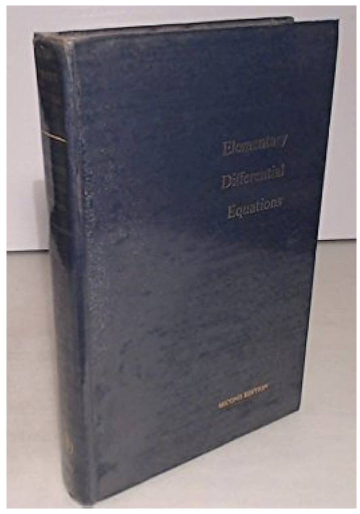
A Solution Manual For

Elementary Differential equations, Chaundy, 1969



Nasser M. Abbasi February 5, 2025 Compiled on February 5, 2025 at 4:04pm

Contents

CHAPTER 1

Lookup tables for all problems in current book

1.1 Exercises 3, page 60

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE
4190	1(a)	$y'y = x$
4191	1(b)	$y' - y = x^3$
4192	1(c)	$y' + y\cot(x) = x$
4193	1(d)	$y' + y\cot(x) = \tan(x)$
4194	1(e)	$y' + y\tan(x) = \cot(x)$
4195	1(f)	$y' + y\ln(x) = x^{-x}$
4196	2(a)	$y + xy' = x$
4197	2(b)	$xy' - y = x^3$
4198	2(c)	$xy' + ny = x^n$
4199	2(d)	$xy' - ny = x^n$
4200	2(e)	$(x^3 + x)\, y' + y = x$
4201	3(a)	$\cot(x)\, y' + y = x$
4202	3(b)	$\cot(x)\, y' + y = \tan(x)$
4203	3(c)	$\tan(x)\, y' + y = \cot(x)$
4204	3(a)	$\tan(x)\, y' = y - \cos(x)$
4205	4(a)	$y' + y\cos(x) = \sin(2x)$
4206	4(b)	$\cos(x)\, y' + y = \sin(2x)$
4207	4(c)	$y' + y\sin(x) = \sin(2x)$
4208	4(d)	$y'\sin(x) + y = \sin(2x)$
4209	5(a)	$\sqrt{x^2+1}\, y' + y = 2x$
4210	5(b)	$\sqrt{x^2+1}\, y' - y = 2\sqrt{x^2+1}$
4211	5(c)	$\sqrt{(x+a)(x+b)}\,(2y' - 3) + y = 0$
4212	5(d)	$\sqrt{(x+a)(x+b)}\, y' + y = \sqrt{x+a} - \sqrt{x+b}$

CHAPTER 2

Book Solved Problems

2.1 Exercises 3, page 60

2.1.1 Problem 1(a)

Internal problem ID [4190]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 1(a)
Date solved : `Monday, January 27, 2025 at 08:41:35 AM`
CAS classification : `[_separable]`

Solve

$$y'y = x$$

Solved as first order separable ode

Time used: 0.130 (sec)

The ode

$$y' = \frac{x}{y} \tag{2.1}$$

is separable as it can be written as

$$\begin{aligned} y' &= \frac{x}{y} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= x \\ g(y) &= \frac{1}{y} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int y\, dy &= \int x\, dx \end{aligned}$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

Solving for y gives

$$\begin{aligned} y &= \sqrt{x^2 + 2c_1} \\ y &= -\sqrt{x^2 + 2c_1} \end{aligned}$$

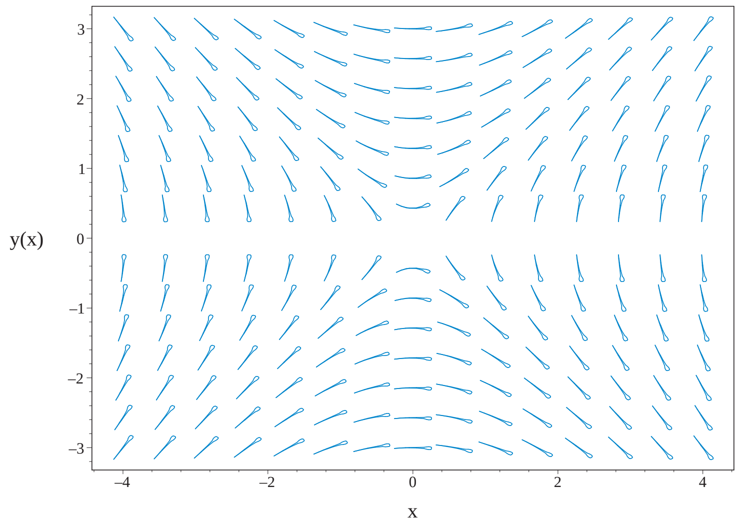


Figure 2.1: Slope field plot
$y'y = x$

Summary of solutions found

$$y = \sqrt{x^2 + 2c_1}$$
$$y = -\sqrt{x^2 + 2c_1}$$

Solved as first order homogeneous class A ode

Time used: 0.484 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x}{y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = x$ and $N = y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dx}x + u &= \frac{1}{u} \\ \frac{du}{dx} &= \frac{\frac{1}{u(x)} - u(x)}{x} \end{aligned}$$

Or

$$u'(x) - \frac{\frac{1}{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\, u(x)\, x + u(x)^2 - 1 = 0$$

Which is now solved as separable in $u(x)$.

The ode

$$u'(x) = -\frac{u(x)^2 - 1}{u(x)\, x} \tag{2.2}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)^2 - 1}{u(x)\, x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{u^2 - 1}{u} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{u}{u^2 - 1}\, du &= \int -\frac{1}{x}\, dx \end{aligned}$$

$$\frac{\ln\left(u(x)^2 - 1\right)}{2} = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2 - 1}{u} = 0$$

for $u(x)$ gives

$$\begin{aligned} u(x) &= -1 \\ u(x) &= 1 \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \frac{\ln\left(u(x)^2 - 1\right)}{2} &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= -1 \\ u(x) &= 1 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= -1 \\ u(x) &= 1 \\ u(x) &= \frac{\sqrt{e^{2c_1} + x^2}}{x} \\ u(x) &= -\frac{\sqrt{e^{2c_1} + x^2}}{x} \end{aligned}$$

