &= \frac{-t^2 + x^2}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 0 \, dR + c_2 \\ S(R) &= c_2 \end{aligned}$$

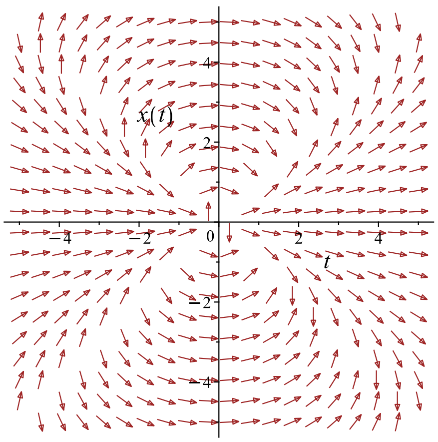
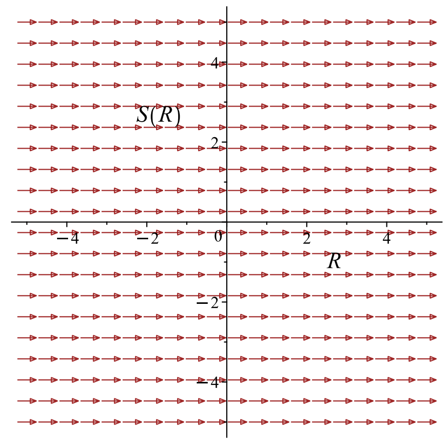
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$\frac{2 \ln(x) x^2 + t^2}{2x^2} = c_2$$

Which gives

$$x = e^{\frac{\text{LambertW}\left(-t^2 e^{-2c_2}\right)}{2} + c_2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{xt}{-t^2+x^2}$ 	$R = t$ $S = \frac{2\ln(x)\, x^2 + t^2}{2x^2}$	$\frac{dS}{dR} = 0$ 

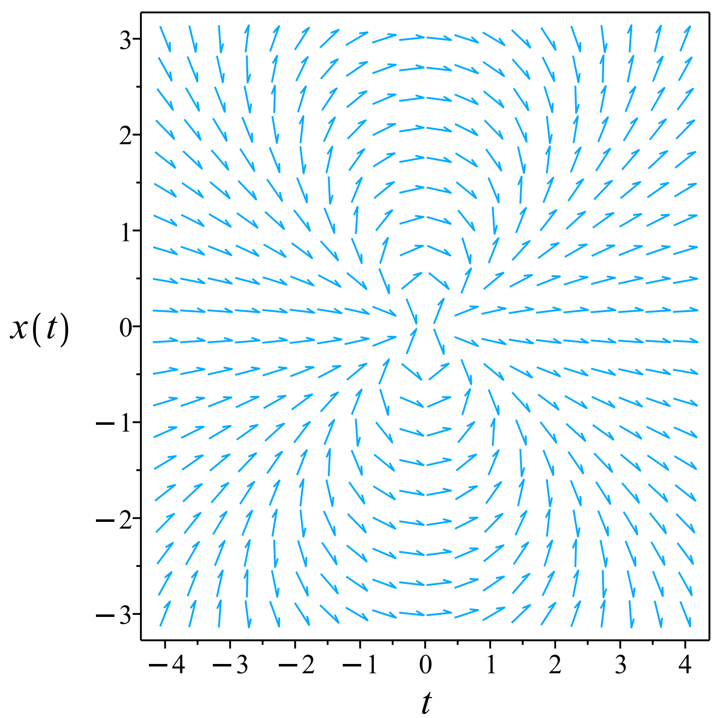


Figure 2.46: Slope field plot
$(t^2 - x^2)\, x' = xt$

Summary of solutions found

$$x = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-t^2 \mathrm{e}^{-2c_2}\right)}{2}+c_2}$$

Maple step by step solution

Let's solve

$(t^2 - x^2)\, x' = xt$

- Highest derivative means the order of the ODE is 1

  x'

- Solve for the highest derivative

  $x' = \frac{xt}{t^2 - x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 19

```
dsolve((t^2-x(t)^2)*diff(x(t),t) = x(t)*t,
       x(t),singsol=all)
```

$$x = \sqrt{-\frac{1}{\mathrm{LambertW}\left(-c_1\, t^2\right)}}\, t$$

Mathematica DSolve solution

Solving time : `7.493` (sec)
Leaf size : 56

```
DSolve[{(t^2-x[t]^2)*D[x[t],t]==t*x[t],{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
x(t) &\to -\frac{it}{\sqrt{W\left(-e^{-2c_1}t^2\right)}} \\
x(t) &\to \frac{it}{\sqrt{W\left(-e^{-2c_1}t^2\right)}} \\
x(t) &\to 0
\end{aligned}
$$

2.1.17 problem 3 (v)

Internal problem ID [18180]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 3 (v)
Date solved : Thursday, December 19, 2024 at 01:52:28 PM
CAS classification : [[_linear, ‘class A‘]]

Solve

$$e^{3t}x' + 3x\,e^{3t} = 2t$$

Solved as first order linear ode

Time used: 0.069 (sec)

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= 3 \\ p(t) &= 2t\,e^{-3t} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= e^{\int 3dt} \\ &= e^{3t} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = (\mu)\left(2t\,\mathrm{e}^{-3t}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x\,\mathrm{e}^{3t}\right) = \left(\mathrm{e}^{3t}\right)\left(2t\,\mathrm{e}^{-3t}\right)$$

$$\mathrm{d}\left(x\,\mathrm{e}^{3t}\right) = \left(2t\,\mathrm{e}^{-3t}\mathrm{e}^{3t}\right)\,\mathrm{d}t$$

Integrating gives

$$\begin{aligned} x\,\mathrm{e}^{3t} &= \int 2t\,\mathrm{e}^{-3t}\mathrm{e}^{3t}\,dt \\ &= t^2 + c_1 \end{aligned}$$

Dividing throughout by the integrating factor e^{3t} gives the final solution

$$x = \mathrm{e}^{-3t}\left(t^2 + c_1\right)$$

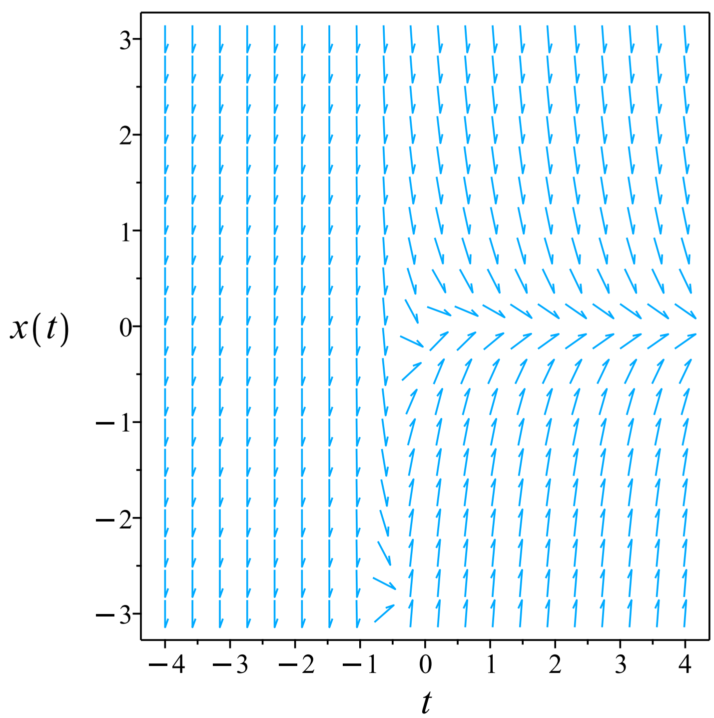


Figure 2.47: Slope field plot
$\mathrm{e}^{3t}x' + 3x\,\mathrm{e}^{3t} = 2t$

Summary of solutions found

$$x = e^{-3t}(t^2 + c_1)$$

Solved as first order Exact ode

Time used: 0.390 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$(e^{3t})\,\mathrm{d}x = (-3x\,e^{3t} + 2t)\,\mathrm{d}t$$
$$(3x\,e^{3t} - 2t)\,\mathrm{d}t + (e^{3t})\,\mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t,x) &= 3x\,\mathrm{e}^{3t} - 2t \\ N(t,x) &= \mathrm{e}^{3t} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(3x\,\mathrm{e}^{3t} - 2t\right) \\ &= 3\,\mathrm{e}^{3t} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\mathrm{e}^{3t}\right) \\ &= 3\,\mathrm{e}^{3t} \end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (2) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int N\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{3t}\,\mathrm{d}x \\ \phi &= x\,\mathrm{e}^{3t} + f(t) \end{aligned} \tag{3}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$\frac{\partial \phi}{\partial t} = 3x\,\mathrm{e}^{3t} + f'(t) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial t} = 3x\,\mathrm{e}^{3t} - 2t$. Therefore equation (4) becomes

$$3x\,\mathrm{e}^{3t} - 2t = 3x\,\mathrm{e}^{3t} + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$f'(t) = -2t$$

Integrating the above w.r.t t gives

$$\int f'(t)\,\mathrm{d}t = \int (-2t)\,\mathrm{d}t$$
$$f(t) = -t^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(t)$ into equation (3) gives ϕ

$$\phi = x\,\mathrm{e}^{3t} - t^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x\,\mathrm{e}^{3t} - t^2$$

Solving for x gives

$$x = \mathrm{e}^{-3t}\left(t^2 + c_1\right)$$

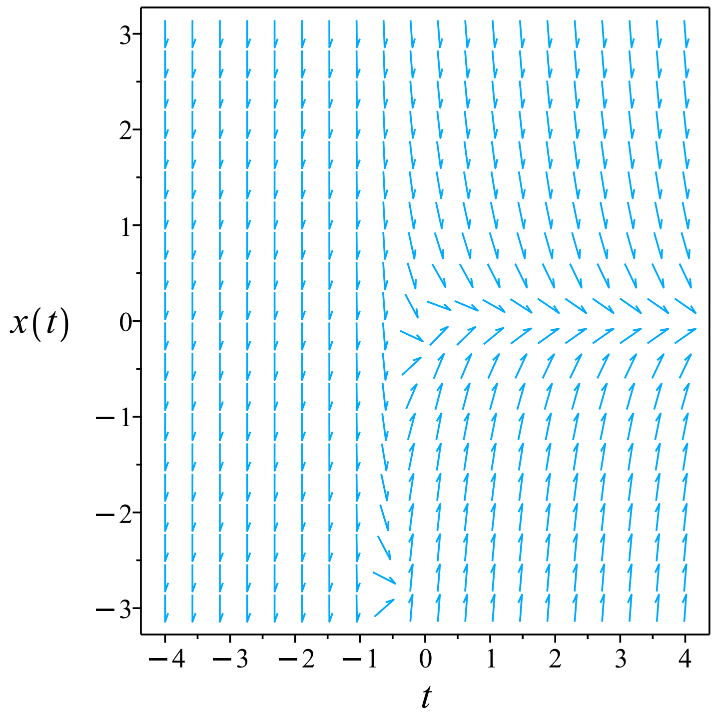


Figure 2.48: Slope field plot
$e^{3t}x' + 3x\,e^{3t} = 2t$

Summary of solutions found

$$x = e^{-3t}\left(t^2 + c_1\right)$$

Solved using Lie symmetry for first order ode

Time used: 1.231 (sec)

Writing the ode as

$$x' = -\left(3x\,e^{3t} - 2t\right)e^{-3t}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = t^2a_4 + txa_5 + x^2a_6 + ta_2 + xa_3 + a_1 \tag{1E}$$
$$\eta = t^2b_4 + txb_5 + x^2b_6 + tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2tb_4 + xb_5 + b_2 - \left(3x\,\mathrm{e}^{3t} - 2t\right)\mathrm{e}^{-3t}(-2ta_4 + tb_5 - xa_5 + 2xb_6 - a_2 + b_3) \\
&- \left(3x\,\mathrm{e}^{3t} - 2t\right)^2 \mathrm{e}^{-6t}(ta_5 + 2xa_6 + a_3) - \left(-\left(9x\,\mathrm{e}^{3t} - 2\right)\mathrm{e}^{-3t}\right. \\
&\left. + 3\left(3x\,\mathrm{e}^{3t} - 2t\right)\mathrm{e}^{-3t}\right)\left(t^2a_4 + txa_5 + x^2a_6 + ta_2 + xa_3 + a_1\right) \\
&+ 3t^2b_4 + 3txb_5 + 3x^2b_6 + 3tb_2 + 3xb_3 + 3b_1 = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}
&\left(-4t^2a_3 - 2\,\mathrm{e}^{3t}a_1 + 3b_1\mathrm{e}^{6t} + b_2\mathrm{e}^{6t} - 9\,\mathrm{e}^{6t}t\,x^2a_5 + 6\,\mathrm{e}^{6t}txa_4 - 8t^2xa_6\right. \\
&+ 6\,\mathrm{e}^{3t}t^3a_4 - 6\,\mathrm{e}^{3t}t^2a_4 + 2\,\mathrm{e}^{3t}t^2b_5 - 2\,\mathrm{e}^{3t}x^2a_6 - 18\,\mathrm{e}^{6t}x^3a_6 + 3t^2b_4\mathrm{e}^{6t} \\
&+ 3\,\mathrm{e}^{6t}x^2a_5 - 3x^2b_6\mathrm{e}^{6t} + 2tb_4\mathrm{e}^{6t} + xb_5\mathrm{e}^{6t} + 6\,\mathrm{e}^{3t}t^2a_2 + 6\,\mathrm{e}^{3t}ta_1 - 4\,\mathrm{e}^{3t}ta_2 \\
&+ 2\,\mathrm{e}^{3t}tb_3 - 2\,\mathrm{e}^{3t}xa_3 - 9\,\mathrm{e}^{6t}x^2a_3 + 3tb_2\mathrm{e}^{6t} + 3\,\mathrm{e}^{6t}xa_2 + 18\,\mathrm{e}^{3t}txa_3 \\
&\left.- 4t^3a_5 + 18\,\mathrm{e}^{3t}t^2xa_5 + 30\,\mathrm{e}^{3t}t\,x^2a_6 - 4\,\mathrm{e}^{3t}txa_5 + 4\,\mathrm{e}^{3t}txb_6\right)\mathrm{e}^{-6t} = 0
\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}
&-4t^2a_3 - 2\,\mathrm{e}^{3t}a_1 + 3b_1\mathrm{e}^{6t} + b_2\mathrm{e}^{6t} - 9\,\mathrm{e}^{6t}t\,x^2a_5 + 6\,\mathrm{e}^{6t}txa_4 - 8t^2xa_6 \\
&+ 6\,\mathrm{e}^{3t}t^3a_4 - 6\,\mathrm{e}^{3t}t^2a_4 + 2\,\mathrm{e}^{3t}t^2b_5 - 2\,\mathrm{e}^{3t}x^2a_6 - 18\,\mathrm{e}^{6t}x^3a_6 + 3t^2b_4\mathrm{e}^{6t} \\
&+ 3\,\mathrm{e}^{6t}x^2a_5 - 3x^2b_6\mathrm{e}^{6t} + 2tb_4\mathrm{e}^{6t} + xb_5\mathrm{e}^{6t} + 6\,\mathrm{e}^{3t}t^2a_2 + 6\,\mathrm{e}^{3t}ta_1 - 4\,\mathrm{e}^{3t}ta_2 \\
&+ 2\,\mathrm{e}^{3t}tb_3 - 2\,\mathrm{e}^{3t}xa_3 - 9\,\mathrm{e}^{6t}x^2a_3 + 3tb_2\mathrm{e}^{6t} + 3\,\mathrm{e}^{6t}xa_2 + 18\,\mathrm{e}^{3t}txa_3 \\
&- 4t^3a_5 + 18\,\mathrm{e}^{3t}t^2xa_5 + 30\,\mathrm{e}^{3t}t\,x^2a_6 - 4\,\mathrm{e}^{3t}txa_5 + 4\,\mathrm{e}^{3t}txb_6 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
&-4t^2a_3 - 2\,\mathrm{e}^{3t}a_1 + 3b_1\mathrm{e}^{6t} + b_2\mathrm{e}^{6t} - 9\,\mathrm{e}^{6t}t\,x^2a_5 + 6\,\mathrm{e}^{6t}txa_4 - 8t^2xa_6 \\
&+ 6\,\mathrm{e}^{3t}t^3a_4 - 6\,\mathrm{e}^{3t}t^2a_4 + 2\,\mathrm{e}^{3t}t^2b_5 - 2\,\mathrm{e}^{3t}x^2a_6 - 18\,\mathrm{e}^{6t}x^3a_6 + 3t^2b_4\mathrm{e}^{6t} \\
&+ 3\,\mathrm{e}^{6t}x^2a_5 - 3x^2b_6\mathrm{e}^{6t} + 2tb_4\mathrm{e}^{6t} + xb_5\mathrm{e}^{6t} + 6\,\mathrm{e}^{3t}t^2a_2 + 6\,\mathrm{e}^{3t}ta_1 - 4\,\mathrm{e}^{3t}ta_2 \\
&+ 2\,\mathrm{e}^{3t}tb_3 - 2\,\mathrm{e}^{3t}xa_3 - 9\,\mathrm{e}^{6t}x^2a_3 + 3tb_2\mathrm{e}^{6t} + 3\,\mathrm{e}^{6t}xa_2 + 18\,\mathrm{e}^{3t}txa_3 \\
&- 4t^3a_5 + 18\,\mathrm{e}^{3t}t^2xa_5 + 30\,\mathrm{e}^{3t}t\,x^2a_6 - 4\,\mathrm{e}^{3t}txa_5 + 4\,\mathrm{e}^{3t}txb_6 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x, \mathrm{e}^{3t}, \mathrm{e}^{6t}\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2, \mathrm{e}^{3t} = v_3, \mathrm{e}^{6t} = v_4\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&6v_3v_1^3a_4 + 18v_3v_1^2v_2a_5 - 9v_4v_1v_2^2a_5 + 30v_3v_1v_2^2a_6 - 18v_4v_2^3a_6 + 6v_3v_1^2a_2 \\
&+ 18v_3v_1v_2a_3 - 9v_4v_2^2a_3 - 6v_3v_1^2a_4 + 6v_4v_1v_2a_4 - 4v_1^3a_5 - 4v_3v_1v_2a_5 \\
&+ 3v_4v_2^2a_5 - 8v_1^2v_2a_6 - 2v_3v_2^2a_6 + 3v_1^2b_4v_4 + 2v_3v_1^2b_5 + 4v_3v_1v_2b_6 \\
&- 3v_2^2b_6v_4 + 6v_3v_1a_1 - 4v_3v_1a_2 + 3v_4v_2a_2 - 4v_1^2a_3 - 2v_3v_2a_3 \\
&+ 3v_1b_2v_4 + 2v_3v_1b_3 + 2v_1b_4v_4 + v_2b_5v_4 - 2v_3a_1 + 3b_1v_4 + b_2v_4 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}
&6v_3v_1^3a_4 - 4v_1^3a_5 + 18v_3v_1^2v_2a_5 - 8v_1^2v_2a_6 + (6a_2 - 6a_4 + 2b_5)\, v_1^2v_3 \\
&+ 3v_1^2b_4v_4 - 4v_1^2a_3 + 30v_3v_1v_2^2a_6 - 9v_4v_1v_2^2a_5 + (18a_3 - 4a_5 + 4b_6)\, v_1v_2v_3 \\
&+ 6v_4v_1v_2a_4 + (6a_1 - 4a_2 + 2b_3)\, v_1v_3 + (3b_2 + 2b_4)\, v_1v_4 - 18v_4v_2^3a_6 - 2v_3v_2^2a_6 \\
&+ (-9a_3 + 3a_5 - 3b_6)\, v_2^2v_4 - 2v_3v_2a_3 + (3a_2 + b_5)\, v_2v_4 - 2v_3a_1 + (3b_1 + b_2)\, v_4 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0\\
-4a_3 &= 0\\
-2a_3 &= 0\\
6a_4 &= 0\\
-9a_5 &= 0\\
-4a_5 &= 0\\
18a_5 &= 0\\
-18a_6 &= 0\\
-8a_6 &= 0\\
-2a_6 &= 0\\
30a_6 &= 0\\
3b_4 &= 0\\
3a_2 + b_5 &= 0\\
3b_1 + b_2 &= 0\\
3b_2 + 2b_4 &= 0\\
6a_1 - 4a_2 + 2b_3 &= 0\\
6a_2 - 6a_4 + 2b_5 &= 0\\
-9a_3 + 3a_5 - 3b_6 &= 0\\
18a_3 - 4a_5 + 4b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{b_5}{3} \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -\frac{2b_5}{3} \\
b_4 &= 0 \\
b_5 &= b_5 \\
b_6 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{t}{3} \\
\eta &= tx - \frac{2}{3}x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t, x)\,\xi \\
&= tx - \frac{2}{3}x - \left(-\left(3x\,\mathrm{e}^{3t} - 2t\right)\mathrm{e}^{-3t}\right)\left(-\frac{t}{3}\right) \\
&= \frac{\left(-2x\,\mathrm{e}^{3t} + 2t^2\right)\mathrm{e}^{-3t}}{3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{(-2x\,\mathrm{e}^{3t}+2t^2)\mathrm{e}^{-3t}}{3}} dy \end{aligned}$$

Which results in

$$S = -\frac{3\ln\left(-x\,\mathrm{e}^{3t}+t^2\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\left(3x\,\mathrm{e}^{3t} - 2t\right)\mathrm{e}^{-3t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= \frac{-9x\,\mathrm{e}^{3t}+6t}{2x\,\mathrm{e}^{3t}-2t^2} \\ S_x &= -\frac{3\,\mathrm{e}^{3t}}{2x\,\mathrm{e}^{3t}-2t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0\, dR + c_2$$
$$S(R) = c_2$$

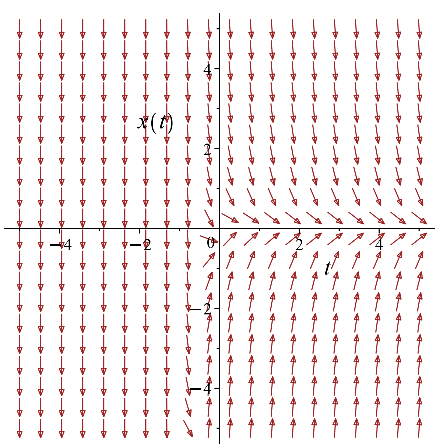
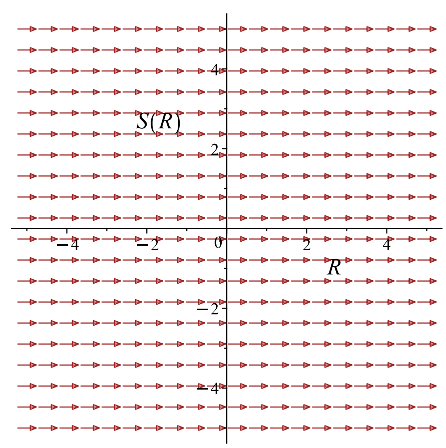
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$-\frac{3\ln\left(-x\,\mathrm{e}^{3t} + t^2\right)}{2} = c_2$$

Which gives

$$x = -\left(\mathrm{e}^{-\frac{2c_2}{3}} - t^2\right)\mathrm{e}^{-3t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -(3x\,\mathrm{e}^{3t} - 2t)\,\mathrm{e}^{-3t}$ 	$R = t$ $S = -\frac{3\ln\left(-x\,\mathrm{e}^{3t} + t^2\right)}{2}$	$\frac{dS}{dR} = 0$ 

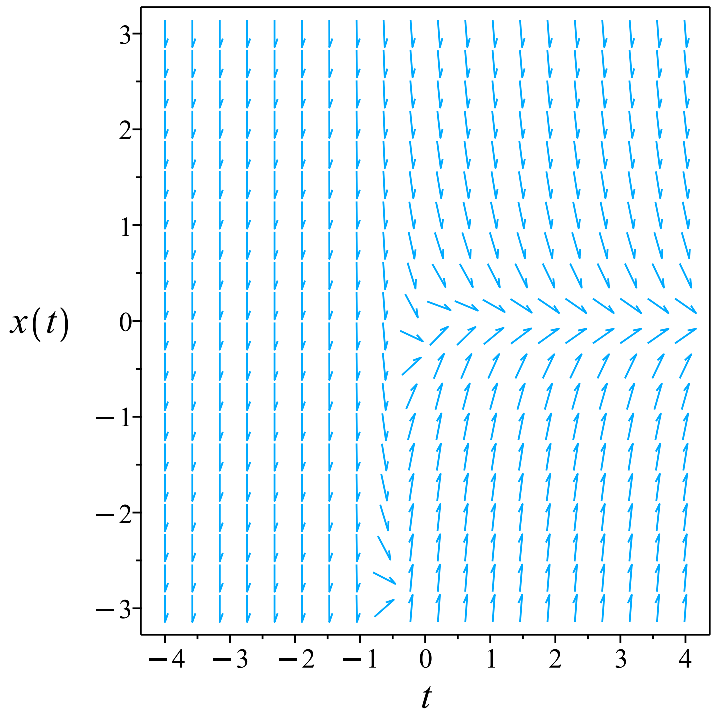


Figure 2.49: Slope field plot
$e^{3t}x' + 3x\,e^{3t} = 2t$

Summary of solutions found

$$x = -\left(e^{-\frac{2c_2}{3}} - t^2\right) e^{-3t}$$

Maple step by step solution

Let's solve
$e^{3t}x' + 3x\,e^{3t} = 2t$

- Highest derivative means the order of the ODE is 1
  x'
- Isolate the derivative
  $x' = -3x + \frac{2t}{e^{3t}}$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE
  $x' + 3x = \frac{2t}{e^{3t}}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)\,(x' + 3x) = \frac{2\mu(t)t}{e^{3t}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(x\mu(t))$
  $\mu(t)(x' + 3x) = x'\mu(t) + x\mu'(t)$
- Isolate $\mu'(t)$
  $\mu'(t) = 3\mu(t)$
- Solve to find the integrating factor
  $\mu(t) = (e^{3t})^2 e^{-3t}$
- Integrate both sides with respect to t
  $\int \left(\frac{d}{dt}(x\mu(t))\right) dt = \int \frac{2\mu(t)t}{e^{3t}} dt + C1$
- Evaluate the integral on the lhs
  $x\mu(t) = \int \frac{2\mu(t)t}{e^{3t}} dt + C1$
- Solve for x
  $x = \frac{\int \frac{2\mu(t)t}{e^{3t}} dt + C1}{\mu(t)}$
- Substitute $\mu(t) = (e^{3t})^2 e^{-3t}$
  $x = \frac{\int 2t\, e^{-3t} e^{3t} dt + C1}{(e^{3t})^2 e^{-3t}}$
- Evaluate the integrals on the rhs
  $x = \frac{t^2 + C1}{(e^{3t})^2 e^{-3t}}$
- Simplify
  $x = e^{-3t}(t^2 + C1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(exp(3*t)*diff(x(t),t)+3*x(t)*exp(3*t) = 2*t,
       x(t),singsol=all)
```

$$x = \mathrm{e}^{-3t}(t^2 + c_1)$$

Mathematica DSolve solution

Solving time : 0.056 (sec)
Leaf size : 17

```
DSolve[{Exp[3*t]*D[x[t],t]+3*x[t]*Exp[3*t]==2*t,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to e^{-3t}(t^2 + c_1)$$

2.1.18 problem 3 (vi)

Internal problem ID [18181]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 3 (vi)
Date solved : `Thursday, December 19, 2024 at 01:52:30 PM`
CAS classification :
`[_exact, _rational, [_1st_order, '_with_symmetry_[F(x),G(x)]'], [_Abel, '2nd type', 'c`

Solve

$$2t + 3x + (3t - x)\, x' = t^2$$

Solved as first order Exact ode

Time used: 0.546 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3t - x)\,\mathrm{d}x &= \left(t^2 - 2t - 3x\right)\mathrm{d}t \\ \left(-t^2 + 2t + 3x\right)\mathrm{d}t + (3t - x)\,\mathrm{d}x &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, x) &= -t^2 + 2t + 3x \\ N(t, x) &= 3t - x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-t^2 + 2t + 3x\right) \\ &= 3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(3t - x) \\ &= 3 \end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -t^2 + 2t + 3x\,\mathrm{d}t$$
$$\phi = -\frac{t(t^2 - 3t - 9x)}{3} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 3t + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = 3t - x$. Therefore equation (4) becomes

$$3t - x = 3t + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -x$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int (-x)\,\mathrm{d}x$$
$$f(x) = -\frac{x^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t(t^2 - 3t - 9x)}{3} - \frac{x^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{t(t^2 - 3t - 9x)}{3} - \frac{x^2}{2}$$

Solving for x gives

$$x = 3t - \frac{\sqrt{-6t^3 + 99t^2 - 18c_1}}{3}$$

$$x = 3t + \frac{\sqrt{-6t^3 + 99t^2 - 18c_1}}{3}$$

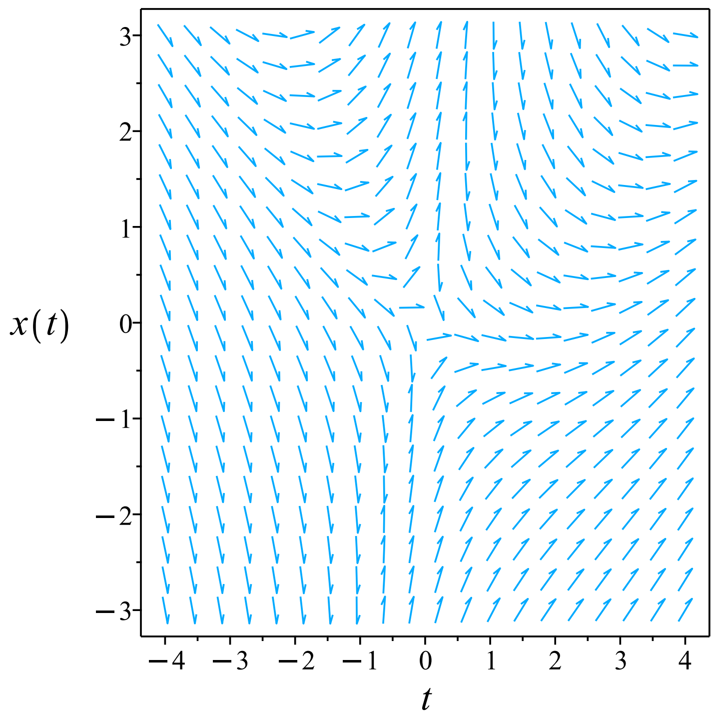


Figure 2.50: Slope field plot
$2t + 3x + (3t - x)\,x' = t^2$

Summary of solutions found

$$x = 3t - \frac{\sqrt{-6t^3 + 99t^2 - 18c_1}}{3}$$

$$x = 3t + \frac{\sqrt{-6t^3 + 99t^2 - 18c_1}}{3}$$

Maple step by step solution

Let's solve

$2t+3x+(3t-x)\,x' = t^2$

- Highest derivative means the order of the ODE is 1

  x'

- □ Check if ODE is exact
  - ○ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(t,x) = 0$
  - ○ Compute derivative of lhs

    $F'(t,x) + \left(\frac{\partial}{\partial x}F(t,x)\right) x' = 0$
  - ○ Evaluate derivatives

    $3 = 3$
  - ○ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(t,x) = C1\,, M(t,x) = F'(t,x)\,, N(t,x) = \frac{\partial}{\partial x}F(t,x)\right]$

- Solve for $F(t,x)$ by integrating $M(t,x)$ with respect to t

  $F(t,x) = \int (-t^2+2t+3x)\,dt + f_1(x)$

- Evaluate integral

  $F(t,x) = -\frac{t^3}{3} + t^2 + 3tx + f_1(x)$

- Take derivative of $F(t,x)$ with respect to x

  $N(t,x) = \frac{\partial}{\partial x}F(t,x)$

- Compute derivative

  $3t - x = 3t + \frac{d}{dx}f_1(x)$

- Isolate for $\frac{d}{dx}f_1(x)$

  $\frac{d}{dx}f_1(x) = -x$

- Solve for $f_1(x)$

  $f_1(x) = -\frac{x^2}{2}$

- Substitute $f_1(x)$ into equation for $F(t,x)$

  $F(t,x) = -\frac{1}{3}t^3 + t^2 + 3tx - \frac{1}{2}x^2$

- Substitute $F(t,x)$ into the solution of the ODE

  $-\frac{1}{3}t^3 + t^2 + 3tx - \frac{1}{2}x^2 = C1$

- Solve for x

  $\left\{x = 3t - \frac{\sqrt{-6t^3+99t^2-18C1}}{3}, x = 3t + \frac{\sqrt{-6t^3+99t^2-18C1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 51

```
dsolve(2*t+3*x(t)+(3*t-x(t))*diff(x(t),t) = t^2,
       x(t),singsol=all)
```

$$x = 3t - \frac{\sqrt{-6t^3 + 99t^2 + 18c_1}}{3}$$
$$x = 3t + \frac{\sqrt{-6t^3 + 99t^2 + 18c_1}}{3}$$

Mathematica DSolve solution

Solving time : 0.161 (sec)
Leaf size : 67

```
DSolve[{(2*t+3*x[t])+(3*t-x[t])*D[x[t],t]==t^2,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to 3t - i\sqrt{\frac{2t^3}{3} - 11t^2 - c_1}$$
$$x(t) \to 3t + i\sqrt{\frac{2t^3}{3} - 11t^2 - c_1}$$

2.1.19 problem 4 (i)

Internal problem ID [18182]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 4 (i)
Date solved : `Thursday, December 19, 2024 at 06:17:34 PM`
CAS classification : `[[_linear, ‘class A‘]]`

Solve

$$x' + 2x = e^t$$

Solved as first order linear ode

Time used: 0.187 (sec)

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = 2$$
$$p(t) = e^t$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt} \\ &= e^{\int 2dt} \\ &= e^{2t}\end{aligned}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = \mu p$$

$$\frac{d}{dt}(\mu x) = (\mu)\left(e^{t}\right)$$

$$\frac{d}{dt}\left(x\,e^{2t}\right) = \left(e^{2t}\right)\left(e^{t}\right)$$

$$d\left(x\,e^{2t}\right) = \left(e^{t}e^{2t}\right)\,dt$$

Integrating gives

$$x\,e^{2t} = \int e^{t}e^{2t}\,dt$$

$$= \frac{e^{3t}}{3} + c_1$$

Dividing throughout by the integrating factor e^{2t} gives the final solution

$$x = \frac{\left(e^{3t} + 3c_1\right)e^{-2t}}{3}$$

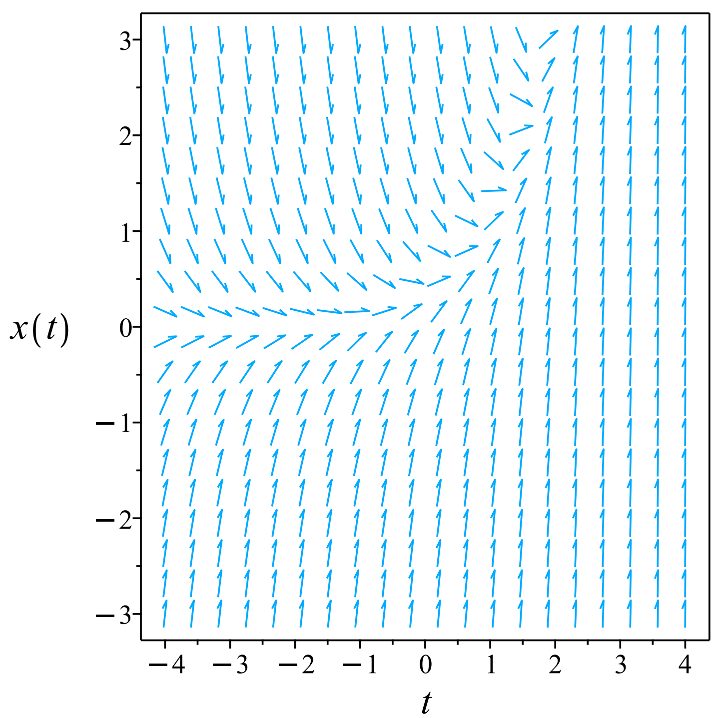


Figure 2.51: Slope field plot
$x' + 2x = e^{t}$

Summary of solutions found

$$x = \frac{(\mathrm{e}^{3t} + 3c_1)\,\mathrm{e}^{-2t}}{3}$$

Solved as first order Exact ode

Time used: 0.182 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = \left(-2x + \mathrm{e}^{t}\right)\mathrm{d}t$$
$$\left(2x - \mathrm{e}^{t}\right)\mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,x) &= 2x - \mathrm{e}^{t} \\
N(t,x) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(2x - \mathrm{e}^{t}\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1((2) - (0)) \\
&= 2
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}t} \\
&= e^{\int 2 \,\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2t} \\
&= \mathrm{e}^{2t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{2t}(2x - \mathrm{e}^t) \\ &= (2x - \mathrm{e}^t)\,\mathrm{e}^{2t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{2t}(1) \\ &= \mathrm{e}^{2t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left((2x - \mathrm{e}^t)\,\mathrm{e}^{2t}\right) + \left(\mathrm{e}^{2t}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{N}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{2t}\,\mathrm{d}x \\ \phi &= x\,\mathrm{e}^{2t} + f(t) \qquad (3)\end{aligned}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$\frac{\partial \phi}{\partial t} = 2x\,\mathrm{e}^{2t} + f'(t) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial t} = (2x - \mathrm{e}^t)\,\mathrm{e}^{2t}$. Therefore equation (4) becomes

$$\left(2x - \mathrm{e}^t\right)\mathrm{e}^{2t} = 2x\,\mathrm{e}^{2t} + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$\begin{aligned} f'(t) &= -\mathrm{e}^t\mathrm{e}^{2t} \\ &= -\mathrm{e}^{3t} \end{aligned}$$

Integrating the above w.r.t t results in

$$\begin{aligned} \int f'(t)\,\mathrm{d}t &= \int \left(-\mathrm{e}^{3t}\right)\mathrm{d}t \\ f(t) &= -\frac{\mathrm{e}^{3t}}{3} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(t)$ into equation (3) gives ϕ

$$\phi = x\,\mathrm{e}^{2t} - \frac{\mathrm{e}^{3t}}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x\,\mathrm{e}^{2t} - \frac{\mathrm{e}^{3t}}{3}$$

Solving for x gives

$$x = \frac{\left(\mathrm{e}^{3t} + 3c_1\right)\mathrm{e}^{-2t}}{3}$$

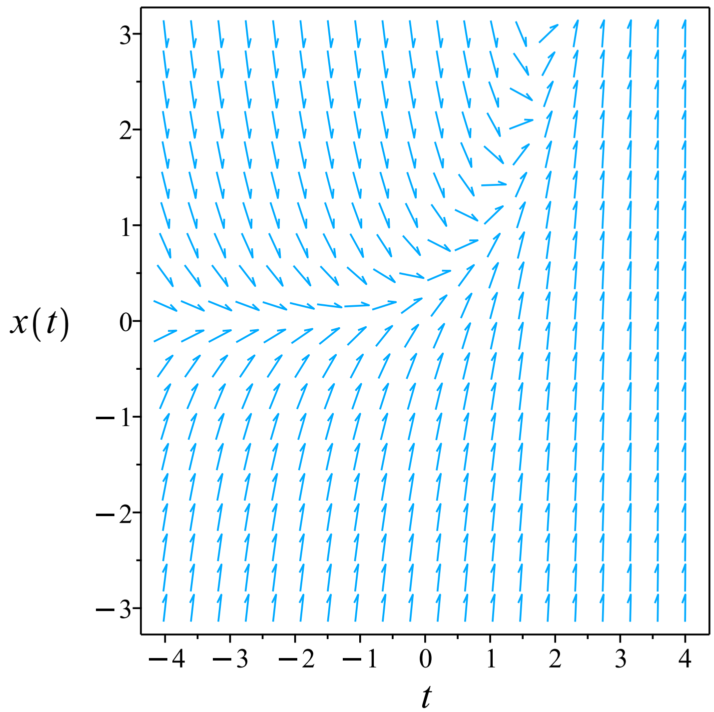


Figure 2.52: Slope field plot
$x' + 2x = e^t$

Summary of solutions found

$$x = \frac{(e^{3t} + 3c_1)\, e^{-2t}}{3}$$

Solved using Lie symmetry for first order ode

Time used: 0.563 (sec)

Writing the ode as

$$x' = -2x + e^t$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = t a_2 + x a_3 + a_1 \tag{1E}$$
$$\eta = t b_2 + x b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(-2x + \mathrm{e}^t\right)(b_3 - a_2) - \left(-2x + \mathrm{e}^t\right)^2 a_3 - \mathrm{e}^t (ta_2 + xa_3 + a_1) + 2tb_2 + 2xb_3 + 2b_1 = 0 \quad \text{(5E)}$$

Putting the above in normal form gives

$$-\mathrm{e}^{2t} a_3 - \mathrm{e}^t t a_2 + 3\,\mathrm{e}^t x a_3 - 4x^2 a_3 - \mathrm{e}^t a_1 - \mathrm{e}^t a_2 + \mathrm{e}^t b_3 + 2tb_2 + 2xa_2 + 2b_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-\mathrm{e}^{2t} a_3 - \mathrm{e}^t t a_2 + 3\,\mathrm{e}^t x a_3 - 4x^2 a_3 - \mathrm{e}^t a_1 - \mathrm{e}^t a_2 + \mathrm{e}^t b_3 + 2tb_2 + 2xa_2 + 2b_1 + b_2 = 0 \quad \text{(6E)}$$

Simplifying the above gives

$$-\mathrm{e}^{2t} a_3 - \mathrm{e}^t t a_2 + 3\,\mathrm{e}^t x a_3 - 4x^2 a_3 - \mathrm{e}^t a_1 - \mathrm{e}^t a_2 + \mathrm{e}^t b_3 + 2tb_2 + 2xa_2 + 2b_1 + b_2 = 0 \quad \text{(6E)}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x, \mathrm{e}^t, \mathrm{e}^{2t}\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2, \mathrm{e}^t = v_3, \mathrm{e}^{2t} = v_4\}$$

The above PDE (6E) now becomes

$$-v_3 v_1 a_2 - 4v_2^2 a_3 + 3v_3 v_2 a_3 - v_3 a_1 + 2v_2 a_2 - v_3 a_2 - v_4 a_3 + 2v_1 b_2 + v_3 b_3 + 2b_1 + b_2 = 0 \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-v_3 v_1 a_2 + 2v_1 b_2 - 4v_2^2 a_3 + 3v_3 v_2 a_3 + 2v_2 a_2 + (-a_1 - a_2 + b_3)\, v_3 - v_4 a_3 + 2b_1 + b_2 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a_2 &= 0 \\
2a_2 &= 0 \\
-4a_3 &= 0 \\
-a_3 &= 0 \\
3a_3 &= 0 \\
2b_2 &= 0 \\
2b_1 + b_2 &= 0 \\
-a_1 - a_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_3 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t, x)\, \xi \\
&= x - \left(-2x + \mathrm{e}^{t}\right)(1) \\
&= 3x - \mathrm{e}^{t} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{3x - \mathrm{e}^{t}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(3x - \mathrm{e}^{t}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -2x + \mathrm{e}^{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= \frac{\mathrm{e}^{t}}{-9x + 3\,\mathrm{e}^{t}} \\ S_x &= \frac{1}{3x - \mathrm{e}^{t}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -\frac{2}{3}\, dR$$
$$S(R) = -\frac{2R}{3} + c_2$$

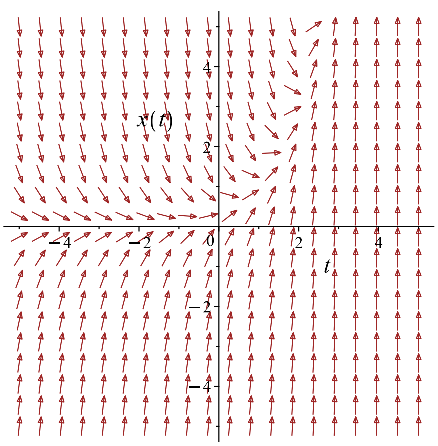
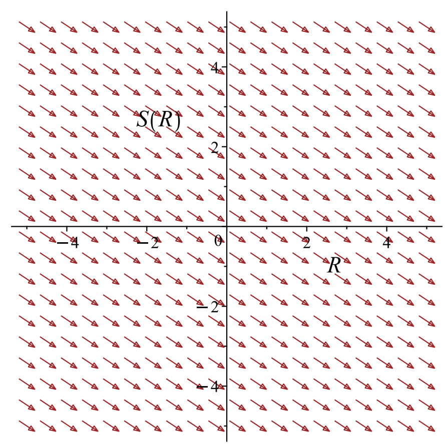
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$\frac{\ln(3x - e^t)}{3} = -\frac{2t}{3} + c_2$$