Converting $u(x) = -1$ back to y gives

$$y = -x$$

Converting $u(x) = 1$ back to y gives

$$y = x$$

Converting $u(x) = \frac{\sqrt{e^{2c_1}+x^2}}{x}$ back to y gives

$$y = \sqrt{e^{2c_1} + x^2}$$

Converting $u(x) = -\frac{\sqrt{e^{2c_1}+x^2}}{x}$ back to y gives

$$y = -\sqrt{e^{2c_1} + x^2}$$

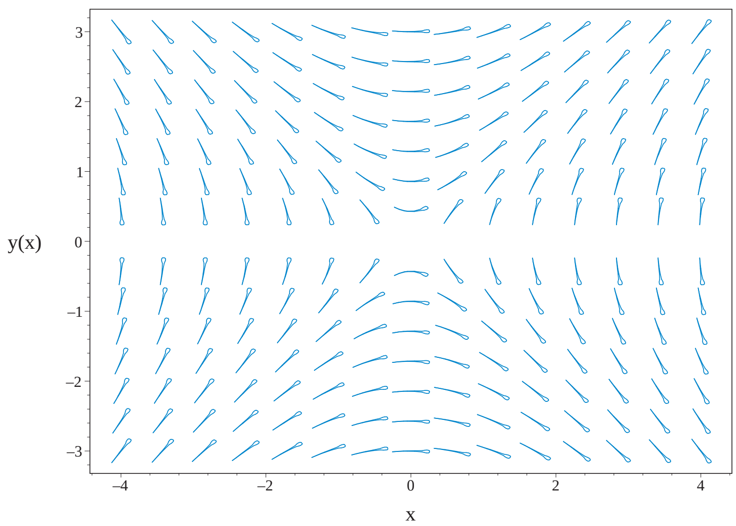


Figure 2.2: Slope field plot
$y'y = x$

Summary of solutions found

$$y = x$$
$$y = \sqrt{e^{2c_1} + x^2}$$
$$y = -x$$
$$y = -\sqrt{e^{2c_1} + x^2}$$

Solved as first order homogeneous class D2 ode

Time used: 0.297 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$(u'(x)\,x + u(x))\,u(x)\,x = x$$

Which is now solved The ode

$$u'(x) = -\frac{u(x)^2 - 1}{u(x)\,x} \tag{2.3}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)^2 - 1}{u(x)\,x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{u^2 - 1}{u} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$
$$\int \frac{u}{u^2-1}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln\left(u(x)^2-1\right)}{2} = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2-1}{u} = 0$$

for $u(x)$ gives

$$u(x) = -1$$
$$u(x) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(x)^2-1\right)}{2} = \ln\left(\frac{1}{x}\right) + c_1$$
$$u(x) = -1$$
$$u(x) = 1$$

Solving for $u(x)$ gives

$$u(x) = -1$$
$$u(x) = 1$$
$$u(x) = \frac{\sqrt{e^{2c_1} + x^2}}{x}$$
$$u(x) = -\frac{\sqrt{e^{2c_1} + x^2}}{x}$$

Converting $u(x) = -1$ back to y gives

$$y = -x$$

Converting $u(x) = 1$ back to y gives

$$y = x$$

Converting $u(x) = \frac{\sqrt{e^{2c_1}+x^2}}{x}$ back to y gives

$$y = \sqrt{e^{2c_1} + x^2}$$

Converting $u(x) = -\frac{\sqrt{e^{2c_1}+x^2}}{x}$ back to y gives

$$y = -\sqrt{e^{2c_1} + x^2}$$

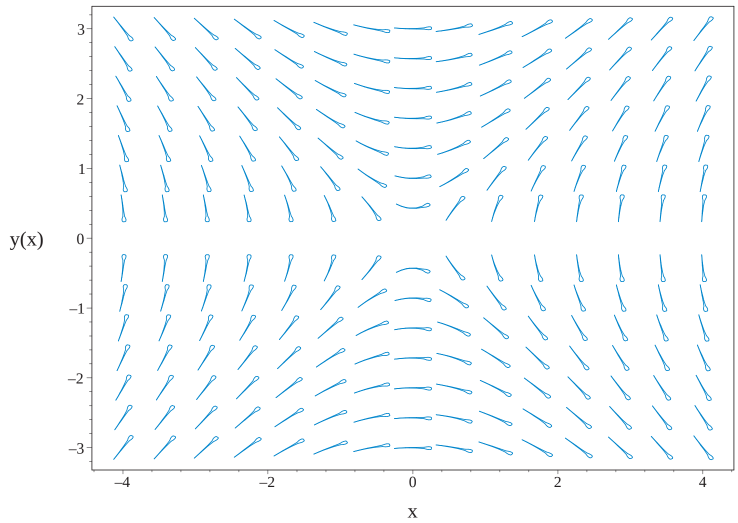


Figure 2.3: Slope field plot
$y'y = x$

Summary of solutions found

$$y = x$$
$$y = \sqrt{e^{2c_1} + x^2}$$
$$y = -x$$
$$y = -\sqrt{e^{2c_1} + x^2}$$

Solved as first order homogeneous class Maple C ode

Time used: 0.440 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{x_0 + X}{Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = \frac{1}{u}$$
$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{\frac{1}{u(X)} - u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) u(X)\, X + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{u(X)^2 - 1}{u\,(X)\,X} \tag{2.4}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{u(X)^2 - 1}{u\,(X)\,X}\\ &= f(X)g(u)\end{aligned}$$

Where

$$f(X) = -\frac{1}{X}$$
$$g(u) = \frac{u^2 - 1}{u}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(X)\, dX$$
$$\int \frac{u}{u^2 - 1}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln\left(u(X)^2 - 1\right)}{2} = \ln\left(\frac{1}{X}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2 - 1}{u} = 0$$

for $u(X)$ gives

$$u(X) = -1$$
$$u(X) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(X)^2-1\right)}{2} = \ln\left(\frac{1}{X}\right) + c_1$$
$$u(X) = -1$$
$$u(X) = 1$$

Solving for $u(X)$ gives

$$u(X) = -1$$
$$u(X) = 1$$
$$u(X) = \frac{\sqrt{e^{2c_1} + X^2}}{X}$$
$$u(X) = -\frac{\sqrt{e^{2c_1} + X^2}}{X}$$