Which gives

$$x = \frac{e^{-2t+3c_2}}{3} + \frac{e^t}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -2x + e^t$ 	$R = t$ $S = \frac{\ln(3x - e^t)}{3}$	$\frac{dS}{dR} = -\frac{2}{3}$ 

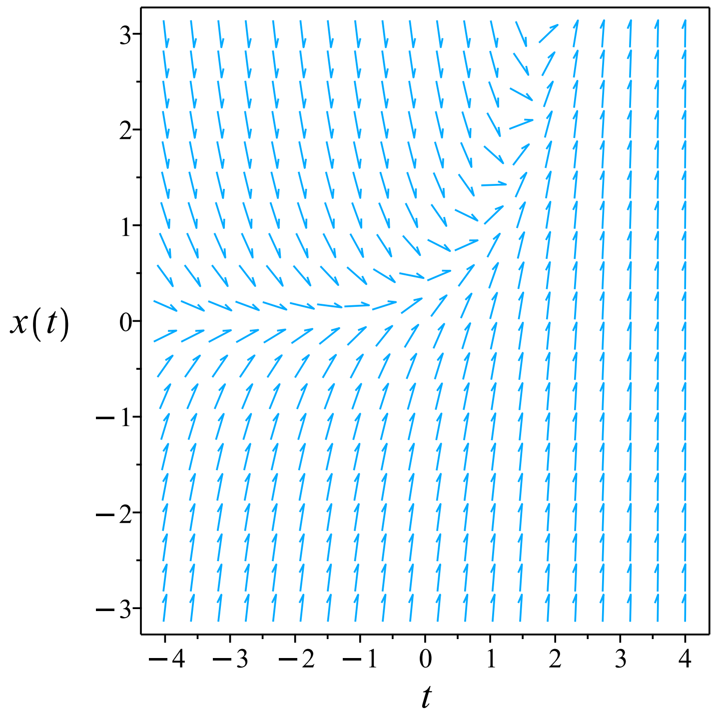


Figure 2.53: Slope field plot
$x' + 2x = e^t$

Summary of solutions found

$$x = \frac{e^{-2t+3c_2}}{3} + \frac{e^t}{3}$$

Maple step by step solution

Let's solve
$x' + 2x = e^t$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = -2x + e^t$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE
  $x' + 2x = e^t$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)(x' + 2x) = \mu(t) e^t$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(x\mu(t))$

$\mu(t)(x'+2x) = x'\mu(t) + x\mu'(t)$

- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = e^{2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(x\mu(t))\right) dt = \int \mu(t)\, e^t dt + C1$
- Evaluate the integral on the lhs

  $x\mu(t) = \int \mu(t)\, e^t dt + C1$
- Solve for x

  $x = \frac{\int \mu(t) e^t dt + C1}{\mu(t)}$
- Substitute $\mu(t) = e^{2t}$

  $x = \frac{\int e^t e^{2t} dt + C1}{e^{2t}}$
- Evaluate the integrals on the rhs

  $x = \frac{\frac{e^{3t}}{3} + C1}{e^{2t}}$
- Simplify

  $x = \frac{(e^{3t} + 3C1)e^{-2t}}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 18

```
dsolve(diff(x(t),t)+2*x(t) = exp(t),
       x(t),singsol=all)
```

$$x = \frac{(e^{3t} + 3c_1)\, e^{-2t}}{3}$$

Mathematica DSolve solution

Solving time : `0.154` (sec)
Leaf size : 21

```
DSolve[{D[x[t],t]+2*x[t]==Exp[t],{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{e^t}{3} + c_1 e^{-2t}$$

2.1.20 problem 4 (ii)

Internal problem ID [18183]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 4 (ii)
Date solved : `Thursday, December 19, 2024 at 06:17:36 PM`
CAS classification : `[_separable]`

Solve

$$x' + x\tan(t) = 0$$

Solved as first order linear ode

Time used: 0.059 (sec)

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \tan(t) \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int \tan(t)dt} \\ &= \sec(t) \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(x \sec(t)) = 0$$

Integrating gives

$$x \sec(t) = \int 0 \, dt + c_1$$
$$= c_1$$

Dividing throughout by the integrating factor $\sec(t)$ gives the final solution

$$x = c_1 \cos(t)$$

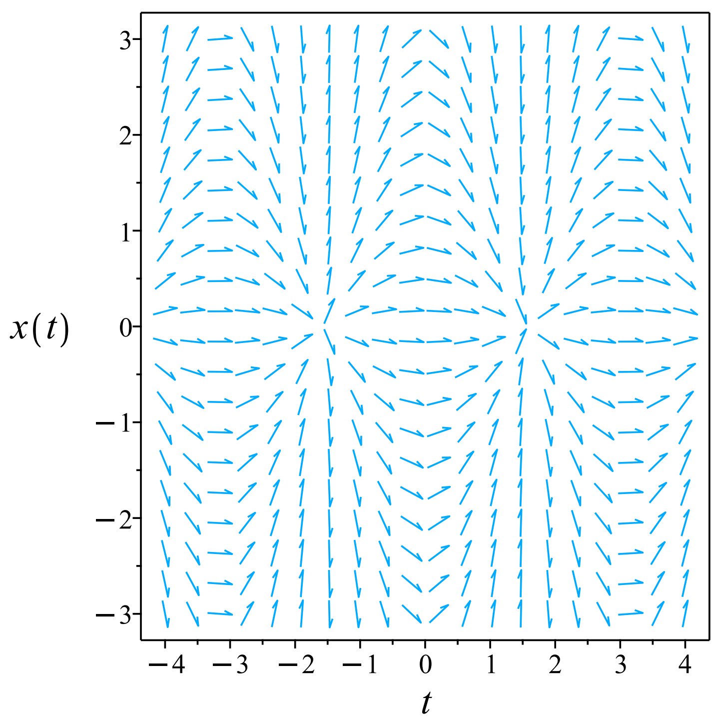


Figure 2.54: Slope field plot
$x' + x\tan(t) = 0$

Summary of solutions found

$$x = c_1 \cos(t)$$

Solved as first order separable ode

Time used: 0.104 (sec)

The ode $x' = -x\tan(t)$ is separable as it can be written as

$$
\begin{aligned}
x' &= -x\tan(t) \\
&= f(t)g(x)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(t) &= -\tan(t) \\
g(x) &= x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(x)}\,dx &= \int f(t)\,dt \\
\int \frac{1}{x}\,dx &= \int -\tan(t)\,dt \\
\ln(x) &= \ln(\cos(t)) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(x)$ is zero, since we had to divide by this above. Solving $g(x) = 0$ or $x = 0$ for x gives

$$
x = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln(x) &= \ln(\cos(t)) + c_1 \\
x &= 0
\end{aligned}
$$

Solving for x gives

$$
\begin{aligned}
x &= 0 \\
x &= \cos(t)\,e^{c_1}
\end{aligned}
$$

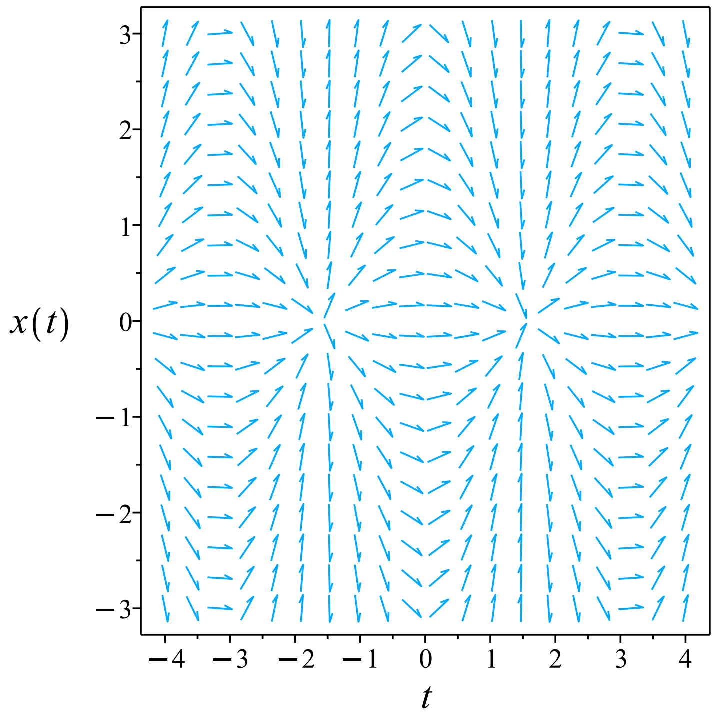


Figure 2.55: Slope field plot
$x' + x\tan(t) = 0$

Summary of solutions found

$$x = 0$$
$$x = \cos(t)\,e^{c_1}$$

Solved as first order homogeneous class D2 ode

Time used: 0.244 (sec)

Applying change of variables $x = u(t)\,t$, then the ode becomes

$$u'(t)\,t + u(t) + u(t)\,t\tan(t) = 0$$

Which is now solved The ode $u'(t) = -\frac{u(t)(\tan(t)t+1)}{t}$ is separable as it can be written as

$$\begin{aligned} u'(t) &= -\frac{u(t)\,(\tan(t)\,t+1)}{t} \\ &= f(t)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= -\frac{\tan(t)\,t+1}{t} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(t)\,dt$$
$$\int \frac{1}{u}\,du = \int -\frac{\tan(t)\,t+1}{t}\,dt$$
$$\ln(u(t)) = \ln\left(\frac{\cos(t)}{t}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(t)$ gives

$$u(t) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(t)) = \ln\left(\frac{\cos(t)}{t}\right) + c_1$$
$$u(t) = 0$$

Solving for $u(t)$ gives

$$u(t) = 0$$
$$u(t) = \frac{e^{c_1}\cos(t)}{t}$$

Converting $u(t) = 0$ back to x gives

$$x = 0$$

Converting $u(t) = \frac{e^{c_1}\cos(t)}{t}$ back to x gives

$$x = \cos(t)\,e^{c_1}$$

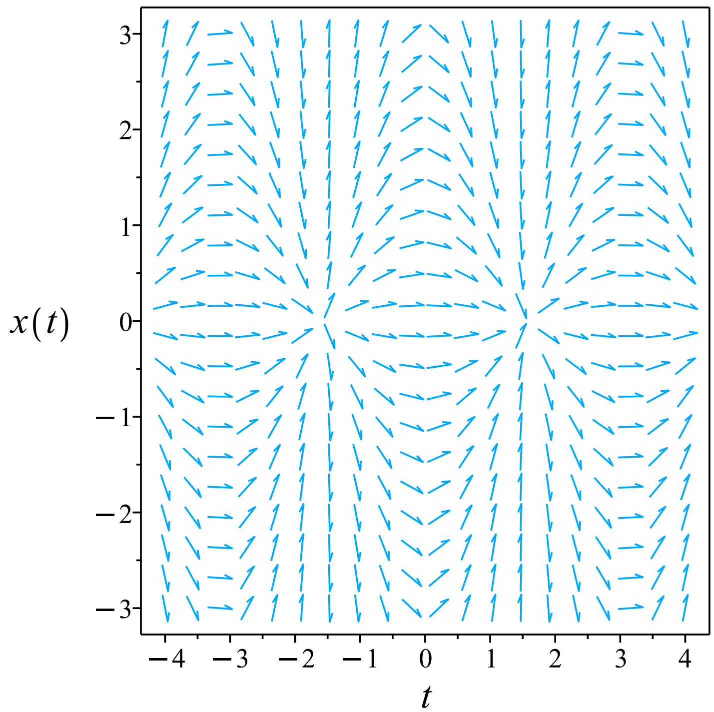


Figure 2.56: Slope field plot
$x' + x\tan(t) = 0$

Summary of solutions found

$$x = 0$$
$$x = \cos(t)\,e^{c_1}$$

Solved as first order Exact ode

Time used: 0.111 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = (-x \tan(t))\,\mathrm{d}t$$
$$(x \tan(t))\,\mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, x) = x \tan(t)$$
$$N(t, x) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(x \tan(t))$$
$$= \tan(t)$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\ &= 1((\tan(t)) - (0)) \\ &= \tan(t) \end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \tan(t) \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(\cos(t))} \\ &= \sec(t) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sec(t)(x \tan(t)) \\ &= x \tan(t) \sec(t) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sec(t)(1) \\ &= \sec(t) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ (x \tan(t) \sec(t)) + (\sec(t))\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{N}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \sec(t)\,\mathrm{d}x$$
$$\phi = x\sec(t) + f(t) \tag{3}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$\frac{\partial\phi}{\partial t} = x\tan(t)\sec(t) + f'(t) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial t} = x\tan(t)\sec(t)$. Therefore equation (4) becomes

$$x\tan(t)\sec(t) = x\tan(t)\sec(t) + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$f'(t) = 0$$

Therefore

$$f(t) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(t)$ into equation (3) gives ϕ

$$\phi = x\sec(t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x\sec(t)$$

Solving for x gives

$$x = \frac{c_1}{\sec(t)}$$

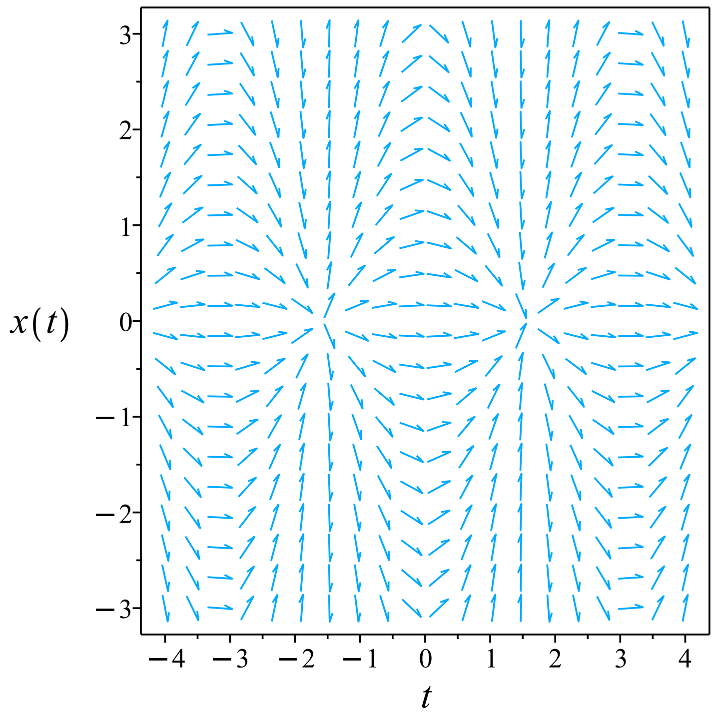


Figure 2.57: Slope field plot
$x' + x\tan(t) = 0$

Summary of solutions found

$$x = \frac{c_1}{\sec(t)}$$

Solved using Lie symmetry for first order ode

Time used: 0.339 (sec)

Writing the ode as

$$x' = -x\tan(t)$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$
$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - x\tan(t)(b_3 - a_2) - x^2\tan(t)^2 a_3 \\ &+ x(1+\tan(t)^2)(ta_2 + xa_3 + a_1) + \tan(t)(tb_2 + xb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} &\tan(t)^2 txa_2 + \tan(t)^2 xa_1 + \tan(t) tb_2 + x\tan(t) a_2 \\ &+ txa_2 + x^2a_3 + \tan(t) b_1 + xa_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} &\tan(t)^2 txa_2 + \tan(t)^2 xa_1 + \tan(t) tb_2 + x\tan(t) a_2 \\ &+ txa_2 + x^2a_3 + \tan(t) b_1 + xa_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x, \tan(t)\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2, \tan(t) = v_3\}$$

The above PDE (6E) now becomes

$$v_3^2 v_1 v_2 a_2 + v_3^2 v_2 a_1 + v_1 v_2 a_2 + v_2 v_3 a_2 + v_2^2 a_3 + v_3 v_1 b_2 + v_2 a_1 + v_3 b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$v_3^2 v_1 v_2 a_2 + v_3^2 v_2 a_1 + v_1 v_2 a_2 + v_2 v_3 a_2 + v_2^2 a_3 + v_3 v_1 b_2 + v_2 a_1 + v_3 b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= x
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x} dy
\end{aligned}
$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -x\tan(t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 0 \\ S_x &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\tan(t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int -\tan(R)\, dR \\ S(R) &= \ln(\cos(R)) + c_2 \end{aligned}$$

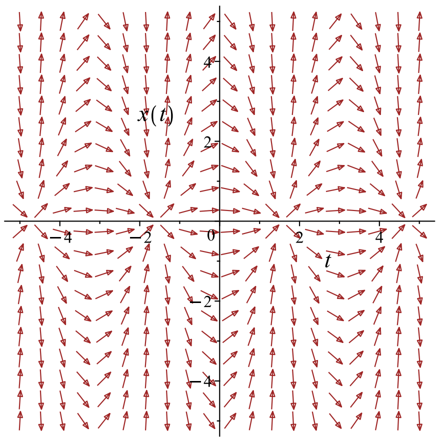
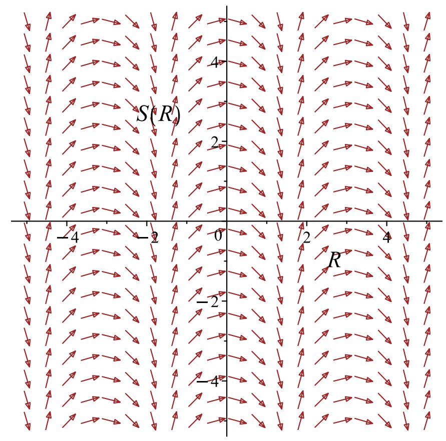
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$\ln(x) = \ln(\cos(t)) + c_2$$

Which gives

$$x = e^{c_2} \cos(t)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -x \tan(t)$ 	$R = t$ $S = \ln(x)$	$\frac{dS}{dR} = -\tan(R)$ 

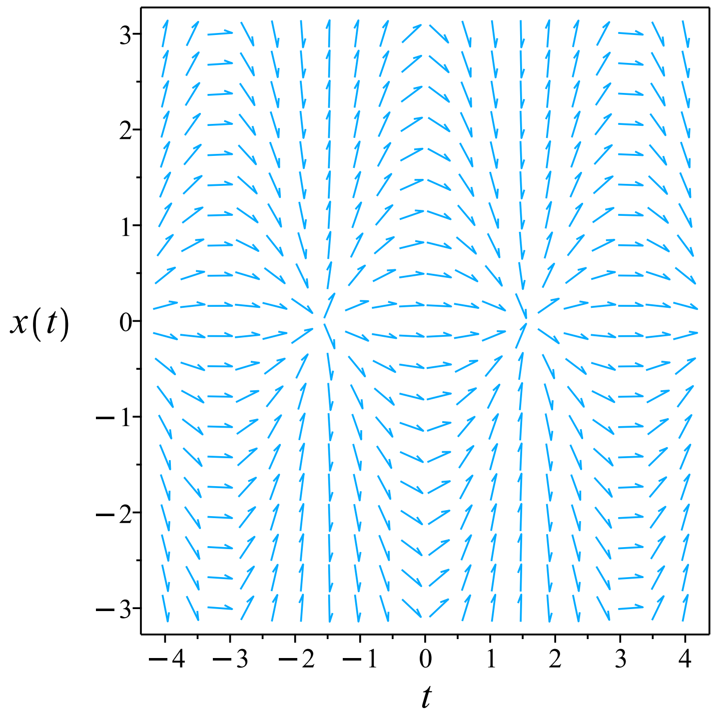


Figure 2.58: Slope field plot
$x' + x\tan(t) = 0$

Summary of solutions found

$$x = e^{c_2}\cos(t)$$

Maple step by step solution

Let's solve
$x' + x\tan(t) = 0$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = -x\tan(t)$
- Separate variables
  $\frac{x'}{x} = -\tan(t)$
- Integrate both sides with respect to t
  $\int \frac{x'}{x} dt = \int -\tan(t)\, dt + C1$
- Evaluate integral
  $\ln(x) = \ln(\cos(t)) + C1$

- Solve for x
  $x = \cos(t)\, e^{C1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 8

```
dsolve(diff(x(t),t)+x(t)*tan(t) = 0,
       x(t),singsol=all)
```

$$x = c_1 \cos(t)$$

Mathematica DSolve solution

Solving time : 0.136 (sec)
Leaf size : 15

```
DSolve[{D[x[t],t]+x[t]*Tan[t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to c_1 \cos(t)$$
$$x(t) \to 0$$

2.1.21 problem 4 (iii)

Internal problem ID [18184]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 4 (iii)
Date solved : `Thursday, December 19, 2024 at 06:17:37 PM`
CAS classification : `[_linear]`

Solve

$$x' - x\tan(t) = 4\sin(t)$$

Solved as first order linear ode

Time used: 0.173 (sec)

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = -\tan(t)$$
$$p(t) = 4\sin(t)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt} \\ &= e^{\int -\tan(t)dt} \\ &= \cos(t)\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = (\mu)\,(4\sin(t))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(x\cos(t)) = (\cos(t))\,(4\sin(t))$$

$$\mathrm{d}(x\cos(t)) = (4\sin(t)\cos(t))\;\mathrm{d}t$$

Integrating gives

$$x\cos(t) = \int 4\sin(t)\cos(t)\;dt$$

$$= -2\cos(t)^2 + c_1$$

Dividing throughout by the integrating factor $\cos(t)$ gives the final solution

$$x = -2\cos(t) + c_1\sec(t)$$

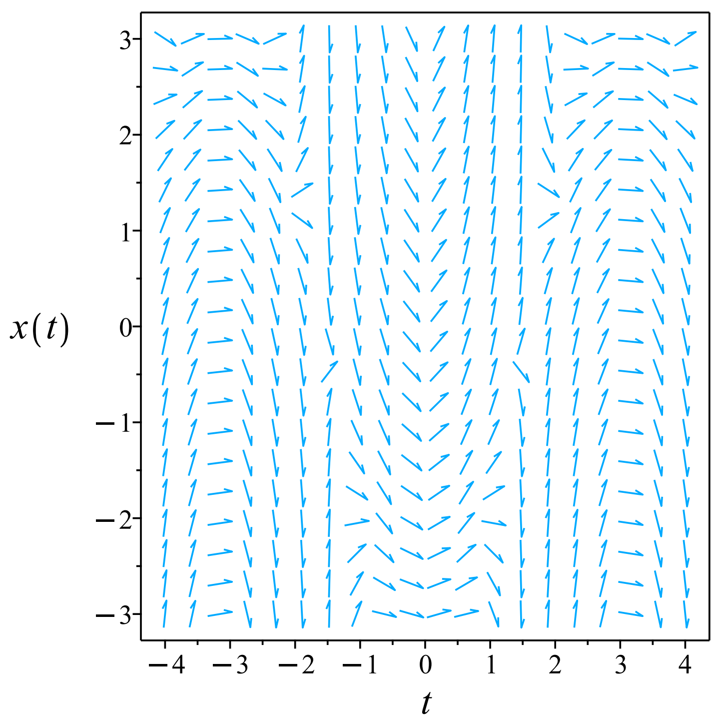


Figure 2.59: Slope field plot
$x' - x\tan(t) = 4\sin(t)$

Summary of solutions found

$$x = -2\cos(t) + c_1 \sec(t)$$

Solved as first order Exact ode

Time used: 0.131 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = (x\tan(t) + 4\sin(t))\,\mathrm{d}t$$
$$(-x\tan(t) - 4\sin(t))\,\mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= -x\tan(t) - 4\sin(t)\\ N(t,x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-x\tan(t) - 4\sin(t))\\ &= -\tan(t)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= 1((-\tan(t)) - (0))\\ &= -\tan(t)\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -\tan(t)\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\cos(t))}\\ &= \cos(t)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \cos(t)(-x\tan(t) - 4\sin(t)) \\ &= \sin(t)(-4\cos(t) - x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \cos(t)(1) \\ &= \cos(t)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ (\sin(t)(-4\cos(t) - x)) + (\cos(t))\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{N}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \cos(t)\,\mathrm{d}x \\ \phi &= x\cos(t) + f(t)\end{aligned} \tag{3}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$\frac{\partial\phi}{\partial t} = -x\sin(t) + f'(t) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial t} = \sin(t)(-4\cos(t) - x)$. Therefore equation (4) becomes

$$\sin(t)(-4\cos(t) - x) = -x\sin(t) + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$f'(t) = -4\sin(t)\cos(t)$$

Integrating the above w.r.t t gives

$$\int f'(t)\,\mathrm{d}t = \int (-2\sin(2t))\,\mathrm{d}t$$
$$f(t) = \cos(2t) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(t)$ into equation (3) gives ϕ

$$\phi = x\cos(t) + \cos(2t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x\cos(t) + \cos(2t)$$

Solving for x gives

$$x = -\frac{\cos(2t) - c_1}{\cos(t)}$$

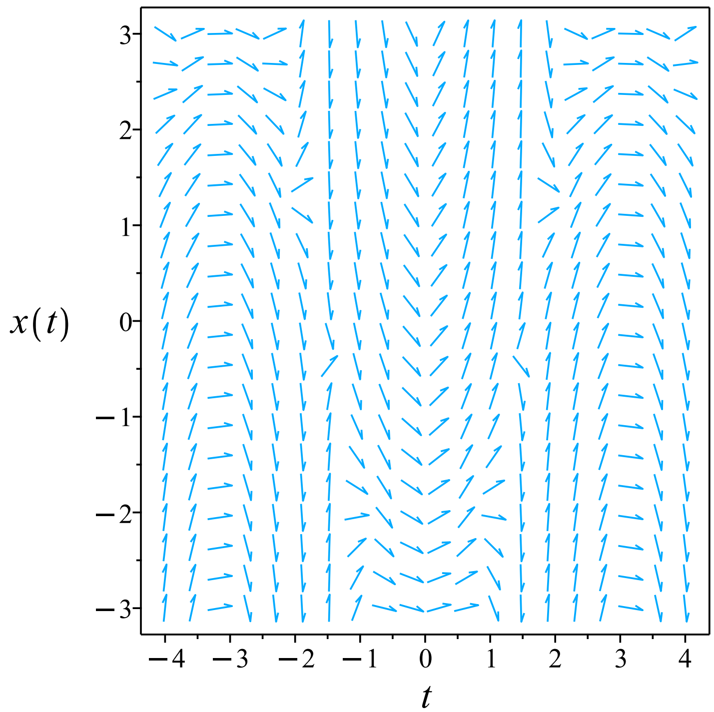


Figure 2.60: Slope field plot
$x' - x\tan(t) = 4\sin(t)$

Summary of solutions found

$$x = -\frac{\cos(2t) - c_1}{\cos(t)}$$

Maple step by step solution

Let's solve
$x' - x\tan(t) = 4\sin(t)$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = x\tan(t) + 4\sin(t)$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE
  $x' - x\tan(t) = 4\sin(t)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)(x' - x\tan(t)) = 4\mu(t)\sin(t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(x\mu(t))$

$\mu(t)(x' - x\tan(t)) = x'\mu(t) + x\mu'(t)$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)\tan(t)$
- Solve to find the integrating factor

  $\mu(t) = \cos(t)$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(x\mu(t))\right) dt = \int 4\mu(t)\sin(t)\, dt + C1$
- Evaluate the integral on the lhs

  $x\mu(t) = \int 4\mu(t)\sin(t)\, dt + C1$
- Solve for x

  $x = \frac{\int 4\mu(t)\sin(t)dt + C1}{\mu(t)}$
- Substitute $\mu(t) = \cos(t)$

  $x = \frac{\int 4\sin(t)\cos(t)dt + C1}{\cos(t)}$
- Evaluate the integrals on the rhs

  $x = \frac{2\sin(t)^2 + C1}{\cos(t)}$
- Simplify

  $x = (2\sin(t)^2 + C1)\sec(t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(diff(x(t),t)-x(t)*tan(t) = 4*sin(t),
       x(t),singsol=all)
```

$$x = -2\cos(t) + c_1\sec(t) + \sec(t)$$

Mathematica DSolve solution

Solving time : 0.065 (sec)
Leaf size : 17

```
DSolve[{D[x[t],t]-x[t]*Tan[t]==4*Sin[t],{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \sec(t)(-\cos(2t) + c_1)$$

2.1.22 problem 4 (iv)

Internal problem ID [18185]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 4 (iv)
Date solved : `Thursday, December 19, 2024 at 06:17:39 PM`
CAS classification : `[_linear]`

Solve

$$t^3x' + \left(-3t^2 + 2\right)x = t^3$$

Solved as first order linear ode

Time used: 0.101 (sec)

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= -\frac{3t^2 - 2}{t^3} \\ p(t) &= 1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int -\frac{3t^2-2}{t^3}dt} \\ &= \frac{\mathrm{e}^{-\frac{1}{t^2}}}{t^3} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = \mu$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x\,\mathrm{e}^{-\frac{1}{t^2}}}{t^3}\right) = \frac{\mathrm{e}^{-\frac{1}{t^2}}}{t^3}$$
$$\mathrm{d}\left(\frac{x\,\mathrm{e}^{-\frac{1}{t^2}}}{t^3}\right) = \frac{\mathrm{e}^{-\frac{1}{t^2}}}{t^3}\mathrm{d}t$$

Integrating gives

$$\frac{x\,\mathrm{e}^{-\frac{1}{t^2}}}{t^3} = \int \frac{\mathrm{e}^{-\frac{1}{t^2}}}{t^3}\,dt$$
$$= \frac{\mathrm{e}^{-\frac{1}{t^2}}}{2} + c_1$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{-\frac{1}{t^2}}}{t^3}$ gives the final solution

$$x = \left(c_1\,\mathrm{e}^{\frac{1}{t^2}} + \frac{1}{2}\right)t^3$$

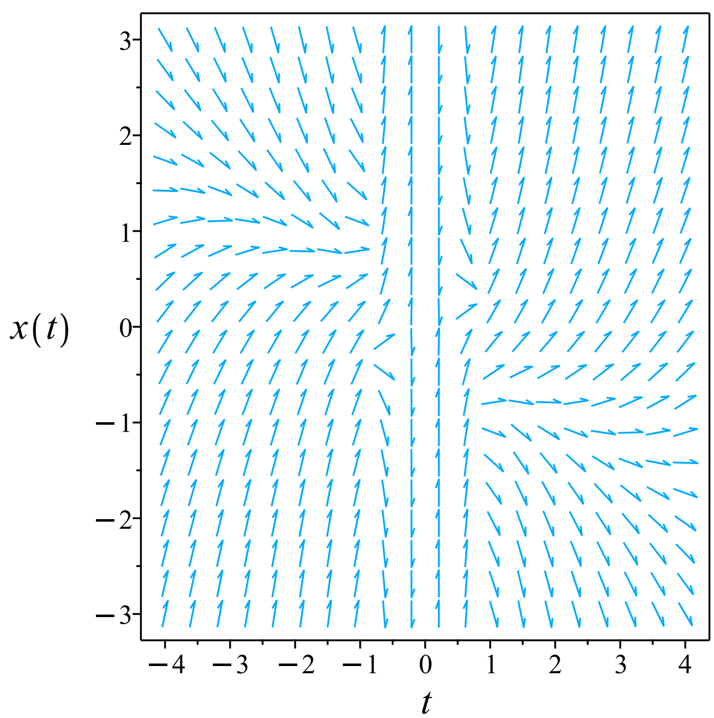


Figure 2.61: Slope field plot
$t^3x' + (-3t^2 + 2)\,x = t^3$

Summary of solutions found

$$x = \left(c_1 \,\mathrm{e}^{\frac{1}{t^2}} + \frac{1}{2}\right) t^3$$

Solved as first order Exact ode

Time used: 0.146 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(t^3\right)\mathrm{d}x &= \left(-\left(-3t^2+2\right)x + t^3\right)\mathrm{d}t\\ \left(\left(-3t^2+2\right)x - t^3\right)\mathrm{d}t + \left(t^3\right)\mathrm{d}x &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t,x) &= \left(-3t^2+2\right)x-t^3 \\ N(t,x) &= t^3 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(\left(-3t^2+2\right)x-t^3\right) \\ &= -3t^2+2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(t^3\right) \\ &= 3t^2 \end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial x}-\frac{\partial N}{\partial t}\right) \\ &= \frac{1}{t^3}\left(\left(-3t^2+2\right)-\left(3t^2\right)\right) \\ &= \frac{-6t^2+2}{t^3} \end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}t} \\ &= e^{\int \frac{-6t^2+2}{t^3}\,\mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{1}{t^2}-6\ln(t)} \\ &= \frac{\mathrm{e}^{-\frac{1}{t^2}}}{t^6} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{e^{-\frac{1}{t^2}}}{t^6}\left(\left(-3t^2+2\right)x - t^3\right)\\ &= -\frac{\left(t^3+3x\,t^2-2x\right)e^{-\frac{1}{t^2}}}{t^6}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{e^{-\frac{1}{t^2}}}{t^6}\left(t^3\right)\\ &= \frac{e^{-\frac{1}{t^2}}}{t^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{dx}{dt} &= 0\\ \left(-\frac{\left(t^3+3x\,t^2-2x\right)e^{-\frac{1}{t^2}}}{t^6}\right)+\left(\frac{e^{-\frac{1}{t^2}}}{t^3}\right)\frac{dx}{dt} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,dx &= \int\overline{N}\,dx\\ \int\frac{\partial\phi}{\partial x}\,dx &= \int\frac{e^{-\frac{1}{t^2}}}{t^3}\,dx\end{aligned}$$

$$\phi = \frac{x\,e^{-\frac{1}{t^2}}}{t^3}+f(t) \tag{3}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= -\frac{3x\,\mathrm{e}^{-\frac{1}{t^2}}}{t^4} + \frac{2x\,\mathrm{e}^{-\frac{1}{t^2}}}{t^6} + f'(t) \\ &= \frac{x(-3t^2+2)\,\mathrm{e}^{-\frac{1}{t^2}}}{t^6} + f'(t)\end{aligned} \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial t} = -\frac{(t^3+3x\,t^2-2x)\,\mathrm{e}^{-\frac{1}{t^2}}}{t^6}$. Therefore equation (4) becomes

$$-\frac{(t^3+3x\,t^2-2x)\,\mathrm{e}^{-\frac{1}{t^2}}}{t^6} = \frac{x(-3t^2+2)\,\mathrm{e}^{-\frac{1}{t^2}}}{t^6} + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$f'(t) = -\frac{\mathrm{e}^{-\frac{1}{t^2}}}{t^3}$$

Integrating the above w.r.t t gives

$$\int f'(t)\,\mathrm{d}t = \int \left(-\frac{\mathrm{e}^{-\frac{1}{t^2}}}{t^3}\right) \mathrm{d}t$$

$$f(t) = -\frac{\mathrm{e}^{-\frac{1}{t^2}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(t)$ into equation (3) gives ϕ

$$\phi = \frac{x\,\mathrm{e}^{-\frac{1}{t^2}}}{t^3} - \frac{\mathrm{e}^{-\frac{1}{t^2}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{x\,\mathrm{e}^{-\frac{1}{t^2}}}{t^3} - \frac{\mathrm{e}^{-\frac{1}{t^2}}}{2}$$

Solving for x gives

$$x = \frac{t^3 \left(e^{-\frac{1}{t^2}} + 2c_1\right) e^{\frac{1}{t^2}}}{2}$$

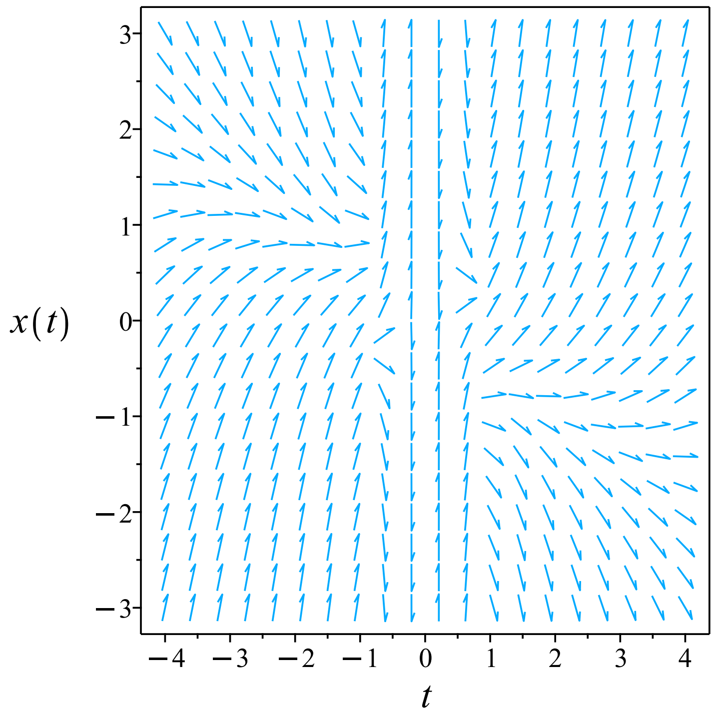


Figure 2.62: Slope field plot
$t^3 x' + (-3t^2 + 2)\, x = t^3$

Summary of solutions found

$$x = \frac{t^3 \left(e^{-\frac{1}{t^2}} + 2c_1\right) e^{\frac{1}{t^2}}}{2}$$

Solved using Lie symmetry for first order ode

Time used: 1.458 (sec)

Writing the ode as

$$x' = \frac{t^3 + 3x\, t^2 - 2x}{t^3}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = t^3 a_7 + x\, t^2 a_8 + t\, x^2 a_9 + x^3 a_{10} + t^2 a_4 + x t a_5 + x^2 a_6 + t a_2 + x a_3 + a_1 \tag{1E}$$

$$\eta = t^3 b_7 + x\, t^2 b_8 + t\, x^2 b_9 + x^3 b_{10} + t^2 b_4 + x t b_5 + x^2 b_6 + t b_2 + x b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&3t^2 b_7 + 2txb_8 + x^2 b_9 + 2tb_4 + xb_5 + b_2 \qquad\qquad (5E)\\
&+ \frac{(t^3 + 3x\, t^2 - 2x)\left(-3t^2 a_7 + t^2 b_8 - 2txa_8 + 2txb_9 - x^2 a_9 + 3x^2 b_{10} - 2ta_4 + tb_5 - xa_5 + 2xb_6 - a_2 + b_3\right)}{t^3}\\
&- \frac{(t^3 + 3x\, t^2 - 2x)^2 \left(t^2 a_8 + 2txa_9 + 3x^2 a_{10} + ta_5 + 2xa_6 + a_3\right)}{t^6}\\
&- \left(\frac{3t^2 + 6xt}{t^3} - \frac{3(t^3 + 3x\, t^2 - 2x)}{t^4}\right)\left(t^3 a_7 + x\, t^2 a_8\right.\\
&\left.+ t\, x^2 a_9 + x^3 a_{10} + t^2 a_4 + xta_5 + x^2 a_6 + ta_2 + xa_3 + a_1\right)\\
&- \frac{(3t^2 - 2)\left(t^3 b_7 + x\, t^2 b_8 + t\, x^2 b_9 + x^3 b_{10} + t^2 b_4 + xtb_5 + x^2 b_6 + tb_2 + xb_3 + b_1\right)}{t^3}\\
&= 0
\end{aligned}$$

Putting the above in normal form gives

$$\begin{aligned}
&-\frac{2t^7 a_4 + t^7 a_5 - t^7 b_5 - 2t^5 b_4 + 8x^3 a_6 + 3t^8 a_7 - t^8 b_8 + t^8 a_8 + 12x^4 a_{10} - 2t^6 b_7 + t^6 a_2 + t^6 a_3 - t^6 b_3 + 3t^5 b_1}{}\\
&= 0
\end{aligned}$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2t^7a_4 - t^7a_5 + t^7b_5 + 2t^5b_4 - 8x^3a_6 - 3t^8a_7 + t^8b_8 - t^8a_8 \\
&- 12x^4a_{10} + 2t^6b_7 - t^6a_2 - t^6a_3 + t^6b_3 - 3t^5b_1 + 2t^4b_2 + 2t^3b_1 \\
&- 4x^2a_3 - 6t^5xa_3 - 6t^4x^2a_3 + 3t^4xa_1 - 4t^3xa_2 + 4t^3xa_3 + 6t^2x^2a_3 \\
&- 6t^2xa_1 - 2b_2t^6 + 2t^7xb_8 + 4x^2b_9t^6 - 6t^7xa_7 - 8t^7xa_8 + 2t^7xb_9 \\
&- 12t^6x^2a_8 - 13t^6x^2a_9 + 3t^6x^2b_{10} - 18t^5x^3a_9 + 6t^5x^3b_{10} + 10t^4x^2a_8 \\
&- 2t^4x^2b_9 + 20t^3x^3a_9 - 4t^3x^3b_{10} - 2t^7xa_9 - 3t^6x^2a_{10} - 18t^5x^3a_{10} \\
&- 24t^4x^4a_{10} + 4t^5xa_8 + 8t^4x^2a_9 + 12t^3x^3a_{10} + 30t^2x^4a_{10} - 4t^2x^2a_8 \\
&- 8t\,x^3a_9 - 3t^6xa_4 - 7t^6xa_5 - 2t^6xa_6 + 2t^6xb_6 - 9t^5x^2a_5 \\
&- 12t^5x^2a_6 + 3t^5x^2b_6 - 15t^4x^3a_6 - 2t^4xa_4 + 4t^4xa_5 + 8t^3x^2a_5 \\
&+ 8t^3x^2a_6 - 2t^3x^2b_6 + 18t^2x^3a_6 - 4t\,x^2a_5 + xb_5t^6 - t^7b_4 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-6v_1^5v_2a_3 - 6v_1^4v_2^2a_3 + 3v_1^4v_2a_1 - 4v_1^3v_2a_2 + 4v_1^3v_2a_3 + 6v_1^2v_2^2a_3 \\
&- 6v_1^2v_2a_1 + 2v_1^7v_2b_8 + 4v_2^2b_9v_1^6 - 6v_1^7v_2a_7 - 8v_1^7v_2a_8 + 2v_1^7v_2b_9 \\
&- 12v_1^6v_2^2a_8 - 13v_1^6v_2^2a_9 + 3v_1^6v_2^2b_{10} - 18v_1^5v_2^3a_9 + 6v_1^5v_2^3b_{10} + 10v_1^4v_2^2a_8 \\
&- 2v_1^4v_2^2b_9 + 20v_1^3v_2^3a_9 - 4v_1^3v_2^3b_{10} - 2v_1^7v_2a_9 - 3v_1^6v_2^2a_{10} - 18v_1^5v_2^3a_{10} \\
&- 24v_1^4v_2^4a_{10} + 4v_1^5v_2a_8 + 8v_1^4v_2^2a_9 + 12v_1^3v_2^3a_{10} + 30v_1^2v_2^4a_{10} - 4v_1^2v_2^2a_8 \\
&- 8v_1v_2^3a_9 - 3v_1^6v_2a_4 - 7v_1^6v_2a_5 - 2v_1^6v_2a_6 + 2v_1^6v_2b_6 - 9v_1^5v_2^2a_5 \\
&- 12v_1^5v_2^2a_6 + 3v_1^5v_2^2b_6 - 15v_1^4v_2^3a_6 - 2v_1^4v_2a_4 + 4v_1^4v_2a_5 + 8v_1^3v_2^2a_5 \\
&+ 8v_1^3v_2^2a_6 - 2v_1^3v_2^2b_6 + 18v_1^2v_2^3a_6 - 4v_1v_2^2a_5 + v_2b_5v_1^6 - 2v_1^7a_4 - v_1^7a_5 \\
&+ v_1^7b_5 + 2v_1^5b_4 - 8v_2^3a_6 - 3v_1^8a_7 + v_1^8b_8 - v_1^8a_8 - 12v_2^4a_{10} + 2v_1^6b_7 - v_1^6a_2 \\
&- v_1^6a_3 + v_1^6b_3 - 3v_1^5b_1 + 2v_1^4b_2 + 2v_1^3b_1 - 4v_2^2a_3 - 2b_2v_1^6 - v_1^7b_4 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-6v_1^2v_2a_1 - 24v_1^4v_2^4a_{10} + 30v_1^2v_2^4a_{10} - 8v_1v_2^3a_9 - 15v_1^4v_2^3a_6 + 18v_1^2v_2^3a_6\\
&- 4v_1v_2^2a_5 - 8v_2^3a_6 - 12v_2^4a_{10} + 2v_1^4b_2 + 2v_1^3b_1 - 4v_2^2a_3 + (-6a_3 + 4a_8)\, v_2v_1^5\\
&+ (-6a_3 + 10a_8 - 2b_9 + 8a_9)\, v_2^2v_1^4 + (3a_1 - 2a_4 + 4a_5)\, v_2v_1^4\\
&+ (-4a_2 + 4a_3)\, v_2v_1^3 + (6a_3 - 4a_8)\, v_2^2v_1^2 + (2b_8 - 6a_7 - 8a_8 + 2b_9 - 2a_9)\, v_2v_1^7\\
&+ (4b_9 - 12a_8 - 13a_9 + 3b_{10} - 3a_{10})\, v_1^6v_2^2 + (-18a_9 + 6b_{10} - 18a_{10})\, v_2^3v_1^5\\
&+ (20a_9 - 4b_{10} + 12a_{10})\, v_2^3v_1^3 + (-3a_4 - 7a_5 - 2a_6 + b_5 + 2b_6)\, v_2v_1^6\\
&+ (-9a_5 - 12a_6 + 3b_6)\, v_2^2v_1^5 + (8a_5 + 8a_6 - 2b_6)\, v_2^2v_1^3 + (-3a_7 + b_8 - a_8)\, v_1^8\\
&+ (-2a_4 - a_5 - b_4 + b_5)\, v_1^7 + (2b_7 - a_2 - a_3 + b_3 - 2b_2)\, v_1^6 + (-3b_1 + 2b_4)\, v_1^5 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-6a_1 &= 0 \\
-4a_3 &= 0 \\
-4a_5 &= 0 \\
-15a_6 &= 0 \\
-8a_6 &= 0 \\
18a_6 &= 0 \\
-8a_9 &= 0 \\
-24a_{10} &= 0 \\
-12a_{10} &= 0 \\
30a_{10} &= 0 \\
2b_1 &= 0 \\
2b_2 &= 0 \\
-4a_2 + 4a_3 &= 0 \\
-6a_3 + 4a_8 &= 0 \\
6a_3 - 4a_8 &= 0 \\
-3b_1 + 2b_4 &= 0 \\
3a_1 - 2a_4 + 4a_5 &= 0 \\
-9a_5 - 12a_6 + 3b_6 &= 0 \\
8a_5 + 8a_6 - 2b_6 &= 0 \\
-3a_7 + b_8 - a_8 &= 0 \\
-18a_9 + 6b_{10} - 18a_{10} &= 0 \\
20a_9 - 4b_{10} + 12a_{10} &= 0 \\
-6a_3 + 10a_8 - 2b_9 + 8a_9 &= 0 \\
-2a_4 - a_5 - b_4 + b_5 &= 0 \\
-3a_4 - 7a_5 - 2a_6 + b_5 + 2b_6 &= 0 \\
2b_7 - a_2 - a_3 + b_3 - 2b_2 &= 0 \\
2b_8 - 6a_7 - 8a_8 + 2b_9 - 2a_9 &= 0 \\
4b_9 - 12a_8 - 13a_9 + 3b_{10} - 3a_{10} &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= \frac{b_8}{3} \\
a_8 &= 0 \\
a_9 &= 0 \\
a_{10} &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -2b_7 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0 \\
b_7 &= b_7 \\
b_8 &= b_8 \\
b_9 &= 0 \\
b_{10} &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= t^3 - 2x
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^3 - 2x} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln (t^3 - 2x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{t^3 + 3x\,t^2 - 2x}{t^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -\frac{3t^2}{2t^3 - 4x} \\ S_x &= \frac{1}{t^3 - 2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-3t^2 + 2}{2t^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-3R^2 + 2}{2R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -\frac{3R^2 - 2}{2R^3}\, dR$$
$$S(R) = -\frac{1}{2R^2} - \frac{3\ln(R)}{2} + c_2$$

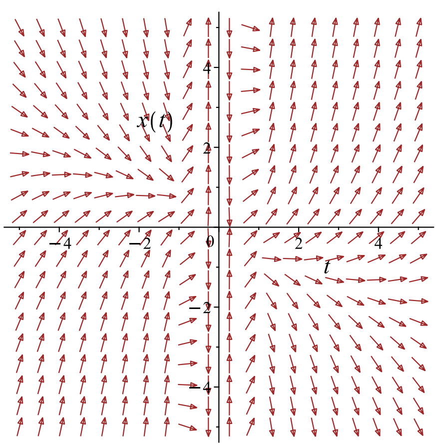
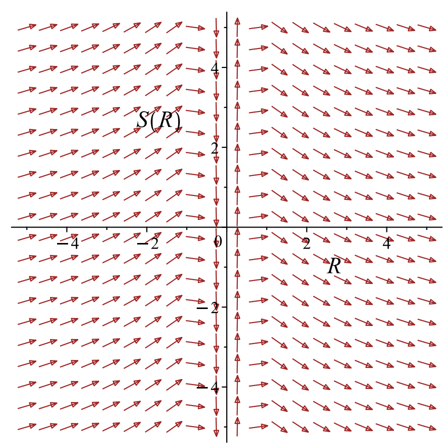
To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$-\frac{\ln(t^3 - 2x)}{2} = -\frac{1}{2t^2} - \frac{3\ln(t)}{2} + c_2$$

Which gives

$$x = -\frac{e^{\frac{3\ln(t)t^2 - 2c_2 t^2 + 1}{t^2}}}{2} + \frac{t^3}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in t, x coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dx}{dt} = \frac{t^3 + 3x\,t^2 - 2x}{t^3}$

</td><td>$R = t$
$S = -\frac{\ln(t^3 - 2x)}{2}$</td><td>$\frac{dS}{dR} = \frac{-3R^2 + 2}{2R^3}$

</td></tr>
</table>

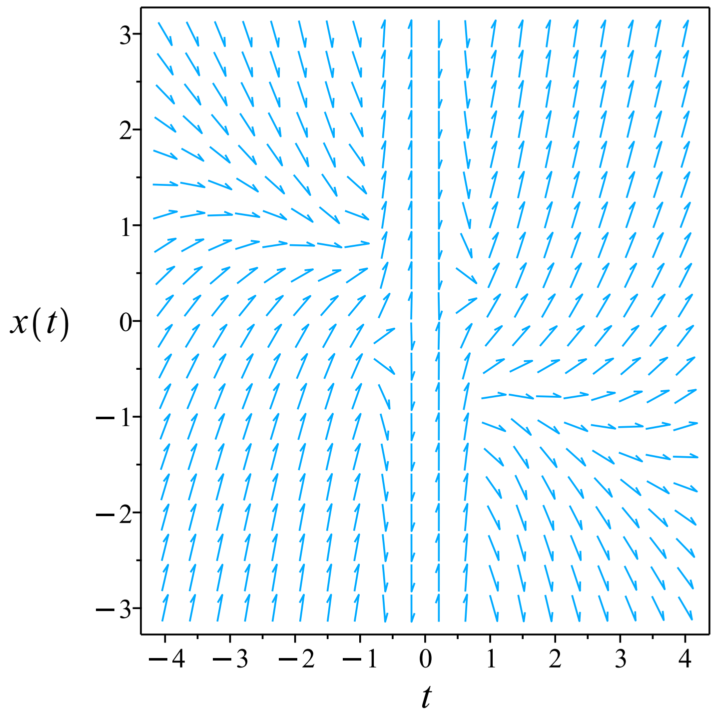


Figure 2.63: Slope field plot
$t^3x' + (-3t^2+2)\,x = t^3$

Summary of solutions found

$$x = -\frac{e^{\frac{3\ln(t)t^2-2c_2\,t^2+1}{t^2}}}{2}+\frac{t^3}{2}$$

Maple step by step solution

Let's solve
$t^3x' + (-3t^2+2)\,x = t^3$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = \frac{-(-3t^2+2)x+t^3}{t^3}$
- Collect w.r.t. x and simplify
  $x' = 1 + \frac{(3t^2-2)x}{t^3}$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

$x' - \frac{(3t^2-2)x}{t^3} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' - \frac{(3t^2-2)x}{t^3}\right) = \mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(x\mu(t))$

  $\mu(t)\left(x' - \frac{(3t^2-2)x}{t^3}\right) = x'\mu(t) + x\mu'(t)$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{\mu(t)(3t^2-2)}{t^3}$

- Solve to find the integrating factor

  $\mu(t) = \frac{e^{-\frac{1}{t^2}}}{t^3}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(x\mu(t))\right) dt = \int \mu(t)\, dt + C1$

- Evaluate the integral on the lhs

  $x\mu(t) = \int \mu(t)\, dt + C1$

- Solve for x

  $x = \frac{\int \mu(t)dt + C1}{\mu(t)}$

- Substitute $\mu(t) = \frac{e^{-\frac{1}{t^2}}}{t^3}$

  $x = \frac{t^3\left(\int \frac{e^{-\frac{1}{t^2}}}{t^3}dt + C1\right)}{e^{-\frac{1}{t^2}}}$

- Evaluate the integrals on the rhs

  $x = \frac{t^3\left(\frac{e^{-\frac{1}{t^2}}}{2} + C1\right)}{e^{-\frac{1}{t^2}}}$

- Simplify

  $x = \left(C1\, e^{\frac{1}{t^2}} + \frac{1}{2}\right) t^3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 16

```
dsolve(t^3*diff(x(t),t)+(-3*t^2+2)*x(t) = t^3,
       x(t),singsol=all)
```

$$x = \left(c_1 \,\mathrm{e}^{\frac{1}{t^2}} + \frac{1}{2}\right) t^3$$

Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 23

```
DSolve[{t^3*D[x[t],t]+(2-3*t^2)*x[t]==t^3,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{2} t^3 \left(1 + 2c_1 e^{\frac{1}{t^2}}\right)$$

2.1.23 problem 4 (v)

Internal problem ID [18186]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 4 (v)
Date solved : `Thursday, December 19, 2024 at 06:17:41 PM`
CAS classification : `[_separable]`

Solve

$$x' + 2xt + tx^4 = 0$$

Solved as first order separable ode

Time used: 0.527 (sec)

The ode $x' = -tx^4 - 2xt$ is separable as it can be written as

$$\begin{aligned} x' &= -tx^4 - 2xt \\ &= f(t)g(x) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= -t \\ g(x) &= x(x^3 + 2) \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(x)}\, dx &= \int f(t)\, dt \\ \int \frac{1}{x\,(x^3+2)}\, dx &= \int -t\, dt \\ \ln\left(\frac{\sqrt{x}}{(x^3+2)^{1/6}}\right) &= -\frac{t^2}{2} + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(x)$ is zero, since we had to divide by this above. Solving $g(x) = 0$ or $x(x^3 + 2) = 0$ for x gives

$$
\begin{aligned}
x &= 0 \\
x &= -2^{1/3} \\
x &= \frac{2^{1/3}}{2} - \frac{i\sqrt{3}\,2^{1/3}}{2} \\
x &= \frac{2^{1/3}}{2} + \frac{i\sqrt{3}\,2^{1/3}}{2}
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(\frac{\sqrt{x}}{(x^3+2)^{1/6}}\right) &= -\frac{t^2}{2} + c_1 \\
x &= 0 \\
x &= -2^{1/3} \\
x &= \frac{2^{1/3}}{2} - \frac{i\sqrt{3}\,2^{1/3}}{2} \\
x &= \frac{2^{1/3}}{2} + \frac{i\sqrt{3}\,2^{1/3}}{2}
\end{aligned}
$$

Solving for x gives

$$
\begin{aligned}
x &= 0 \\
x &= \frac{\left(-2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)^2 \mathrm{e}^{-3t^2+6c_1}\right)^{1/3}}{\mathrm{e}^{-3t^2+6c_1}-1} \\
x &= -2^{1/3} \\
x &= \frac{2^{1/3}}{2} - \frac{i\sqrt{3}\,2^{1/3}}{2} \\
x &= \frac{2^{1/3}}{2} + \frac{i\sqrt{3}\,2^{1/3}}{2} \\
x &= -\frac{\left(-2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)^2 \mathrm{e}^{-3t^2+6c_1}\right)^{1/3}}{2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)} - \frac{i\sqrt{3}\left(-2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)^2 \mathrm{e}^{-3t^2+6c_1}\right)^{1/3}}{2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)}
\end{aligned}
$$

$$x = -\frac{\left(-2\left(e^{-3t^2+6c_1}-1\right)^2 e^{-3t^2+6c_1}\right)^{1/3}}{2\left(e^{-3t^2+6c_1}-1\right)} + \frac{i\sqrt{3}\left(-2\left(e^{-3t^2+6c_1}-1\right)^2 e^{-3t^2+6c_1}\right)^{1/3}}{2\,e^{-3t^2+6c_1}-2}$$

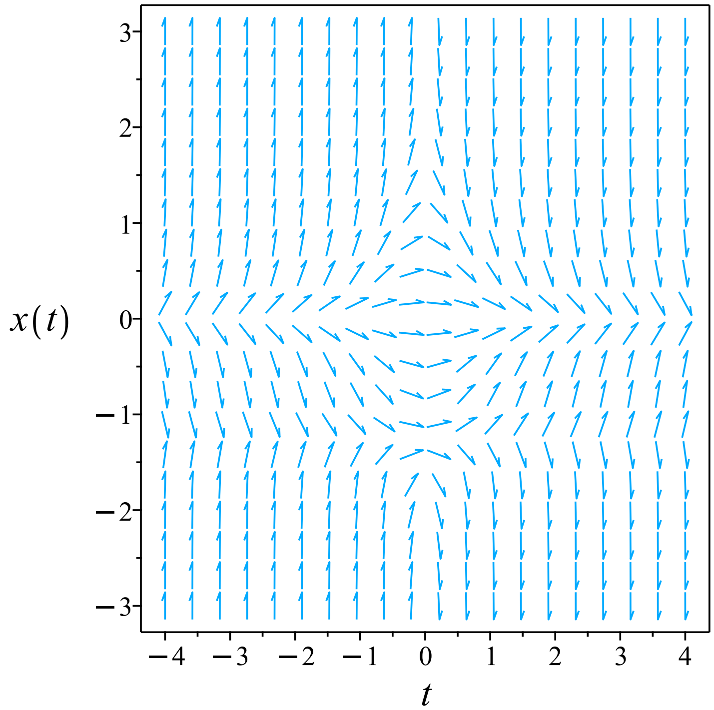


Figure 2.64: Slope field plot
$x' + 2xt + tx^4 = 0$

Summary of solutions found

$$x = 0$$

$$x = \frac{\left(-2\left(e^{-3t^2+6c_1}-1\right)^2 e^{-3t^2+6c_1}\right)^{1/3}}{e^{-3t^2+6c_1}-1}$$

$$x = -2^{1/3}$$

$$x = \frac{2^{1/3}}{2} - \frac{i\sqrt{3}\,2^{1/3}}{2}$$

$$x = \frac{2^{1/3}}{2} + \frac{i\sqrt{3}\,2^{1/3}}{2}$$

$$x = -\frac{\left(-2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)^2 \mathrm{e}^{-3t^2+6c_1}\right)^{1/3}}{2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)} - \frac{i\sqrt{3}\left(-2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)^2 \mathrm{e}^{-3t^2+6c_1}\right)^{1/3}}{2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)}$$

$$x = -\frac{\left(-2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)^2 \mathrm{e}^{-3t^2+6c_1}\right)^{1/3}}{2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)} + \frac{i\sqrt{3}\left(-2\left(\mathrm{e}^{-3t^2+6c_1}-1\right)^2 \mathrm{e}^{-3t^2+6c_1}\right)^{1/3}}{2\,\mathrm{e}^{-3t^2+6c_1}-2}$$

Solved as first order Bernoulli ode

Time used: 0.130 (sec)

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= -t\,x^4 - 2xt \end{aligned}$$

This is a Bernoulli ODE.

$$x' = (-2t)\,x + (-t)\,x^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(t)x + f_1(t)x^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= -2t \\ f_1 &= -t \end{aligned}$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(t)x^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $v = x^{1-n}$ in equation (3) which generates a new ODE in $v(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= -2t \\ f_1(t) &= -t \\ n &= 4 \end{aligned}$$

Dividing both sides of ODE (1) by $x^n = x^4$ gives

$$x'\frac{1}{x^4} = -\frac{2t}{x^3} - t \tag{4}$$

Let

$$\begin{aligned} v &= x^{1-n} \\ &= \frac{1}{x^3} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$v' = -\frac{3}{x^4}x' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{v'(t)}{3} &= -2v(t)\,t - t \\ v' &= 6tv + 3t \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(t)$ which is now solved.

In canonical form a linear first order is

$$v'(t) + q(t)v(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= -6t \\ p(t) &= 3t \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int -6tdt} \\ &= \mathrm{e}^{-3t^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu v) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mu v) &= (\mu)\,(3t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(v\,\mathrm{e}^{-3t^2}\right) &= \left(\mathrm{e}^{-3t^2}\right)(3t) \\ \mathrm{d}\left(v\,\mathrm{e}^{-3t^2}\right) &= \left(3t\,\mathrm{e}^{-3t^2}\right)\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$
\begin{aligned}
v\,\mathrm{e}^{-3t^2} &= \int 3t\,\mathrm{e}^{-3t^2}\,dt \\
&= -\frac{\mathrm{e}^{-3t^2}}{2} + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor e^{-3t^2} gives the final solution

$$
v(t) = c_1\,\mathrm{e}^{3t^2} - \frac{1}{2}
$$

The substitution $v = x^{1-n}$ is now used to convert the above solution back to x which results in

$$
\frac{1}{x^3} = c_1\,\mathrm{e}^{3t^2} - \frac{1}{2}
$$

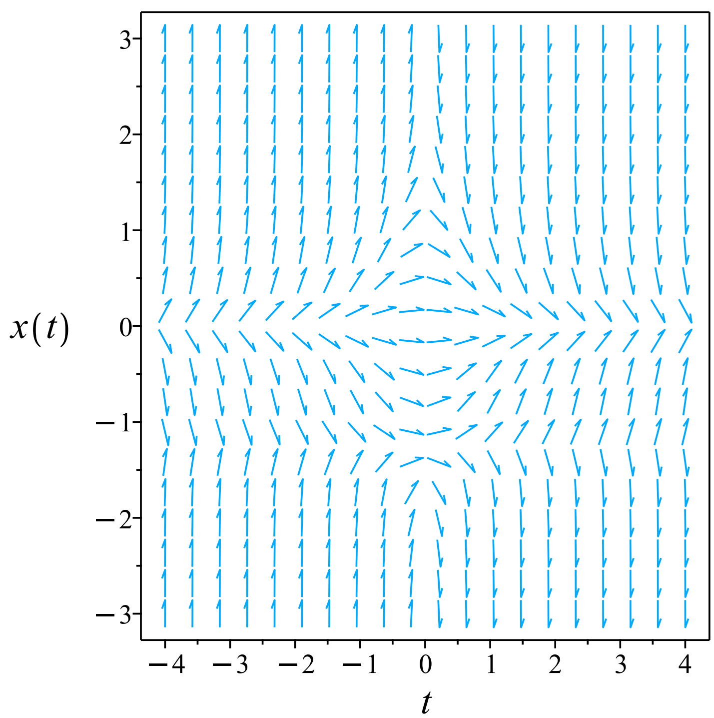


Figure 2.65: Slope field plot
$x' + 2xt + tx^4 = 0$

<u>Summary of solutions found</u>

$$
\frac{1}{x^3} = c_1\,\mathrm{e}^{3t^2} - \frac{1}{2}
$$

Solved as first order Exact ode

Time used: 0.221 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x &= \left(-t\,x^4 - 2xt\right)\mathrm{d}t\\ \left(t\,x^4 + 2xt\right)\mathrm{d}t + \mathrm{d}x &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= t\,x^4 + 2xt\\ N(t,x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(t\,x^4 + 2xt\right)\\ &= 4t\,x^3 + 2t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(4t\,x^3 + 2t\right) - (0)\right)\\ &= 4t\,x^3 + 2t\end{aligned}$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right)\\ &= \frac{1}{tx\,(x^3+2)}\left((0) - \left(4t\,x^3 + 2t\right)\right)\\ &= \frac{-4x^3 - 2}{x\,(x^3+2)}\end{aligned}$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}x}\\ &= e^{\int \frac{-4x^3-2}{x\left(x^3+2\right)}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln\left(x\left(x^3+2\right)\right)}\\ &= \frac{1}{x\,(x^3+2)}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x\left(x^3+2\right)}\left(t\,x^4+2xt\right) \\ &= t\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x\left(x^3+2\right)}(1) \\ &= \frac{1}{x\left(x^3+2\right)}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ (t)+\left(\frac{1}{x\left(x^3+2\right)}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int\overline{M}\,\mathrm{d}t \\ \int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int t\,\mathrm{d}t \\ \phi &= \frac{t^2}{2}+f(x)\end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = 0+f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{x(x^3+2)}$. Therefore equation (4) becomes

$$\frac{1}{x\left(x^{3}+2\right)} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \frac{1}{x\left(x^{3}+2\right)}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(\frac{1}{x\left(x^{3}+2\right)}\right)\mathrm{d}x$$

$$f(x) = -\frac{\ln\left(x^{3}+2\right)}{6} + \frac{\ln(x)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{t^{2}}{2} - \frac{\ln\left(x^{3}+2\right)}{6} + \frac{\ln(x)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{t^{2}}{2} - \frac{\ln\left(x^{3}+2\right)}{6} + \frac{\ln(x)}{2}$$

Solving for x gives

$$x = \frac{\left(-2\left(\mathrm{e}^{-3t^{2}+6c_1}-1\right)^{2}\mathrm{e}^{-3t^{2}+6c_1}\right)^{1/3}}{\mathrm{e}^{-3t^{2}+6c_1}-1}$$

$$x = -\frac{\left(-2\left(\mathrm{e}^{-3t^{2}+6c_1}-1\right)^{2}\mathrm{e}^{-3t^{2}+6c_1}\right)^{1/3}}{2\left(\mathrm{e}^{-3t^{2}+6c_1}-1\right)} - \frac{i\sqrt{3}\left(-2\left(\mathrm{e}^{-3t^{2}+6c_1}-1\right)^{2}\mathrm{e}^{-3t^{2}+6c_1}\right)^{1/3}}{2\left(\mathrm{e}^{-3t^{2}+6c_1}-1\right)}$$

$$x = -\frac{\left(-2\left(\mathrm{e}^{-3t^{2}+6c_1}-1\right)^{2}\mathrm{e}^{-3t^{2}+6c_1}\right)^{1/3}}{2\left(\mathrm{e}^{-3t^{2}+6c_1}-1\right)} + \frac{i\sqrt{3}\left(-2\left(\mathrm{e}^{-3t^{2}+6c_1}-1\right)^{2}\mathrm{e}^{-3t^{2}+6c_1}\right)^{1/3}}{2\,\mathrm{e}^{-3t^{2}+6c_1}-2}$$

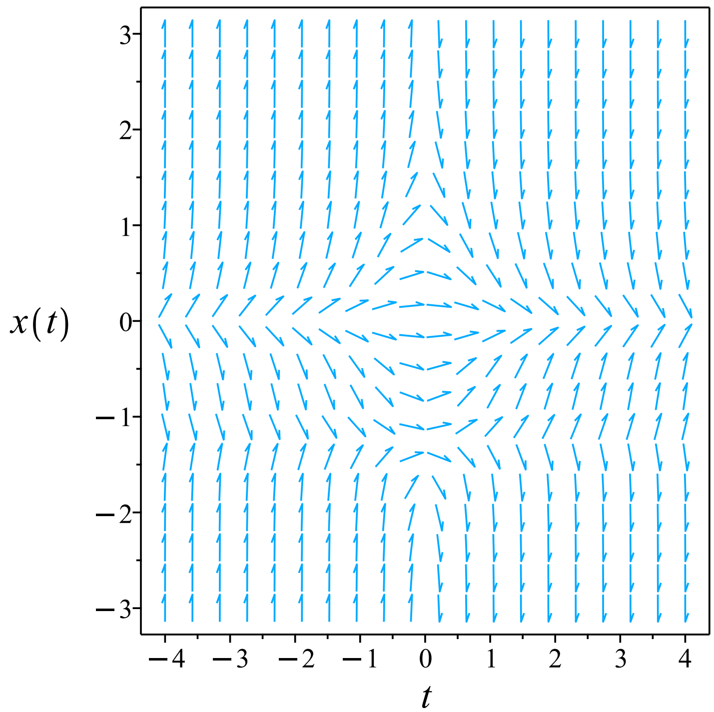


Figure 2.66: Slope field plot
$x' + 2xt + tx^4 = 0$

Summary of solutions found

$$x = \frac{\left(-2\left(e^{-3t^2+6c_1}-1\right)^2 e^{-3t^2+6c_1}\right)^{1/3}}{e^{-3t^2+6c_1}-1}$$

$$x = -\frac{\left(-2\left(e^{-3t^2+6c_1}-1\right)^2 e^{-3t^2+6c_1}\right)^{1/3}}{2\left(e^{-3t^2+6c_1}-1\right)} - \frac{i\sqrt{3}\left(-2\left(e^{-3t^2+6c_1}-1\right)^2 e^{-3t^2+6c_1}\right)^{1/3}}{2\left(e^{-3t^2+6c_1}-1\right)}$$

$$x = -\frac{\left(-2\left(e^{-3t^2+6c_1}-1\right)^2 e^{-3t^2+6c_1}\right)^{1/3}}{2\left(e^{-3t^2+6c_1}-1\right)} + \frac{i\sqrt{3}\left(-2\left(e^{-3t^2+6c_1}-1\right)^2 e^{-3t^2+6c_1}\right)^{1/3}}{2\,e^{-3t^2+6c_1}-2}$$

Maple step by step solution

Let's solve

$x' + 2xt + tx^4 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Solve for the highest derivative

  $x' = -tx^4 - 2xt$

- Separate variables

  $\frac{x'}{x(x^3+2)} = -t$

- Integrate both sides with respect to t

  $\int \frac{x'}{x(x^3+2)}dt = \int -tdt + C1$

- Evaluate integral

  $-\frac{\ln(x^3+2)}{6} + \frac{\ln(x)}{2} = -\frac{t^2}{2} + C1$

- Solve for x

  $x = \frac{\left(-2\left(\mathrm{e}^{-3t^2+6C1}-1\right)^2 \mathrm{e}^{-3t^2+6C1}\right)^{1/3}}{\mathrm{e}^{-3t^2+6C1}-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 121

```
dsolve(diff(x(t),t)+2*x(t)*t+t*x(t)^4 = 0,
       x(t),singsol=all)
```

$$x = \frac{2^{1/3}\left(\left(2c_1\,\mathrm{e}^{3t^2}-1\right)^2\right)^{1/3}}{2c_1\,\mathrm{e}^{3t^2}-1}$$
$$x = -\frac{\left(1+i\sqrt{3}\right)2^{1/3}\left(\left(2c_1\,\mathrm{e}^{3t^2}-1\right)^2\right)^{1/3}}{4c_1\,\mathrm{e}^{3t^2}-2}$$
$$x = \frac{\left(i\sqrt{3}-1\right)2^{1/3}\left(\left(2c_1\,\mathrm{e}^{3t^2}-1\right)^2\right)^{1/3}}{4c_1\,\mathrm{e}^{3t^2}-2}$$

Mathematica DSolve solution

Solving time : `11.147` (sec)
Leaf size : 177

```
DSolve[{D[x[t],t]+2*t*x[t]+t*x[t]^4==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t)\to -\frac{\sqrt[3]{-2}e^{2c_1}}{\sqrt[3]{e^{3t^2}-e^{6c_1}}}$$
$$x(t)\to \frac{\sqrt[3]{2}e^{2c_1}}{\sqrt[3]{e^{3t^2}-e^{6c_1}}}$$
$$x(t)\to \frac{(-1)^{2/3}\sqrt[3]{2}e^{2c_1}}{\sqrt[3]{e^{3t^2}-e^{6c_1}}}$$
$$x(t)\to 0$$
$$x(t)\to \sqrt[3]{-2}$$
$$x(t)\to -\sqrt[3]{2}$$
$$x(t)\to -(-1)^{2/3}\sqrt[3]{2}$$
$$x(t)\to \frac{1-i\sqrt{3}}{2^{2/3}}$$

2.1.24 problem 4 (vi)

Internal problem ID [18187]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 4 (vi)
Date solved : `Thursday, December 19, 2024 at 06:17:43 PM`
CAS classification : `[_linear]`

Solve

$$t x' + x \ln(t) = t^2$$

Solved as first order linear ode

Time used: 0.484 (sec)

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{\ln(t)}{t} \\ p(t) &= t \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= e^{\int \frac{\ln(t)}{t} dt} \\ &= e^{\frac{\ln(t)^2}{2}} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = (\mu)\,(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x\,\mathrm{e}^{\frac{\ln(t)^2}{2}}\right) = \left(\mathrm{e}^{\frac{\ln(t)^2}{2}}\right)(t)$$
$$\mathrm{d}\left(x\,\mathrm{e}^{\frac{\ln(t)^2}{2}}\right) = \left(t\,\mathrm{e}^{\frac{\ln(t)^2}{2}}\right)\,\mathrm{d}t$$

Integrating gives

$$x\,\mathrm{e}^{\frac{\ln(t)^2}{2}} = \int t\,\mathrm{e}^{\frac{\ln(t)^2}{2}}\,dt$$
$$= \int t\,\mathrm{e}^{\frac{\ln(t)^2}{2}}\,dt + c_1$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{\ln(t)^2}{2}}$ gives the final solution

$$x = \mathrm{e}^{-\frac{\ln(t)^2}{2}}\left(\int t\,\mathrm{e}^{\frac{\ln(t)^2}{2}}\,dt + c_1\right)$$

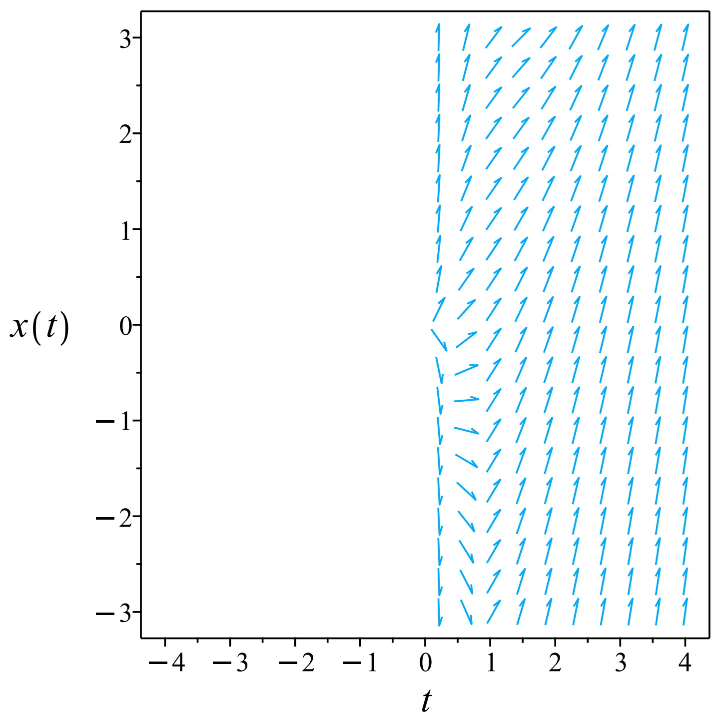


Figure 2.67: Slope field plot
$tx' + x\ln(t) = t^2$

Summary of solutions found

$$x = \mathrm{e}^{-\frac{\ln(t)^2}{2}} \left(\int t\,\mathrm{e}^{\frac{\ln(t)^2}{2}} dt + c_1\right)$$

Solved as first order Exact ode

Time used: 0.176 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$(t)\,\mathrm{d}x = \left(-x\ln(t) + t^2\right)\mathrm{d}t$$
$$\left(x\ln(t) - t^2\right)\mathrm{d}t + (t)\,\mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,x) = x\ln(t) - t^2$$
$$N(t,x) = t$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(x\ln(t) - t^2\right)\\ &= \ln(t)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= \frac{1}{t}((\ln(t)) - (1))\\ &= \frac{\ln(t) - 1}{t}\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t}\\ &= e^{\int \frac{\ln(t)-1}{t} \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{\ln(t)^2}{2} - \ln(t)}\\ &= t^{\frac{\ln(t)}{2} - 1}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= t^{\frac{\ln(t)}{2}-1}\left(x\ln(t) - t^2\right) \\
&= -\left(-x\ln(t) + t^2\right) t^{\frac{\ln(t)}{2}-1}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= t^{\frac{\ln(t)}{2}-1}(t) \\
&= \mathrm{e}^{\frac{\ln(t)^2}{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(-\left(-x\ln(t) + t^2\right) t^{\frac{\ln(t)}{2}-1}\right) + \left(\mathrm{e}^{\frac{\ln(t)^2}{2}}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}
$$

Integrating (2) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{N}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{\frac{\ln(t)^2}{2}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x\,\mathrm{e}^{\frac{\ln(t)^2}{2}} + f(t) \tag{3}
$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$
\frac{\partial \phi}{\partial t} = \frac{x\ln(t)\,\mathrm{e}^{\frac{\ln(t)^2}{2}}}{t} + f'(t) \tag{4}
$$

But equation (1) says that $\frac{\partial \phi}{\partial t} = -(-x\ln(t) + t^2)\, t^{\frac{\ln(t)}{2}-1}$. Therefore equation (4) becomes

$$-\left(-x\ln(t) + t^2\right) t^{\frac{\ln(t)}{2}-1} = \frac{x\ln(t)\,\mathrm{e}^{\frac{\ln(t)^2}{2}}}{t} + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$\begin{aligned} f'(t) &= -\frac{-\ln(t)\, t^{\frac{\ln(t)}{2}-1} tx + t^{\frac{\ln(t)}{2}-1} t^3 + x\ln(t)\,\mathrm{e}^{\frac{\ln(t)^2}{2}}}{t} \\ &= -t^{1+\frac{\ln(t)}{2}} \end{aligned}$$

Integrating the above w.r.t t results in

$$\begin{aligned} \int f'(t)\,\mathrm{d}t &= \int \left(-t^{1+\frac{\ln(t)}{2}}\right) \mathrm{d}t \\ f(t) &= \int_0^t -\tau^{1+\frac{\ln(\tau)}{2}} d\tau + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(t)$ into equation (3) gives ϕ

$$\phi = x\,\mathrm{e}^{\frac{\ln(t)^2}{2}} + \int_0^t -\tau^{1+\frac{\ln(\tau)}{2}} d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x\,\mathrm{e}^{\frac{\ln(t)^2}{2}} + \int_0^t -\tau^{1+\frac{\ln(\tau)}{2}} d\tau$$

Solving for x gives

$$x = -\left(\int_0^t -\tau^{1+\frac{\ln(\tau)}{2}} d\tau - c_1\right) \mathrm{e}^{-\frac{\ln(t)^2}{2}}$$

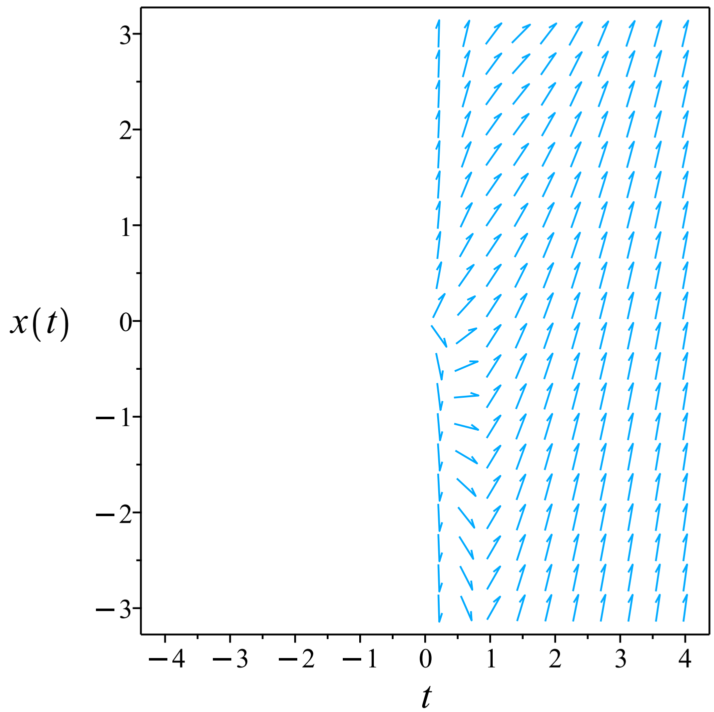


Figure 2.68: Slope field plot
$tx' + x\ln(t) = t^2$

Summary of solutions found

$$x = -\left(\int_0^t -\tau^{1+\frac{\ln(\tau)}{2}}d\tau - c_1\right)\mathrm{e}^{-\frac{\ln(t)^2}{2}}$$

Maple step by step solution

Let's solve
$tx' + x\ln(t) = t^2$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = \frac{-x\ln(t)+t^2}{t}$
- Collect w.r.t. x and simplify
  $x' = -\frac{x\ln(t)}{t} + t$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE
  $x' + \frac{x\ln(t)}{t} = t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)\left(x'+\frac{x\ln(t)}{t}\right)=\mu(t)\,t$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(x\mu(t))$
  $\mu(t)\left(x'+\frac{x\ln(t)}{t}\right)=x'\mu(t)+x\mu'(t)$
- Isolate $\mu'(t)$
  $\mu'(t)=\frac{\mu(t)\ln(t)}{t}$
- Solve to find the integrating factor
  $\mu(t)=\mathrm{e}^{\frac{\ln(t)^2}{2}}$
- Integrate both sides with respect to t
  $\int\left(\frac{d}{dt}(x\mu(t))\right)dt=\int\mu(t)\,tdt+C1$
- Evaluate the integral on the lhs
  $x\mu(t)=\int\mu(t)\,tdt+C1$
- Solve for x
  $x=\frac{\int\mu(t)tdt+C1}{\mu(t)}$
- Substitute $\mu(t)=\mathrm{e}^{\frac{\ln(t)^2}{2}}$
  $x=\frac{\int t\,\mathrm{e}^{\frac{\ln(t)^2}{2}}dt+C1}{\mathrm{e}^{\frac{\ln(t)^2}{2}}}$
- Simplify
  $x=\mathrm{e}^{-\frac{\ln(t)^2}{2}}\left(\int t\,\mathrm{e}^{\frac{\ln(t)^2}{2}}dt+C1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 25

```
dsolve(t*diff(x(t),t)+x(t)*ln(t) = t^2,
       x(t),singsol=all)
```

$$x = \mathrm{e}^{-\frac{\ln(t)^2}{2}} \left(\int t\,\mathrm{e}^{\frac{\ln(t)^2}{2}} dt + c_1\right)$$

Mathematica DSolve solution

Solving time : 0.145 (sec)
Leaf size : 48

```
DSolve[{t*D[x[t],t]+x[t]*Log[t]==t^2,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{2} e^{-\frac{1}{2}\log^2(t)-2} \left(\sqrt{2\pi}\,\mathrm{erfi}\left(\frac{\log(t)+2}{\sqrt{2}}\right) + 2e^2 c_1\right)$$

2.1.25 problem 5

Internal problem ID [18188]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 5
Date solved : `Thursday, December 19, 2024 at 06:17:45 PM`
CAS classification : `[_linear]`

Solve

$$tx' + xg(t) = h(t)$$

Solved as first order linear ode

Time used: 0.060 (sec)

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = \frac{g(t)}{t}$$
$$p(t) = \frac{h(t)}{t}$$

The integrating factor μ is

$$\mu = e^{\int \frac{g(t)}{t} dt}$$

Therefore the solution is

$$x = \left(\int \frac{h(t)\, e^{\int \frac{g(t)}{t} dt}}{t} dt + c_1 \right) e^{-\int \frac{g(t)}{t} dt}$$

Summary of solutions found

$$x = \left(\int \frac{h(t)\,\mathrm{e}^{\int \frac{g(t)}{t}dt}}{t}dt + c_1\right)\mathrm{e}^{-\int \frac{g(t)}{t}dt}$$

Solved as first order Exact ode

Time used: 0.374 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$(t)\,\mathrm{d}x = (-xg(t) + h(t))\,\mathrm{d}t$$
$$(xg(t) - h(t))\,\mathrm{d}t + (t)\,\mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= xg(t) - h(t)\\ N(t,x) &= t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(xg(t) - h(t))\\ &= g(t)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= \frac{1}{t}((g(t)) - (1))\\ &= \frac{g(t) - 1}{t}\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int \frac{g(t)-1}{t}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\int \frac{g(t)-1}{t}dt}\\ &= \mathrm{e}^{\int \frac{g(t)-1}{t}dt}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{\int \frac{g(t)-1}{t}dt}(xg(t) - h(t)) \\ &= (xg(t) - h(t))\,\mathrm{e}^{\int \frac{g(t)-1}{t}dt}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{\int \frac{g(t)-1}{t}dt}(t) \\ &= t\,\mathrm{e}^{\int \frac{g(t)-1}{t}dt}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left((xg(t) - h(t))\,\mathrm{e}^{\int \frac{g(t)-1}{t}dt}\right) + \left(t\,\mathrm{e}^{\int \frac{g(t)-1}{t}dt}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{N}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int t\,\mathrm{e}^{\int \frac{g(t)-1}{t}dt}\,\mathrm{d}x\end{aligned}$$

$$\phi = t\,\mathrm{e}^{\int \frac{g(t)-1}{t}dt}x + f(t) \tag{3}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$\frac{\partial \phi}{\partial t} = \mathrm{e}^{\int \frac{g(t)-1}{t}dt}x + (g(t)-1)\,\mathrm{e}^{\int \frac{g(t)-1}{t}dt}x + f'(t) \tag{4}$$

$$= \mathrm{e}^{\int \frac{g(t)-1}{t}dt}xg(t) + f'(t)$$

But equation (1) says that $\frac{\partial \phi}{\partial t} = (xg(t) - h(t))\,\mathrm{e}^{\int \frac{g(t)-1}{t} dt}$. Therefore equation (4) becomes

$$(xg(t) - h(t))\,\mathrm{e}^{\int \frac{g(t)-1}{t} dt} = \mathrm{e}^{\int \frac{g(t)-1}{t} dt} xg(t) + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$f'(t) = -\mathrm{e}^{\int \frac{g(t)-1}{t} dt} h(t)$$

Integrating the above w.r.t t gives

$$\int f'(t)\,\mathrm{d}t = \int \left(-\mathrm{e}^{\int \frac{g(t)-1}{t} dt} h(t)\right) \mathrm{d}t$$

$$f(t) = \int_0^t -\mathrm{e}^{\int \frac{g(\tau)-1}{\tau} d\tau} h(\tau)\,d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(t)$ into equation (3) gives ϕ

$$\phi = t\,\mathrm{e}^{\int \frac{g(t)-1}{t} dt} x + \int_0^t -\mathrm{e}^{\int \frac{g(\tau)-1}{\tau} d\tau} h(\tau)\,d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = t\,\mathrm{e}^{\int \frac{g(t)-1}{t} dt} x + \int_0^t -\mathrm{e}^{\int \frac{g(\tau)-1}{\tau} d\tau} h(\tau)\,d\tau$$

Solving for x gives

$$x = -\frac{\left(\int_0^t -\mathrm{e}^{\int \frac{g(\tau)-1}{\tau} d\tau} h(\tau)\,d\tau - c_1\right) \mathrm{e}^{\int -\frac{g(t)-1}{t} dt}}{t}$$

Summary of solutions found

$$x = -\frac{\left(\int_0^t -\mathrm{e}^{\int \frac{g(\tau)-1}{\tau} d\tau} h(\tau)\,d\tau - c_1\right) \mathrm{e}^{\int -\frac{g(t)-1}{t} dt}}{t}$$

Maple step by step solution

Let's solve

$tx' + xg(t) = h(t)$

- Highest derivative means the order of the ODE is 1

  x'

- Solve for the highest derivative

  $x' = \frac{-xg(t)+h(t)}{t}$

- Collect w.r.t. x and simplify

  $x' = -\frac{xg(t)}{t} + \frac{h(t)}{t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{xg(t)}{t} = \frac{h(t)}{t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + \frac{xg(t)}{t}\right) = \frac{\mu(t)h(t)}{t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(x\mu(t))$

  $\mu(t)\left(x' + \frac{xg(t)}{t}\right) = x'\mu(t) + x\mu'(t)$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)g(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{\int \frac{g(t)}{t}dt}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(x\mu(t))\right)dt = \int \frac{\mu(t)h(t)}{t}dt + C1$

- Evaluate the integral on the lhs

  $x\mu(t) = \int \frac{\mu(t)h(t)}{t}dt + C1$

- Solve for x

  $x = \frac{\int \frac{\mu(t)h(t)}{t}dt+C1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{\int \frac{g(t)}{t}dt}$

  $x = \frac{\int \frac{h(t)\mathrm{e}^{\int \frac{g(t)}{t}dt}}{t}dt+C1}{\mathrm{e}^{\int \frac{g(t)}{t}dt}}$

- Simplify

  $x = \left(\int \frac{h(t)\mathrm{e}^{\int \frac{g(t)}{t}dt}}{t}dt + C1\right)\mathrm{e}^{-\int \frac{g(t)}{t}dt}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 35