Converting $u(X) = -1$ back to $Y(X)$ gives

$$Y(X) = -X$$

Converting $u(X) = 1$ back to $Y(X)$ gives

$$Y(X) = X$$

Converting $u(X) = \frac{\sqrt{e^{2c_1}+X^2}}{X}$ back to $Y(X)$ gives

$$Y(X) = \sqrt{e^{2c_1} + X^2}$$

Converting $u(X) = -\frac{\sqrt{e^{2c_1}+X^2}}{X}$ back to $Y(X)$ gives

$$Y(X) = -\sqrt{e^{2c_1} + X^2}$$

Using the solution for $Y(X)$

$$Y(X) = X \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = x$$

Using the solution for $Y(X)$

$$Y(X) = \sqrt{e^{2c_1} + X^2} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = \sqrt{e^{2c_1} + x^2}$$

Using the solution for $Y(X)$

$$Y(X) = -X \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = -x$$

Using the solution for $Y(X)$

$$Y(X) = -\sqrt{e^{2c_1} + X^2} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = -\sqrt{e^{2c_1} + x^2}$$

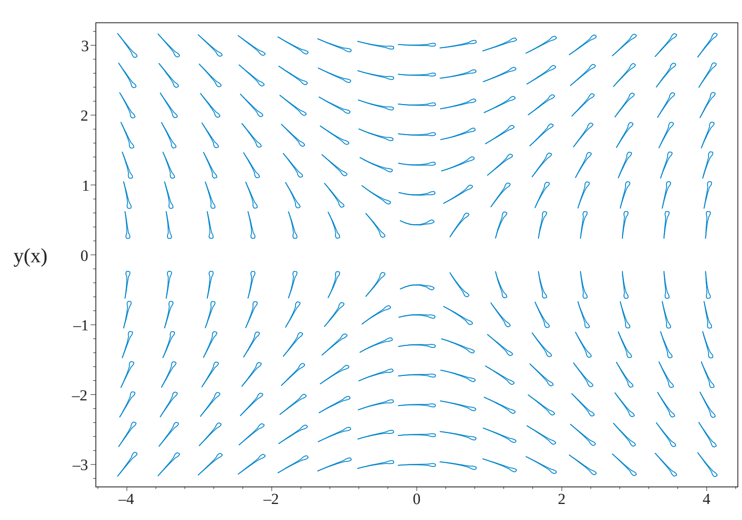


Figure 2.4: Slope field plot
$y'y = x$

Solved as first order Bernoulli ode

Time used: 0.064 (sec)

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= \frac{x}{y}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = (x)\frac{1}{y} \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

Comparing this to (1) shows that

$$
\begin{aligned}
f_0 &= 0 \\
f_1 &= x
\end{aligned}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_1(x) \tag{3}
$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= 0 \\
f_1(x) &= x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = 0 + x \tag{4}
$$

Let

$$
\begin{aligned}
v &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
v' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{v'(x)}{2} &= x \\
v' &= 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $v(x)$ which is now solved.

Since the ode has the form $v'(x) = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dv &= \int 2x\,dx \\
v(x) &= x^2 + c_1
\end{aligned}
$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$y^2 = x^2 + c_1$$

Solving for y gives

$$y = \sqrt{x^2 + c_1}$$
$$y = -\sqrt{x^2 + c_1}$$

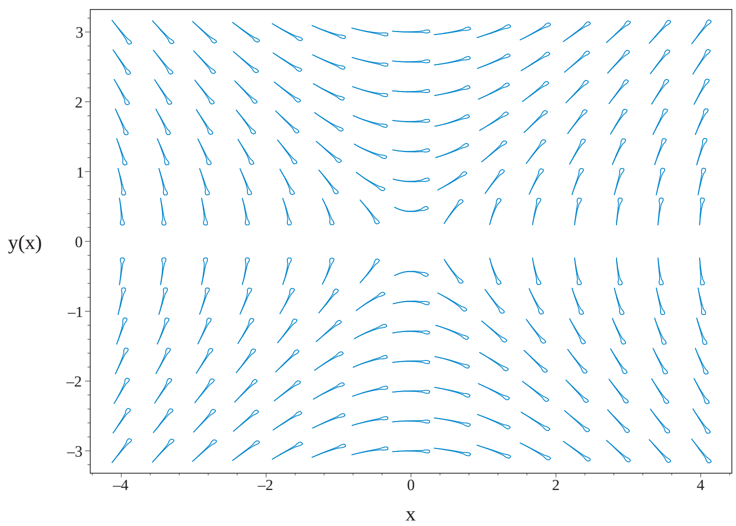


Figure 2.5: Slope field plot
$y'y = x$

Summary of solutions found

$$y = \sqrt{x^2 + c_1}$$
$$y = -\sqrt{x^2 + c_1}$$

Solved as first order Exact ode

Time used: 0.083 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y)\,\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+(y)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(y)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int M\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int -x\,\mathrm{d}x\end{aligned}$$

$$\phi=-\frac{x^2}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=y$. Therefore equation (4) becomes

$$y=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y)\,\mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^2}{2}$$

Solving for y gives

$$y = \sqrt{x^2 + 2c_1}$$
$$y = -\sqrt{x^2 + 2c_1}$$

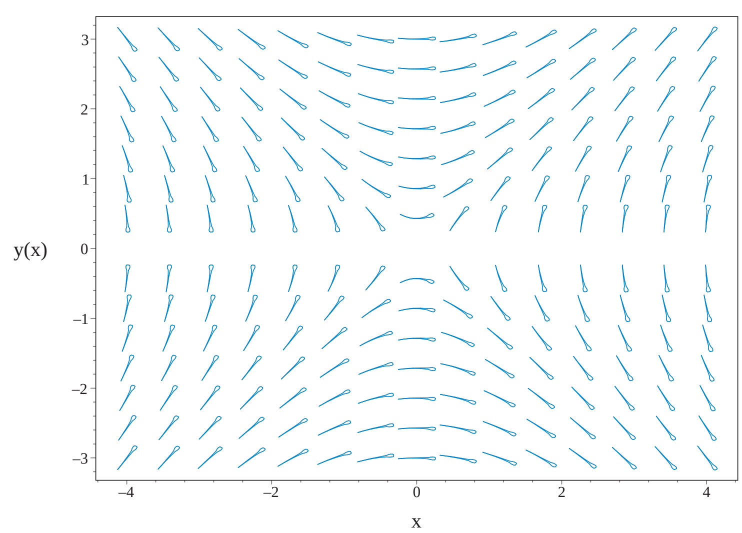


Figure 2.6: Slope field plot
$y'y = x$

Summary of solutions found

$$y = \sqrt{x^2 + 2c_1}$$
$$y = -\sqrt{x^2 + 2c_1}$$

Solved as first order isobaric ode

Time used: 0.330 (sec)