```
dsolve(t*diff(x(t),t)+x(t)*g(t) = h(t),
       x(t),singsol=all)
```

$$x = \left(\int \frac{h(t)\, e^{\int \frac{g(t)}{t}dt}}{t}dt + c_1\right) e^{-\int \frac{g(t)}{t}dt}$$

Mathematica DSolve solution

Solving time : 0.084 (sec)
Leaf size : 63

```
DSolve[{t*D[x[t],t]+x[t]*g[t]==h[t],{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \exp\left(\int_1^t -\frac{g(K[1])}{K[1]}dK[1]\right)\left(\int_1^t \frac{\exp\left(-\int_1^{K[2]} -\frac{g(K[1])}{K[1]}dK[1]\right)h(K[2])}{K[2]}dK[2] + c_1\right)$$

2.1.26 problem 6

Internal problem ID [18189]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 3. Solutions of first-order equations. Exercises at page 47
Problem number : 6
Date solved : `Thursday, December 19, 2024 at 06:17:47 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$t^2x'' - 6tx' + 12x = 0$$

Solved as second order Euler type ode

Time used: 0.087 (sec)

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 6trt^{r-1} + 12t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 6r\,t^r + 12t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 6r + 12 = 0$$

Or

$$r^2 - 7r + 12 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 3 \\ r_2 &= 4 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$x = c_1 x_1 + c_2 x_2$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$x = c_2 t^4 + c_1 t^3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_2 t^4 + c_1 t^3$$

Solved as second order solved by an integrating factor

Time used: 0.031 (sec)

The ode satisfies this form

$$x'' + p(t)\, x' + \frac{\left(p(t)^2 + p'(t)\right) x}{2} = f(t)$$

Where $p(t) = -\frac{6}{t}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -\frac{6}{t}\, dx} \\ &= \frac{1}{t^3} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(\frac{x}{t^3}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\frac{x}{t^3}\right)' = c_1$$

Integrating again gives

$$\left(\frac{x}{t^3}\right) = c_1 t + c_2$$

Hence the solution is

$$x = \frac{c_1 t + c_2}{\frac{1}{t^3}}$$

Or

$$x = t^4 c_1 + c_2 t^3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = t^4 c_1 + c_2 t^3$$

Solved as second order ode using change of variable on x method 2

Time used: 0.321 (sec)

In normal form the ode

$$t^2 x'' - 6t x' + 12x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$p(t) = -\frac{6}{t}$$
$$q(t) = \frac{12}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2} x(\tau) + p_1 \left(\frac{d}{d\tau} x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\int p(t)dt} dt \\
&= \int \mathrm{e}^{-\int -\frac{6}{t}dt} dt \\
&= \int e^{6\ln(t)}\, dt \\
&= \int t^6 dt \\
&= \frac{t^7}{7}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{12}{t^2}}{t^{12}} \\
&= \frac{12}{t^{14}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) + \frac{12x(\tau)}{t^{14}} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{12}{t^{14}} = \frac{12}{49\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}x(\tau)+\frac{12x(\tau)}{49\tau^2}=0$$

The above ode is now solved for $x(\tau)$. Writing the ode as

$$\frac{d^2}{d\tau^2}x(\tau)+\frac{12x(\tau)}{49\tau^2}=0 \tag{1}$$

$$A\frac{d^2}{d\tau^2}x(\tau)+B\frac{d}{d\tau}x(\tau)+Cx(\tau)=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=0\\ C&=\frac{12}{49\tau^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(\tau)=x(\tau)\,e^{\int \frac{B}{2A}\,d\tau}$$

Then (2) becomes

$$z''(\tau)=rz(\tau) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-12}{49\tau^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-12\\ t&=49\tau^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(\tau)=\left(-\frac{12}{49\tau^2}\right)z(\tau) \tag{7}$$

Equation (7) is now solved. After finding $z(\tau)$ then $x(\tau)$ is found using the inverse transformation

$$x(\tau) = z(\tau)\, e^{-\int \frac{B}{2A}\, d\tau}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.26: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 49\tau^2$. There is a pole at $\tau = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{12}{49\tau^2}$$

For the pole at $\tau = 0$ let b be the coefficient of $\frac{1}{\tau^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{12}{49}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{4}{7} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{7}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{\tau^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{12}{49\tau^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{12}{49}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{4}{7} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{7}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{12}{49\tau^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{4}{7}$	$\frac{3}{7}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{4}{7}$	$\frac{3}{7}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{3}{7}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= \frac{3}{7} - \left(\frac{3}{7}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{\tau - c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{\tau - c_1}\right) + (-)[\sqrt{r}]_\infty\\
&= \frac{3}{7\tau} + (-)\,(0)\\
&= \frac{3}{7\tau}\\
&= \frac{3}{7\tau}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(\tau)$ of degree $d = 0$ to solve the ode. The polynomial $p(\tau)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(\tau) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{3}{7\tau}\right)(0) + \left(\left(-\frac{3}{7\tau^2}\right) + \left(\frac{3}{7\tau}\right)^2 - \left(-\frac{12}{49\tau^2}\right)\right) &= 0\\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(\tau) &= p e^{\int \omega \, d\tau}\\
&= \mathrm{e}^{\int \frac{3}{7\tau} d\tau}\\
&= \tau^{3/7}
\end{aligned}
$$

The first solution to the original ode in $x(\tau)$ is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,d\tau}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \tau^{3/7} \end{aligned}$$

Which simplifies to

$$x_1 = \tau^{3/7}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,d\tau}}{x_1^2}\,d\tau$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\,d\tau \\ &= \tau^{3/7} \int \frac{1}{\tau^{6/7}}\,d\tau \\ &= \tau^{3/7}\left(7\tau^{1/7}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x(\tau) &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(\tau^{3/7}\right) + c_2\left(\tau^{3/7}\left(7\tau^{1/7}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to x using (6) which results in

$$x = \frac{c_1 7^{4/7}\left(t^7\right)^{3/7}}{7} + c_2 7^{3/7}\left(t^7\right)^{4/7}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{c_1 7^{4/7}\left(t^7\right)^{3/7}}{7} + c_2 7^{3/7}\left(t^7\right)^{4/7}$$

Solved as second order ode using change of variable on x method 1

Time used: 0.136 (sec)

In normal form the ode

$$t^2x'' - 6tx' + 12x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{6}{t} \\ q(t) &= \frac{12}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'\,(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'\,(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{3}\,\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{2\sqrt{3}}{c\sqrt{\frac{1}{t^2}}\,t^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{2\sqrt{3}}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{6}{t}\frac{2\sqrt{3}\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{2\sqrt{3}\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= -\frac{7c\sqrt{3}}{6}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) - \frac{7c\sqrt{3}\left(\frac{d}{d\tau}x(\tau)\right)}{6} + c^2 x(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
x(\tau) = \mathrm{e}^{\frac{7\sqrt{3}\,c\tau}{12}}\left(c_1 \cosh\left(\frac{\sqrt{3}\,c\tau}{12}\right) + ic_2 \sinh\left(\frac{\sqrt{3}\,c\tau}{12}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int 2\sqrt{3}\,\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{2\sqrt{3}\,\ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
x = t^{7/2}\left(c_1 \cosh\left(\frac{\ln(t)}{2}\right) + ic_2 \sinh\left(\frac{\ln(t)}{2}\right)\right)
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
x = t^{7/2}\left(c_1 \cosh\left(\frac{\ln(t)}{2}\right) + ic_2 \sinh\left(\frac{\ln(t)}{2}\right)\right)
$$

Solved as second order ode using change of variable on y method 1

Time used: 0.169 (sec)

In normal form the given ode is written as

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{6}{t} \\ q(t) &= \frac{12}{t^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{12}{t^2} - \frac{\left(-\frac{6}{t}\right)'}{2} - \frac{\left(-\frac{6}{t}\right)^2}{4} \\ &= \frac{12}{t^2} - \frac{\left(\frac{6}{t^2}\right)}{2} - \frac{\left(\frac{36}{t^2}\right)}{4} \\ &= \frac{12}{t^2} - \left(\frac{3}{t^2}\right) - \frac{9}{t^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable t then the transformation

$$x = v(t)\, z(t) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(t)$ is given by

$$\begin{aligned} z(t) &= \mathrm{e}^{-\int \frac{p(t)}{2} dt} \\ &= e^{-\int \frac{-\frac{6}{t}}{2}} \\ &= t^3 \end{aligned} \tag{5}$$

Hence (3) becomes

$$x = v(t)\, t^3 \tag{4}$$

Applying this change of variable to the original ode results in

$$t^5 v''(t) = 0$$

Which is now solved for $v(t)$.

The above ode can be simplified to

$$v''(t) = 0$$

Integrating twice gives the solution

$$v(t) = c_1 t + c_2$$

Will add steps showing solving for IC soon.

Now that $v(t)$ is known, then

$$\begin{aligned} x &= v(t)\, z(t) \\ &= (c_1 t + c_2)\,(z(t)) \end{aligned} \tag{7}$$

But from (5)

$$z(t) = t^3$$

Hence (7) becomes

$$x = (c_1 t + c_2)\, t^3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = (c_1 t + c_2)\, t^3$$

Solved as second order ode using change of variable on y method 2

Time used: 0.119 (sec)

In normal form the ode

$$t^2 x'' - 6tx' + 12x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{6}{t} \\ q(t) &= \frac{12}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $x = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} - \frac{6n}{t^2} + \frac{12}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 4 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{2v'(t)}{t} &= 0 \\ v''(t) + \frac{2v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{2u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form a linear first order is

$$u'(t) + q(t)u(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{2}{t} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt} \\ &= e^{\int \frac{2}{t}dt} \\ &= t^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(u\,t^2\right) &= 0\end{aligned}$$

Integrating gives

$$\begin{aligned}u\,t^2 &= \int 0\,dt + c_1 \\ &= c_1\end{aligned}$$

Dividing throughout by the integrating factor t^2 gives the final solution

$$u(t) = \frac{c_1}{t^2}$$

Now that $u(t)$ is known, then

$$\begin{aligned}v'(t) &= u(t) \\ v(t) &= \int u(t)\,dt + c_2 \\ &= -\frac{c_1}{t} + c_2\end{aligned}$$

Hence

$$\begin{aligned}x &= v(t)\,t^n \\ &= \left(-\frac{c_1}{t} + c_2\right)t^4 \\ &= (c_2 t - c_1)\,t^3\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \left(-\frac{c_1}{t} + c_2\right)t^4$$

Solved as second order ode using Kovacic algorithm

Time used: 0.048 (sec)

Writing the ode as

$$t^2x'' - 6tx' + 12x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -6t \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.27: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-6t}{t^2}\, dt} \\
&= z_1 e^{3\ln(t)} \\
&= z_1 \left(t^3\right)
\end{aligned}
$$

Which simplifies to

$$x_1 = t^3$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{-6t}{t^2}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{6\ln(t)}}{(x_1)^2}\, dt \\ &= x_1(t) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(t^3\right) + c_2\left(t^3(t)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_2\, t^4 + c_1\, t^3$$

Solved as second order ode adjoint method

Time used: 0.154 (sec)

In normal form the ode

$$t^2 x'' - 6tx' + 12x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{6}{t} \\ q(t) &= \frac{12}{t^2} \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(-\frac{6\xi(t)}{t}\right)' + \left(\frac{12\xi(t)}{t^2}\right) &= 0 \\ \xi''(t) + \frac{6\xi(t)}{t^2} + \frac{6\xi'(t)}{t} &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is Euler second order ODE. Let the solution be $\xi = t^r$, then $\xi' = rt^{r-1}$ and $\xi'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 6trt^{r-1} + 6t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 6r\,t^r + 6t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 6r + 6 = 0$$

Or

$$r^2 + 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -3 \\ r_2 &= -2 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$\xi = c_1\xi_1 + c_2\xi_2$$

Where $\xi_1 = t^{r_1}$ and $\xi_2 = t^{r_2}$. Hence

$$\xi = \frac{c_1}{t^3} + \frac{c_2}{t^2}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(t)\, x' - x\xi'(t) + \xi(t)\, p(t)\, x &= \int \xi(t)\, r(t)\, dt \\
x' + x\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi\,(t)}
\end{aligned}
$$

Or

$$
x' + x\left(-\frac{6}{t} - \frac{-\frac{3c_1}{t^4} - \frac{2c_2}{t^3}}{\frac{c_1}{t^3} + \frac{c_2}{t^2}}\right) = 0
$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$
x' + q(t)x = p(t)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{4c_2t + 3c_1}{t\,(c_2t + c_1)} \\
p(t) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dt} \\
&= \mathrm{e}^{\int -\frac{4c_2t+3c_1}{t(c_2t+c_1)} dt} \\
&= \frac{1}{t^3\,(c_2t + c_1)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{t^3\,(c_2t + c_1)}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{x}{t^3\,(c_2t + c_1)} &= \int 0\, dt + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{t^3(c_2t+c_1)}$ gives the final solution

$$x = t^3(c_2t + c_1)\, c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = t^3(c_2t + c_1)\, c_3$$

The constants can be merged to give

$$x = t^3(c_2t + c_1)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = t^3(c_2t + c_1)$$

Maple step by step solution

Let's solve
$t^2x'' - 6tx' + 12x = 0$

- Highest derivative means the order of the ODE is 2
  x''
- Isolate 2nd derivative
  $x'' = -\frac{12x}{t^2} + \frac{6x'}{t}$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $x'' - \frac{6x'}{t} + \frac{12x}{t^2} = 0$
- Multiply by denominators of the ODE
  $t^2x'' - 6tx' + 12x = 0$
- Make a change of variables
  $s = \ln(t)$

□ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of x with respect to t , using the chain rule
    $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$
  - Compute derivative
    $x' = \frac{\frac{d}{ds}x(s)}{t}$
  - Calculate the 2nd derivative of x with respect to t , using the chain rule

$x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}x(s)\right)$

- ○ Compute derivative

  $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) - 6\frac{d}{ds}x(s) + 12x(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}x(s) - 7\frac{d}{ds}x(s) + 12x(s) = 0$
- Characteristic polynomial of ODE

  $r^2 - 7r + 12 = 0$
- Factor the characteristic polynomial

  $(r-3)(r-4) = 0$
- Roots of the characteristic polynomial

  $r = (3, 4)$
- 1st solution of the ODE

  $x_1(s) = e^{3s}$
- 2nd solution of the ODE

  $x_2(s) = e^{4s}$
- General solution of the ODE

  $x(s) = C1 x_1(s) + C2 x_2(s)$
- Substitute in solutions

  $x(s) = C1\, e^{3s} + C2\, e^{4s}$
- Change variables back using $s = \ln(t)$

  $x = C2\, t^4 + C1\, t^3$
- Simplify

  $x = t^3(C2t + C1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(t^2*diff(diff(x(t),t),t)-6*t*diff(x(t),t)+12*x(t) = 0,
       x(t),singsol=all)
```

$$x = t^3(c_2t + c_1)$$

Mathematica DSolve solution

Solving time : 0.017 (sec)
Leaf size : 16

```
DSolve[{t^2*D[x[t],{t,2}]-6*t*D[x[t],t]+12*x[t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to t^3(c_2t + c_1)$$

2.2 Chapter 4. Autonomous systems. Exercises at page 69

2.2.1 problem 1

Internal problem ID [18190]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 1
Date solved : `Thursday, December 19, 2024 at 06:17:48 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = -\lambda x$$

Solved as first order autonomous ode

Time used: 0.148 (sec)

Integrating gives

$$\int -\frac{1}{\lambda x} dx = dt$$
$$-\frac{\ln(x)}{\lambda} = t + c_1$$

Singular solutions are found by solving

$$-\lambda x = 0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x = 0$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

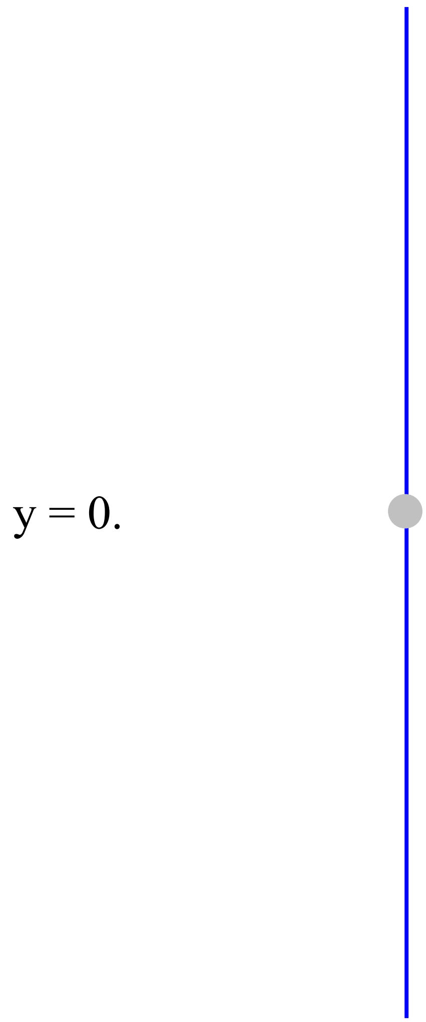


Figure 2.69: Phase line diagram

Solving for x gives

$$x = 0$$

$$x = e^{-c_1\lambda - t\lambda}$$

<u>Summary of solutions found</u>

$$x = 0$$

$$x = e^{-c_1\lambda - t\lambda}$$

Solved as first order homogeneous class D2 ode

Time used: 0.121 (sec)

Applying change of variables $x = u(t)\,t$, then the ode becomes

$$u'(t)\,t + u(t) = -\lambda u(t)\,t$$

Which is now solved The ode $u'(t) = -\frac{u(t)(t\lambda+1)}{t}$ is separable as it can be written as

$$\begin{aligned} u'(t) &= -\frac{u(t)\,(t\lambda+1)}{t} \\ &= f(t)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= -\frac{t\lambda+1}{t} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(t)\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{t\lambda+1}{t}\,dt \\ \ln(u(t)) &= -t\lambda + \ln\left(\frac{1}{t}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(t)$ gives

$$u(t) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(t)) &= -t\lambda + \ln\left(\frac{1}{t}\right) + c_1 \\ u(t) &= 0 \end{aligned}$$

Solving for $u(t)$ gives

$$u(t) = 0$$
$$u(t) = \frac{e^{-t\lambda+c_1}}{t}$$

Converting $u(t) = 0$ back to x gives

$$x = 0$$

Converting $u(t) = \frac{e^{-t\lambda+c_1}}{t}$ back to x gives

$$x = e^{-t\lambda+c_1}$$

Summary of solutions found

$$x = 0$$
$$x = e^{-t\lambda+c_1}$$

Solved as first order Exact ode

Time used: 0.106 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x &= (-\lambda x)\,\mathrm{d}t\\ (\lambda x)\,\mathrm{d}t + \mathrm{d}x &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= \lambda x\\ N(t,x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(\lambda x)\\ &= \lambda\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= 1((\lambda) - (0))\\ &= \lambda\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int \lambda\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{t\lambda}\\ &= \mathrm{e}^{t\lambda}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{t\lambda}(\lambda x)\\ &= \lambda x\,\mathrm{e}^{t\lambda}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{t\lambda}(1)\\ &= \mathrm{e}^{t\lambda}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\\ \left(\lambda x\,\mathrm{e}^{t\lambda}\right) + \left(\mathrm{e}^{t\lambda}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{N}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{t\lambda}\,\mathrm{d}x\end{aligned}$$

$$\phi = \mathrm{e}^{t\lambda}x + f(t) \tag{3}$$

Where $f(t)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t t gives

$$\frac{\partial \phi}{\partial t} = \lambda x \,\mathrm{e}^{t\lambda} + f'(t) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial t} = \lambda x \,\mathrm{e}^{t\lambda}$. Therefore equation (4) becomes

$$\lambda x \,\mathrm{e}^{t\lambda} = \lambda x \,\mathrm{e}^{t\lambda} + f'(t) \tag{5}$$

Solving equation (5) for $f'(t)$ gives

$$f'(t) = 0$$

Therefore

$$f(t) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(t)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{t\lambda} x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{t\lambda} x$$

Solving for x gives

$$x = c_1 \,\mathrm{e}^{-t\lambda}$$

Summary of solutions found

$$x = c_1 \,\mathrm{e}^{-t\lambda}$$

Solved using Lie symmetry for first order ode

Time used: 0.349 (sec)

Writing the ode as

$$\begin{aligned} x' &= -\lambda x \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$

$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \lambda x(b_3 - a_2) - \lambda^2 x^2 a_3 + \lambda(tb_2 + xb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\lambda^2 x^2 a_3 + \lambda t b_2 + \lambda x a_2 + \lambda b_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-\lambda^2 x^2 a_3 + \lambda t b_2 + \lambda x a_2 + \lambda b_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$-\lambda^2 a_3 v_2^2 + \lambda a_2 v_2 + \lambda b_2 v_1 + \lambda b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-\lambda^2 a_3 v_2^2 + \lambda a_2 v_2 + \lambda b_2 v_1 + \lambda b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}\lambda a_2 &= 0\\ \lambda b_2 &= 0\\ -\lambda^2 a_3 &= 0\\ \lambda b_1 + b_2 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= a_1\\ a_2 &= 0\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 0\\ \eta &= x\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = -\lambda x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 0 \\ S_x &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\lambda \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\lambda$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -\lambda\, dR$$
$$S(R) = -\lambda R + c_2$$

To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$\ln(x) = -t\lambda + c_2$$

Which gives

$$x = e^{-t\lambda + c_2}$$

Summary of solutions found

$$x = e^{-t\lambda + c_2}$$

Maple step by step solution

Let's solve
$x' = -\lambda x$

- Highest derivative means the order of the ODE is 1
  x'
- Solve for the highest derivative
  $x' = -\lambda x$
- Separate variables
  $\frac{x'}{x} = -\lambda$
- Integrate both sides with respect to t
  $\int \frac{x'}{x} dt = \int -\lambda dt + C1$
- Evaluate integral
  $\ln(x) = -t\lambda + C1$
- Solve for x
  $x = e^{-t\lambda + C1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 11

```
dsolve(diff(x(t),t) = -lambda*x(t),
       x(t),singsol=all)
```

$$x = c_1 \, e^{-t\lambda}$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 18

```
DSolve[{D[x[t],t]==\[Lambda]*x[t],{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to c_1 e^{\lambda t}$$
$$x(t) \to 0$$

2.2.2 problem 2

Internal problem ID [18191]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 2
Date solved : `Thursday, December 19, 2024 at 06:17:49 PM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' &= x \\
y' &= x + 2y
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} & 0 \\ \mathrm{e}^{2t} - \mathrm{e}^{t} & \mathrm{e}^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t & 0 \\ e^{2t} - e^t & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ (e^{2t} - e^t) c_1 + e^{2t} c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ (c_1 + c_2) e^{2t} - e^t c_1 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1 - \lambda)(2 - \lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{2t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \, e^t \\ c_1 \, e^t + c_2 \, e^{2t} \end{bmatrix}$$

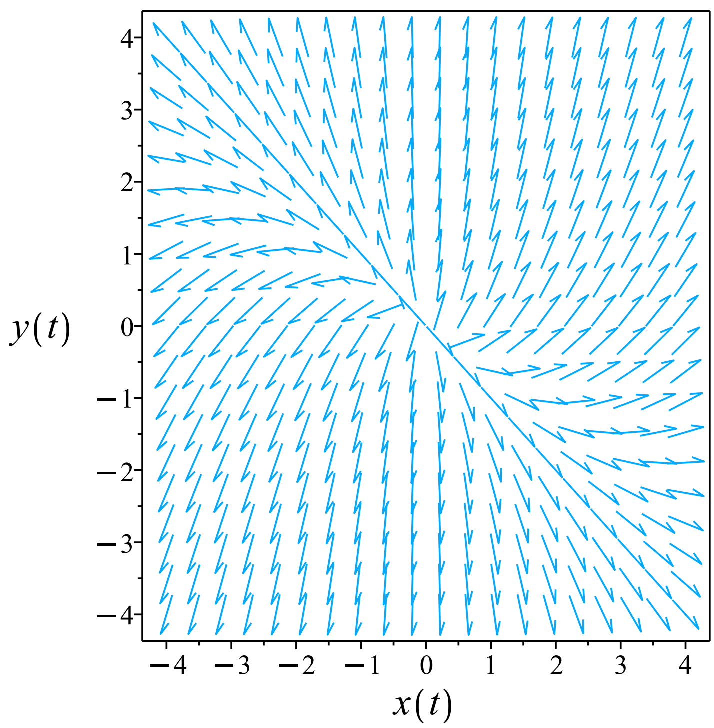


Figure 2.70: Phase plot

Maple step by step solution

Let's solve

$[x' = x, y' = x + 2y]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix}$

- Convert system into a vector equation

$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = C1 \vec{x}_1 + C2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = C1\, \mathrm{e}^{t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C2\, \mathrm{e}^{2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -C1\, \mathrm{e}^{t} \\ C1\, \mathrm{e}^{t} + C2\, \mathrm{e}^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x = -C1\, \mathrm{e}^{t}, y = C1\, \mathrm{e}^{t} + C2\, \mathrm{e}^{2t}\}$$

Maple dsolve solution

Solving time : 0.081 (sec)
Leaf size : 23

```
dsolve([diff(x(t),t) = x(t), diff(y(t),t) = x(t)+2*y(t)]
       ,{op([x(t), y(t)])})
```

$$x = e^{t} c_2$$
$$y = -e^{t} c_2 + c_1 e^{2t}$$

Mathematica DSolve solution

Solving time : 0.009 (sec)
Leaf size : 33

```
DSolve[{{D[x[t],t]==x[t],D[y[t],t]==x[t]+2*y[t]},{}},
       {x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to c_1 e^t$$
$$y(t) \to e^t \left(c_1 \left(e^t - 1\right) + c_2 e^t\right)$$

2.2.3 problem 3

Internal problem ID [18192]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 3
Date solved : `Thursday, December 19, 2024 at 06:17:50 PM`
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]`

Solve

$$t^2x'' - 2tx' + 2x = 0$$

Solved as second order Euler type ode

Time used: 0.090 (sec)

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 2trt^{r-1} + 2t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 2r\,t^r + 2t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 2r + 2 = 0$$

Or

$$r^2 - 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$x = c_1x_1 + c_2x_2$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$x = c_2\,t^2 + c_1t$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_2\,t^2 + c_1t$$

Solved as second order solved by an integrating factor

Time used: 0.029 (sec)

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = -\frac{2}{t}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{2}{t}\,dx} \\ &= \frac{1}{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(\frac{x}{t}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\frac{x}{t}\right)' = c_1$$

Integrating again gives

$$\left(\frac{x}{t}\right) = c_1 t + c_2$$

Hence the solution is

$$x = \frac{c_1 t + c_2}{\frac{1}{t}}$$

Or

$$x = t^2 c_1 + c_2 t$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = t^2 c_1 + c_2 t$$

Solved as second order ode using change of variable on x method 2

Time used: 0.335 (sec)

In normal form the ode

$$t^2 x'' - 2tx' + 2x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= -\frac{2}{t} \\
q(t) &= \frac{2}{t^2}
\end{aligned}
$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$
\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\tau''(t) + p(t)\,\tau'(t) = 0
$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\int p(t)dt} dt \\
&= \int e^{-\int -\frac{2}{t}dt} dt \\
&= \int e^{2\ln(t)}\, dt \\
&= \int t^2 dt \\
&= \frac{t^3}{3}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{2}{t^2}}{t^4} \\
&= \frac{2}{t^6}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) &= 0\\ \frac{d^2}{d\tau^2}x(\tau) + \frac{2x(\tau)}{t^6} &= 0\end{aligned}$$

But in terms of τ

$$\frac{2}{t^6} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}x(\tau) + \frac{2x(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $x(\tau)$. Writing the ode as

$$\frac{d^2}{d\tau^2}x(\tau) + \frac{2x(\tau)}{9\tau^2} = 0 \tag{1}$$

$$A\frac{d^2}{d\tau^2}x(\tau) + B\frac{d}{d\tau}x(\tau) + Cx(\tau) = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1\\ B &= 0\\ C &= \frac{2}{9\tau^2}\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(\tau) = x(\tau)\, e^{\int \frac{B}{2A}\, d\tau}$$

Then (2) becomes

$$z''(\tau) = rz(\tau) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{9\tau^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 9\tau^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(\tau) = \left(-\frac{2}{9\tau^2}\right) z(\tau) \tag{7}$$

Equation (7) is now solved. After finding $z(\tau)$ then $x(\tau)$ is found using the inverse transformation

$$x(\tau) = z(\tau)\, e^{-\int \frac{B}{2A}\, d\tau}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.31: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9\tau^2$. There is a pole at $\tau = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\tau^2}$$

For the pole at $\tau = 0$ let b be the coefficient of $\frac{1}{\tau^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{\tau^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{9\tau^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{9\tau^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{\tau - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{\tau - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{3\tau} + (-)(0) \\ &= \frac{1}{3\tau} \\ &= \frac{1}{3\tau} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(\tau)$ of degree $d = 0$ to solve the ode. The polynomial $p(\tau)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(\tau) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{3\tau}\right)(0) + \left(\left(-\frac{1}{3\tau^2}\right) + \left(\frac{1}{3\tau}\right)^2 - \left(-\frac{2}{9\tau^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(\tau) &= pe^{\int \omega \, d\tau} \\ &= e^{\int \frac{1}{3\tau} d\tau} \\ &= \tau^{1/3} \end{aligned}$$

The first solution to the original ode in $x(\tau)$ is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, d\tau}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \tau^{1/3} \end{aligned}$$

Which simplifies to

$$x_1 = \tau^{1/3}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, d\tau}}{x_1^2}\, d\tau$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, d\tau \\ &= \tau^{1/3} \int \frac{1}{\tau^{2/3}}\, d\tau \\ &= \tau^{1/3}\left(3\tau^{1/3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x(\tau) &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\tau^{1/3}\right) + c_2\left(\tau^{1/3}\left(3\tau^{1/3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to x using (6) which results in

$$
x = \frac{c_1 3^{2/3} (t^3)^{1/3}}{3} + c_2 3^{1/3} \left(t^3\right)^{2/3}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
x = \frac{c_1 3^{2/3} (t^3)^{1/3}}{3} + c_2 3^{1/3} \left(t^3\right)^{2/3}
$$

Solved as second order ode using change of variable on x method 1

Time used: 0.116 (sec)

In normal form the ode

$$
t^2 x'' - 2t x' + 2x = 0 \tag{1}
$$

Becomes

$$
x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(t) &= -\frac{2}{t} \\
q(t) &= \frac{2}{t^2}
\end{aligned}
$$

Applying change of variables $\tau = g(t)$ to (2) results

$$
\frac{d^2}{d\tau^2} x(\tau) + p_1 \left(\frac{d}{d\tau} x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{2}\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2}\\ &= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{2}{t}\frac{\sqrt{2}\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{2}\sqrt{\frac{1}{t^2}}}{c}\right)^2}\\ &= -\frac{3c\sqrt{2}}{2}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0\\ \frac{d^2}{d\tau^2}x(\tau) - \frac{3c\sqrt{2}\left(\frac{d}{d\tau}x(\tau)\right)}{2} + c^2 x(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$x(\tau) = \mathrm{e}^{\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1\cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2\sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\, dt \\ &= \frac{\int \sqrt{2}\sqrt{\frac{1}{t^2}}dt}{c} \\ &= \frac{\sqrt{2}\,\ln(t)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$x = t^{3/2}\left(c_1 \cosh\left(\frac{\ln(t)}{2}\right) + ic_2 \sinh\left(\frac{\ln(t)}{2}\right)\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = t^{3/2}\left(c_1 \cosh\left(\frac{\ln(t)}{2}\right) + ic_2 \sinh\left(\frac{\ln(t)}{2}\right)\right)$$

Solved as second order ode using change of variable on y method 1

Time used: 0.063 (sec)

In normal form the given ode is written as

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$\begin{aligned}p(t) &= -\frac{2}{t} \\ q(t) &= \frac{2}{t^2}\end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{t^2} - \frac{\left(-\frac{2}{t}\right)'}{2} - \frac{\left(-\frac{2}{t}\right)^2}{4} \\ &= \frac{2}{t^2} - \frac{\left(\frac{2}{t^2}\right)}{2} - \frac{\left(\frac{4}{t^2}\right)}{4} \\ &= \frac{2}{t^2} - \left(\frac{1}{t^2}\right) - \frac{1}{t^2} \\ &= 0\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable t then the transformation

$$x = v(t)\, z(t) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(t)$ is given by

$$\begin{aligned} z(t) &= \mathrm{e}^{-\int \frac{p(t)}{2} dt} \\ &= e^{-\int \frac{-\frac{2}{t}}{2}} \\ &= t \end{aligned} \tag{5}$$

Hence (3) becomes

$$x = v(t)\, t \tag{4}$$

Applying this change of variable to the original ode results in

$$t^3 v''(t) = 0$$

Which is now solved for $v(t)$.

The above ode can be simplified to

$$v''(t) = 0$$

Integrating twice gives the solution

$$v(t) = c_1 t + c_2$$

Will add steps showing solving for IC soon.

Now that $v(t)$ is known, then

$$\begin{aligned} x &= v(t)\, z(t) \\ &= (c_1 t + c_2)\,(z(t)) \end{aligned} \tag{7}$$

But from (5)

$$z(t) = t$$

Hence (7) becomes

$$x = (c_1 t + c_2)\, t$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = (c_1 t + c_2)\, t$$

Solved as second order ode using change of variable on y method 2

Time used: 0.227 (sec)

In normal form the ode

$$t^2x'' - 2tx' + 2x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{2}{t} \\ q(t) &= \frac{2}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $x = v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} - \frac{2n}{t^2} + \frac{2}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{2v'(t)}{t} &= 0 \\ v''(t) + \frac{2v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{2u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form a linear first order is

$$u'(t) + q(t)u(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{2}{t} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dt} \\ &= \mathrm{e}^{\int \frac{2}{t} dt} \\ &= t^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(u\, t^2\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} u\, t^2 &= \int 0\, dt + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor t^2 gives the final solution

$$u(t) = \frac{c_1}{t^2}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\, dt + c_2 \\ &= -\frac{c_1}{t} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
x &= v(t)\, t^n \\
&= \left(-\frac{c_1}{t} + c_2\right) t^2 \\
&= (c_2 t - c_1)\, t
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
x = \left(-\frac{c_1}{t} + c_2\right) t^2
$$

Solved as second order ode using non constant coeff transformation on B method

Time used: 0.059 (sec)

Given an ode of the form

$$
Ax'' + Bx' + Cx = F(t)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
x = Bv
$$

This results in

$$
\begin{aligned}
x' &= B'v + v'B \\
x'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $x = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= t^2 \\
B &= -2t \\
C &= 2 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (t^2)(0) + (-2t)(-2) + (2)(-2t) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2t^3v'' + (0)v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2t^3u'(t) = 0$$

Which is now solved for u. Since the ode has the form $u'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\begin{aligned}
\int du &= \int 0\,dt + c_1 \\
u(t) &= c_1
\end{aligned}$$

The ode for v now becomes

$$v'(t) = c_1$$

Which is now solved for v. Since the ode has the form $v'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\begin{aligned}
\int dv &= \int c_1\,dt \\
v(t) &= c_1t + c_2
\end{aligned}$$

Replacing $v(t)$ above by $-\frac{x}{2t}$, then the solution becomes

$$\begin{aligned} x(t) &= Bv \\ &= -2(c_1 t + c_2)\, t \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = -2(c_1 t + c_2)\, t$$

Solved as second order ode using Kovacic algorithm

Time used: 0.044 (sec)

Writing the ode as

$$t^2 x'' - 2t x' + 2x = 0 \tag{1}$$

$$A x'' + B x' + C x = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -2t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.32: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{t^2}\,dt} \\
&= z_1 e^{\ln(t)} \\
&= z_1(t)
\end{aligned}$$

Which simplifies to

$$x_1 = t$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-2t}{t^2}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{2\ln(t)}}{(x_1)^2}\,dt \\
&= x_1(t)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(t) + c_2(t(t))
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_2 t^2 + c_1 t$$

Solved as second order ode adjoint method

Time used: 0.276 (sec)

In normal form the ode

$$t^2x'' - 2tx' + 2x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{2}{t} \\ q(t) &= \frac{2}{t^2} \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - \left(-\frac{2\xi(t)}{t}\right)' + \left(\frac{2\xi(t)}{t^2}\right) &= 0 \\ \xi''(t) + \frac{2\xi'(t)}{t} &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is second order ode with missing dependent variable ξ. Let

$$p(t) = \xi'$$

Then

$$p'(t) = \xi''$$

Hence the ode becomes

$$p'(t) + \frac{2p(t)}{t} = 0$$

Which is now solve for $p(t)$ as first order ode. In canonical form a linear first order is

$$p'(t) + q(t)p(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{2}{t} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int \frac{2}{t}dt} \\ &= t^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu p &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(p\,t^2\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} p\,t^2 &= \int 0\,dt + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor t^2 gives the final solution

$$p(t) = \frac{c_1}{t^2}$$

For solution (1) found earlier, since $p = \xi'$ then we now have a new first order ode to solve which is

$$\xi' = \frac{c_1}{t^2}$$

Since the ode has the form $\xi' = f(t)$, then we only need to integrate $f(t)$.

$$\begin{aligned} \int d\xi &= \int \frac{c_1}{t^2}\,dt \\ \xi &= -\frac{c_1}{t} + c_2 \end{aligned}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(t)\,x' - x\xi'(t) + \xi(t)\,p(t)\,x &= \int \xi(t)\,r(t)\,dt \\ x' + x\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\,r(t)\,dt}{\xi\,(t)} \end{aligned}$$

Or

$$x' + x\left(-\frac{2}{t} - \frac{c_1}{t^2\left(-\frac{c_1}{t} + c_2\right)}\right) = 0$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = -\frac{-2c_2t + c_1}{t\left(-c_2t + c_1\right)}$$
$$p(t) = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt}\\ &= \mathrm{e}^{\int -\frac{-2c_2t+c_1}{t(-c_2t+c_1)}dt}\\ &= \frac{1}{\left(c_2t - c_1\right)t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{\left(c_2t - c_1\right)t}\right) &= 0\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{x}{\left(c_2t - c_1\right)t} &= \int 0\,dt + c_3\\ &= c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{(c_2t-c_1)t}$ gives the final solution

$$x = \left(c_2t - c_1\right)tc_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = \left(c_2t - c_1\right)tc_3$$

The constants can be merged to give

$$x = (c_2 t - c_1)\, t$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = (c_2 t - c_1)\, t$$

Maple step by step solution

Let's solve
$t^2 x'' - 2t x' + 2x = 0$

- Highest derivative means the order of the ODE is 2
x''
- Isolate 2nd derivative
$x'' = -\frac{2x}{t^2} + \frac{2x'}{t}$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
$x'' - \frac{2x'}{t} + \frac{2x}{t^2} = 0$
- Multiply by denominators of the ODE
$t^2 x'' - 2t x' + 2x = 0$
- Make a change of variables
$s = \ln(t)$

□ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of x with respect to t , using the chain rule
  $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$
  - Compute derivative
  $x' = \frac{\frac{d}{ds}x(s)}{t}$
  - Calculate the 2nd derivative of x with respect to t , using the chain rule
  $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t)\left(\frac{d}{ds}x(s)\right)$
  - Compute derivative
  $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

Substitute the change of variables back into the ODE
$t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) - 2\frac{d}{ds}x(s) + 2x(s) = 0$

- Simplify
  $\frac{d^2}{ds^2}x(s) - 3\frac{d}{ds}x(s) + 2x(s) = 0$
- Characteristic polynomial of ODE
  $r^2 - 3r + 2 = 0$
- Factor the characteristic polynomial
  $(r-1)(r-2) = 0$
- Roots of the characteristic polynomial
  $r = (1, 2)$
- 1st solution of the ODE
  $x_1(s) = e^s$
- 2nd solution of the ODE
  $x_2(s) = e^{2s}$
- General solution of the ODE
  $x(s) = C1x_1(s) + C2x_2(s)$
- Substitute in solutions
  $x(s) = C1\,e^s + C2\,e^{2s}$
- Change variables back using $s = \ln(t)$
  $x = C2\,t^2 + C1t$
- Simplify
  $x = t(C2t + C1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 11

```
dsolve(t^2*diff(diff(x(t),t),t)-2*t*diff(x(t),t)+2*x(t) = 0,
       x(t),singsol=all)
```

$$x = t(c_2t + c_1)$$

Mathematica DSolve solution

Solving time : 0.16 (sec)
Leaf size : 133

```
DSolve[{t^2*D[x[t],{t,2}]-2*D[x[t],t]+2*x[t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to 2^{-\frac{1}{2}i\left(\sqrt{7}-i\right)}t^{\frac{1}{2}-\frac{i\sqrt{7}}{2}}\left(c_2t^{i\sqrt{7}}\,\text{Hypergeometric1F1}\left(-\frac{1}{2}-\frac{i\sqrt{7}}{2},1-i\sqrt{7},-\frac{2}{t}\right)\right.$$
$$\left.+\,2^{i\sqrt{7}}c_1\,\text{Hypergeometric1F1}\left(\frac{1}{2}i\left(i+\sqrt{7}\right),1+i\sqrt{7},-\frac{2}{t}\right)\right)$$

2.2.4 problem 5 (i)

Internal problem ID [18193]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 5 (i)
Date solved : `Thursday, December 19, 2024 at 06:17:52 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$x'' - 5x' + 6x = 0$$

Solved as second order linear constant coeff ode

Time used: 0.059 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} - 5\lambda\, e^{t\lambda} + 6\, e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(6)} \\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{5}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= 2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x &= c_1 e^{(3)t} + c_2 e^{(2)t}\end{aligned}$$

Or

$$x = c_1 \, \mathrm{e}^{3t} + c_2 \, \mathrm{e}^{2t}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_1 \, \mathrm{e}^{3t} + c_2 \, \mathrm{e}^{2t}$$

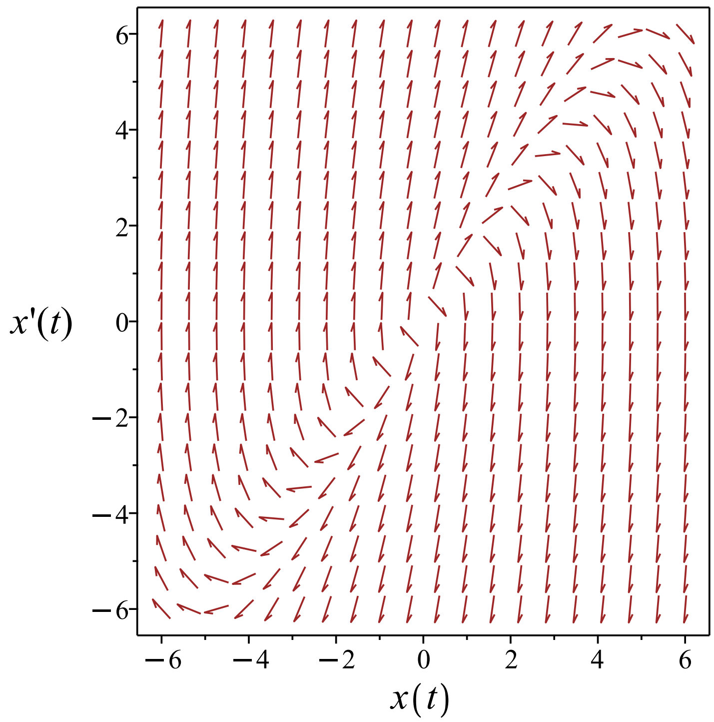


Figure 2.71: Slope field plot
$x'' - 5x' + 6x = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.055 (sec)

Writing the ode as

$$x'' - 5x' + 6x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.34: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dt} \\ &= z_1 e^{\frac{5t}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{5t}{2}}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{2t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{-5}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{5t}}{(x_1)^2}\,dt \\ &= x_1\left(\mathrm{e}^{5t}\mathrm{e}^{-4t}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(e^{2t}\right) + c_2 \left(e^{2t} \left(e^{5t} e^{-4t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
x = c_1 \, e^{2t} + c_2 \, e^{3t}
$$

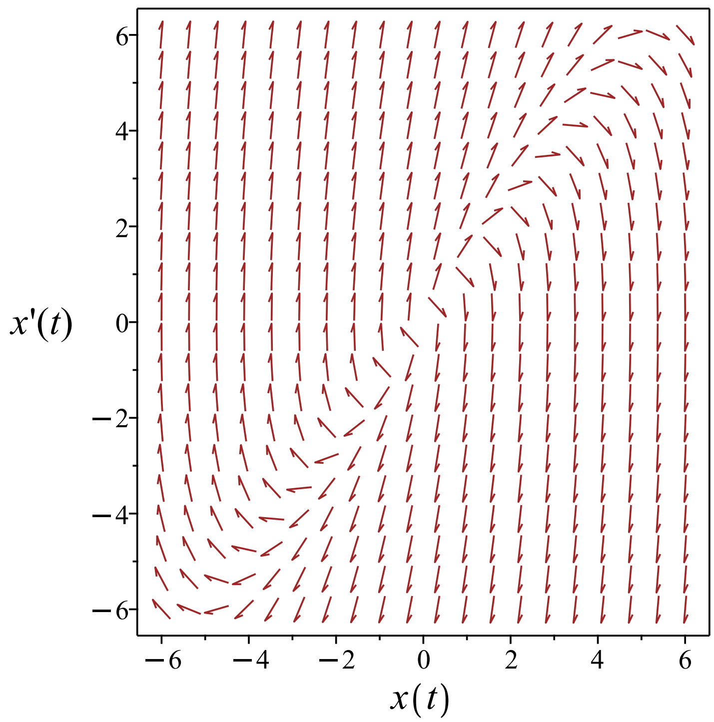


Figure 2.72: Slope field plot
$x'' - 5x' + 6x = 0$

Solved as second order ode adjoint method

Time used: 0.293 (sec)

In normal form the ode

$$
x'' - 5x' + 6x = 0 \tag{1}
$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= -5 \\ q(t) &= 6 \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (-5\xi(t))' + (6\xi(t)) &= 0 \\ \xi''(t) + 5\xi'(t) + 6\xi(t) &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} + 5\lambda\,\mathrm{e}^{t\lambda} + 6\,\mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{5^2 - (4)\,(1)\,(6)} \\ &= -\frac{5}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\lambda_1 = -\frac{5}{2} + \frac{1}{2}$$

$$\lambda_2 = -\frac{5}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = -2$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$\xi = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$\xi = c_1 e^{(-2)t} + c_2 e^{(-3)t}$$

Or

$$\xi = c_1 \, \mathrm{e}^{-2t} + c_2 \, \mathrm{e}^{-3t}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(t)\, x' - x\xi'(t) + \xi(t)\, p(t)\, x = \int \xi(t)\, r(t)\, dt$$
$$x' + x\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) = \frac{\int \xi(t)\, r(t)\, dt}{\xi\,(t)}$$

Or

$$x' + x\left(-5 - \frac{-2c_1 \, \mathrm{e}^{-2t} - 3c_2 \, \mathrm{e}^{-3t}}{c_1 \, \mathrm{e}^{-2t} + c_2 \, \mathrm{e}^{-3t}}\right) = 0$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = -\frac{2c_2 \, \mathrm{e}^{-t} + 3c_1}{c_2 \, \mathrm{e}^{-t} + c_1}$$
$$p(t) = 0$$

The integrating factor μ is

$$\mu = e^{\int q \, dt}$$
$$= \mathrm{e}^{\int -\frac{2c_2 \, \mathrm{e}^{-t}+3c_1}{c_2 \, \mathrm{e}^{-t}+c_1} dt}$$
$$= \frac{\mathrm{e}^{-3t}}{c_2 \, \mathrm{e}^{-t} + c_1}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}\mu x &= 0 \\
\frac{d}{dt}\left(\frac{x\,e^{-3t}}{c_2\,e^{-t}+c_1}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{x\,e^{-3t}}{c_2\,e^{-t}+c_1} &= \int 0\,dt + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{e^{-3t}}{c_2\,e^{-t}+c_1}$ gives the final solution

$$
x = e^{2t}\left(c_1\,e^{t}+c_2\right)c_3
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
x = e^{2t}\left(c_1\,e^{t}+c_2\right)c_3
$$

The constants can be merged to give

$$
x = e^{2t}\left(c_1\,e^{t}+c_2\right)
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
x = e^{2t}\left(c_1\,e^{t}+c_2\right)
$$

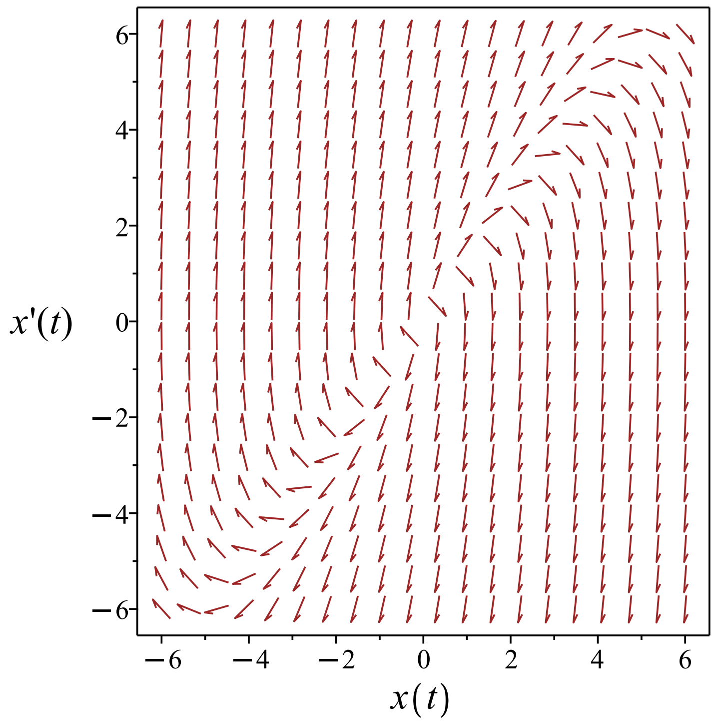


Figure 2.73: Slope field plot
$x'' - 5x' + 6x = 0$

Maple step by step solution

Let's solve
$x'' - 5x' + 6x = 0$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of ODE
  $r^2 - 5r + 6 = 0$
- Factor the characteristic polynomial
  $(r - 2)(r - 3) = 0$
- Roots of the characteristic polynomial
  $r = (2, 3)$
- 1st solution of the ODE
  $x_1(t) = e^{2t}$
- 2nd solution of the ODE
  $x_2(t) = e^{3t}$
- General solution of the ODE

$$x = C1x_1(t) + C2x_2(t)$$

- Substitute in solutions

$$x = C1\,e^{2t} + C2\,e^{3t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(diff(diff(x(t),t),t)-5*diff(x(t),t)+6*x(t) = 0,
       x(t),singsol=all)
```

$$x = c_1\,e^{3t} + c_2\,e^{2t}$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 20

```
DSolve[{D[x[t],{t,2}]-5*D[x[t],t]+6*x[t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to e^{2t}\left(c_2e^t + c_1\right)$$

2.2.5 problem 5 (ii)

Internal problem ID [18194]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 5 (ii)
Date solved : `Thursday, December 19, 2024 at 06:17:52 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$x'' - 4x' + 4x = 0$$

Solved as second order linear constant coeff ode

Time used: 0.061 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} - 4\lambda\, e^{t\lambda} + 4\, e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\lambda_{1,2} = \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)}$$
$$= 2$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$x = c_1 e^{2t} + c_2 t\, e^{2t} \tag{1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_1 e^{2t} + c_2 t\, e^{2t}$$

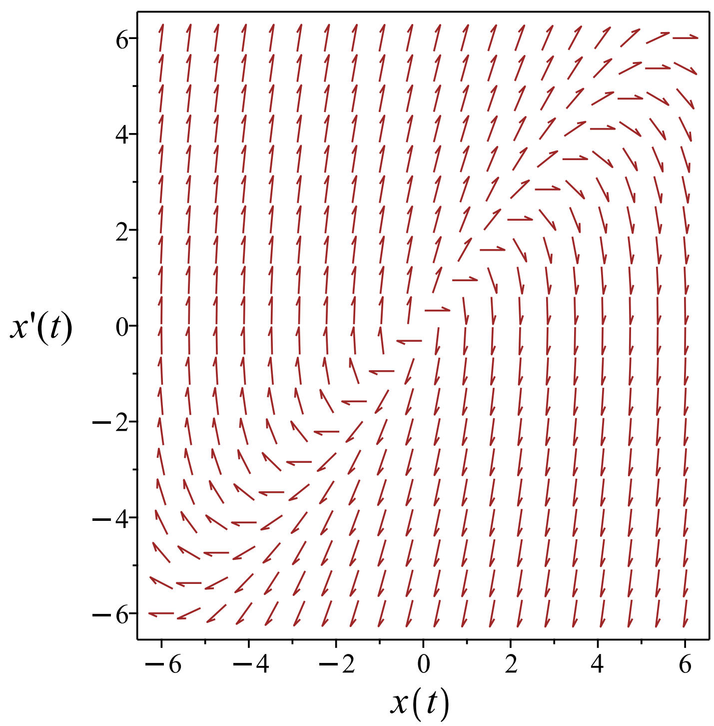


Figure 2.74: Slope field plot
$x'' - 4x' + 4x = 0$

Solved as second order solved by an integrating factor

Time used: 0.033 (sec)

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= \mathrm{e}^{-2t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(\mathrm{e}^{-2t}x\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-2t}x\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-2t}x\right) = c_1 t + c_2$$

Hence the solution is

$$x = \frac{c_1 t + c_2}{\mathrm{e}^{-2t}}$$

Or

$$x = c_1 t\,\mathrm{e}^{2t} + c_2\,\mathrm{e}^{2t}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_1 t\,\mathrm{e}^{2t} + c_2\,\mathrm{e}^{2t}$$

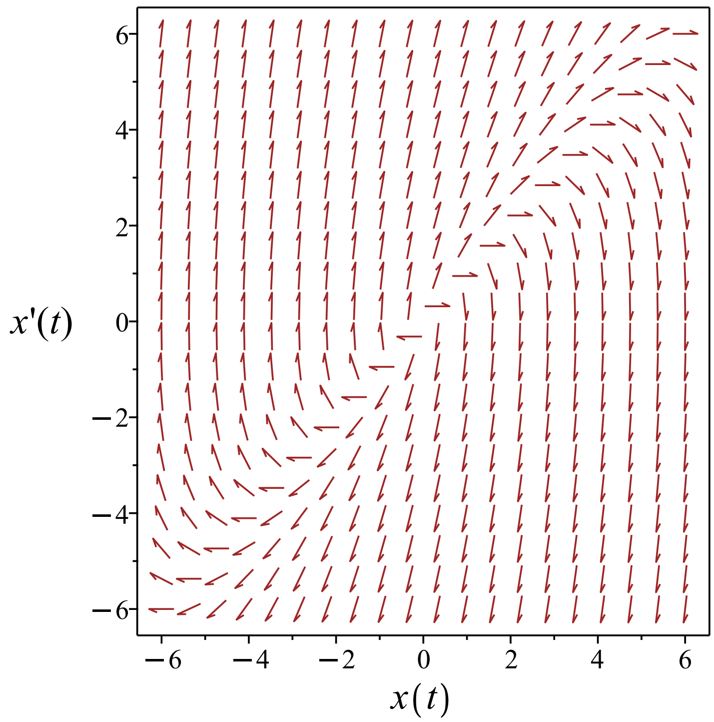


Figure 2.75: Slope field plot
$x'' - 4x' + 4x = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.047 (sec)

Writing the ode as

$$x'' - 4x' + 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.36: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dt} \\ &= z_1 e^{2t} \\ &= z_1 \left(e^{2t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = e^{2t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{-4}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{4t}}{(x_1)^2}\,dt \\ &= x_1(t) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(e^{2t}\right) + c_2\left(e^{2t}(t)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_1 e^{2t} + c_2 t e^{2t}$$

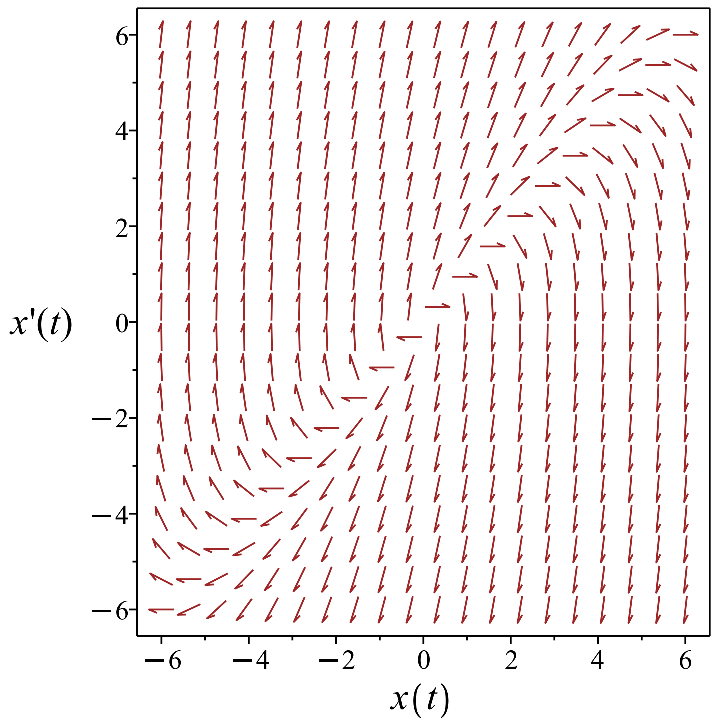


Figure 2.76: Slope field plot
$x'' - 4x' + 4x = 0$

Solved as second order ode adjoint method

Time used: 0.148 (sec)

In normal form the ode

$$x'' - 4x' + 4x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 4 \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (-4\xi(t))' + (4\xi(t)) &= 0 \\ \xi''(t) + 4\xi'(t) + 4\xi(t) &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} + 4\lambda\,\mathrm{e}^{t\lambda} + 4\,\mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)} \\ &= -2 \end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$\xi = c_1 \mathrm{e}^{-2t} + c_2 t\,\mathrm{e}^{-2t} \tag{1}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(t)\,x' - x\xi'(t) + \xi(t)\,p(t)\,x &= \int \xi(t)\,r(t)\,dt\\ x' + x\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\,r(t)\,dt}{\xi\,(t)}\end{aligned}$$

Or

$$x' + x\left(-4 - \frac{-2c_1\,\mathrm{e}^{-2t} + c_2\,\mathrm{e}^{-2t} - 2c_2t\,\mathrm{e}^{-2t}}{c_1\,\mathrm{e}^{-2t} + c_2t\,\mathrm{e}^{-2t}}\right) = 0$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(t) &= -\frac{2c_2t + 2c_1 + c_2}{c_2t + c_1}\\ p(t) &= 0\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt}\\ &= \mathrm{e}^{\int -\frac{2c_2t+2c_1+c_2}{c_2t+c_1}dt}\\ &= \frac{\mathrm{e}^{-2t}}{c_2t + c_1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x\,\mathrm{e}^{-2t}}{c_2t + c_1}\right) &= 0\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{x\,\mathrm{e}^{-2t}}{c_2t + c_1} &= \int 0\,dt + c_3\\ &= c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{e^{-2t}}{c_2 t + c_1}$ gives the final solution

$$x = (c_2 t + c_1) e^{2t} c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = (c_2 t + c_1) e^{2t} c_3$$

The constants can be merged to give

$$x = (c_2 t + c_1) e^{2t}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = (c_2 t + c_1) e^{2t}$$

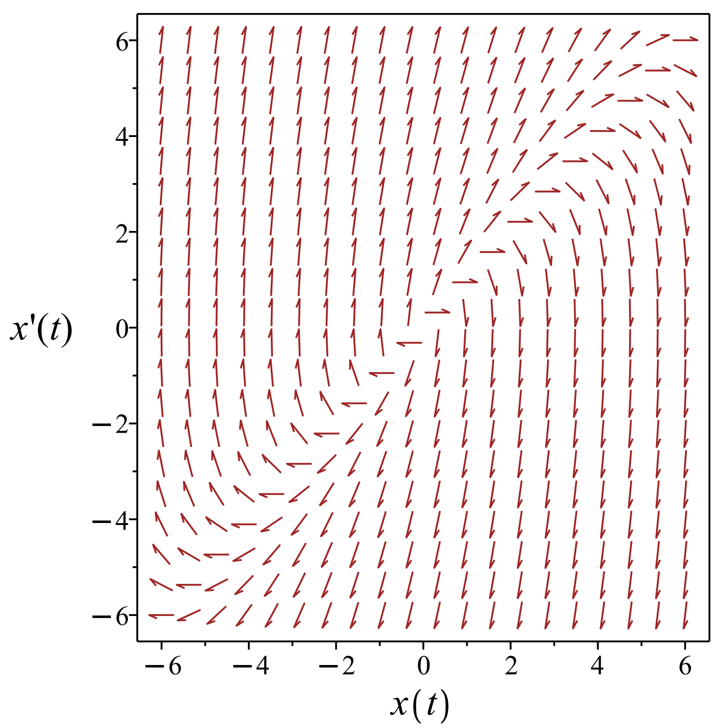


Figure 2.77: Slope field plot
$x'' - 4x' + 4x = 0$

Maple step by step solution

Let's solve

$x'' - 4x' + 4x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the ODE

  $x_1(t) = e^{2t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t\,e^{2t}$

- General solution of the ODE

  $x = C1 x_1(t) + C2 x_2(t)$

- Substitute in solutions

  $x = C1\,e^{2t} + C2 t\,e^{2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(x(t),t),t)-4*diff(x(t),t)+4*x(t) = 0,
       x(t),singsol=all)
```

$$x = (c_2 t + c_1)\,e^{2t}$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 18

```
DSolve[{D[x[t],{t,2}]-4*D[x[t],t]+4*x[t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to e^{2t}(c_2 t + c_1)$$

2.2.6 problem 5 (iiI=i)

Internal problem ID [18195]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 5 (iiI=i)
Date solved : `Thursday, December 19, 2024 at 06:17:53 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$x'' - 4x' + 5x = 0$$

Solved as second order linear constant coeff ode

Time used: 0.090 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -4, C = 5$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} - 4\lambda\, e^{t\lambda} + 5\, e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 5$ into the above gives

$$\lambda_{1,2} = \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-4^2 - (4)(1)(5)}$$
$$= 2 \pm i$$

Hence

$$\lambda_1 = 2 + i$$
$$\lambda_2 = 2 - i$$

Which simplifies to

$$\lambda_1 = 2 + i$$
$$\lambda_2 = 2 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{2t}(c_1 \cos(t) + c_2 \sin(t))$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \mathrm{e}^{2t}(c_1 \cos(t) + c_2 \sin(t))$$

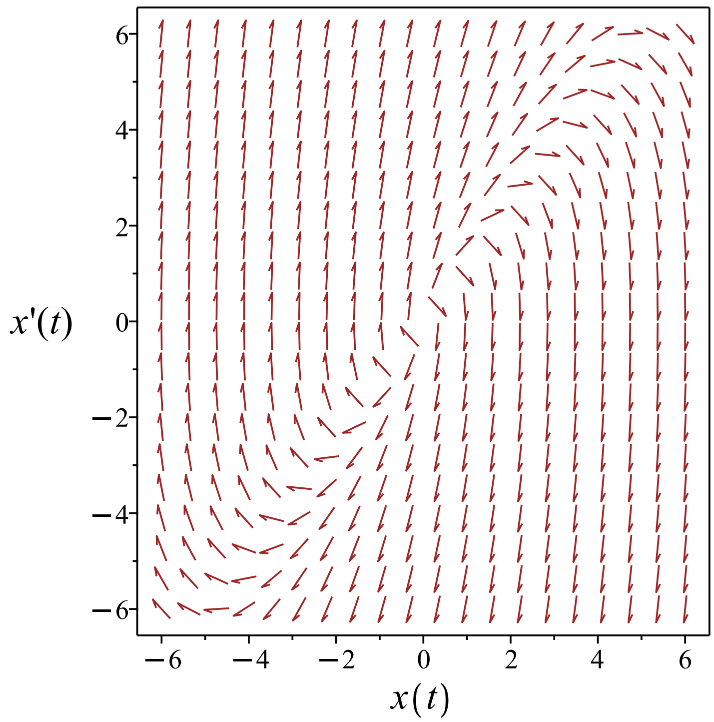


Figure 2.78: Slope field plot
$x'' - 4x' + 5x = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.105 (sec)

Writing the ode as

$$x'' - 4x' + 5x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.38: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\, dt} \\ &= z_1 e^{2t} \\ &= z_1\left(\mathrm{e}^{2t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{2t}\cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{-4}{1}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{4t}}{(x_1)^2}\, dt \\ &= x_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(e^{2t} \cos(t)\right) + c_2 \left(e^{2t} \cos(t) (\tan(t))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_1 e^{2t} \cos(t) + c_2 e^{2t} \sin(t)$$

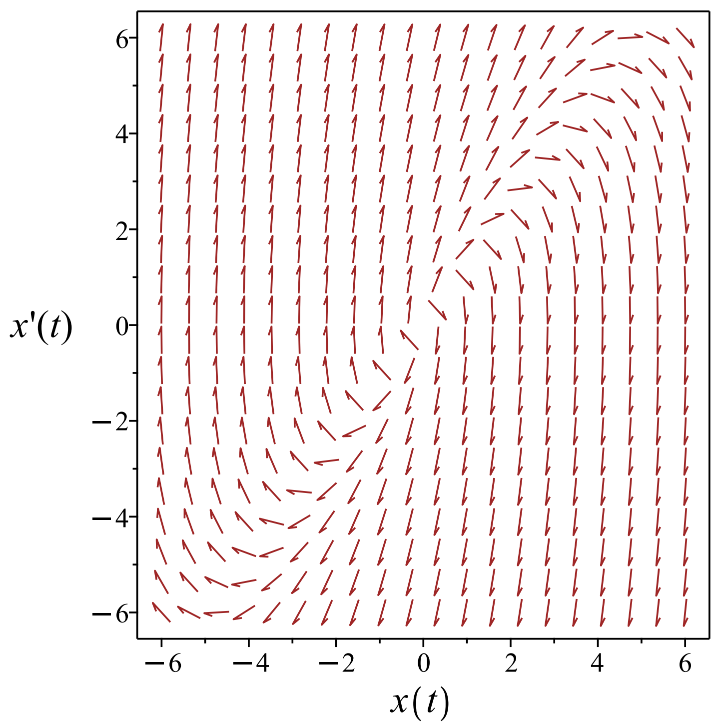


Figure 2.79: Slope field plot
$x'' - 4x' + 5x = 0$

Solved as second order ode adjoint method

Time used: 0.615 (sec)

In normal form the ode

$$x'' - 4x' + 5x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 5 \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (-4\xi(t))' + (5\xi(t)) &= 0 \\ \xi''(t) + 4\xi'(t) + 5\xi(t) &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} + 4\lambda\,\mathrm{e}^{t\lambda} + 5\,\mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)} \\ &= -2 \pm i \end{aligned}$$

Hence

$$\lambda_1 = -2 + i$$
$$\lambda_2 = -2 - i$$

Which simplifies to

$$\lambda_1 = -2 + i$$
$$\lambda_2 = -2 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$\xi = e^{-2t}(c_1 \cos(t) + c_2 \sin(t))$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(t)\, x' - x\xi'(t) + \xi(t)\, p(t)\, x = \int \xi(t)\, r(t)\, dt$$
$$x' + x\left(p(t) - \frac{\xi'(t)}{\xi(t)}\right) = \frac{\int \xi(t)\, r(t)\, dt}{\xi(t)}$$

Or

$$x' + x\left(-4 - \frac{(-2\,\mathrm{e}^{-2t}(c_1\cos(t) + c_2\sin(t)) + \mathrm{e}^{-2t}(-c_1\sin(t) + c_2\cos(t)))\,\mathrm{e}^{2t}}{c_1\cos(t) + c_2\sin(t)}\right) = 0$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = -\frac{(2c_1 + c_2)\cos(t) - \sin(t)(c_1 - 2c_2)}{c_1\cos(t) + c_2\sin(t)}$$
$$p(t) = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt}\\ &= \mathrm{e}^{\int -\frac{(2c_1+c_2)\cos(t)-\sin(t)(c_1-2c_2)}{c_1\cos(t)+c_2\sin(t)}dt}\\ &= \mathrm{e}^{-\ln(\tan(t)c_2+c_1)+\frac{\ln\left(1+\tan(t)^2\right)}{2}-2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(x\,\mathrm{e}^{-\ln(\tan(t)c_2+c_1)+\frac{\ln\left(1+\tan(t)^2\right)}{2}-2t}\right) &= 0\end{aligned}$$

Integrating gives

$$\begin{aligned}x\,\mathrm{e}^{-\ln(\tan(t)c_2+c_1)+\frac{\ln\left(1+\tan(t)^2\right)}{2}-2t} &= \int 0\,dt + c_3\\ &= c_3\end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{-\ln(\tan(t)c_2+c_1)+\frac{\ln\left(1+\tan(t)^2\right)}{2}-2t}$ gives the final solution

$$x = (\tan(t)\,c_2 + c_1)\,\mathrm{e}^{\ln\left(\frac{1}{\sqrt{1+\tan(t)^2}}\right)+2t}c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = (\tan(t)\,c_2 + c_1)\,\mathrm{e}^{\ln\left(\frac{1}{\sqrt{1+\tan(t)^2}}\right)+2t}c_3$$

The constants can be merged to give

$$x = (\tan(t)\,c_2 + c_1)\,\mathrm{e}^{\ln\left(\frac{1}{\sqrt{1+\tan(t)^2}}\right)+2t}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = (\tan(t)\,c_2 + c_1)\,\mathrm{e}^{\ln\left(\frac{1}{\sqrt{1+\tan(t)^2}}\right)+2t}$$

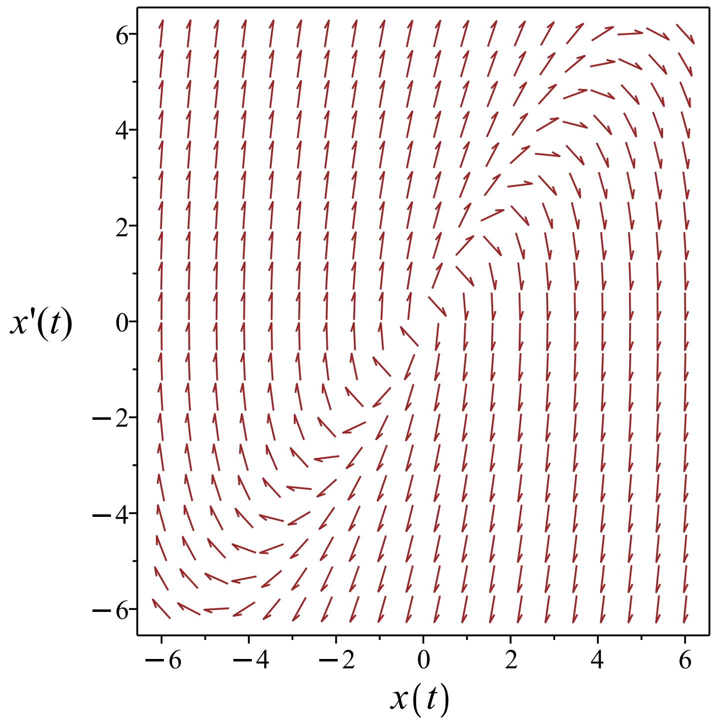


Figure 2.80: Slope field plot
$x'' - 4x' + 5x = 0$

Maple step by step solution

Let's solve
$x'' - 4x' + 5x = 0$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of ODE
  $r^2 - 4r + 5 = 0$
- Use quadratic formula to solve for r
  $r = \frac{4 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (2 - \mathrm{I}, 2 + \mathrm{I})$
- 1st solution of the ODE
  $x_1(t) = \mathrm{e}^{2t} \cos(t)$
- 2nd solution of the ODE
  $x_2(t) = \mathrm{e}^{2t} \sin(t)$
- General solution of the ODE

$$x = C1x_1(t) + C2x_2(t)$$

- Substitute in solutions

$$x = C1\,e^{2t}\cos(t) + C2\,e^{2t}\sin(t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 18

```
dsolve(diff(diff(x(t),t),t)-4*diff(x(t),t)+5*x(t) = 0,
       x(t),singsol=all)
```

$$x = e^{2t}(c_1\sin(t) + c_2\cos(t))$$

Mathematica DSolve solution

Solving time : 0.015 (sec)
Leaf size : 22

```
DSolve[{D[x[t],{t,2}]-4*D[x[t],t]+5*x[t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to e^{2t}(c_2\cos(t) + c_1\sin(t))$$

2.2.7 problem 5 (iv)

Internal problem ID [18196]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 5 (iv)
Date solved : `Thursday, December 19, 2024 at 06:17:54 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$x'' + 3x' = 0$$

Solved as second order linear constant coeff ode

Time used: 0.065 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 3, C = 0$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} + 3\lambda\, e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(0)} \\ &= -\frac{3}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{3}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x &= c_1 e^{(0)t} + c_2 e^{(-3)t}\end{aligned}$$

Or

$$x = c_1 + c_2\,\mathrm{e}^{-3t}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_1 + c_2\,\mathrm{e}^{-3t}$$

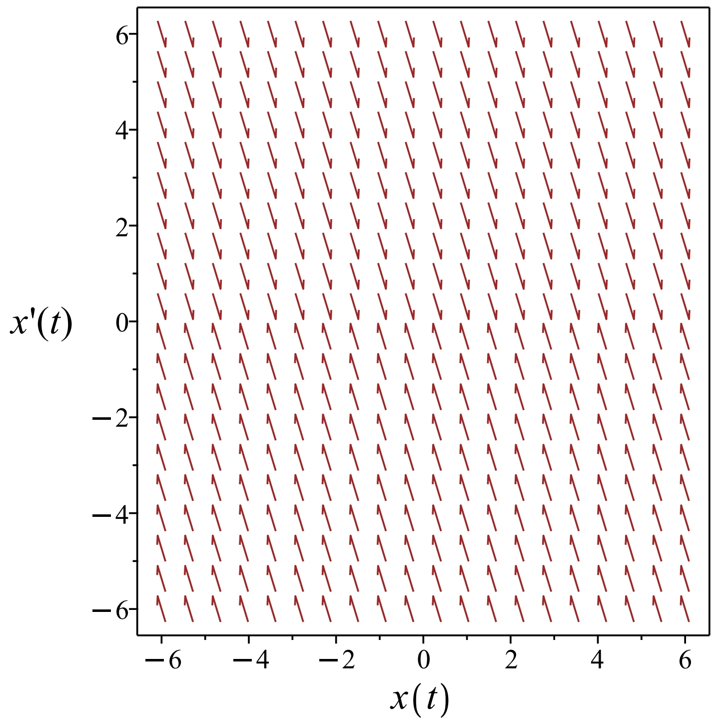


Figure 2.81: Slope field plot
$x'' + 3x' = 0$

Solved as second order linear exact ode

Time used: 0.125 (sec)

An ode of the form

$$p(t)\,x'' + q(t)\,x' + r(t)\,x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 3 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,x'+(q(t)-p'(t))\,x)'=s(x)$$

Integrating gives

$$p(t)\,x'+(q(t)-p'(t))\,x=\int s(t)\;dt$$

Substituting the above values for p,q,r,s gives

$$x'+3x=c_1$$

We now have a first order ode to solve which is

$$x'+3x=c_1$$

Integrating gives

$$\int \frac{1}{-3x+c_1}dx=dt$$
$$-\frac{\ln(-3x+c_1)}{3}=t+c_2$$

Applying the exponential to both sides gives

$$e^{\ln\left(\frac{1}{(-3x+c_1)^{1/3}}\right)}=e^{t+c_2}$$
$$\frac{1}{(-3x+c_1)^{1/3}}=\mathrm{e}^{t}c_2$$

Singular solutions are found by solving

$$-3x+c_1=0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x=\frac{c_1}{3}$$

Solving for x gives

$$x=\frac{c_1}{3}$$
$$x=\frac{(c_1\,\mathrm{e}^{3t}c_2^3-1)\,\mathrm{e}^{-3t}}{3c_2^3}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{c_1}{3}$$

$$x = \frac{(c_1\, e^{3t} c_2^3 - 1)\, e^{-3t}}{3c_2^3}$$

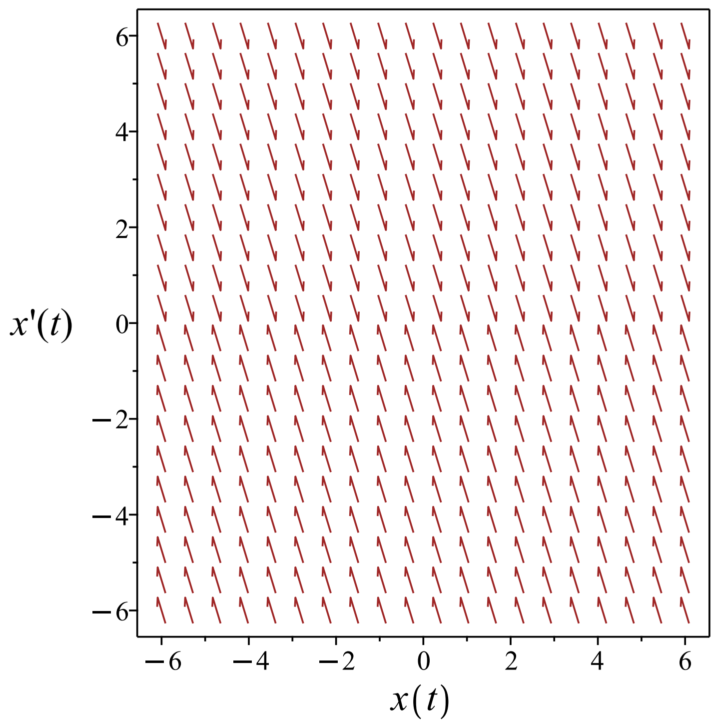


Figure 2.82: Slope field plot
$x'' + 3x' = 0$

Solved as second order missing y ode

Time used: 0.260 (sec)

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) + 3p(t) = 0$$

Which is now solve for $p(t)$ as first order ode. Integrating gives

$$\int -\frac{1}{3p} dp = dt$$
$$-\frac{\ln(p)}{3} = t + c_1$$

Applying the exponential to both sides gives

$$e^{\ln\left(\frac{1}{p^{1/3}}\right)} = e^{t+c_1}$$
$$\frac{1}{p(t)^{1/3}} = c_1 \, \mathrm{e}^{t}$$

Singular solutions are found by solving

$$-3p = 0$$

for $p(t)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(t) = 0$$

Solving for $p(t)$ gives

$$p(t) = 0$$
$$p(t) = c_1 \, \mathrm{e}^{-3t}$$

For solution (1) found earlier, since $p = x'$ then we now have a new first order ode to solve which is

$$x' = 0$$

Since the ode has the form $x' = f(t)$, then we only need to integrate $f(t)$.

$$\int dx = \int 0 \, dt + c_2$$
$$x = c_2$$

For solution (2) found earlier, since $p = x'$ then we now have a new first order ode to solve which is

$$x' = c_1 \, \mathrm{e}^{-3t}$$

Since the ode has the form $x' = f(t)$, then we only need to integrate $f(t)$.

$$\int dx = \int c_1\,\mathrm{e}^{-3t}\,dt$$
$$x = -\frac{c_1\,\mathrm{e}^{-3t}}{3} + c_3$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$x = c_2$$
$$x = -\frac{c_1\,\mathrm{e}^{-3t}}{3} + c_3$$

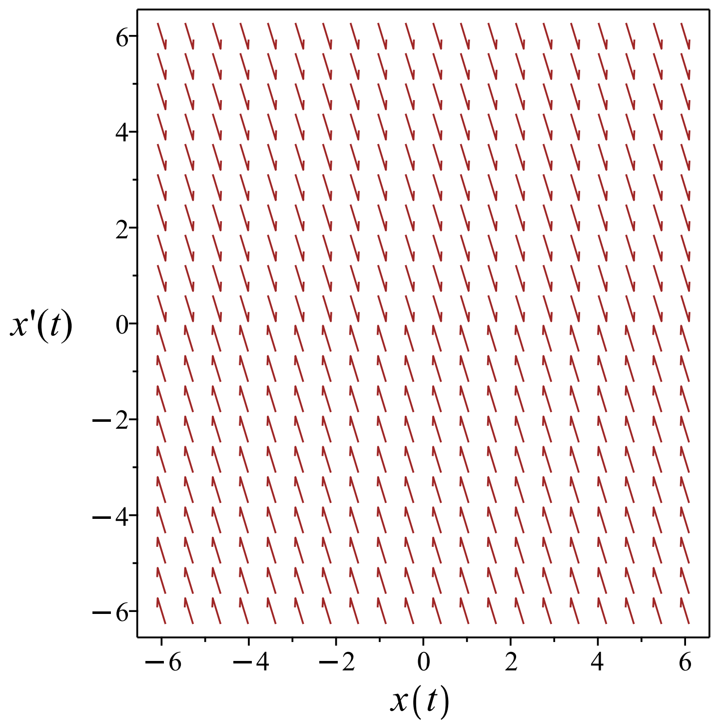


Figure 2.83: Slope field plot
$x'' + 3x' = 0$

Solved as second order integrable as is ode

Time used: 0.061 (sec)

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' + 3x')\, dt = 0$$
$$x' + 3x = c_1$$

Which is now solved for x. Integrating gives

$$\int \frac{1}{-3x + c_1} dx = dt$$
$$-\frac{\ln(-3x + c_1)}{3} = t + c_2$$

Applying the exponential to both sides gives

$$e^{\ln\left(\frac{1}{(-3x+c_1)^{1/3}}\right)} = e^{t+c_2}$$
$$\frac{1}{(-3x + c_1)^{1/3}} = \mathrm{e}^t c_2$$

Singular solutions are found by solving

$$-3x + c_1 = 0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x = \frac{c_1}{3}$$

Solving for x gives

$$x = \frac{c_1}{3}$$
$$x = \frac{(c_1\, \mathrm{e}^{3t} c_2^3 - 1)\, \mathrm{e}^{-3t}}{3c_2^3}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{c_1}{3}$$
$$x = \frac{(c_1\, \mathrm{e}^{3t} c_2^3 - 1)\, \mathrm{e}^{-3t}}{3c_2^3}$$

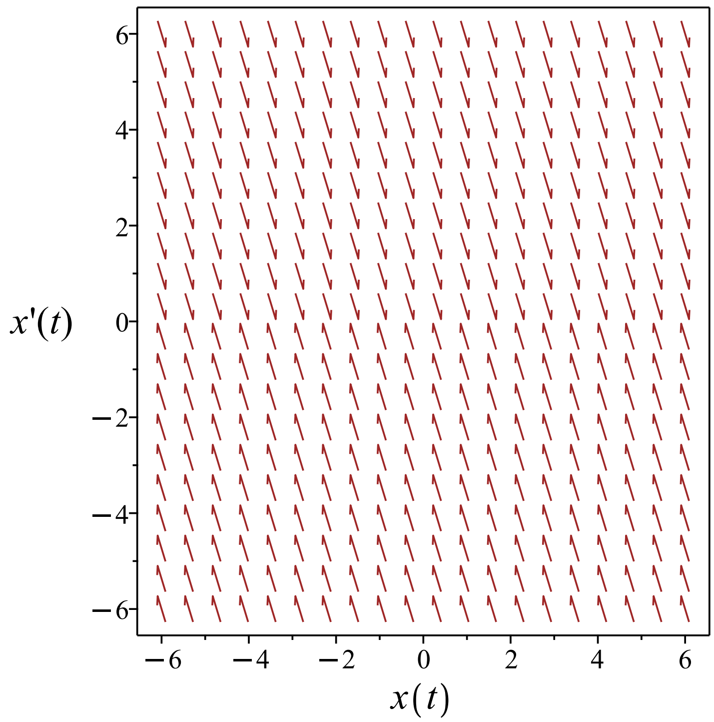


Figure 2.84: Slope field plot
$x'' + 3x' = 0$

Solved as second order integrable as is ode (ABC method)

Time used: 0.056 (sec)

Writing the ode as

$$x'' + 3x' = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' + 3x')\, dt = 0$$
$$x' + 3x = c_1$$

Which is now solved for x. Integrating gives

$$\int \frac{1}{-3x + c_1} dx = dt$$
$$-\frac{\ln(-3x + c_1)}{3} = t + c_2$$

Applying the exponential to both sides gives

$$e^{\ln\left(\frac{1}{(-3x+c_1)^{1/3}}\right)} = e^{t+c_2}$$
$$\frac{1}{(-3x+c_1)^{1/3}} = \mathrm{e}^t c_2$$

Singular solutions are found by solving

$$-3x + c_1 = 0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x = \frac{c_1}{3}$$

Solving for x gives

$$x = \frac{c_1}{3}$$
$$x = \frac{(c_1\,\mathrm{e}^{3t} c_2^3 - 1)\,\mathrm{e}^{-3t}}{3c_2^3}$$

Will add steps showing solving for IC soon.