Solving for y' gives

$$y' = \frac{x}{y} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{x}{y} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = \frac{1}{u(x)}$$

The ode

$$u'(x) = -\frac{u(x)^2 - 1}{u(x) x} \tag{2.5}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)^2 - 1}{u(x) x} \\ &= f(x) g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{u^2 - 1}{u} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)} du = \int f(x) dx$$

$$\int \frac{u}{u^2 - 1} du = \int -\frac{1}{x} dx$$

$$\frac{\ln\left(u(x)^2 - 1\right)}{2} = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2 - 1}{u} = 0$$

for $u(x)$ gives

$$
\begin{aligned}
u(x) &= -1 \\
u(x) &= 1
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\frac{\ln \left(u(x)^2 - 1\right)}{2} &= \ln \left(\frac{1}{x}\right) + c_1 \\
u(x) &= -1 \\
u(x) &= 1
\end{aligned}
$$

Solving for $u(x)$ gives

$$
\begin{aligned}
u(x) &= -1 \\
u(x) &= 1 \\
u(x) &= \frac{\sqrt{e^{2c_1} + x^2}}{x} \\
u(x) &= -\frac{\sqrt{e^{2c_1} + x^2}}{x}
\end{aligned}
$$

Converting $u(x) = -1$ back to y gives

$$
\frac{y}{x} = -1
$$

Converting $u(x) = 1$ back to y gives

$$
\frac{y}{x} = 1
$$

Converting $u(x) = \frac{\sqrt{e^{2c_1}+x^2}}{x}$ back to y gives

$$
\frac{y}{x} = \frac{\sqrt{e^{2c_1} + x^2}}{x}
$$

Converting $u(x) = -\frac{\sqrt{e^{2c_1}+x^2}}{x}$ back to y gives

$$
\frac{y}{x} = -\frac{\sqrt{e^{2c_1} + x^2}}{x}
$$

Solving for y gives

$$
\begin{aligned}
y &= x \\
y &= \sqrt{e^{2c_1} + x^2} \\
y &= -x \\
y &= -\sqrt{e^{2c_1} + x^2}
\end{aligned}
$$

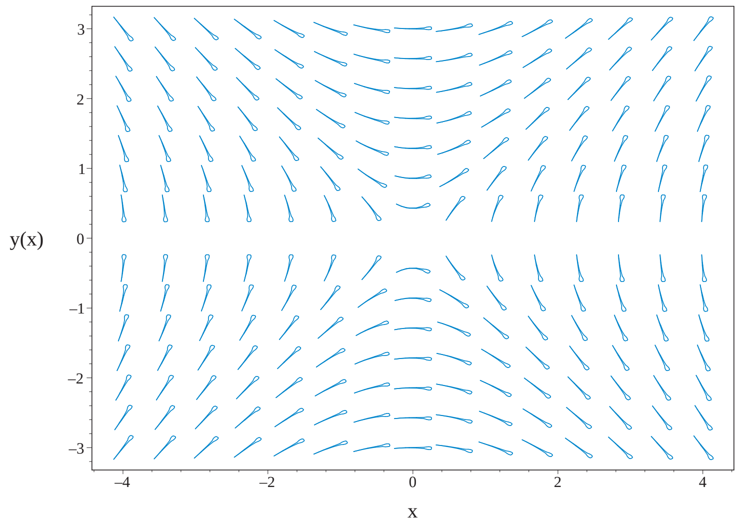


Figure 2.7: Slope field plot
$y'y = x$

Summary of solutions found

$$y = x$$
$$y = \sqrt{e^{2c_1} + x^2}$$
$$y = -x$$
$$y = -\sqrt{e^{2c_1} + x^2}$$

Solved using Lie symmetry for first order ode

Time used: 0.593 (sec)

Writing the ode as

$$y' = \frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{x(b_3 - a_2)}{y} - \frac{x^2 a_3}{y^2} - \frac{x a_2 + y a_3 + a_1}{y} + \frac{x(x b_2 + y b_3 + b_1)}{y^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2 a_3 - x^2 b_2 + 2yx a_2 - 2yx b_3 + y^2 a_3 - b_2 y^2 - x b_1 + y a_1}{y^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_3 + x^2b_2 - 2yxa_2 + 2yxb_3 - y^2a_3 + b_2y^2 + xb_1 - ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1v_2 - a_3v_1^2 - a_3v_2^2 + b_2v_1^2 + b_2v_2^2 + 2b_3v_1v_2 - a_1v_2 + b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_3 + b_2)\, v_1^2 + (-2a_2 + 2b_3)\, v_1v_2 + b_1v_1 + (-a_3 + b_2)\, v_2^2 - a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ -2a_2 + 2b_3 &= 0 \\ -a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{x}{y}\right)(x) \\ &= \frac{-x^2 + y^2}{y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2+y^2}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2+y^2)}{2}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{x^2-y^2} \\ S_y &= -\frac{y}{x^2-y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R,S.

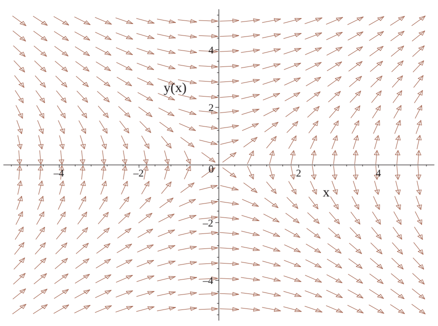
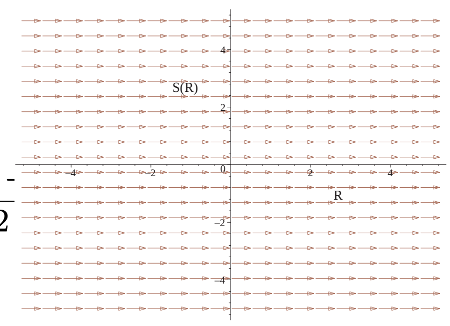
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 0\, dR + c_2 \\ S(R) &= c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(-x+y)}{2} + \frac{\ln(x+y)}{2} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{y}$ 	$R = x$ $S = \frac{\ln(-x+y)}{2} + \frac{\ln(x}{2}$	$\frac{dS}{dR} = 0$ 

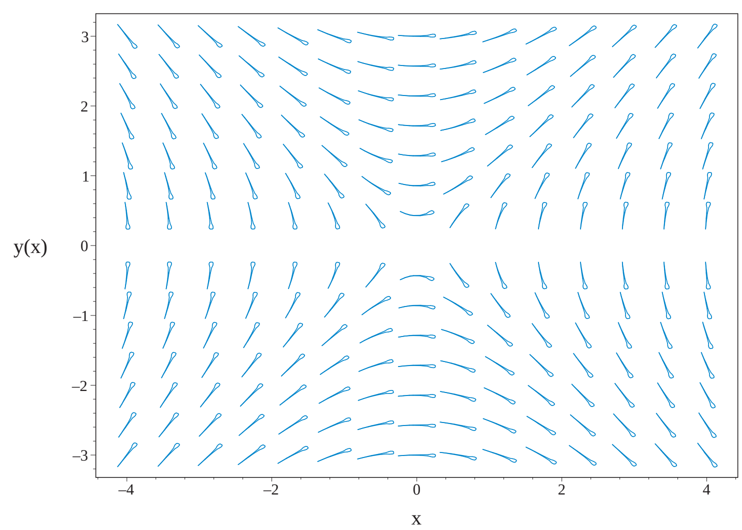


Figure 2.8: Slope field plot
$y'y = x$

<u>Summary of solutions found</u>

$$\frac{\ln(-x+y)}{2} + \frac{\ln(x+y)}{2} = c_2$$

Maple step by step solution

Let's solve
$y(x)\left(\frac{d}{dx}y(x)\right) = x$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Integrate both sides with respect to x
  $\int y(x)\left(\frac{d}{dx}y(x)\right)dx = \int x dx + C1$
- Evaluate integral
  $\frac{y(x)^2}{2} = \frac{x^2}{2} + C1$
- Solve for $y(x)$

$$\left\{y(x)=\sqrt{x^2+2\,C1}\,,y(x)=-\sqrt{x^2+2\,C1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 23

```
dsolve(y(x)*diff(y(x),x) = x,y(x),singsol=all)
```