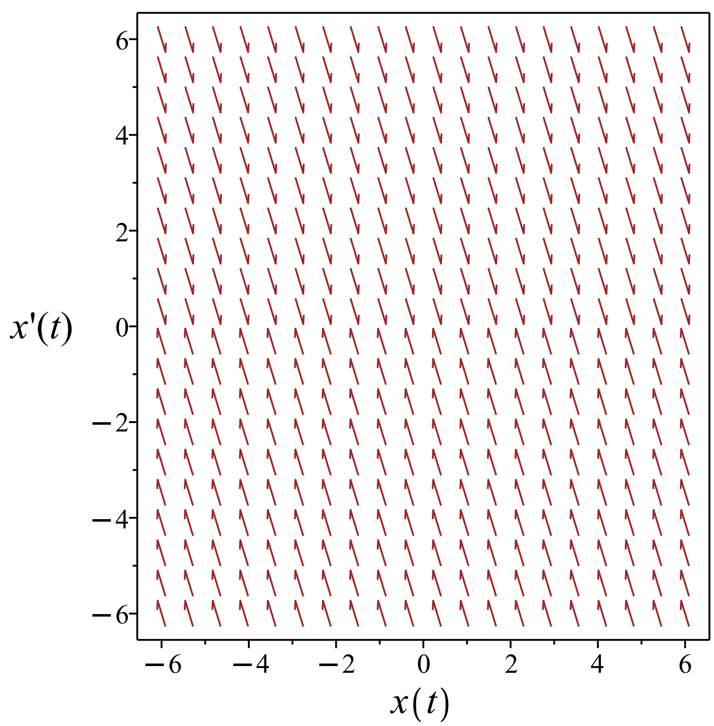


Figure 2.85: Slope field plot
$x'' + 3x' = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.054 (sec)

Writing the ode as

$$x'' + 3x' = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.40: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = e^{-\frac{3t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt} \\
&= z_1 e^{-\frac{3t}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-3t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-3t}}{(x_1)^2}\,dt \\
&= x_1\left(\frac{\mathrm{e}^{3t}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-3t}\right) + c_2\left(\mathrm{e}^{-3t}\left(\frac{\mathrm{e}^{3t}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
x = c_1\,\mathrm{e}^{-3t} + \frac{c_2}{3}
$$

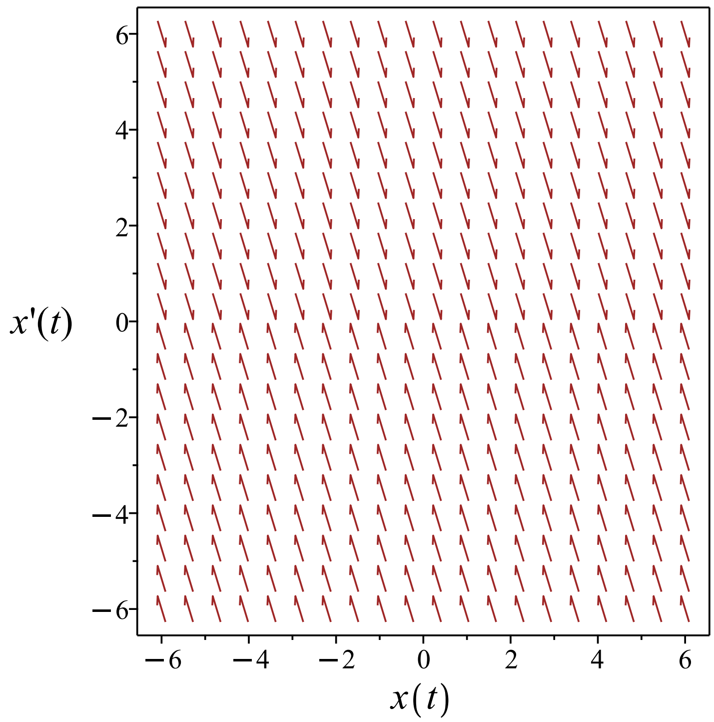


Figure 2.86: Slope field plot
$x'' + 3x' = 0$

Solved as second order ode adjoint method

Time used: 0.267 (sec)

In normal form the ode

$$x'' + 3x' = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = r(t) \tag{2}$$

Where

$$\begin{aligned}p(t) &= 3\\ q(t) &= 0\\ r(t) &= 0\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi'' - (\xi\, p)' + \xi q &= 0\\ \xi'' - (3\xi(t))' + (0) &= 0\\ \xi''(t) - 3\xi'(t) &= 0\end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = -3, C = 0$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} - 3\lambda\, \mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 3\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(0)}\\ &= \frac{3}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{3}{2}\\ \lambda_2 &= \frac{3}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}\xi &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ \xi &= c_1 e^{(3)t} + c_2 e^{(0)t}\end{aligned}$$

Or

$$\xi = c_1\, \mathrm{e}^{3t} + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(t)\,x' - x\xi'(t) + \xi(t)\,p(t)\,x &= \int \xi(t)\,r(t)\,dt\\ x' + x\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\,r(t)\,dt}{\xi\,(t)}\end{aligned}$$

Or

$$x' + x\left(3 - \frac{3c_1\,\mathrm{e}^{3t}}{c_1\,\mathrm{e}^{3t} + c_2}\right) = 0$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(t) &= \frac{3c_2}{c_1\,\mathrm{e}^{3t} + c_2}\\ p(t) &= 0\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt}\\ &= \mathrm{e}^{\int \frac{3c_2}{c_1\,\mathrm{e}^{3t}+c_2}dt}\\ &= \frac{\mathrm{e}^{3t}}{c_1\,\mathrm{e}^{3t} + c_2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x\,\mathrm{e}^{3t}}{c_1\,\mathrm{e}^{3t} + c_2}\right) &= 0\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{x\,\mathrm{e}^{3t}}{c_1\,\mathrm{e}^{3t} + c_2} &= \int 0\,dt + c_3\\ &= c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{e^{3t}}{c_1 e^{3t}+c_2}$ gives the final solution

$$x = c_3\left(c_1 + c_2 e^{-3t}\right)$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = c_3\left(c_1 + c_2 e^{-3t}\right)$$

The constants can be merged to give

$$x = c_1 + c_2 e^{-3t}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = c_1 + c_2 e^{-3t}$$

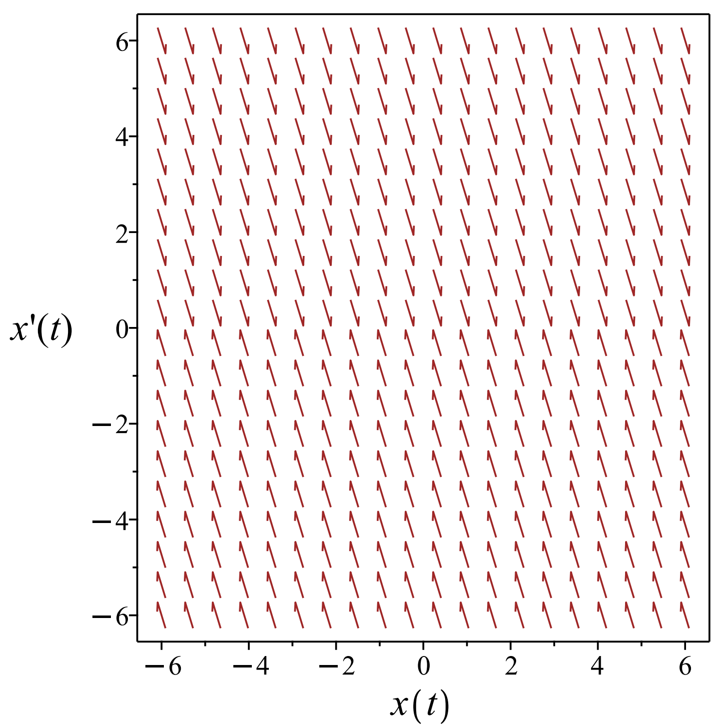


Figure 2.87: Slope field plot
$x'' + 3x' = 0$

Maple step by step solution

Let's solve
$x'' + 3x' = 0$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of ODE
  $r^2 + 3r = 0$
- Factor the characteristic polynomial
  $r(r + 3) = 0$
- Roots of the characteristic polynomial
  $r = (-3, 0)$
- 1st solution of the ODE
  $x_1(t) = e^{-3t}$
- 2nd solution of the ODE
  $x_2(t) = 1$
- General solution of the ODE
  $x = C1 x_1(t) + C2 x_2(t)$
- Substitute in solutions
  $x = C1\, e^{-3t} + C2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 12

```
dsolve(diff(diff(x(t),t),t)+3*diff(x(t),t) = 0,
       x(t),singsol=all)
```

$$x = c_1 + c_2\, e^{-3t}$$

Mathematica DSolve solution

Solving time : 0.015 (sec)
Leaf size : 19

```
DSolve[{D[x[t],{t,2}]+3*D[x[t],t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to c_2 - \frac{1}{3} c_1 e^{-3t}$$

2.2.8 problem 6 (i)

Internal problem ID [18197]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 6 (i)
Date solved : `Thursday, December 19, 2024 at 06:17:55 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$x'' - 3x' + 2x = 0$$

With initial conditions

$$x(0) = 0$$
$$x'(0) = 1$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$p(t) = -3$$
$$q(t) = 2$$
$$F = 0$$

Hence the ode is

$$x'' - 3x' + 2x = 0$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.111 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} - 3\lambda\,\mathrm{e}^{t\lambda} + 2\,\mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$

$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{(2)t} + c_2 e^{(1)t}$$

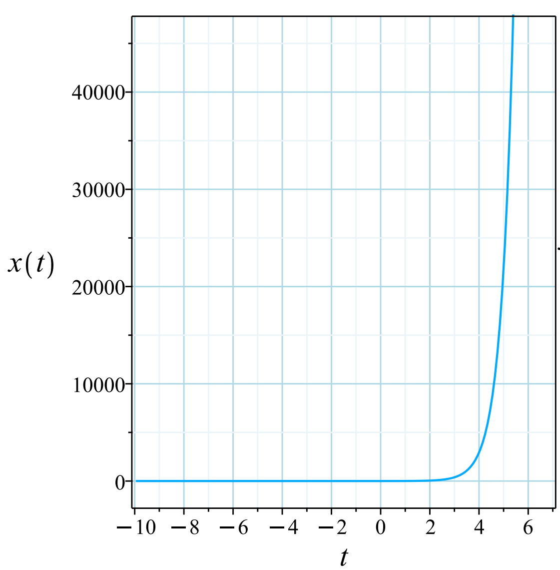
Or

$$x = c_1 \, \mathrm{e}^{2t} + \mathrm{e}^{t} c_2$$

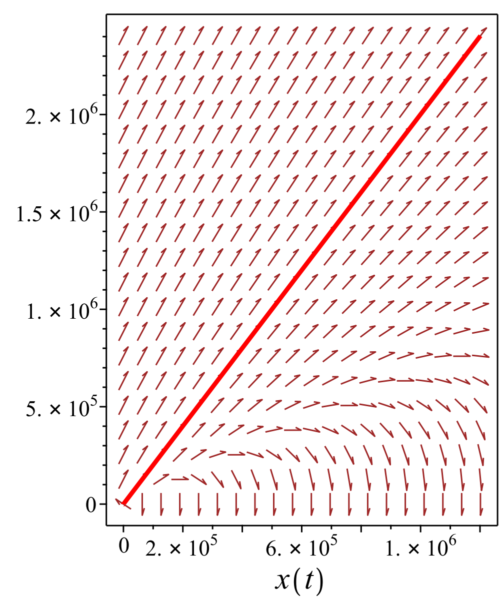
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \mathrm{e}^{2t} - \mathrm{e}^{t}$$



(a) Solution plot
$x = \mathrm{e}^{2t} - \mathrm{e}^{t}$



(b) Slope field plot
$x'' - 3x' + 2x = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.062 (sec)

Writing the ode as

$$x'' - 3x' + 2x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.42: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dt} \\ &= z_1 e^{\frac{3t}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{3t}{2}}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^t$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{-3}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{3t}}{(x_1)^2}\,dt \\ &= x_1\left(\mathrm{e}^{3t}\mathrm{e}^{-2t}\right) \end{aligned}$$

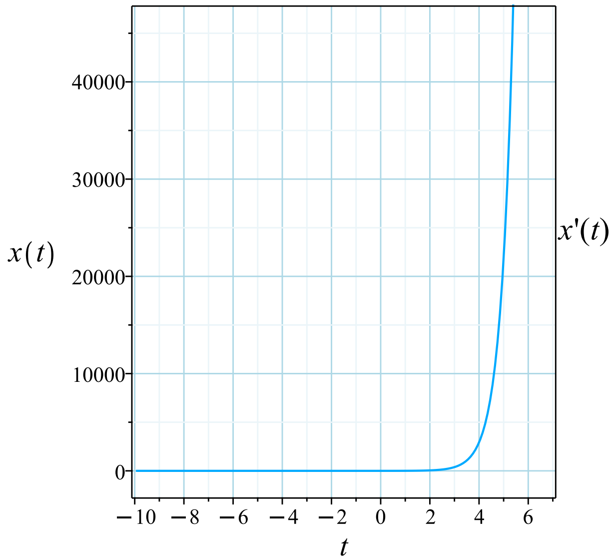
Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(\mathrm{e}^{3t}\mathrm{e}^{-2t}\right)\right) \end{aligned}$$

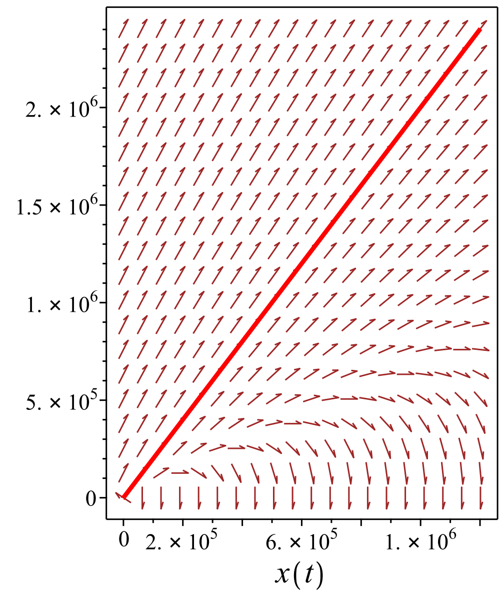
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \mathrm{e}^{2t} - \mathrm{e}^t$$



(a) Solution plot
$x = e^{2t} - e^{t}$



(b) Slope field plot
$x'' - 3x' + 2x = 0$

Solved as second order ode adjoint method

Time used: 0.311 (sec)

In normal form the ode

$$x'' - 3x' + 2x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= -3 \\ q(t) &= 2 \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (-3\xi(t))' + (2\xi(t)) &= 0 \\ \xi''(t) + 3\xi'(t) + 2\xi(t) &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} + 3\lambda\, e^{t\lambda} + 2\, e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}\xi &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ \xi &= c_1 e^{(-1)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$\xi = c_1\, e^{-t} + c_2\, e^{-2t}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(t)\,x' - x\xi'(t) + \xi(t)\,p(t)\,x = \int \xi(t)\,r(t)\,dt$$
$$x' + x\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) = \frac{\int \xi(t)\,r(t)\,dt}{\xi\,(t)}$$

Or

$$x' + x\left(-3 - \frac{-c_1\,\mathrm{e}^{-t} - 2c_2\,\mathrm{e}^{-2t}}{c_1\,\mathrm{e}^{-t} + c_2\,\mathrm{e}^{-2t}}\right) = 0$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = -\frac{c_2\,\mathrm{e}^{-t} + 2c_1}{c_2\,\mathrm{e}^{-t} + c_1}$$
$$p(t) = 0$$

The integrating factor μ is

$$\mu = e^{\int q\,dt}$$
$$= \mathrm{e}^{\int -\frac{c_2\,\mathrm{e}^{-t}+2c_1}{c_2\,\mathrm{e}^{-t}+c_1}dt}$$
$$= \frac{\mathrm{e}^{-2t}}{c_2\,\mathrm{e}^{-t} + c_1}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x\,\mathrm{e}^{-2t}}{c_2\,\mathrm{e}^{-t} + c_1}\right) = 0$$

Integrating gives

$$\frac{x\,\mathrm{e}^{-2t}}{c_2\,\mathrm{e}^{-t} + c_1} = \int 0\,dt + c_3$$
$$= c_3$$

Dividing throughout by the integrating factor $\frac{e^{-2t}}{c_2 e^{-t} + c_1}$ gives the final solution

$$x = e^t (c_1 e^t + c_2) c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = e^t (c_1 e^t + c_2) c_3$$

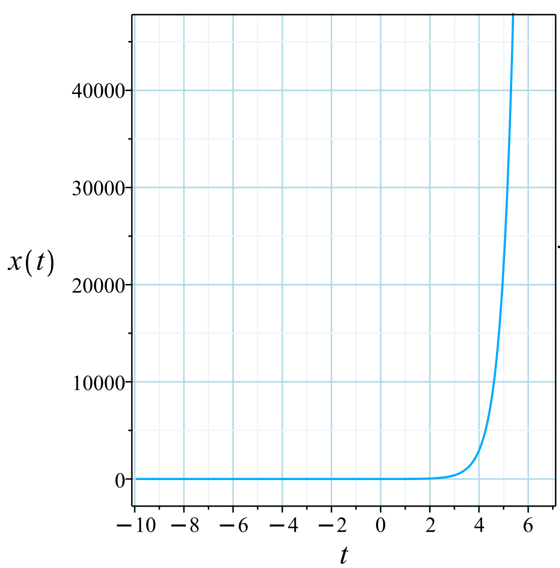
The constants can be merged to give

$$x = e^t (c_1 e^t + c_2)$$

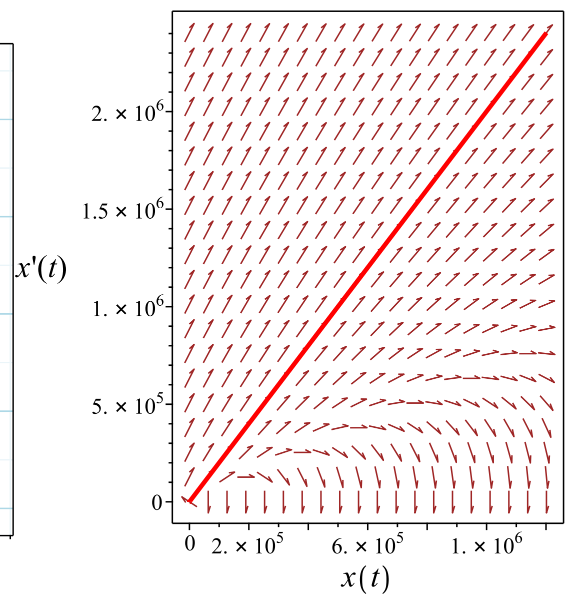
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = e^t (e^t - 1)$$



(a) Solution plot
$x = e^t (e^t - 1)$



(b) Slope field plot
$x'' - 3x' + 2x = 0$

Maple step by step solution

Let's solve

$\left[x'' - 3x' + 2x = 0, x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of ODE
  $r^2 - 3r + 2 = 0$
- Factor the characteristic polynomial
  $(r-1)(r-2) = 0$
- Roots of the characteristic polynomial
  $r = (1, 2)$
- 1st solution of the ODE
  $x_1(t) = e^t$
- 2nd solution of the ODE
  $x_2(t) = e^{2t}$
- General solution of the ODE
  $x = C1 x_1(t) + C2 x_2(t)$
- Substitute in solutions
  $x = C1\, e^t + C2\, e^{2t}$

☐ Check validity of solution $x = \_C1 e^t + \_C2 e^{2t}$

  - Use initial condition $x(0) = 0$
    $0 = \_C1 + \_C2$
  - Compute derivative of the solution
    $x' = \_C1\, e^t + 2\_C2\, e^{2t}$
  - Use the initial condition $x'\Big|_{\{t=0\}} = 1$
    $1 = \_C1 + 2\_C2$
  - Solve for $\_C1$ and $\_C2$
    $\{\_C1 = -1, \_C2 = 1\}$
  - Substitute constant values into general solution and simplify
    $x = e^{2t} - e^t$

- Solution to the IVP
  $x = e^{2t} - e^t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 13

```
dsolve([diff(diff(x(t),t),t)-3*diff(x(t),t)+2*x(t) = 0,
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = e^{2t} - e^{t}$$

Mathematica DSolve solution

Solving time : 0.018 (sec)
Leaf size : 14

```
DSolve[{D[x[t],{t,2}]-3*D[x[t],t]+2*x[t]==0,{x[0]==0,Derivative[1][x][0] == 1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to e^{t}\left(e^{t} - 1\right)$$

2.2.9 problem 6 (ii)

Internal problem ID [18198]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 6 (ii)
Date solved : `Thursday, December 19, 2024 at 06:17:56 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$x'' + x = 0$$

With initial conditions

$$x(0) = 0$$
$$x'(0) = 1$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$p(t) = 0$$
$$q(t) = 1$$
$$F = 0$$

Hence the ode is

$$x'' + x = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.097 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} + \mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(t) + c_2 \sin(t))$$

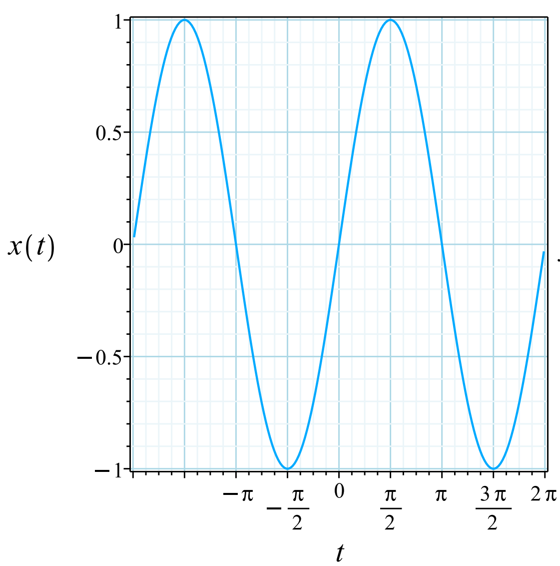
Or

$$x = c_1 \cos(t) + c_2 \sin(t)$$

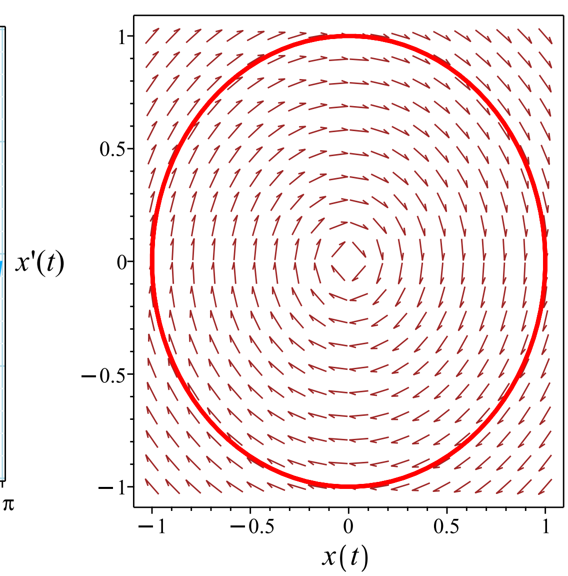
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \sin(t)$$



(a) Solution plot
$x = \sin(t)$



(b) Slope field plot
$x'' + x = 0$

Solved as second order can be made integrable

Time used: 0.879 (sec)

Multiplying the ode by x' gives

$$x'x'' + x'x = 0$$

Integrating the above w.r.t t gives

$$\int (x'x'' + x'x)\, dt = 0$$
$$\frac{x'^2}{2} + \frac{x^2}{2} = c_1$$

Which is now solved for x. Solving for the derivative gives these ODE's to solve

$$x' = \sqrt{-x^2 + 2c_1} \tag{1}$$
$$x' = -\sqrt{-x^2 + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{-x^2 + 2c_1}} dx = dt$$
$$\arctan\left(\frac{x}{\sqrt{-x^2 + 2c_1}}\right) = t + c_2$$

Singular solutions are found by solving

$$\sqrt{-x^2 + 2c_1} = 0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x = \sqrt{2}\,\sqrt{c_1}$$
$$x = -\sqrt{2}\,\sqrt{c_1}$$

Solving for x gives

$$x = \sqrt{2}\,\sqrt{c_1}$$
$$x = \tan(t + c_2)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(t + c_2)^2 + 1}}$$
$$x = -\sqrt{2}\,\sqrt{c_1}$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{-x^2+2c_1}}dx = dt$$
$$-\arctan\left(\frac{x}{\sqrt{-x^2+2c_1}}\right) = t+c_3$$

Singular solutions are found by solving

$$-\sqrt{-x^2+2c_1} = 0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x = \sqrt{2}\,\sqrt{c_1}$$
$$x = -\sqrt{2}\,\sqrt{c_1}$$

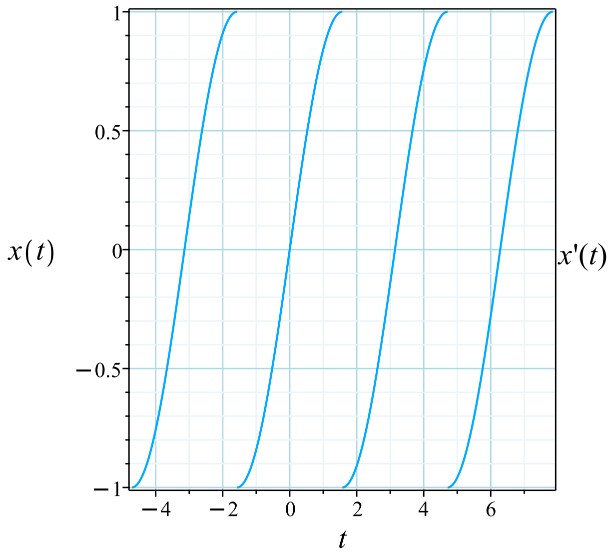
Solving for x gives

$$x = \sqrt{2}\,\sqrt{c_1}$$
$$x = -\sqrt{2}\,\sqrt{c_1}$$
$$x = -\tan(t+c_3)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(t+c_3)^2+1}}$$

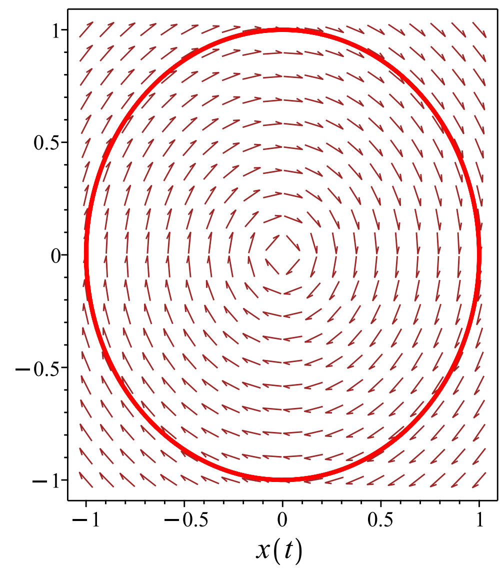
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \tan(t)\sqrt{\frac{1}{1+\tan(t)^2}}$$



(a) Solution plot
$x = \tan(t)\sqrt{\frac{1}{1+\tan(t)^2}}$



(b) Slope field plot
$x'' + x = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.077 (sec)

Writing the ode as

$$x'' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \cos(t) \end{aligned}$$

Which simplifies to

$$x_1 = \cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\ &= \cos(t) \int \frac{1}{\cos(t)^2}\,dt \\ &= \cos(t)\,(\tan(t)) \end{aligned}$$

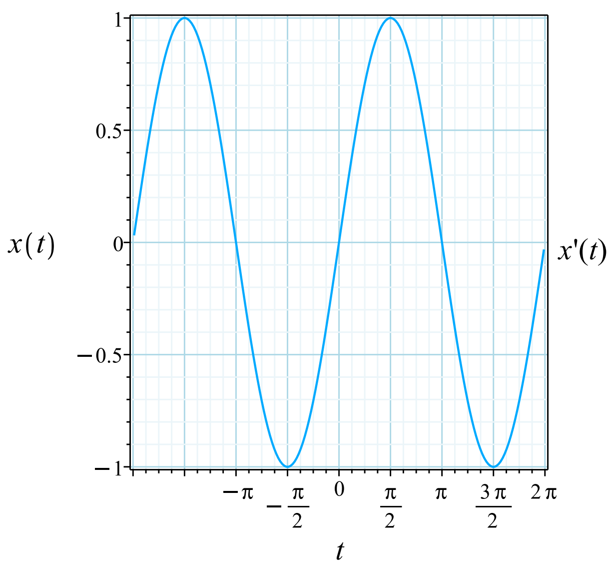
Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(\cos(t)) + c_2(\cos(t)(\tan(t)))
\end{aligned}
$$

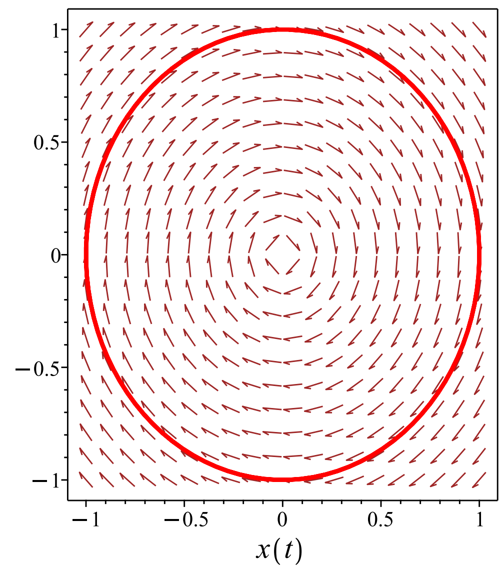
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \sin(t)$$



(a) Solution plot
$x = \sin(t)$



(b) Slope field plot
$x'' + x = 0$

Solved as second order ode adjoint method

Time used: 0.393 (sec)

In normal form the ode

$$x'' + x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = r(t) \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= 0 \\
q(t) &= 1 \\
r(t) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi'' - (\xi\, p)' + \xi q &= 0\\ \xi'' - (0)' + (\xi(t)) &= 0\\ \xi''(t) + \xi(t) &= 0\end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} + \mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$\xi = e^{0}(c_1 \cos(t) + c_2 \sin(t))$$

Or

$$\xi = c_1 \cos(t) + c_2 \sin(t)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(t)\, x' - x\xi'(t) + \xi(t)\, p(t)\, x &= \int \xi(t)\, r(t)\, dt \\ x' + x\left(p(t) - \frac{\xi'(t)}{\xi(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi(t)} \end{aligned}$$

Or

$$x' - \frac{x(-c_1 \sin(t) + c_2 \cos(t))}{c_1 \cos(t) + c_2 \sin(t)} = 0$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= -\frac{-c_1 \sin(t) + c_2 \cos(t)}{c_1 \cos(t) + c_2 \sin(t)} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dt} \\ &= e^{\int -\frac{-c_1 \sin(t) + c_2 \cos(t)}{c_1 \cos(t) + c_2 \sin(t)} dt} \\ &= \frac{1}{c_1 \cos(t) + c_2 \sin(t)} \end{aligned}$$

The ode becomes

$$\frac{d}{dt}\mu x = 0$$
$$\frac{d}{dt}\left(\frac{x}{c_1\cos(t) + c_2\sin(t)}\right) = 0$$

Integrating gives

$$\frac{x}{c_1\cos(t) + c_2\sin(t)} = \int 0\,dt + c_3$$
$$= c_3$$

Dividing throughout by the integrating factor $\frac{1}{c_1\cos(t)+c_2\sin(t)}$ gives the final solution

$$x = (c_1\cos(t) + c_2\sin(t))\,c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = (c_1\cos(t) + c_2\sin(t))\,c_3$$

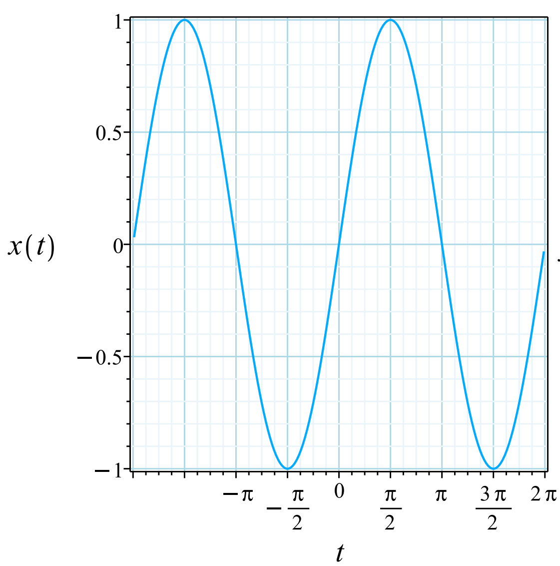
The constants can be merged to give

$$x = c_1\cos(t) + c_2\sin(t)$$

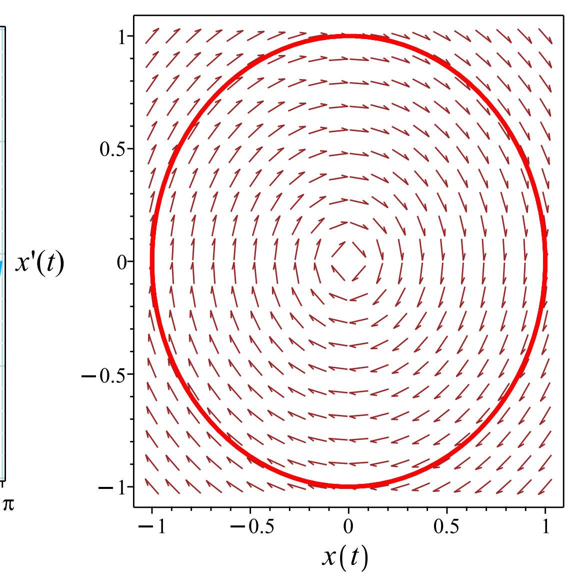
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \sin(t)$$



(a) Solution plot
$x = \sin(t)$



(b) Slope field plot
$x'' + x = 0$

Maple step by step solution

Let's solve

$$\left[x''+x=0, x(0)=0, x'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2+1=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $x_1(t)=\cos(t)$

- 2nd solution of the ODE

  $x_2(t)=\sin(t)$

- General solution of the ODE

  $x=C1x_1(t)+C2x_2(t)$

- Substitute in solutions

  $x=C1\cos(t)+C2\sin(t)$

☐ Check validity of solution $x=\_C1\cos(t)+\_C2\sin(t)$

  ◦ Use initial condition $x(0)=0$

    $0=\_C1$

  ◦ Compute derivative of the solution

    $x'=-\_C1\sin(t)+\_C2\cos(t)$

  ◦ Use the initial condition $x'\Big|_{\{t=0\}}=1$

    $1=\_C2$

  ◦ Solve for $\_C1$ and $\_C2$

    $\{\_C1=0, \_C2=1\}$

  ◦ Substitute constant values into general solution and simplify

    $x=\sin(t)$

- Solution to the IVP

  $x=\sin(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 6

```
dsolve([diff(diff(x(t),t),t)+x(t) = 0,
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = \sin(t)$$

Mathematica DSolve solution

Solving time : 0.01 (sec)
Leaf size : 7

```
DSolve[{D[x[t],{t,2}]+x[t]==0,{x[0]==0,Derivative[1][x][0] == 1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \sin(t)$$

2.2.10 problem 6 (iii)

Internal problem ID [18199]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 6 (iii)
Date solved : `Thursday, December 19, 2024 at 06:17:59 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$x'' + 2x' + x = 0$$

With initial conditions

$$\begin{aligned} x(0) &= 0 \\ x'(0) &= 1 \end{aligned}$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 2x' + x = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.098 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} + 2\lambda\, e^{t\lambda} + e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)}\\ &= -1\end{aligned}$$

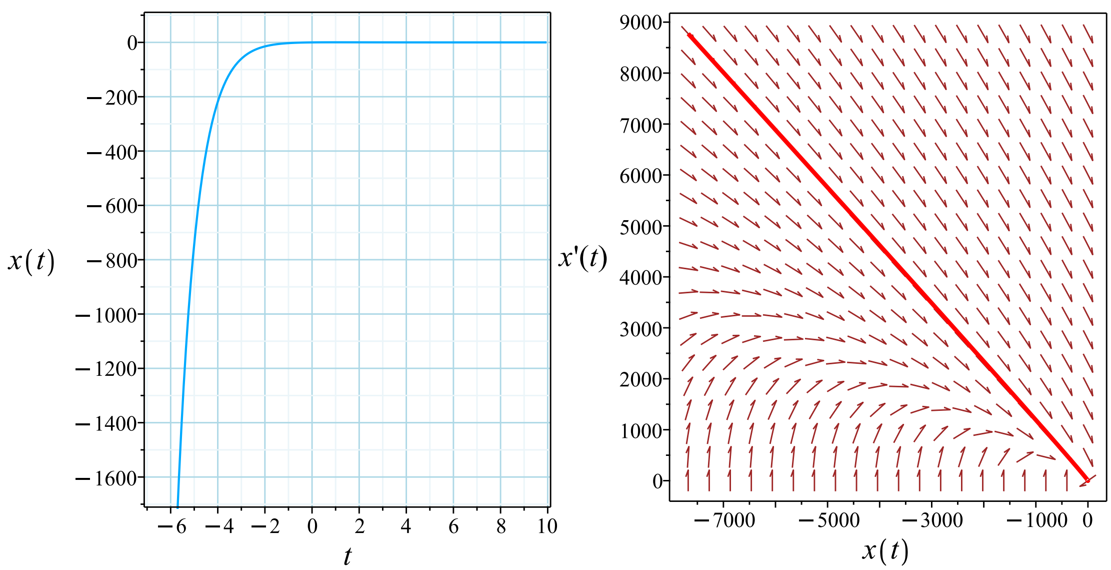
Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$x = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = t\,\mathrm{e}^{-t}$$



(a) Solution plot
$x = t\,\mathrm{e}^{-t}$

(b) Slope field plot
$x'' + 2x' + x = 0$

Solved as second order solved by an integrating factor

Time used: 0.197 (sec)

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= \mathrm{e}^{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(\mathrm{e}^{t}x\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(e^{t} x\right)' = c_1$$

Integrating again gives

$$\left(e^{t} x\right) = c_1 t + c_2$$

Hence the solution is

$$x = \frac{c_1 t + c_2}{e^{t}}$$

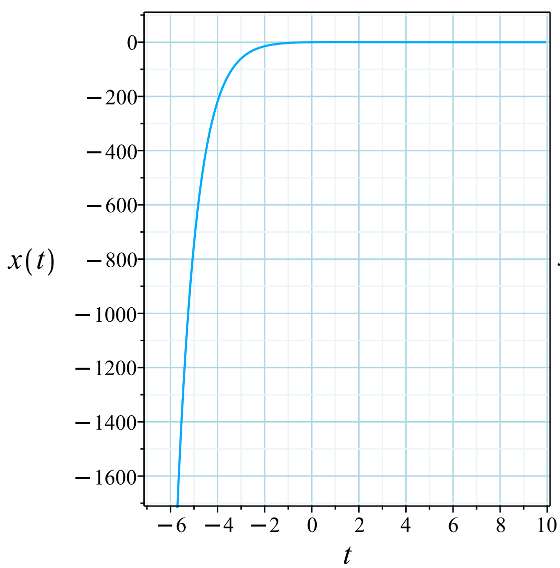
Or

$$x = c_1 t\, e^{-t} + c_2\, e^{-t}$$

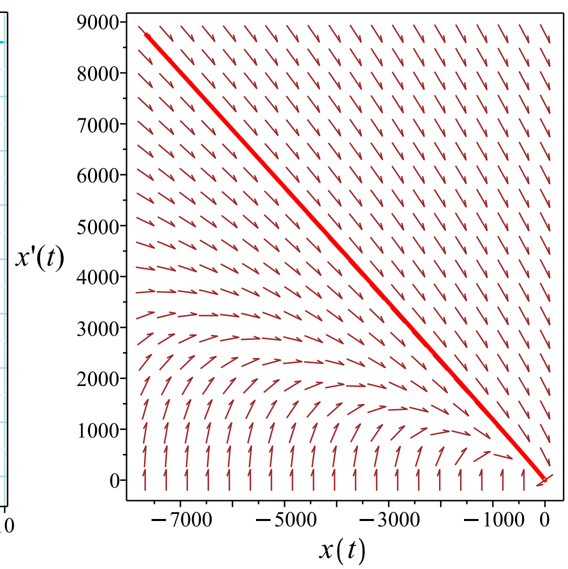
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = t\, e^{-t}$$



(a) Solution plot
$x = t\, e^{-t}$



(b) Slope field plot
$x'' + 2x' + x = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.049 (sec)

Writing the ode as

$$x'' + 2x' + x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\
&= z_1 e^{-t} \\
&= z_1\left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-2t}}{(x_1)^2}\,dt \\ &= x_1(t) \end{aligned}$$

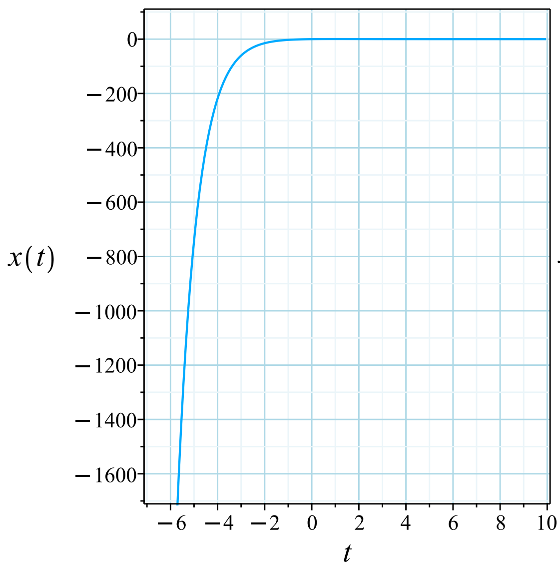
Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(\mathrm{e}^{-t}\right) + c_2\left(\mathrm{e}^{-t}(t)\right) \end{aligned}$$

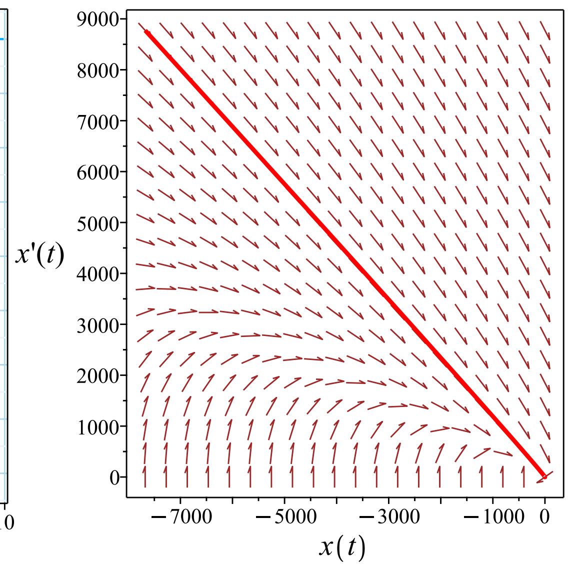
Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$x = t\,\mathrm{e}^{-t}$$



(a) Solution plot
$x = t\,\mathrm{e}^{-t}$



(b) Slope field plot
$x'' + 2x' + x = 0$

Solved as second order ode adjoint method

Time used: 0.143 (sec)

In normal form the ode

$$x'' + 2x' + x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = r(t) \tag{2}$$

Where

$$\begin{aligned}p(t) &= 2\\ q(t) &= 1\\ r(t) &= 0\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi'' - (\xi\,p)' + \xi q &= 0\\ \xi'' - (2\xi(t))' + (\xi(t)) &= 0\\ \xi''(t) - 2\xi'(t) + \xi(t) &= 0\end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} - 2\lambda\, \mathrm{e}^{t\lambda} + \mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2)^2 - (4)\,(1)\,(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$\xi = c_1 \mathrm{e}^t + c_2 t\, \mathrm{e}^t \tag{1}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(t)\, x' - x\xi'(t) + \xi(t)\, p(t)\, x &= \int \xi(t)\, r(t)\, dt\\ x' + x\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi\,(t)}\end{aligned}$$

Or

$$x' + x\left(2 - \frac{c_1\, \mathrm{e}^t + \mathrm{e}^t c_2 + c_2 t\, \mathrm{e}^t}{c_1\, \mathrm{e}^t + c_2 t\, \mathrm{e}^t}\right) = 0$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(t) &= -\frac{-c_2 t - c_1 + c_2}{c_2 t + c_1} \\
p(t) &= 0
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q \, dt} \\
&= \mathrm{e}^{\int -\frac{-c_2 t - c_1 + c_2}{c_2 t + c_1} dt} \\
&= \frac{\mathrm{e}^t}{c_2 t + c_1}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mu x &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x\,\mathrm{e}^t}{c_2 t + c_1}\right) &= 0
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{x\,\mathrm{e}^t}{c_2 t + c_1} &= \int 0 \, dt + c_3 \\
&= c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^t}{c_2 t + c_1}$ gives the final solution

$$x = (c_2 t + c_1)\,\mathrm{e}^{-t} c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = (c_2 t + c_1)\,\mathrm{e}^{-t} c_3$$

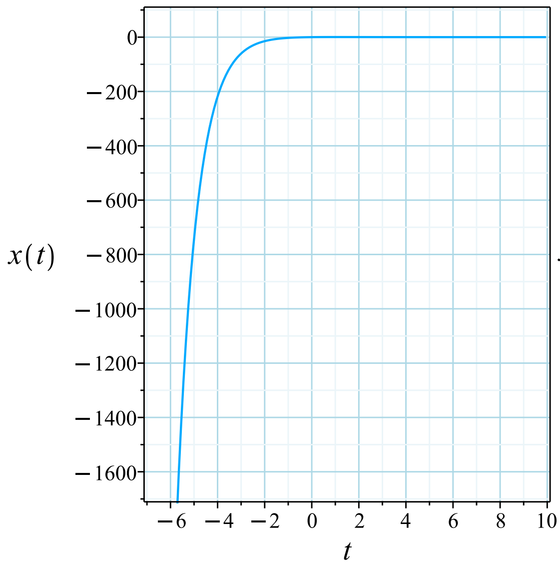
The constants can be merged to give

$$x = (c_2 t + c_1)\,\mathrm{e}^{-t}$$

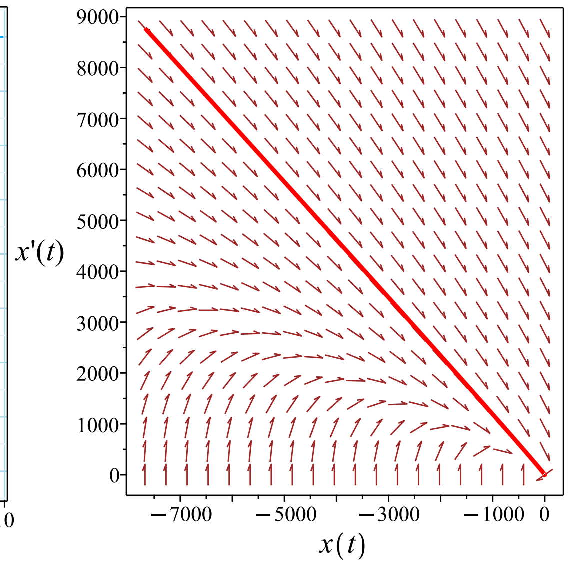
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = t\,\mathrm{e}^{-t}$$



(a) Solution plot
$x = t\,\mathrm{e}^{-t}$



(b) Slope field plot
$x'' + 2x' + x = 0$

Maple step by step solution

Let's solve

$$\left[x'' + 2x' + x = 0, x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of ODE
  $r^2 + 2r + 1 = 0$
- Factor the characteristic polynomial
  $(r+1)^2 = 0$
- Root of the characteristic polynomial
  $r = -1$
- 1st solution of the ODE
  $x_1(t) = \mathrm{e}^{-t}$
- Repeated root, multiply $x_1(t)$ by t to ensure linear independence
  $x_2(t) = t\,\mathrm{e}^{-t}$
- General solution of the ODE
  $x = C1 x_1(t) + C2 x_2(t)$
- Substitute in solutions
  $x = C1\,\mathrm{e}^{-t} + C2 t\,\mathrm{e}^{-t}$

- ☐ Check validity of solution $x = \_C1 e^{-t} + \_C2 t e^{-t}$
  - ∘ Use initial condition $x(0) = 0$

    $0 = \_C1$
  - ∘ Compute derivative of the solution

    $x' = -\_C1\, e^{-t} + \_C2\, e^{-t} - \_C2 t\, e^{-t}$
  - ∘ Use the initial condition $x'\Big|_{\{t=0\}} = 1$

    $1 = -\_C1 + \_C2$
  - ∘ Solve for $\_C1$ and $\_C2$

    $\{\_C1 = 0, \_C2 = 1\}$
  - ∘ Substitute constant values into general solution and simplify

    $x = t\, e^{-t}$
- • Solution to the IVP

  $x = t\, e^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 10