$$y(x)=\sqrt{x^2+c_1}$$
$$y(x)=-\sqrt{x^2+c_1}$$

Mathematica DSolve solution

Solving time : 0.078 (sec)
Leaf size : 35

```
DSolve[{D[y[x],x]*y[x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to-\sqrt{x^2+2c_1}$$
$$y(x)\to\sqrt{x^2+2c_1}$$

2.1.2 Problem 1(b)

Internal problem ID [4191]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 1(b)
Date solved : `Monday, January 27, 2025 at 08:41:39 AM`
CAS classification : `[[_linear, 'class A']]`

Solve

$$y' - y = x^3$$

Solved as first order linear ode

Time used: 0.075 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -1 \\ p(x) &= x^3 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int (-1)dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x^3\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-x}\right) &= \left(\mathrm{e}^{-x}\right)\left(x^3\right) \\ \mathrm{d}\left(y\,\mathrm{e}^{-x}\right) &= \left(x^3\mathrm{e}^{-x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{-x} &= \int x^3\mathrm{e}^{-x}\,dx \\ &= -\left(x^3 + 3x^2 + 6x + 6\right)\mathrm{e}^{-x} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor e^{-x} gives the final solution

$$y = -x^3 + c_1\,\mathrm{e}^{x} - 3x^2 - 6x - 6$$

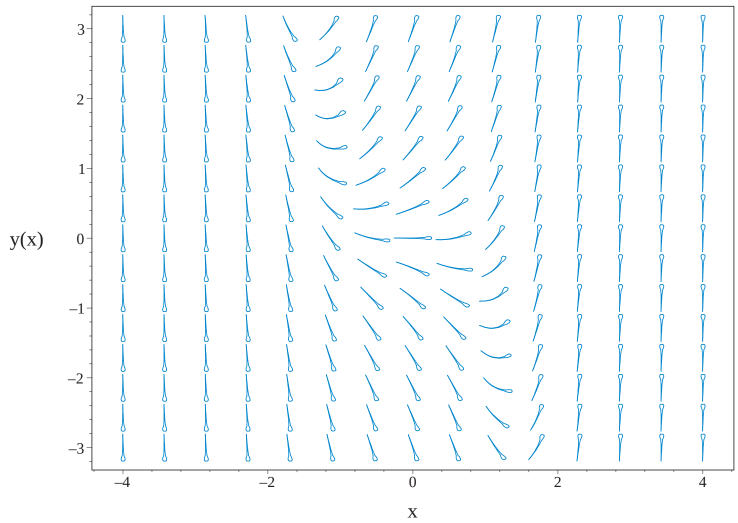


Figure 2.9: Slope field plot
$y' - y = x^3$

Summary of solutions found

$$y = -x^3 + c_1 e^x - 3x^2 - 6x - 6$$

Solved as first order Exact ode

Time used: 0.102 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(x^3 + y\right)\mathrm{d}x$$
$$\left(-x^3 - y\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -x^3 - y \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 - y\right) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}\left(-x^3 - y\right) \\ &= -\left(x^3 + y\right)\mathrm{e}^{-x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\left(x^3 + y\right)\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \overline{N} \,\mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \mathrm{e}^{-x} \,\mathrm{d}y$$

$$\phi = y\,\mathrm{e}^{-x} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -y\,\mathrm{e}^{-x} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -(x^3 + y)\,\mathrm{e}^{-x}$. Therefore equation (4) becomes

$$-\left(x^3 + y\right)\mathrm{e}^{-x} = -y\,\mathrm{e}^{-x} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -x^3\mathrm{e}^{-x}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-x^3\mathrm{e}^{-x}\right)\mathrm{d}x$$

$$f(x) = \left(x^3 + 3x^2 + 6x + 6\right)\mathrm{e}^{-x} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\,\mathrm{e}^{-x} + \left(x^3 + 3x^2 + 6x + 6\right)\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{-x} + \left(x^3 + 3x^2 + 6x + 6\right)\mathrm{e}^{-x}$$

Solving for y gives

$$y = -\left(x^3\mathrm{e}^{-x} + 3x^2\mathrm{e}^{-x} + 6x\,\mathrm{e}^{-x} + 6\,\mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

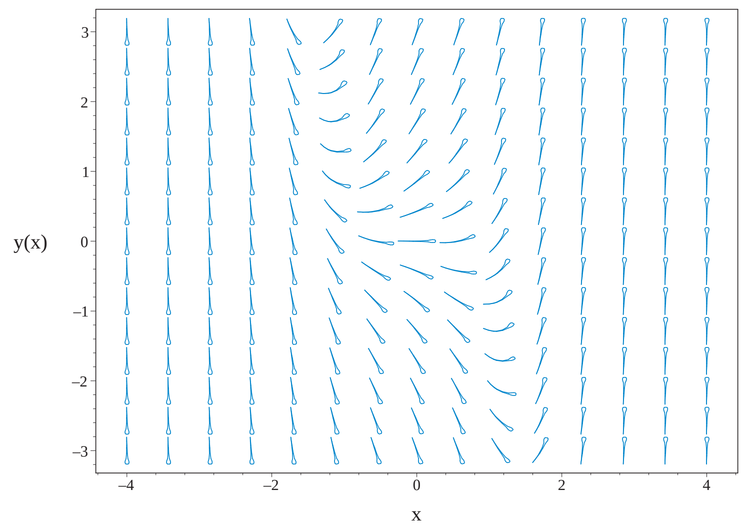


Figure 2.10: Slope field plot
$y' - y = x^3$

Summary of solutions found

$$y = -\left(x^3 e^{-x} + 3x^2 e^{-x} + 6x\, e^{-x} + 6\, e^{-x} - c_1\right) e^x$$

Solved using Lie symmetry for first order ode

Time used: 0.757 (sec)

Writing the ode as

$$y' = x^3 + y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + xya_5 + y^2 a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^2 b_4 + xyb_5 + y^2 b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &2xb_4 + yb_5 + b_2 + \left(x^3 + y\right)\left(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3\right) \\ &- \left(x^3 + y\right)^2 \left(xa_5 + 2ya_6 + a_3\right) - 3x^2\left(x^2 a_4 + xya_5 + y^2 a_6 + xa_2 + ya_3 + a_1\right) \\ &- x^2 b_4 - xyb_5 - y^2 b_6 - xb_2 - yb_3 - b_1 = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} &-x^7 a_5 - 2x^6 ya_6 - x^6 a_3 - 2x^4 ya_5 - 4x^3 y^2 a_6 - 5x^4 a_4 + x^4 b_5 - 2x^3 ya_3 - 4x^3 ya_5 \\ &+ 2x^3 yb_6 - 3x^2 y^2 a_6 - 4x^3 a_2 + x^3 b_3 - 3x^2 ya_3 - x\, y^2 a_5 - 2y^3 a_6 - 3x^2 a_1 \\ &- x^2 b_4 - 2xya_4 - y^2 a_3 - y^2 a_5 + y^2 b_6 - xb_2 + 2xb_4 - ya_2 + yb_5 - b_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^7a_5 - 2x^6ya_6 - x^6a_3 - 2x^4ya_5 - 4x^3y^2a_6 - 5x^4a_4 + x^4b_5 - 2x^3ya_3 - 4x^3ya_5 \\
&+ 2x^3yb_6 - 3x^2y^2a_6 - 4x^3a_2 + x^3b_3 - 3x^2ya_3 - x\,y^2a_5 - 2y^3a_6 - 3x^2a_1 \\
&- x^2b_4 - 2xya_4 - y^2a_3 - y^2a_5 + y^2b_6 - xb_2 + 2xb_4 - ya_2 + yb_5 - b_1 + b_2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_5v_1^7 - 2a_6v_1^6v_2 - a_3v_1^6 - 2a_5v_1^4v_2 - 4a_6v_1^3v_2^2 - 2a_3v_1^3v_2 - 5a_4v_1^4 - 4a_5v_1^3v_2 \\
&- 3a_6v_1^2v_2^2 + b_5v_1^4 + 2b_6v_1^3v_2 - 4a_2v_1^3 - 3a_3v_1^2v_2 - a_5v_1v_2^2 - 2a_6v_2^3 + b_3v_1^3 - 3a_1v_1^2 \\
&- a_3v_2^2 - 2a_4v_1v_2 - a_5v_2^2 - b_4v_1^2 + b_6v_2^2 - a_2v_2 - b_2v_1 + 2b_4v_1 + b_5v_2 - b_1 + b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-a_5v_1^7 - 2a_6v_1^6v_2 - a_3v_1^6 - 2a_5v_1^4v_2 + (-5a_4 + b_5)\,v_1^4 - 4a_6v_1^3v_2^2 \\
&+ (-2a_3 - 4a_5 + 2b_6)\,v_1^3v_2 + (-4a_2 + b_3)\,v_1^3 - 3a_6v_1^2v_2^2 \\
&- 3a_3v_1^2v_2 + (-3a_1 - b_4)\,v_1^2 - a_5v_1v_2^2 - 2a_4v_1v_2 + (-b_2 + 2b_4)\,v_1 \\
&- 2a_6v_2^3 + (-a_3 - a_5 + b_6)\,v_2^2 + (-a_2 + b_5)\,v_2 - b_1 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-3a_3 &= 0 \\
-a_3 &= 0 \\
-2a_4 &= 0 \\
-2a_5 &= 0 \\
-a_5 &= 0 \\
-4a_6 &= 0 \\
-3a_6 &= 0 \\
-2a_6 &= 0 \\
-3a_1 - b_4 &= 0 \\
-4a_2 + b_3 &= 0 \\
-a_2 + b_5 &= 0 \\
-5a_4 + b_5 &= 0 \\
-b_1 + b_2 &= 0 \\
-b_2 + 2b_4 &= 0 \\
-2a_3 - 4a_5 + 2b_6 &= 0 \\
-a_3 - a_5 + b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -\frac{b_4}{3} \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= 2b_4 \\
b_2 &= 2b_4 \\
b_3 &= 0 \\
b_4 &= b_4 \\
b_5 &= 0 \\
b_6 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{1}{3} \\
\eta &= x^2 + 2x + 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x^2 + 2x + 2 - \left(x^3 + y\right)\left(-\frac{1}{3}\right) \\
&= x^2 + 2x + 2 + \frac{1}{3}x^3 + \frac{1}{3}y \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x^2 + 2x + 2 + \frac{1}{3}x^3 + \frac{1}{3}y} dy
\end{aligned}
$$

Which results in

$$
S = 3\ln\left(x^3 + 3x^2 + 6x + y + 6\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x^3 + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{9x^2 + 18x + 18}{x^3 + 3x^2 + 6x + y + 6} \\ S_y &= \frac{3}{x^3 + 3x^2 + 6x + y + 6} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 3\, dR \\ S(R) &= 3R + c_2 \end{aligned}$$

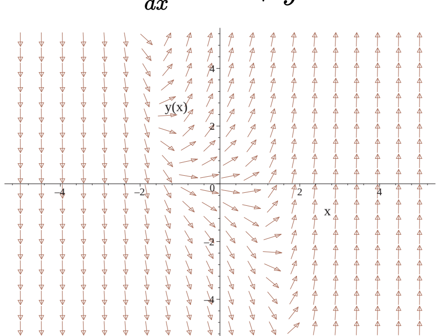
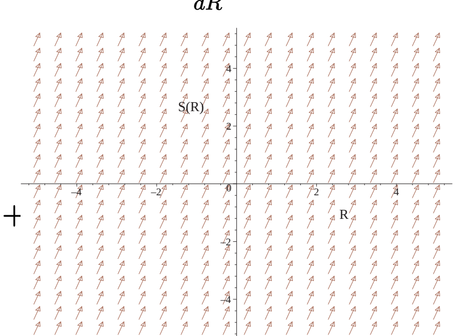
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$3\ln\left(x^3 + 3x^2 + 6x + y + 6\right) = 3x + c_2$$

Which gives

$$y = e^{x + \frac{c_2}{3}} - x^3 - 3x^2 - 6x - 6$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^3 + y$ 	$R = x$ $S = 3\ln\left(x^3 + 3x^2 + 6x +\right.$	$\frac{dS}{dR} = 3$ 

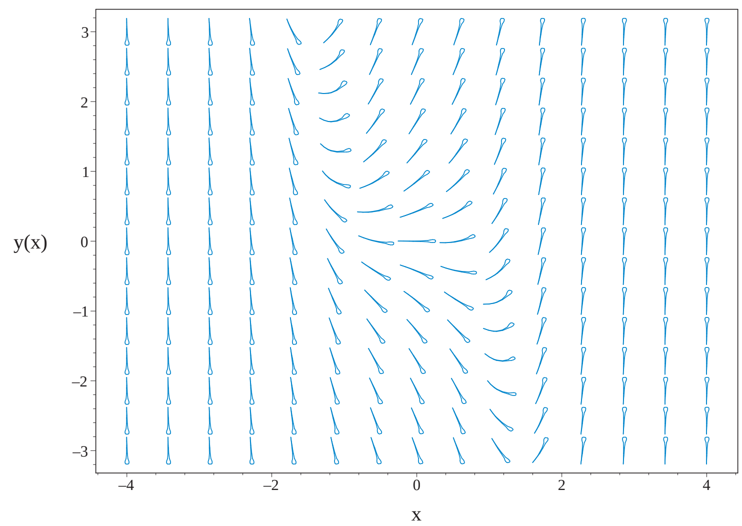


Figure 2.11: Slope field plot
$y' - y = x^3$

Summary of solutions found

$$y = e^{x+\frac{c_2}{3}} - x^3 - 3x^2 - 6x - 6$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) - y(x) = x^3$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = y(x) + x^3$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
$\frac{d}{dx}y(x) - y(x) = x^3$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$\mu(x)\left(\frac{d}{dx}y(x) - y(x)\right) = \mu(x)\,x^3$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
$\mu(x)\left(\frac{d}{dx}y(x) - y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
$\frac{d}{dx}\mu(x) = -\mu(x)$
- Solve to find the integrating factor
$\mu(x) = e^{-x}$
- Integrate both sides with respect to x
$\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \mu(x)\,x^3 dx + C1$
- Evaluate the integral on the lhs
$y(x)\,\mu(x) = \int \mu(x)\,x^3 dx + C1$
- Solve for $y(x)$
$y(x) = \frac{\int \mu(x)x^3 dx + C1}{\mu(x)}$
- Substitute $\mu(x) = e^{-x}$
$y(x) = \frac{\int x^3 e^{-x} dx + C1}{e^{-x}}$
- Evaluate the integrals on the rhs
$y(x) = \frac{-(x^3+3x^2+6x+6)e^{-x} + C1}{e^{-x}}$
- Simplify

$$y(x) = -x^3 + C1\,e^x - 3x^2 - 6x - 6$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 23