```
dsolve([diff(diff(x(t),t),t)+2*diff(x(t),t)+x(t) = 0,
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = t\, e^{-t}$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 12

```
DSolve[{D[x[t],{t,2}]+2*D[x[t],t]+x[t]==0,{x[0]==0,Derivative[1][x][0] == 1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to e^{-t}t$$

2.2.11 problem 6 (iv)

Internal problem ID [18200]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 4. Autonomous systems. Exercises at page 69
Problem number : 6 (iv)
Date solved : `Thursday, December 19, 2024 at 06:18:00 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$x'' - 2x' + 2x = 0$$

With initial conditions

$$x(0) = 0$$
$$x'(0) = 1$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$p(t) = -2$$
$$q(t) = 2$$
$$F = 0$$

Hence the ode is

$$x'' - 2x' + 2x = 0$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.116 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} - 2\lambda\, e^{t\lambda} + 2\, e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(2)} \\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

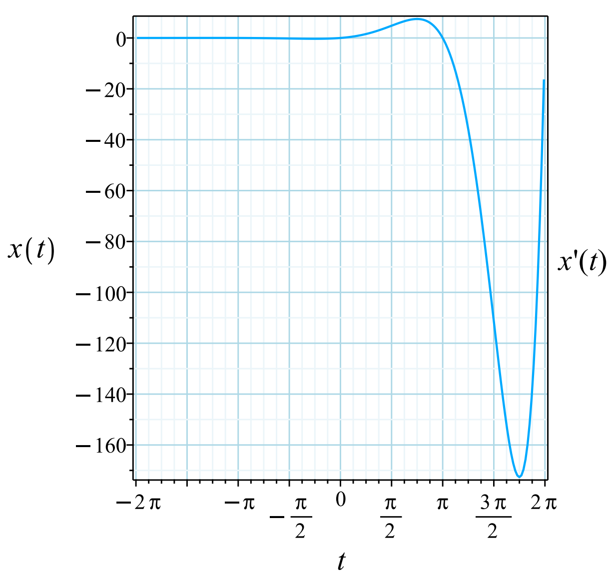
Which becomes

$$x = e^{t}(c_1 \cos(t) + c_2 \sin(t))$$

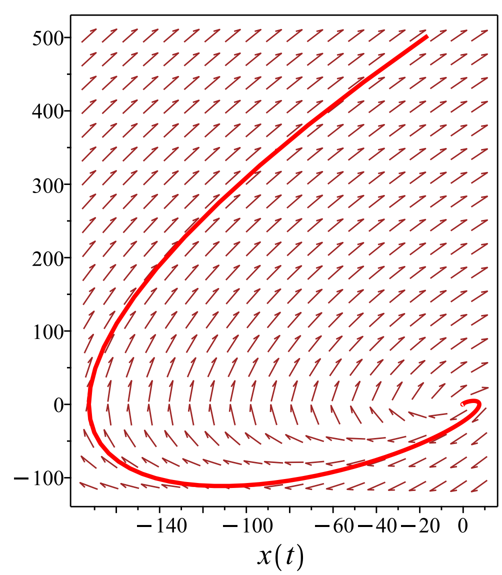
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = e^{t} \sin(t)$$



(a) Solution plot
$x = e^{t} \sin(t)$



(b) Slope field plot
$x'' - 2x' + 2x = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.067 (sec)

Writing the ode as

$$x'' - 2x' + 2x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.48: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dt} \\ &= z_1 e^{t} \\ &= z_1\left(e^{t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = e^t \cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} dt}}{x_1^2} dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{-2}{1} dt}}{(x_1)^2} dt \\ &= x_1 \int \frac{e^{2t}}{(x_1)^2} dt \\ &= x_1(\tan(t)) \end{aligned}$$

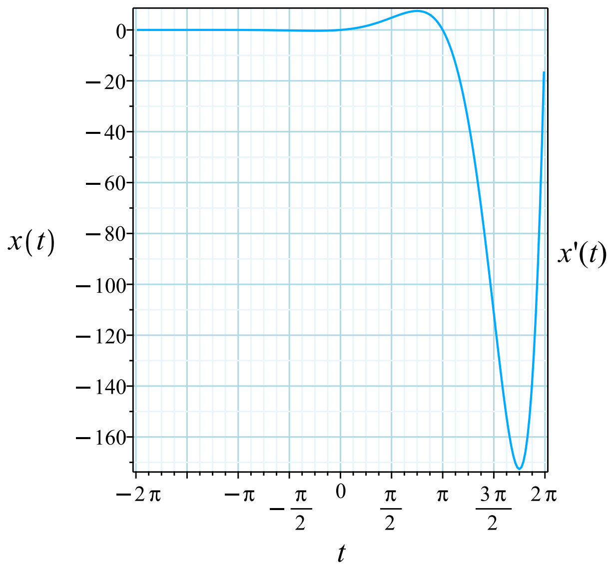
Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(e^t \cos(t)\right) + c_2 \left(e^t \cos(t) (\tan(t))\right) \end{aligned}$$

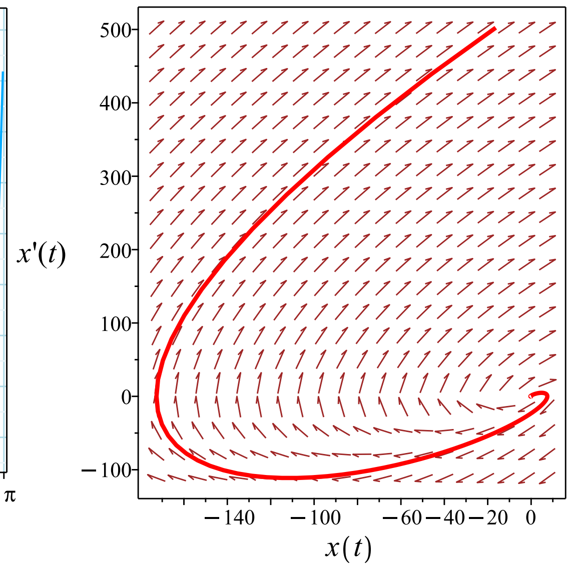
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = e^t \sin(t)$$



(a) Solution plot
$x = e^t \sin(t)$



(b) Slope field plot
$x'' - 2x' + 2x = 0$

Maple step by step solution

Let's solve

$$\left[x'' - 2x' + 2x = 0, x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of ODE
  $r^2 - 2r + 2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{2 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (1 - I, 1 + I)$
- 1st solution of the ODE
  $x_1(t) = e^t \cos(t)$
- 2nd solution of the ODE
  $x_2(t) = e^t \sin(t)$
- General solution of the ODE
  $x = C1 x_1(t) + C2 x_2(t)$
- Substitute in solutions

$$x = C1\,e^t \cos(t) + C2\,e^t \sin(t)$$

- □ Check validity of solution $x = \_C1 e^t \cos(t) + \_C2 e^t \sin(t)$
  - ∘ Use initial condition $x(0) = 0$
    $$0 = \_C1$$
  - ∘ Compute derivative of the solution
    $$x' = \_C1\,e^t \cos(t) - \_C1\,e^t \sin(t) + \_C2\,e^t \sin(t) + \_C2\,e^t \cos(t)$$
  - ∘ Use the initial condition $x'\Big|_{\{t=0\}} = 1$
    $$1 = \_C1 + \_C2$$
  - ∘ Solve for $\_C1$ and $\_C2$
    $$\{\_C1 = 0, \_C2 = 1\}$$
  - ∘ Substitute constant values into general solution and simplify
    $$x = e^t \sin(t)$$
- • Solution to the IVP
  $$x = e^t \sin(t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 9

```
dsolve([diff(diff(x(t),t),t)-2*diff(x(t),t)+2*x(t) = 0,
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = e^t \sin(t)$$

Mathematica DSolve solution

Solving time : 0.012 (sec)
Leaf size : 11

```
DSolve[{D[x[t],{t,2}]-2*D[x[t],t]+2*x[t]==0,{x[0]==0,Derivative[1][x][0] == 1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to e^t \sin(t)$$

2.3 Chapter 5. Linear equations. Exercises at page 85

2.3.1 problem 7 (i)

Internal problem ID [18201]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 5. Linear equations. Exercises at page 85
Problem number : 7 (i)
Date solved : `Thursday, December 19, 2024 at 06:18:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x'' - x = t^2$$

With initial conditions

$$x(0) = 0$$
$$x'(0) = 1$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$p(t) = 0$$
$$q(t) = -1$$
$$F = t^2$$

Hence the ode is

$$x'' - x = t^2$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.288 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = -1, f(t) = t^2$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} - e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{(1)t} + c_2 e^{(-1)t}\end{aligned}$$

Or

$$x = c_1 \, \mathrm{e}^{t} + c_2 \, \mathrm{e}^{-t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \, \mathrm{e}^{t} + c_2 \, \mathrm{e}^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^t, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_3t^2 + A_2t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3t^2 - A_2t - A_1 + 2A_3 = t^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -t^2 - 2$$

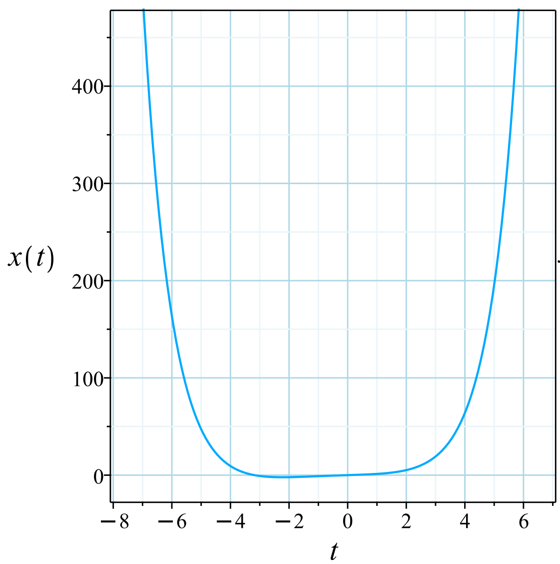
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^t + c_2 e^{-t}\right) + \left(-t^2 - 2\right) \end{aligned}$$

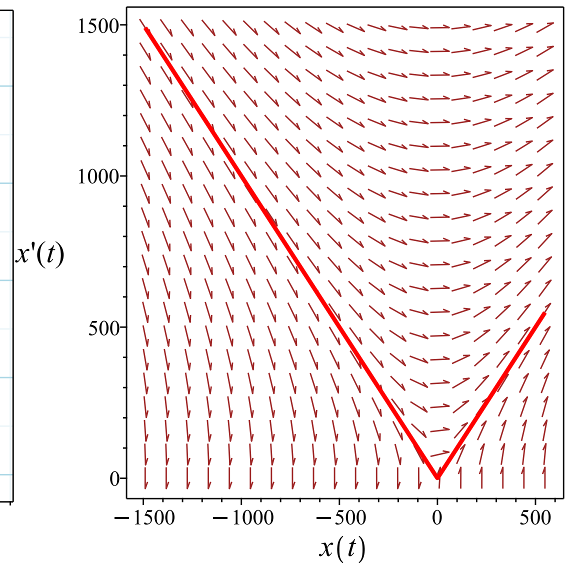
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = -t^2 - 2 + \frac{3 e^t}{2} + \frac{e^{-t}}{2}$$



(a) Solution plot
$x = -t^2 - 2 + \frac{3\,\mathrm{e}^{t}}{2} + \frac{\mathrm{e}^{-t}}{2}$



(b) Slope field plot
$x'' - x = t^2$

Solved as second order ode using Kovacic algorithm

Time used: 0.085 (sec)

Writing the ode as

$$x'' - x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \mathrm{e}^{-t} \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\ &= \mathrm{e}^{-t} \int \frac{1}{\mathrm{e}^{-2t}}\,dt \\ &= \mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \,\mathrm{e}^{-t} + \frac{\mathrm{e}^t c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
t^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, t, t^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^t}{2}, \mathrm{e}^{-t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_3 t^2 + A_2 t + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-A_3 t^2 - A_2 t - A_1 + 2A_3 = t^2
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -t^2 - 2$$

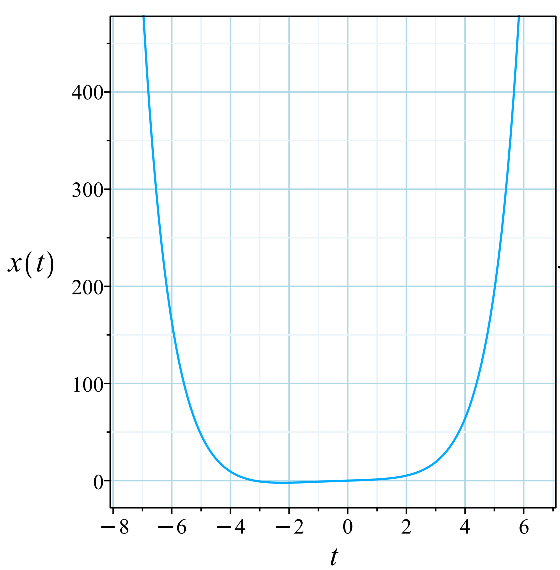
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \, \mathrm{e}^{-t} + \frac{\mathrm{e}^t c_2}{2}\right) + (-t^2 - 2) \end{aligned}$$

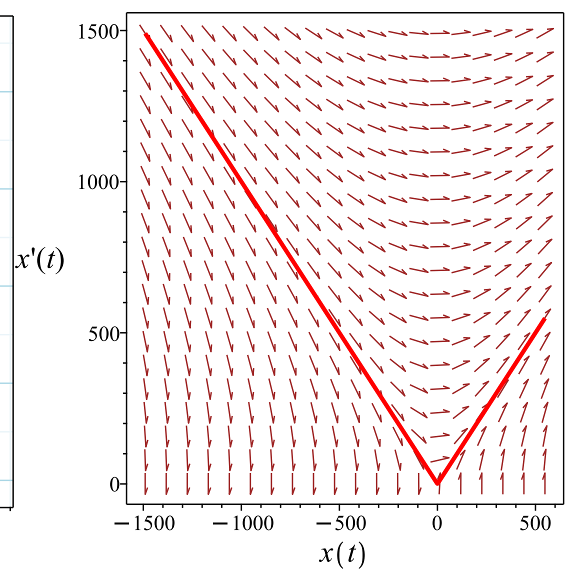
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = -t^2 - 2 + \frac{3\,\mathrm{e}^t}{2} + \frac{\mathrm{e}^{-t}}{2}$$



(a) Solution plot
$x = -t^2 - 2 + \frac{3\,\mathrm{e}^t}{2} + \frac{\mathrm{e}^{-t}}{2}$



(b) Slope field plot
$x'' - x = t^2$

Maple step by step solution

Let's solve

$$\left[x'' - x = t^2, x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of homogeneous ODE
  $r^2 - 1 = 0$
- Factor the characteristic polynomial
  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial
  $r = (-1, 1)$
- 1st solution of the homogeneous ODE
  $x_1(t) = \mathrm{e}^{-t}$
- 2nd solution of the homogeneous ODE
  $x_2(t) = \mathrm{e}^{t}$
- General solution of the ODE
  $x = C1 x_1(t) + C2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE
  $x = C1\,\mathrm{e}^{-t} + \mathrm{e}^{t} C2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function
  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t),x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t),x_2(t))} dt\right), f(t) = t^2\right]$
- ○ Wronskian of solutions of the homogeneous equation
  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{t} \end{bmatrix}$
- ○ Compute Wronskian
  $W(x_1(t), x_2(t)) = 2$
- ○ Substitute functions into equation for $x_p(t)$
  $x_p(t) = -\frac{\mathrm{e}^{-t}\left(\int t^2 \mathrm{e}^{t} dt\right)}{2} + \frac{\mathrm{e}^{t}\left(\int t^2 \mathrm{e}^{-t} dt\right)}{2}$
- ○ Compute integrals
  $x_p(t) = -t^2 - 2$

- Substitute particular solution into general solution to ODE
  $x = \mathrm{e}^{t} C2 + C1\,\mathrm{e}^{-t} - t^2 - 2$

□ Check validity of solution $x = e^t\_C2 + \_C1e^{-t} - t^2 - 2$

- Use initial condition $x(0) = 0$

  $0 = \_C2 + \_C1 - 2$
- Compute derivative of the solution

  $x' = e^t\_C2 - \_C1\,e^{-t} - 2t$
- Use the initial condition $x'\Big|_{\{t=0\}} = 1$

  $1 = -\_C1 + \_C2$
- Solve for $\_C1$ and $\_C2$

  $\left\{\_C1 = \frac{1}{2}, \_C2 = \frac{3}{2}\right\}$
- Substitute constant values into general solution and simplify

  $x = -t^2 - 2 + \frac{3\,e^t}{2} + \frac{e^{-t}}{2}$

• Solution to the IVP

  $x = -t^2 - 2 + \frac{3\,e^t}{2} + \frac{e^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.014` (sec)
Leaf size : 21

```
dsolve([diff(diff(x(t),t),t)-x(t) = t^2,
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = -t^2 - 2 + \frac{3\,e^t}{2} + \frac{e^{-t}}{2}$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 27

```
DSolve[{D[x[t],{t,2}]-x[t]==t^2,{x[0]==0,Derivative[1][x][0] == 1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{2}\left(-2\left(t^2+2\right)+e^{-t}+3e^t\right)$$

2.3.2 problem 7 (ii)

Internal problem ID [18202]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 5. Linear equations. Exercises at page 85
Problem number : 7 (ii)
Date solved : `Thursday, December 19, 2024 at 06:18:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x'' - x = e^t$$

With initial conditions

$$x(0) = 0$$
$$x'(0) = 1$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$p(t) = 0$$
$$q(t) = -1$$
$$F = e^t$$

Hence the ode is

$$x'' - x = e^t$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.301 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = -1, f(t) = e^t$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} - e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{(1)t} + c_2 e^{(-1)t}\end{aligned}$$

Or

$$x = c_1\,\mathrm{e}^{t} + c_2\,\mathrm{e}^{-t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1\,\mathrm{e}^{t} + c_2\,\mathrm{e}^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^t, e^{-t}\}$$

Since e^t is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,e^t\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\,e^t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^t = e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t\,e^t}{2}$$

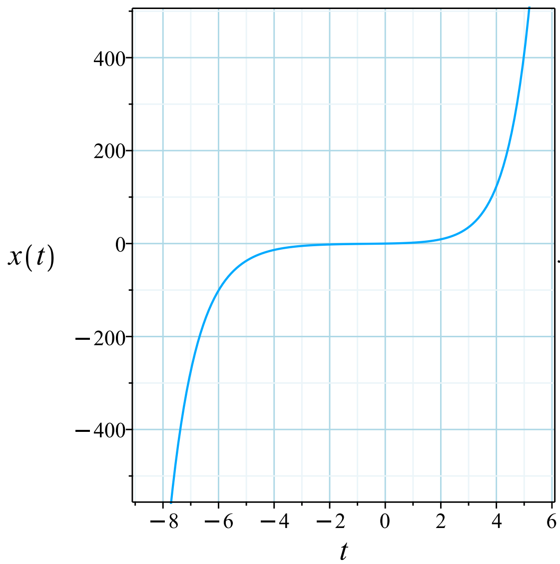
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^t + c_2 e^{-t}\right) + \left(\frac{t\,e^t}{2}\right) \end{aligned}$$

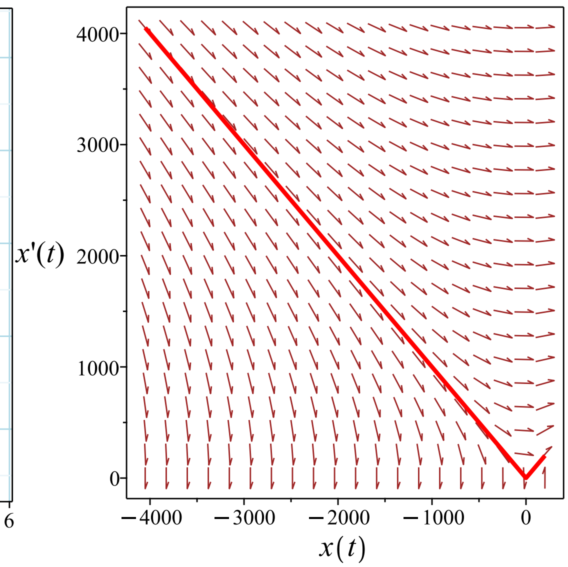
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{t\,e^t}{2} + \frac{e^t}{4} - \frac{e^{-t}}{4}$$



(a) Solution plot
$x = \frac{t\,\mathrm{e}^t}{2} + \frac{\mathrm{e}^t}{4} - \frac{\mathrm{e}^{-t}}{4}$



(b) Slope field plot
$x'' - x = \mathrm{e}^t$

Solved as second order ode using Kovacic algorithm

Time used: 0.155 (sec)

Writing the ode as

$$x'' - x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \mathrm{e}^{-t} \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\ &= \mathrm{e}^{-t} \int \frac{1}{\mathrm{e}^{-2t}}\,dt \\ &= \mathrm{e}^{-t}\left(\frac{\mathrm{e}^{2t}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}x &= c_1x_1 + c_2x_2 \\ &= c_1\left(\mathrm{e}^{-t}\right) + c_2\left(\mathrm{e}^{-t}\left(\frac{\mathrm{e}^{2t}}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{t}c_2}{2}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1x_1 + u_2x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}x_1 &= \mathrm{e}^{-t} \\ x_2 &= \frac{\mathrm{e}^{t}}{2}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-t} & \frac{e^t}{2} \\ \frac{d}{dt}\left(e^{-t}\right) & \frac{d}{dt}\left(\frac{e^t}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-t} & \frac{e^t}{2} \\ -e^{-t} & \frac{e^t}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-t}\right)\left(\frac{e^t}{2}\right) - \left(\frac{e^t}{2}\right)\left(-e^{-t}\right)$$

Which simplifies to

$$W = e^{-t} e^t$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2t}}{2}}{1} dt$$

Which simplifies to

$$u_1 = -\int \frac{e^{2t}}{2} dt$$

Hence

$$u_1 = -\frac{e^{2t}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-t} e^t}{1} dt$$

Which simplifies to

$$u_2 = \int 1dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{e^{2t} e^{-t}}{4} + \frac{t\, e^t}{2}$$

Which simplifies to

$$x_p(t) = \frac{e^t(-1 + 2t)}{4}$$

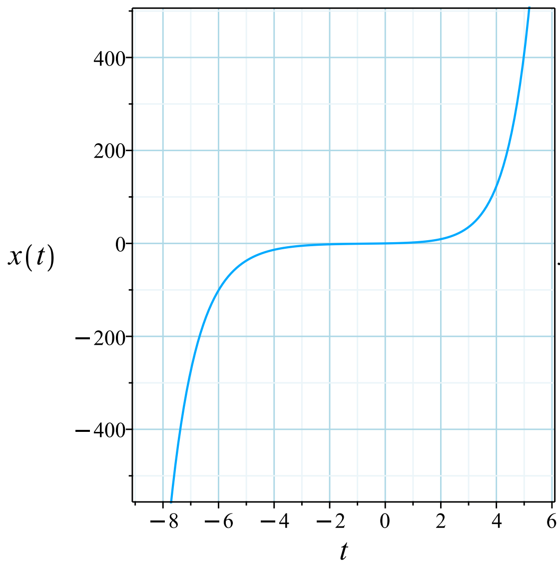
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1\, e^{-t} + \frac{e^t c_2}{2}\right) + \left(\frac{e^t(-1 + 2t)}{4}\right) \end{aligned}$$

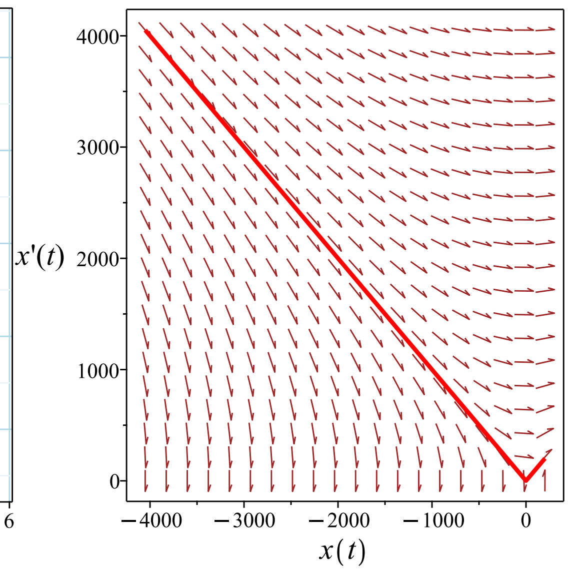
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = -\frac{e^{-t}}{4} + \frac{e^t}{2} + \frac{e^t(-1 + 2t)}{4}$$



(a) Solution plot
$x = -\frac{e^{-t}}{4} + \frac{e^{t}}{2} + \frac{e^{t}(-1+2t)}{4}$



(b) Slope field plot
$x'' - x = e^{t}$

Solved as second order ode adjoint method

Time used: 0.403 (sec)

In normal form the ode

$$x'' - x = e^{t} \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -1 \\ r(t) &= e^{t} \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (-\xi(t)) &= 0 \\ \xi''(t) - \xi(t) &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} - e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}\xi &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ \xi &= c_1 e^{(1)t} + c_2 e^{(-1)t}\end{aligned}$$

Or

$$\xi = c_1 e^t + c_2 e^{-t}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(t)\,x' - x\xi'(t) + \xi(t)\,p(t)\,x = \int \xi(t)\,r(t)\,dt$$
$$x' + x\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) = \frac{\int \xi(t)\,r(t)\,dt}{\xi\,(t)}$$

Or

$$x' - \frac{x(c_1\,\mathrm{e}^{t} - c_2\,\mathrm{e}^{-t})}{c_1\,\mathrm{e}^{t} + c_2\,\mathrm{e}^{-t}} = \frac{c_2 t + \frac{c_1\,\mathrm{e}^{2t}}{2}}{c_1\,\mathrm{e}^{t} + c_2\,\mathrm{e}^{-t}}$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = -\frac{c_1\,\mathrm{e}^{2t} - c_2}{c_1\,\mathrm{e}^{2t} + c_2}$$
$$p(t) = \frac{\mathrm{e}^{t}(c_1\,\mathrm{e}^{2t} + 2c_2 t)}{2c_1\,\mathrm{e}^{2t} + 2c_2}$$

The integrating factor μ is

$$\mu = e^{\int q\,dt}$$
$$= \mathrm{e}^{\int -\frac{c_1\,\mathrm{e}^{2t} - c_2}{c_1\,\mathrm{e}^{2t} + c_2}dt}$$
$$= \frac{\sqrt{\mathrm{e}^{2t}}}{c_1\,\mathrm{e}^{2t} + c_2}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = (\mu)\left(\frac{\mathrm{e}^{t}(c_1\,\mathrm{e}^{2t} + 2c_2 t)}{2c_1\,\mathrm{e}^{2t} + 2c_2}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x\sqrt{\mathrm{e}^{2t}}}{c_1\,\mathrm{e}^{2t} + c_2}\right) = \left(\frac{\sqrt{\mathrm{e}^{2t}}}{c_1\,\mathrm{e}^{2t} + c_2}\right)\left(\frac{\mathrm{e}^{t}(c_1\,\mathrm{e}^{2t} + 2c_2 t)}{2c_1\,\mathrm{e}^{2t} + 2c_2}\right)$$
$$\mathrm{d}\left(\frac{x\sqrt{\mathrm{e}^{2t}}}{c_1\,\mathrm{e}^{2t} + c_2}\right) = \left(\frac{\mathrm{e}^{t}(c_1\,\mathrm{e}^{2t} + 2c_2 t)\sqrt{\mathrm{e}^{2t}}}{(2c_1\,\mathrm{e}^{2t} + 2c_2)\,(c_1\,\mathrm{e}^{2t} + c_2)}\right)\mathrm{d}t$$

Integrating gives

$$\frac{x\sqrt{e^{2t}}}{c_1 e^{2t}+c_2} = \int \frac{e^{t}(c_1 e^{2t}+2c_2 t)\sqrt{e^{2t}}}{(2c_1 e^{2t}+2c_2)(c_1 e^{2t}+c_2)}\, dt$$
$$= \frac{\sqrt{e^{2t}}\, e^{-t} t}{2c_1} - \frac{\sqrt{e^{2t}}\, e^{-t} c_2(-1+2t)}{4c_1(c_1 e^{2t}+c_2)} + c_3$$

Dividing throughout by the integrating factor $\frac{\sqrt{e^{2t}}}{c_1 e^{2t}+c_2}$ gives the final solution

$$x = \frac{(e^{2t})^{3/2} e^{-t} c_1 t + \frac{\sqrt{e^{2t}}\, e^{-t} c_2}{2} + 2c_3 c_1 (c_1 e^{2t}+c_2)}{2\sqrt{e^{2t}}\, c_1}$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = \frac{(e^{2t})^{3/2} e^{-t} c_1 t + \frac{\sqrt{e^{2t}}\, e^{-t} c_2}{2} + 2c_3 c_1 (c_1 e^{2t}+c_2)}{2\sqrt{e^{2t}}\, c_1}$$

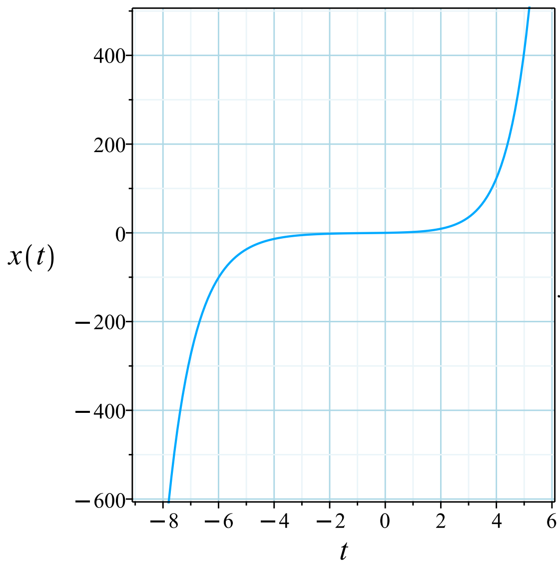
The constants can be merged to give

$$x = \frac{(e^{2t})^{3/2} e^{-t} c_1 t + \frac{\sqrt{e^{2t}}\, e^{-t} c_2}{2} + 2c_1 (c_1 e^{2t}+c_2)}{2\sqrt{e^{2t}}\, c_1}$$

Will add steps showing solving for IC soon.

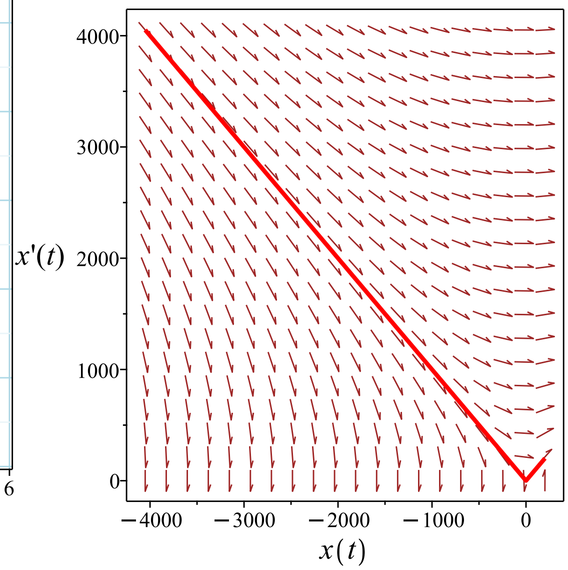
Summary of solutions found

$$x = \frac{\frac{(e^{2t})^{3/2} e^{-t} t}{2} - \frac{\sqrt{e^{2t}}\, e^{-t}}{8} + \frac{e^{2t}}{4} - \frac{1}{8}}{\sqrt{e^{2t}}}$$



(a) Solution plot

$$x = \frac{\frac{\left(e^{2t}\right)^{3/2} e^{-t} t}{2} - \frac{\sqrt{e^{2t}}\, e^{-t}}{8} + \frac{e^{2t}}{4} - \frac{1}{8}}{\sqrt{e^{2t}}}$$



(b) Slope field plot

$x'' - x = e^t$

Maple step by step solution

Let's solve

$$\left[x'' - x = e^t, x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of homogeneous ODE
  $r^2 - 1 = 0$
- Factor the characteristic polynomial
  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial
  $r = (-1, 1)$
- 1st solution of the homogeneous ODE
  $x_1(t) = e^{-t}$
- 2nd solution of the homogeneous ODE
  $x_2(t) = e^t$
- General solution of the ODE
  $x = C1 x_1(t) + C2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

$x = C1\,\mathrm{e}^{-t} + \mathrm{e}^{t} C2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \mathrm{e}^{t}\right]$

- Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{t} \end{bmatrix}$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = 2$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\mathrm{e}^{-t}\left(\int \mathrm{e}^{2t}dt\right)}{2} + \frac{\mathrm{e}^{t}\left(\int 1dt\right)}{2}$

- Compute integrals

  $x_p(t) = \frac{\mathrm{e}^{t}(-1+2t)}{4}$

• Substitute particular solution into general solution to ODE

  $x = C1\,\mathrm{e}^{-t} + \mathrm{e}^{t} C2 + \frac{\mathrm{e}^{t}(-1+2t)}{4}$

□ Check validity of solution $x = \_C1\mathrm{e}^{-t} + \mathrm{e}^{t}\_C2 + \frac{\mathrm{e}^{t}(-1+2t)}{4}$

- Use initial condition $x(0) = 0$

  $0 = \_C1 + \_C2 - \frac{1}{4}$

- Compute derivative of the solution

  $x' = -\_C1\,\mathrm{e}^{-t} + \mathrm{e}^{t}\_C2 + \frac{\mathrm{e}^{t}(-1+2t)}{4} + \frac{\mathrm{e}^{t}}{2}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 1$

  $1 = -\_C1 + \_C2 + \frac{1}{4}$

- Solve for $\_C1$ and $\_C2$

  $\left\{\_C1 = -\frac{1}{4}, \_C2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $x = -\frac{\mathrm{e}^{-t}}{4} + \frac{(2t+1)\mathrm{e}^{t}}{4}$

• Solution to the IVP

  $x = -\frac{\mathrm{e}^{-t}}{4} + \frac{(2t+1)\mathrm{e}^{t}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.015 (sec)
Leaf size : 20

```
dsolve([diff(diff(x(t),t),t)-x(t) = exp(t),
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = -\frac{e^{-t}}{4} + \frac{(2t+1)\, e^{t}}{4}$$

Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 27

```
DSolve[{D[x[t],{t,2}]-x[t]==Exp[t],{x[0]==0,Derivative[1][x][0] == 1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{4} e^{-t} \left(e^{2t}(2t+1) - 1\right)$$

2.3.3 problem 7 (iii)

Internal problem ID [18203]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 5. Linear equations. Exercises at page 85
Problem number : 7 (iii)
Date solved : `Thursday, December 19, 2024 at 06:18:04 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$x'' + 2x' + 4x = e^{t} \cos(2t)$$

With initial conditions

$$x(0) = 0$$
$$x'(0) = 1$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$p(t) = 2$$
$$q(t) = 4$$
$$F = e^{t} \cos(2t)$$

Hence the ode is

$$x'' + 2x' + 4x = e^{t} \cos(2t)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^t \cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.376 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 2, C = 4, f(t) = e^t \cos(2t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2x' + 4x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} + 2\lambda e^{t\lambda} + 4 e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(4)}\\ &= -1 \pm i\sqrt{3}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i\sqrt{3}\\ \lambda_2 &= -1 - i\sqrt{3}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{3} - 1\\ \lambda_2 &= -1 - i\sqrt{3}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = \sqrt{3}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-t}\left(c_1 \cos\left(\sqrt{3}\,t\right) + c_2 \sin\left(\sqrt{3}\,t\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-t}\left(c_1 \cos\left(\sqrt{3}\,t\right) + c_2 \sin\left(\sqrt{3}\,t\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{t} \cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^t \cos(2t), e^t \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-t} \cos\left(\sqrt{3}\, t\right), e^{-t} \sin\left(\sqrt{3}\, t\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^t \cos(2t) + A_2 e^t \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^t \cos(2t) - 8A_1 e^t \sin(2t) + 3A_2 e^t \sin(2t) + 8A_2 e^t \cos(2t) = e^t \cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{73}, A_2 = \frac{8}{73}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{3\, e^t \cos(2t)}{73} + \frac{8\, e^t \sin(2t)}{73}$$

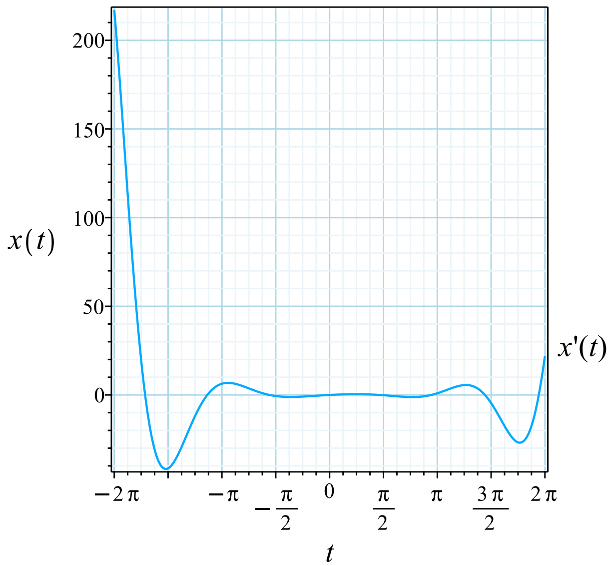
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-t}\left(c_1 \cos\left(\sqrt{3}\, t\right) + c_2 \sin\left(\sqrt{3}\, t\right)\right)\right) + \left(\frac{3\, e^t \cos(2t)}{73} + \frac{8\, e^t \sin(2t)}{73}\right) \end{aligned}$$

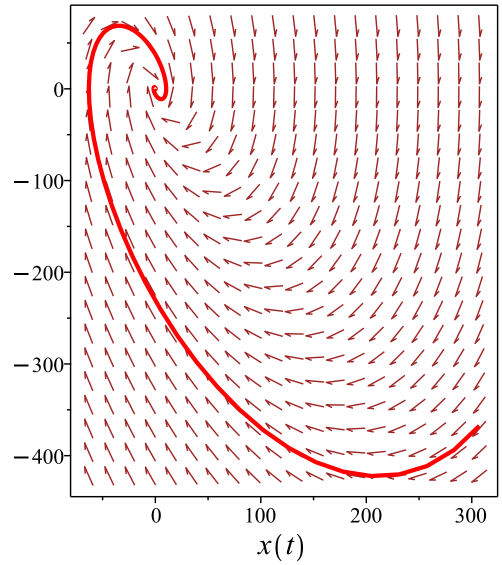
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{3\, e^t \cos(2t)}{73} + \frac{8\, e^t \sin(2t)}{73} + e^{-t}\left(-\frac{3 \cos\left(\sqrt{3}\, t\right)}{73} + \frac{17\sqrt{3}\, \sin\left(\sqrt{3}\, t\right)}{73}\right)$$



(a) Solution plot
$x = \frac{3\,e^{t}\cos(2t)}{73} + \frac{8\,e^{t}\sin(2t)}{73} + e^{-t}\left(-\frac{3\cos(\sqrt{3}\,t)}{73} + \frac{17\sqrt{3}\,\sin(\sqrt{3}\,t)}{73}\right)$



(b) Slope field plot
$x'' + 2x' + 4x = e^{t}\cos(2t)$

Solved as second order ode using Kovacic algorithm

Time used: 0.393 (sec)

Writing the ode as

$$x'' + 2x' + 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -3z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -3$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\sqrt{3}\, t\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dt} \\ &= z_1 e^{-t} \\ &= z_1\left(\mathrm{e}^{-t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t} \cos\left(\sqrt{3}\, t\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-2t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{\sqrt{3}\, \tan\left(\sqrt{3}\, t\right)}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right)\right) + c_2\left(\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right)\left(\frac{\sqrt{3}\,\tan\left(\sqrt{3}\,t\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$
x'' + 2x' + 4x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1\,\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right) + \frac{c_2 \sin\left(\sqrt{3}\,t\right)\mathrm{e}^{-t}\sqrt{3}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{t}\cos\left(2t\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{t}\cos\left(2t\right), \mathrm{e}^{t}\sin\left(2t\right)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right), \frac{\sin\left(\sqrt{3}\,t\right)\mathrm{e}^{-t}\sqrt{3}}{3}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1\mathrm{e}^{t}\cos\left(2t\right) + A_2\mathrm{e}^{t}\sin\left(2t\right)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1\mathrm{e}^t \cos(2t) - 8A_1\mathrm{e}^t \sin(2t) + 3A_2\mathrm{e}^t \sin(2t) + 8A_2\mathrm{e}^t \cos(2t) = \mathrm{e}^t \cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{73}, A_2 = \frac{8}{73}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{3\,\mathrm{e}^t \cos(2t)}{73} + \frac{8\,\mathrm{e}^t \sin(2t)}{73}$$

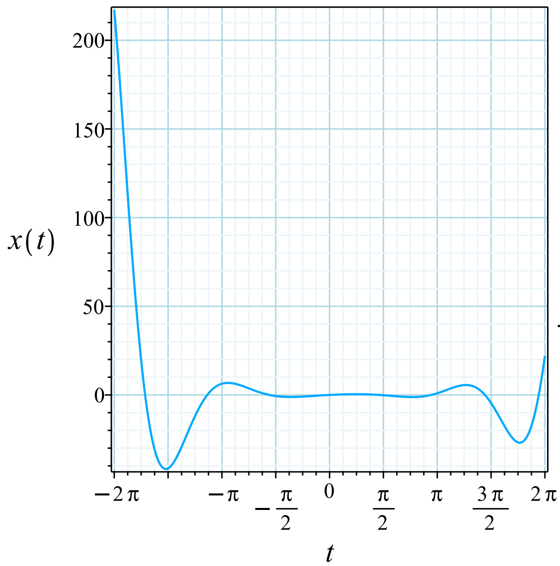
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1\,\mathrm{e}^{-t} \cos\left(\sqrt{3}\,t\right) + \frac{c_2 \sin\left(\sqrt{3}\,t\right) \mathrm{e}^{-t}\sqrt{3}}{3}\right) + \left(\frac{3\,\mathrm{e}^t \cos(2t)}{73} + \frac{8\,\mathrm{e}^t \sin(2t)}{73}\right) \end{aligned}$$

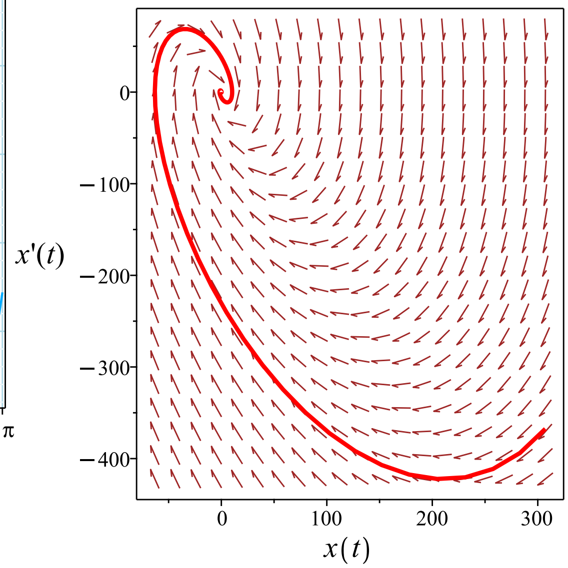
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = -\frac{3\,\mathrm{e}^{-t} \cos\left(\sqrt{3}\,t\right)}{73} + \frac{17 \sin\left(\sqrt{3}\,t\right) \mathrm{e}^{-t}\sqrt{3}}{73} + \frac{3\,\mathrm{e}^t \cos(2t)}{73} + \frac{8\,\mathrm{e}^t \sin(2t)}{73}$$



(a) Solution plot
$x = -\frac{3\,\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right)}{73} + \frac{17\sin\left(\sqrt{3}\,t\right)\mathrm{e}^{-t}\sqrt{3}}{73} + \frac{3\,\mathrm{e}^{t}\cos(2t)}{73} + \frac{8\,\mathrm{e}^{t}\sin(2t)}{73}$



(b) Slope field plot
$x'' + 2x' + 4x = \mathrm{e}^{t}\cos(2t)$

Maple step by step solution

Let's solve

$$\left[x'' + 2x' + 4x = \mathrm{e}^{t}\cos(2t),\, x(0) = 0,\, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  x''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 4 = 0$
- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm\left(\sqrt{-12}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(-1 - \mathrm{I}\sqrt{3}, \mathrm{I}\sqrt{3} - 1\right)$
- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right)$
- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-t}\sin\left(\sqrt{3}\,t\right)$
- General solution of the ODE

$x = C1x_1(t) + C2x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = C1\,\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right) + \mathrm{e}^{-t}\sin\left(\sqrt{3}\,t\right) C2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \mathrm{e}^{t}\cos(2t)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right) & \mathrm{e}^{-t}\sin\left(\sqrt{3}\,t\right) \\ -\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right) - \sin\left(\sqrt{3}\,t\right)\mathrm{e}^{-t}\sqrt{3} & -\mathrm{e}^{-t}\sin\left(\sqrt{3}\,t\right) + \mathrm{e}^{-t}\sqrt{3}\cos\left(\sqrt{3}\,t\right) \end{bmatrix}$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \sqrt{3}\,\mathrm{e}^{-2t}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{\sqrt{3}\,\mathrm{e}^{-t}\left(-\cos\left(\sqrt{3}\,t\right)\left(\int \sin\left(\sqrt{3}\,t\right)\cos(2t)\mathrm{e}^{2t}dt\right) + \sin\left(\sqrt{3}\,t\right)\left(\int \cos\left(\sqrt{3}\,t\right)\cos(2t)\mathrm{e}^{2t}dt\right)\right)}{3}$

- Compute integrals

  $x_p(t) = \frac{\mathrm{e}^{t}(3\cos(2t)+8\sin(2t))}{73}$

- Substitute particular solution into general solution to ODE

  $x = \mathrm{e}^{-t}\sin\left(\sqrt{3}\,t\right) C2 + C1\,\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right) + \frac{\mathrm{e}^{t}(3\cos(2t)+8\sin(2t))}{73}$

□ Check validity of solution $x = \mathrm{e}^{-t}\sin\left(\sqrt{3}\,t\right)\_C2 + \_C1\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right) + \frac{\mathrm{e}^{t}(3\cos(2t)+8\sin(2t))}{73}$

- Use initial condition $x(0) = 0$

  $0 = \frac{3}{73} + \_C1$

- Compute derivative of the solution

  $x' = -\mathrm{e}^{-t}\sin\left(\sqrt{3}\,t\right)\_C2 + \mathrm{e}^{-t}\sqrt{3}\cos\left(\sqrt{3}\,t\right)\_C2 - \_C1\,\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right) - \_C1\,\mathrm{e}^{-t}\sqrt{3}\sin\left(\sqrt{3}\,t\right)$

- Use the initial condition $x'\Big|_{\{t=0\}} = 1$

  $1 = \frac{19}{73} + \sqrt{3}\_C2 - \_C1$

- Solve for $\_C1$ and $\_C2$

  $\left\{\_C1 = -\frac{3}{73}, \_C2 = \frac{17\sqrt{3}}{73}\right\}$

- Substitute constant values into general solution and simplify

  $x = -\frac{3\,\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right)}{73} + \frac{17\sin\left(\sqrt{3}\,t\right)\mathrm{e}^{-t}\sqrt{3}}{73} + \frac{3\,\mathrm{e}^{t}\left(\cos(2t)+\frac{8\sin(2t)}{3}\right)}{73}$

- Solution to the IVP

$$x = -\frac{3\,\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right)}{73}+\frac{17\sin\left(\sqrt{3}\,t\right)\mathrm{e}^{-t}\sqrt{3}}{73}+\frac{3\,\mathrm{e}^{t}\left(\cos(2t)+\frac{8\sin(2t)}{3}\right)}{73}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 47

```
dsolve([diff(diff(x(t),t),t)+2*diff(x(t),t)+4*x(t) = exp(t)*cos(2*t),
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = -\frac{3\,\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right)}{73}+\frac{17\sin\left(\sqrt{3}\,t\right)\mathrm{e}^{-t}\sqrt{3}}{73}+\frac{3\,\mathrm{e}^{t}\left(\cos\left(2t\right)+\frac{8\sin(2t)}{3}\right)}{73}$$

Mathematica DSolve solution

Solving time : 1.045 (sec)
Leaf size : 62

```
DSolve[{D[x[t],{t,2}]+2*D[x[t],t]+4*x[t]==Exp[t]*Cos[2*t],{x[0]==0,Derivative[1][x][0] == 1}
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t)\to \frac{1}{73}e^{-t}\left(8e^{2t}\sin(2t)+17\sqrt{3}\sin\left(\sqrt{3}t\right)+3e^{2t}\cos(2t)-3\cos\left(\sqrt{3}t\right)\right)$$

2.3.4 problem 7 (iv)

Internal problem ID [18204]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 5. Linear equations. Exercises at page 85
Problem number : 7 (iv)
Date solved : `Thursday, December 19, 2024 at 06:18:45 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$x'' - x' + x = \sin(2t)$$

With initial conditions

$$x(0) = 0$$
$$x'(0) = 1$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$p(t) = -1$$
$$q(t) = 1$$
$$F = \sin(2t)$$

Hence the ode is

$$x'' - x' + x = \sin(2t)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.227 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -1, C = 1, f(t) = \sin(2t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} - \lambda e^{t\lambda} + e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(1)}\\ &= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2t) - 3A_2 \sin(2t) + 2A_1 \sin(2t) - 2A_2 \cos(2t) = \sin(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{13}, A_2 = -\frac{3}{13}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$$

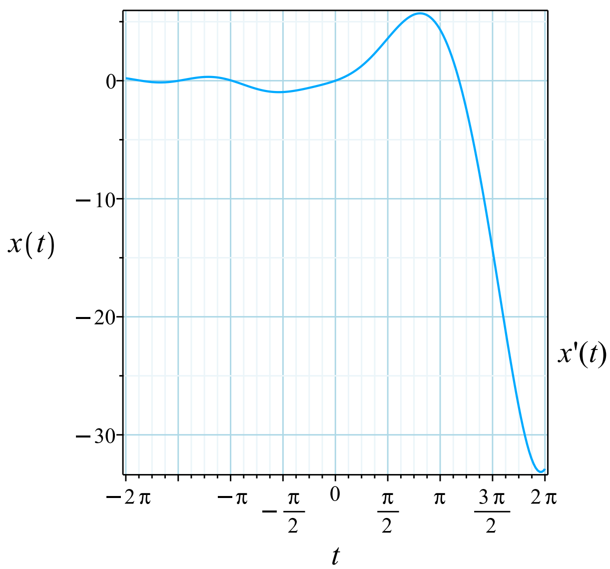
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)\right) + \left(\frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}\right) \end{aligned}$$

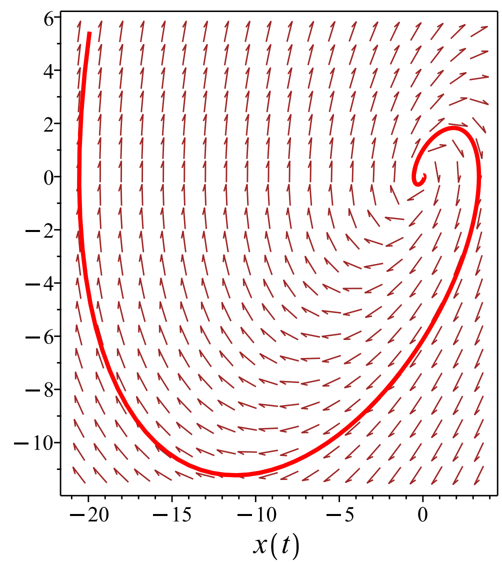
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13} + e^{\frac{t}{2}}\left(-\frac{2\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{13} + \frac{40\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{39}\right)$$



(a) Solution plot
$x = \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13} + e^{\frac{t}{2}}\left(-\frac{2\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{13} + \frac{40\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{39}\right)$



(b) Slope field plot
$x'' - x' + x = \sin(2t)$

Solved as second order ode using Kovacic algorithm

Time used: 0.279 (sec)

Writing the ode as

$$x'' - x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dt} \\ &= z_1 e^{\frac{t}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-1}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{t}}{(x_1)^2}\, dt \\
&= x_1 \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, t}{2}\right)}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) \right) + c_2 \left(\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, t}{2}\right)}{3} \right) \right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$
x'' - x' + x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1\, \mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) \mathrm{e}^{\frac{t}{2}} \sqrt{3}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(2t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(2t), \sin(2t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right), \frac{2\sin\left(\frac{\sqrt{3}\,t}{2}\right) e^{\frac{t}{2}} \sqrt{3}}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2t) - 3A_2 \sin(2t) + 2A_1 \sin(2t) - 2A_2 \cos(2t) = \sin(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{13}, A_2 = -\frac{3}{13}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$$

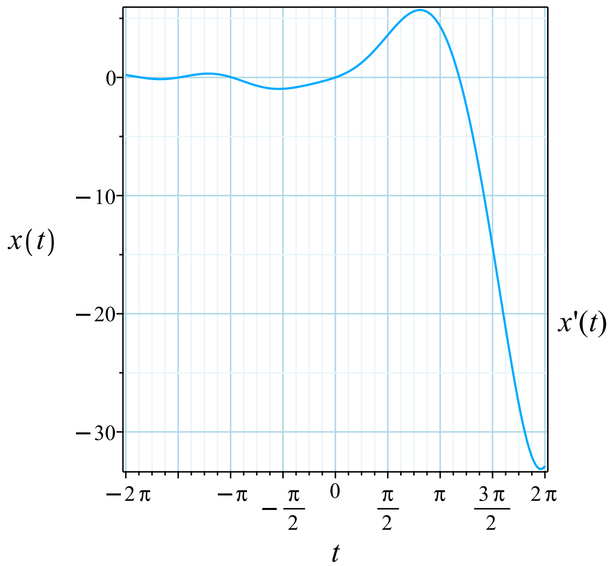
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right) e^{\frac{t}{2}} \sqrt{3}}{3}\right) + \left(\frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}\right) \end{aligned}$$

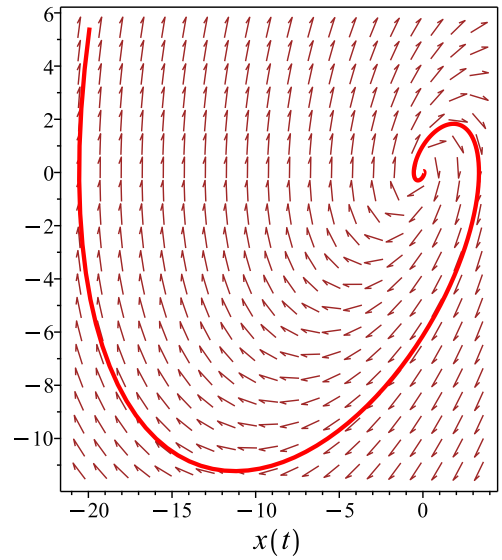
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = -\frac{2 e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)}{13} + \frac{40 \sin\left(\frac{\sqrt{3}\,t}{2}\right) e^{\frac{t}{2}} \sqrt{3}}{39} + \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$$



(a) Solution plot
$x = -\frac{2\,\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{13} + \frac{40\sin\left(\frac{\sqrt{3}\,t}{2}\right)\mathrm{e}^{\frac{t}{2}}\sqrt{3}}{39} + \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$



(b) Slope field plot
$x'' - x' + x = \sin(2t)$

Maple step by step solution

Let's solve

$$\left[x'' - x' + x = \sin(2t), x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of homogeneous ODE
  $r^2 - r + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{1 \pm (\sqrt{-3})}{2}$
- Roots of the characteristic polynomial
  $r = \left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$
- 1st solution of the homogeneous ODE
  $x_1(t) = \mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)$
- 2nd solution of the homogeneous ODE
  $x_2(t) = \mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE
  $x = C1\, x_1(t) + C2\, x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE
  $x = C1\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) C2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function
  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \sin(2t)\right]$
- Wronskian of solutions of the homogeneous equation
  $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) & e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \\ \frac{e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\, t}{2}\right) e^{\frac{t}{2}} \sqrt{3}}{2} & \frac{e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2} + \frac{e^{\frac{t}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2} \end{bmatrix}$
- Compute Wronskian
  $W(x_1(t), x_2(t)) = \frac{\sqrt{3}\, e^{t}}{2}$
- Substitute functions into equation for $x_p(t)$
  $x_p(t) = -\frac{2\sqrt{3}\, e^{\frac{t}{2}}\left(\cos\left(\frac{\sqrt{3}\, t}{2}\right)\left(\int \sin\left(\frac{\sqrt{3}\, t}{2}\right) \sin(2t) e^{-\frac{t}{2}} dt\right) - \sin\left(\frac{\sqrt{3}\, t}{2}\right)\left(\int \cos\left(\frac{\sqrt{3}\, t}{2}\right) \sin(2t) e^{-\frac{t}{2}} dt\right)\right)}{3}$
- Compute integrals
  $x_p(t) = \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$

- Substitute particular solution into general solution to ODE
  $x = e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) C2 + C1\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$

□ Check validity of solution $x = e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \_C2 + \_C1 e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$

- Use initial condition $x(0) = 0$
  $0 = \frac{2}{13} + \_C1$
- Compute derivative of the solution
  $x' = \frac{e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \_C2}{2} + \frac{e^{\frac{t}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\, t}{2}\right) \_C2}{2} + \frac{\_C1\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2} - \frac{\_C1\, e^{\frac{t}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2} - \frac{4\sin(2t)}{13} - \frac{6\cos(2}{13}$
- Use the initial condition $x'\Big|_{\{t=0\}} = 1$
  $1 = -\frac{6}{13} + \frac{\sqrt{3}\, \_C2}{2} + \frac{\_C1}{2}$
- Solve for $\_C1$ and $\_C2$
  $\left\{\_C1 = -\frac{2}{13}, \_C2 = \frac{40\sqrt{3}}{39}\right\}$
- Substitute constant values into general solution and simplify

$$x = -\frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{13} + \frac{40\sin\left(\frac{\sqrt{3}\,t}{2}\right)e^{\frac{t}{2}}\sqrt{3}}{39} + \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$$

- Solution to the IVP

$$x = -\frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{13} + \frac{40\sin\left(\frac{\sqrt{3}\,t}{2}\right)e^{\frac{t}{2}}\sqrt{3}}{39} + \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 46

```
dsolve([diff(diff(x(t),t),t)-diff(x(t),t)+x(t) = sin(2*t),
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = -\frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{13} + \frac{40\sin\left(\frac{\sqrt{3}\,t}{2}\right)e^{\frac{t}{2}}\sqrt{3}}{39} + \frac{2\cos(2t)}{13} - \frac{3\sin(2t)}{13}$$

Mathematica DSolve solution

Solving time : 1.62 (sec)
Leaf size : 67

```
DSolve[{D[x[t],{t,2}]-D[x[t],t]+x[t]==Sin[2*t],{x[0]==0,Derivative[1][x][0] == 1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{39}\left(-9\sin(2t) + 40\sqrt{3}e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right) + 6\cos(2t) - 6e^{t/2}\cos\left(\frac{\sqrt{3}t}{2}\right)\right)$$

2.3.5 problem 7 (v)

Internal problem ID [18205]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 5. Linear equations. Exercises at page 85
Problem number : 7 (v)
Date solved : `Thursday, December 19, 2024 at 06:20:09 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$x'' + 4x' + 3x = t\sin(t)$$

With initial conditions

$$x(0) = 0$$
$$x'(0) = 1$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$p(t) = 4$$
$$q(t) = 3$$
$$F = t\sin(t)$$

Hence the ode is

$$x'' + 4x' + 3x = t\sin(t)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t\sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.519 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 4, C = 3, f(t) = t\sin(t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 4x' + 3x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} + 4\lambda\, e^{t\lambda} + 3\, e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(3)} \\ &= -2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 1 \\ \lambda_2 &= -2 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x &= c_1 e^{(-1)t} + c_2 e^{(-3)t}\end{aligned}$$

Or

$$x = c_1 \,\mathrm{e}^{-t} + c_2 \,\mathrm{e}^{-3t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \,\mathrm{e}^{-t} + c_2 \,\mathrm{e}^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t \sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{t \sin(t), \cos(t)\, t, \cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 t \sin(t) + A_2 \cos(t) t + A_3 \cos(t) + A_4 \sin(t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1 \cos(t) + 2A_1 t \sin(t) + 2A_2 \cos(t) t - 2A_2 \sin(t)\\&+ 2A_3 \cos(t) + 2A_4 \sin(t) + 4A_1 \sin(t) + 4A_1 t \cos(t)\\&- 4A_2 \sin(t) t + 4A_2 \cos(t) - 4A_3 \sin(t) + 4A_4 \cos(t) = t \sin(t)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = -\frac{1}{5}, A_3 = \frac{11}{50}, A_4 = \frac{1}{25}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t \sin(t)}{10} - \frac{\cos(t) t}{5} + \frac{11 \cos(t)}{50} + \frac{\sin(t)}{25}$$

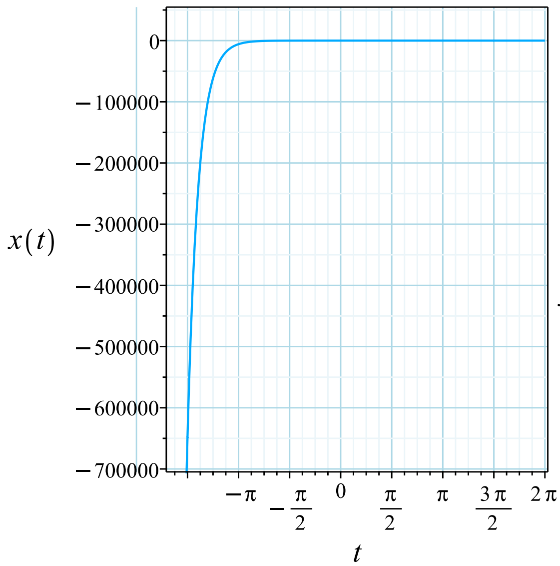
Therefore the general solution is

$$\begin{aligned}x &= x_h + x_p\\&= \left(c_1 e^{-t} + c_2 e^{-3t}\right) + \left(\frac{t \sin(t)}{10} - \frac{\cos(t) t}{5} + \frac{11 \cos(t)}{50} + \frac{\sin(t)}{25}\right)\end{aligned}$$

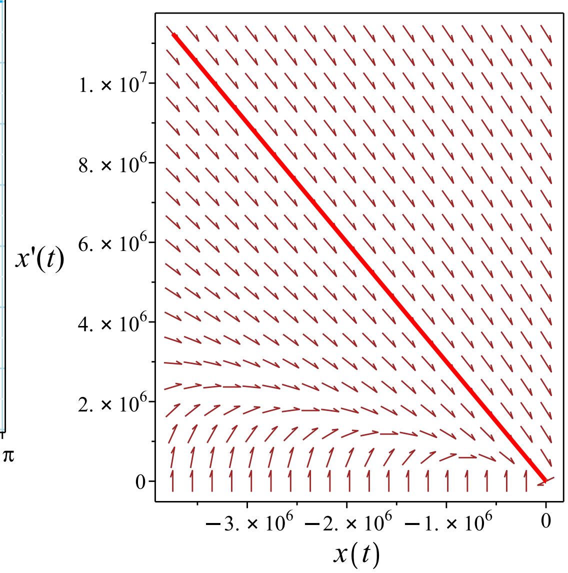
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{t \sin(t)}{10} - \frac{\cos(t) t}{5} + \frac{11 \cos(t)}{50} + \frac{\sin(t)}{25} + \frac{e^{-t}}{4} - \frac{47 e^{-3t}}{100}$$



(a) Solution plot
$x = \frac{t\sin(t)}{10} - \frac{\cos(t)t}{5} + \frac{11\cos(t)}{50} + \frac{\sin(t)}{25} + \frac{e^{-t}}{4} - \frac{47\,e^{-3t}}{100}$



(b) Slope field plot
$x'' + 4x' + 3x = t\sin(t)$

Solved as second order ode using Kovacic algorithm

Time used: 0.153 (sec)

Writing the ode as

$$x'' + 4x' + 3x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\ &= z_1 e^{-2t} \\ &= z_1\left(\mathrm{e}^{-2t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-3t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-4t}}{(x_1)^2}\,dt \\ &= x_1\left(\frac{\mathrm{e}^{-4t}\mathrm{e}^{6t}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t} \left(\frac{\mathrm{e}^{-4t} \mathrm{e}^{6t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 4x' + 3x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \,\mathrm{e}^{-3t} + \frac{c_2 \,\mathrm{e}^{-t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
t \sin(t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{t \sin(t), \cos(t)\, t, \cos(t), \sin(t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{-t}}{2}, \mathrm{e}^{-3t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 t \sin(t) + A_2 \cos(t)\, t + A_3 \cos(t) + A_4 \sin(t)
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&2A_1 \cos(t) + 2A_1 t \sin(t) + 2A_2 \cos(t) t - 2A_2 \sin(t) \\
&+ 2A_3 \cos(t) + 2A_4 \sin(t) + 4A_1 \sin(t) + 4A_1 t \cos(t) \\
&- 4A_2 \sin(t) t + 4A_2 \cos(t) - 4A_3 \sin(t) + 4A_4 \cos(t) = t \sin(t)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = -\frac{1}{5}, A_3 = \frac{11}{50}, A_4 = \frac{1}{25}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t \sin(t)}{10} - \frac{\cos(t) t}{5} + \frac{11 \cos(t)}{50} + \frac{\sin(t)}{25}$$

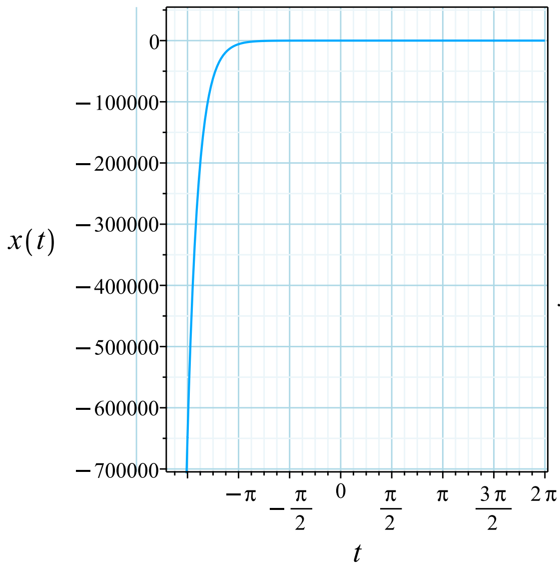
Therefore the general solution is

$$\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \mathrm{e}^{-3t} + \frac{c_2 \mathrm{e}^{-t}}{2}\right) + \left(\frac{t \sin(t)}{10} - \frac{\cos(t) t}{5} + \frac{11 \cos(t)}{50} + \frac{\sin(t)}{25}\right)
\end{aligned}$$

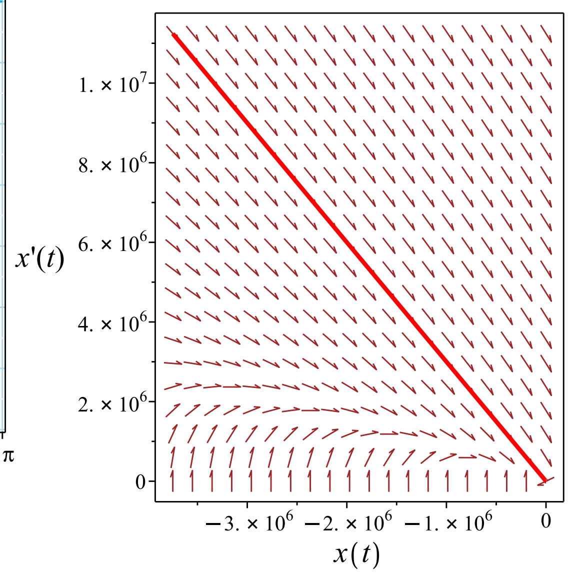
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{t \sin(t)}{10} - \frac{\cos(t) t}{5} + \frac{11 \cos(t)}{50} + \frac{\sin(t)}{25} + \frac{\mathrm{e}^{-t}}{4} - \frac{47 \mathrm{e}^{-3t}}{100}$$



(a) Solution plot
$x = \frac{t\sin(t)}{10} - \frac{\cos(t)t}{5} + \frac{11\cos(t)}{50} + \frac{\sin(t)}{25} + \frac{e^{-t}}{4} - \frac{47\,e^{-3t}}{100}$



(b) Slope field plot
$x'' + 4x' + 3x = t\sin(t)$

Maple step by step solution

Let's solve

$$\left[x'' + 4x' + 3x = t\sin(t), x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 4r + 3 = 0$
- Factor the characteristic polynomial
  $(r+3)(r+1) = 0$
- Roots of the characteristic polynomial
  $r = (-3, -1)$
- 1st solution of the homogeneous ODE
  $x_1(t) = e^{-3t}$
- 2nd solution of the homogeneous ODE
  $x_2(t) = e^{-t}$
- General solution of the ODE
  $x = C1x_1(t) + C2x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

$x = C1\,\mathrm{e}^{-3t} + C2\,\mathrm{e}^{-t} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = t\sin(t)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t} & \mathrm{e}^{-t} \\ -3\,\mathrm{e}^{-3t} & -\mathrm{e}^{-t} \end{bmatrix}$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = 2\,\mathrm{e}^{-4t}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\mathrm{e}^{-3t}\left(\int t\sin(t)\mathrm{e}^{3t}dt\right)}{2} + \frac{\mathrm{e}^{-t}\left(\int t\sin(t)\mathrm{e}^{t}dt\right)}{2}$

- Compute integrals

  $x_p(t) = \frac{(-10t+11)\cos(t)}{50} + \frac{\sin(t)(5t+2)}{50}$

• Substitute particular solution into general solution to ODE

  $x = C2\,\mathrm{e}^{-t} + C1\,\mathrm{e}^{-3t} + \frac{(-10t+11)\cos(t)}{50} + \frac{\sin(t)(5t+2)}{50}$

□ Check validity of solution $x = \_C2\mathrm{e}^{-t} + \_C1\mathrm{e}^{-3t} + \frac{(-10t+11)\cos(t)}{50} + \frac{\sin(t)(5t+2)}{50}$

- Use initial condition $x(0) = 0$

  $0 = \_C2 + \_C1 + \frac{11}{50}$

- Compute derivative of the solution

  $x' = -\_C2\mathrm{e}^{-t} - 3\_C1\,\mathrm{e}^{-3t} - \frac{\cos(t)}{5} - \frac{(-10t+11)\sin(t)}{50} + \frac{\cos(t)(5t+2)}{50} + \frac{\sin(t)}{10}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 1$

  $1 = -\_C2 - 3\_C1 - \frac{4}{25}$

- Solve for $\_C1$ and $\_C2$

  $\left\{\_C1 = -\frac{47}{100}, \_C2 = \frac{1}{4}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{\mathrm{e}^{-t}}{4} - \frac{47\,\mathrm{e}^{-3t}}{100} + \frac{(-10t+11)\cos(t)}{50} + \frac{\sin(t)(5t+2)}{50}$

• Solution to the IVP

  $x = \frac{\mathrm{e}^{-t}}{4} - \frac{47\,\mathrm{e}^{-3t}}{100} + \frac{(-10t+11)\cos(t)}{50} + \frac{\sin(t)(5t+2)}{50}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 35

```
dsolve([diff(diff(x(t),t),t)+4*diff(x(t),t)+3*x(t) = t*sin(t),
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = \frac{e^{-t}}{4} - \frac{47\,e^{-3t}}{100} + \frac{(-10t+11)\cos(t)}{50} + \frac{\sin(t)\,(5t+2)}{50}$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 42

```
DSolve[{D[x[t],{t,2}]+4*D[x[t],t]+3*x[t]==t*Sin[t],{x[0]==0,Derivative[1][x][0] == 1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{100}\left(e^{-3t}(25e^{2t}-47)+2(5t+2)\sin(t)+(22-20t)\cos(t)\right)$$

2.3.6 problem 7 (vi)

Internal problem ID [18206]
Book : Elementary Differential Equations. By R.L.E. Schwarzenberger. Chapman and Hall. London. First Edition (1969)
Section : Chapter 5. Linear equations. Exercises at page 85
Problem number : 7 (vi)
Date solved : `Thursday, December 19, 2024 at 06:20:11 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$x'' + x = \cos(t)$$

With initial conditions

$$\begin{aligned} x(0) &= 0 \\ x'(0) &= 1 \end{aligned}$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \cos(t) \end{aligned}$$

Hence the ode is

$$x'' + x = \cos(t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.218 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 1, f(t) = \cos(t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} + e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(t) + c_2 \sin(t))$$

Or

$$x = c_1 \cos(t) + c_2 \sin(t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(t), \sin(t)\}$$

Since $\cos(t)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\sin(t), \cos(t)\, t\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\sin(t) + A_2\cos(t)\, t$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\cos(t) - 2A_2\sin(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t\sin(t)}{2}$$

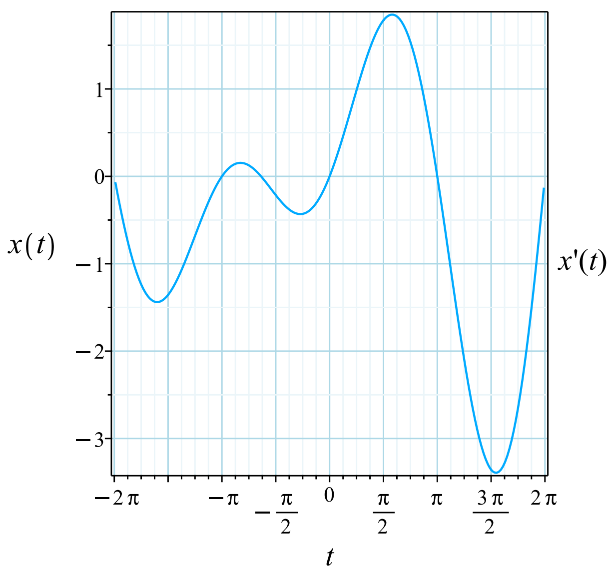
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1\cos(t) + c_2\sin(t)) + \left(\frac{t\sin(t)}{2}\right) \end{aligned}$$

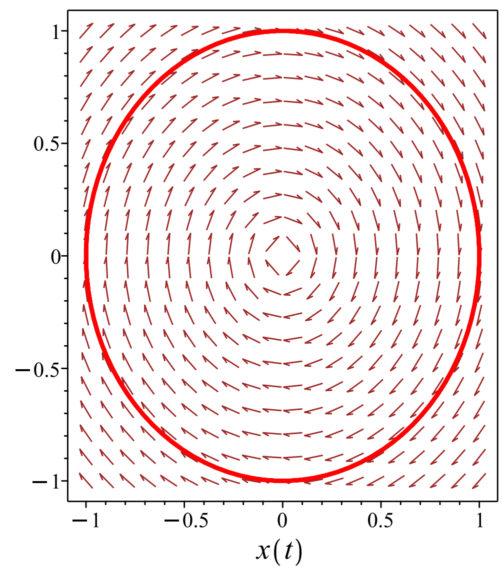
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{t \sin(t)}{2} + \sin(t)$$



(a) Solution plot
$x = \frac{t\sin(t)}{2} + \sin(t)$



(b) Slope field plot
$x'' + x = \cos(t)$

Solved as second order ode using Kovacic algorithm

Time used: 0.094 (sec)

Writing the ode as

$$x'' + x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.60: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \cos(t) \end{aligned}$$

Which simplifies to

$$x_1 = \cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\ &= \cos(t) \int \frac{1}{\cos(t)^2}\,dt \\ &= \cos(t)\,(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1x_1 + c_2x_2 \\ &= c_1(\cos(t)) + c_2(\cos(t)(\tan(t))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(t), \sin(t)\}$$

Since $\cos(t)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t \sin(t), \cos(t)\, t\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t \sin(t) + A_2 \cos(t)\, t$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(t) - 2A_2 \sin(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t \sin(t)}{2}$$

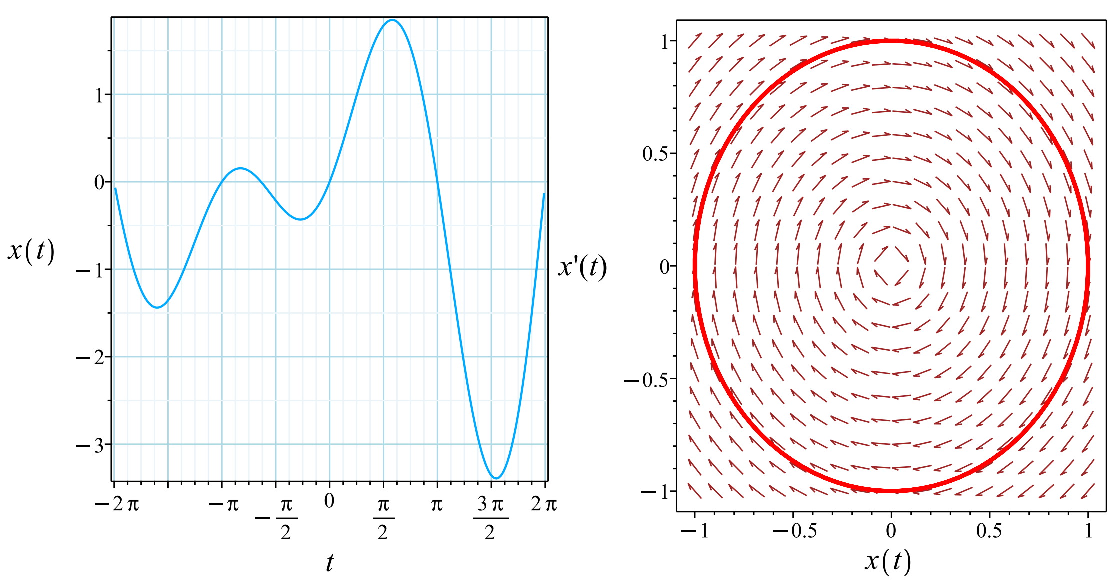
Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(t) + c_2 \sin(t)) + \left(\frac{t \sin(t)}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \frac{t \sin(t)}{2} + \sin(t)$$



(a) Solution plot $x = \frac{t \sin(t)}{2} + \sin(t)$

(b) Slope field plot $x'' + x = \cos(t)$

Solved as second order ode adjoint method

Time used: 5.898 (sec)

In normal form the ode

$$x'' + x = \cos(t) \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ r(t) &= \cos(t) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi^{''} - (\xi\,p)' + \xi q &= 0 \\ \xi^{''} - (0)' + (\xi(t)) &= 0 \\ \xi''(t) + \xi(t) &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(t) + B\xi'(t) + C\xi(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} + \mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$\xi = e^{0}(c_1 \cos(t) + c_2 \sin(t))$$

Or

$$\xi = c_1 \cos(t) + c_2 \sin(t)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(t)\, x' - x\xi'(t) + \xi(t)\, p(t)\, x = \int \xi(t)\, r(t)\, dt$$
$$x' + x\left(p(t) - \frac{\xi'(t)}{\xi(t)}\right) = \frac{\int \xi(t)\, r(t)\, dt}{\xi(t)}$$

Or

$$x' - \frac{x(-c_1 \sin(t) + c_2 \cos(t))}{c_1 \cos(t) + c_2 \sin(t)} = \frac{-\frac{\cos(t)^2 c_2}{2} + c_1\left(\frac{\sin(t)\cos(t)}{2} + \frac{t}{2}\right)}{c_1 \cos(t) + c_2 \sin(t)}$$

Which is now a first order ode. This is now solved for x. In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{-c_1 \sin(t) + c_2 \cos(t)}{c_1 \cos(t) + c_2 \sin(t)} \\
p(t) &= \frac{-\cos(t)^2 c_2 + c_1 \sin(t) \cos(t) + c_1 t}{2c_1 \cos(t) + 2c_2 \sin(t)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dt} \\
&= e^{\int -\frac{-c_1 \sin(t)+c_2 \cos(t)}{c_1 \cos(t)+c_2 \sin(t)} dt} \\
&= \frac{1}{c_1 \cos(t) + c_2 \sin(t)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = \mu p \\
&\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) = (\mu) \left(\frac{-\cos(t)^2 c_2 + c_1 \sin(t) \cos(t) + c_1 t}{2c_1 \cos(t) + 2c_2 \sin(t)} \right) \\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{c_1 \cos(t) + c_2 \sin(t)} \right) \\
&\quad = \left(\frac{1}{c_1 \cos(t) + c_2 \sin(t)} \right) \left(\frac{-\cos(t)^2 c_2 + c_1 \sin(t) \cos(t) + c_1 t}{2c_1 \cos(t) + 2c_2 \sin(t)} \right) \\
&\mathrm{d}\left(\frac{x}{c_1 \cos(t) + c_2 \sin(t)} \right) = \left(\frac{-\cos(t)^2 c_2 + c_1 \sin(t) \cos(t) + c_1 t}{(2c_1 \cos(t) + 2c_2 \sin(t))(c_1 \cos(t) + c_2 \sin(t))} \right) \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{x}{c_1 \cos(t) + c_2 \sin(t)} &= \int \frac{-\cos(t)^2 c_2 + c_1 \sin(t) \cos(t) + c_1 t}{(2c_1 \cos(t) + 2c_2 \sin(t))(c_1 \cos(t) + c_2 \sin(t))} \, dt \\
&= \frac{-t \tan\left(\frac{t}{2}\right) - 2t \tan\left(\frac{t}{2}\right)^3 - t \tan\left(\frac{t}{2}\right)^5 - \frac{1}{2} - \frac{\tan\left(\frac{t}{2}\right)^2}{2} + \frac{\tan\left(\frac{t}{2}\right)^4}{2} + \frac{\tan\left(\frac{t}{2}\right)^6}{2}}{\left(1 + \tan\left(\frac{t}{2}\right)^2\right)^2 \left(c_1 \tan\left(\frac{t}{2}\right)^2 - 2\tan\left(\frac{t}{2}\right) c_2 - c_1\right)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(t)+c_2 \sin(t)}$ gives the final solution

$$
x = -\frac{1}{2} + (2c_1 c_3 + 1) \cos\left(\frac{t}{2}\right)^2 + \sin\left(\frac{t}{2}\right) (2c_2 c_3 + t) \cos\left(\frac{t}{2}\right) - c_1 c_3
$$

Hence, the solution found using Lagrange adjoint equation method is

$$x = -\frac{1}{2} + (2c_1c_3 + 1)\cos\left(\frac{t}{2}\right)^2 + \sin\left(\frac{t}{2}\right)(2c_2c_3 + t)\cos\left(\frac{t}{2}\right) - c_1c_3$$

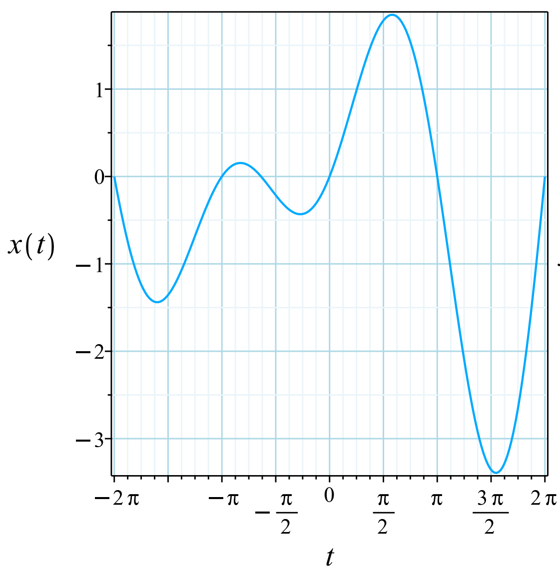
The constants can be merged to give

$$x = -\frac{1}{2} + (2c_1 + 1)\cos\left(\frac{t}{2}\right)^2 + \sin\left(\frac{t}{2}\right)(2c_2 + t)\cos\left(\frac{t}{2}\right) - c_1$$

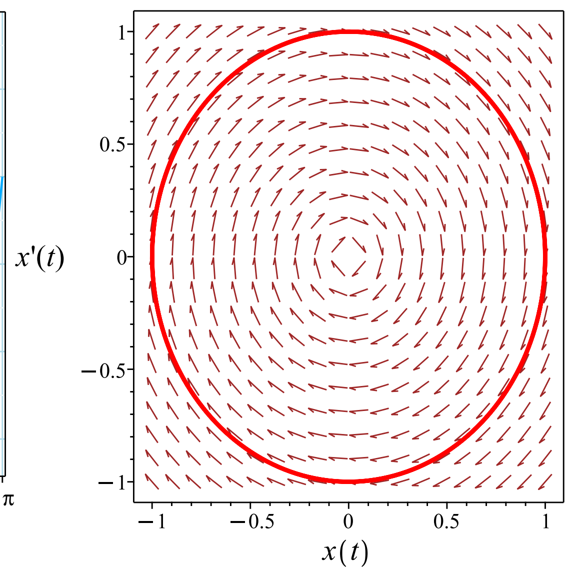
Will add steps showing solving for IC soon.

Summary of solutions found

$$x = \sin\left(\frac{t}{2}\right)(2 + t)\cos\left(\frac{t}{2}\right)$$



(a) Solution plot
$x = \sin\left(\frac{t}{2}\right)(2 + t)\cos\left(\frac{t}{2}\right)$



(b) Slope field plot
$x'' + x = \cos(t)$

Maple step by step solution

Let's solve

$\left[x''+x=\cos\left(t\right), x(0)=0, x'\Big|_{\{t=0\}}=1\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2+1=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t)=\cos\left(t\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t)=\sin\left(t\right)$

- General solution of the ODE

  $x=C1x_1(t)+C2x_2(t)+x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x=C1\cos\left(t\right)+C2\sin\left(t\right)+x_p(t)$

- □ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=\cos\left(t\right)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t))=\left[\begin{array}{cc}\cos\left(t\right) & \sin\left(t\right)\\ -\sin\left(t\right) & \cos\left(t\right)\end{array}\right]$

  - Compute Wronskian

    $W(x_1(t), x_2(t))=1$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t)=-\frac{\cos(t)\left(\int\sin(2t)dt\right)}{2}+\sin\left(t\right)\left(\int\cos\left(t\right)^2dt\right)$

  - Compute integrals

    $x_p(t)=\frac{\cos(t)}{4}+\frac{t\sin(t)}{2}$

- Substitute particular solution into general solution to ODE

$x = C1\cos(t) + C2\sin(t) + \frac{\cos(t)}{4} + \frac{t\sin(t)}{2}$

- ☐ Check validity of solution $x = \_C1\cos(t) + \_C2\sin(t) + \frac{\cos(t)}{4} + \frac{t\sin(t)}{2}$
  - ◦ Use initial condition $x(0) = 0$

    $0 = \_C1 + \frac{1}{4}$
  - ◦ Compute derivative of the solution

    $x' = -\_C1\sin(t) + \_C2\cos(t) + \frac{\sin(t)}{4} + \frac{\cos(t)t}{2}$
  - ◦ Use the initial condition $x'\Big|_{\{t=0\}} = 1$

    $1 = \_C2$
  - ◦ Solve for $\_C1$ and $\_C2$

    $\left\{\_C1 = -\frac{1}{4}, \_C2 = 1\right\}$
  - ◦ Substitute constant values into general solution and simplify

    $x = \sin(t)\left(1 + \frac{t}{2}\right)$
- • Solution to the IVP

  $x = \sin(t)\left(1 + \frac{t}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 12

```
dsolve([diff(diff(x(t),t),t)+x(t) = cos(t),
       op([x(0) = 0, D(x)(0) = 1])],x(t),singsol=all)
```

$$x = \sin(t)\left(1+\frac{t}{2}\right)$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 14

```
DSolve[{D[x[t],{t,2}]+x[t]==Cos[t],{x[0]==0,Derivative[1][x][0] == 1}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t)\to \frac{1}{2}(t+2)\sin(t)$$