```
dsolve(diff(y(x),x)-y(x) = x^3,y(x),singsol=all)
```

$$y(x) = -x^3 - 3x^2 - 6x - 6 + e^x c_1$$

Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 26

```
DSolve[{D[y[x],x]-y[x]==x^3,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -x^3 - 3x^2 - 6x + c_1 e^x - 6$$

2.1.3 Problem 1(c)

Internal problem ID [4192]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 1(c)
Date solved : `Monday, January 27, 2025 at 08:41:41 AM`
CAS classification : `[_linear]`

Solve

$$y' + y\cot(x) = x$$

Solved as first order linear ode

Time used: 0.085 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \cot(x) \\ p(x) &= x \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \cot(x)dx} \\ &= \sin(x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\sin(x)) &= (\sin(x))(x) \\ \mathrm{d}(y\sin(x)) &= (x\sin(x))\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\sin(x) &= \int x\sin(x)\,dx \\ &= \sin(x) - x\cos(x) + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\sin(x)$ gives the final solution

$$y = 1 - x\cot(x) + c_1\csc(x)$$

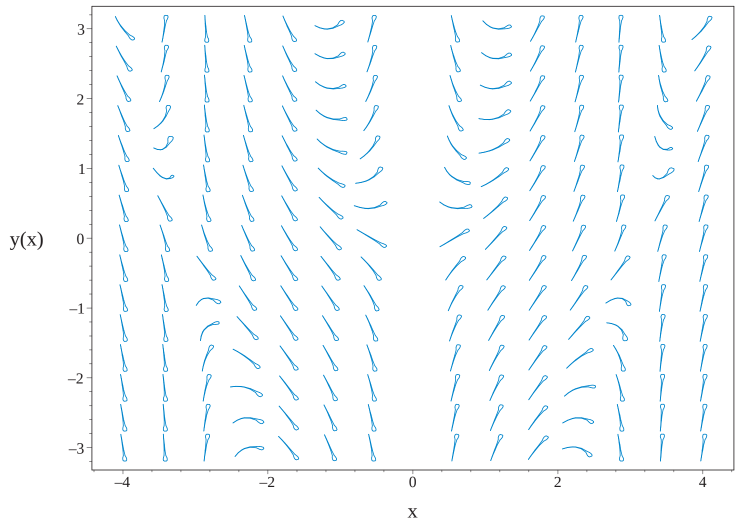


Figure 2.12: Slope field plot
$y' + y\cot(x) = x$

Summary of solutions found

$$y = 1 - x\cot(x) + c_1\csc(x)$$

Solved as first order Exact ode

Time used: 0.157 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y\cot(x) + x)\,\mathrm{d}x$$
$$(y\cot(x) - x)\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y \cot(x) - x \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y \cot(x) - x) \\
&= \cot(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((\cot(x)) - (0)) \\
&= \cot(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \cot(x) \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\sin(x))} \\
&= \sin(x)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \sin(x)(y \cot(x) - x) \\
&= y \cos(x) - x \sin(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sin(x)(1) \\
&= \sin(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(y \cos(x) - x \sin(x)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \overline{N} \,\mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \sin(x) \,\mathrm{d}y$$

$$\phi = y \sin(x) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = y \cos(x) + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = y \cos(x) - x \sin(x)$. Therefore equation (4) becomes

$$y \cos(x) - x \sin(x) = y \cos(x) + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -x \sin(x)$$

Integrating the above w.r.t x gives

$$\int f'(x) \,\mathrm{d}x = \int (-x \sin(x)) \,\mathrm{d}x$$

$$f(x) = -\sin(x) + x \cos(x) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y \sin(x) - \sin(x) + x \cos(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y \sin(x) - \sin(x) + x \cos(x)$$

Solving for y gives

$$y = -\frac{-\sin(x) + x \cos(x) - c_1}{\sin(x)}$$

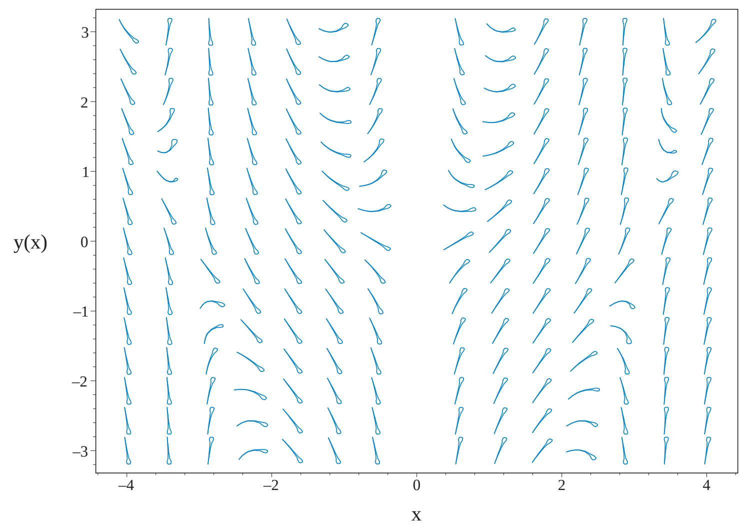


Figure 2.13: Slope field plot
$y' + y\cot(x) = x$

Summary of solutions found

$$y = -\frac{-\sin(x) + x\cos(x) - c_1}{\sin(x)}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) + y(x)\cot(x) = x$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = -y(x)\cot(x) + x$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + y(x)\cot(x) = x$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\cot(x)\right) = \mu(x)\,x$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\cot(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \mu(x)\cot(x)$
- Solve to find the integrating factor
  $\mu(x) = \sin(x)$
- Integrate both sides with respect to x
  $\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \mu(x)\,x dx + C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x) = \int \mu(x)\,x dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int \mu(x)x dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \sin(x)$
  $y(x) = \frac{\int \sin(x)x dx + C1}{\sin(x)}$
- Evaluate the integrals on the rhs
  $y(x) = \frac{\sin(x) - x\cos(x) + C1}{\sin(x)}$

- Simplify
  $y(x) = 1 - x\cot(x) + C1\csc(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 15

```
dsolve(diff(y(x),x)+y(x)*cot(x) = x,y(x),singsol=all)
```

$$y(x) = 1 - \cot(x)\, x + \csc(x)\, c_1$$

Mathematica DSolve solution

Solving time : `0.04` (sec)
Leaf size : 17

```
DSolve[{D[y[x],x]+y[x]*Cot[x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -x\cot(x) + c_1\csc(x) + 1$$

2.1.4 Problem 1(d)

Internal problem ID [4193]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 1(d)
Date solved : `Monday, January 27, 2025 at 08:41:43 AM`
CAS classification : `[_linear]`

Solve

$$y' + y\cot(x) = \tan(x)$$

Solved as first order linear ode

Time used: 0.122 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \cot(x)$$
$$p(x) = \tan(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= e^{\int \cot(x)dx} \\ &= \sin(x)\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)(\tan(x))$$
$$\frac{d}{dx}(y\sin(x)) = (\sin(x))(\tan(x))$$
$$d(y\sin(x)) = (\tan(x)\sin(x))\, dx$$

Integrating gives

$$\begin{aligned}y\sin(x) &= \int \tan(x)\sin(x)\, dx \\ &= -\sin(x) + \ln(\sec(x) + \tan(x)) + c_1\end{aligned}$$

Dividing throughout by the integrating factor $\sin(x)$ gives the final solution

$$y = (-\sin(x) + \ln(\sec(x) + \tan(x)) + c_1)\csc(x)$$

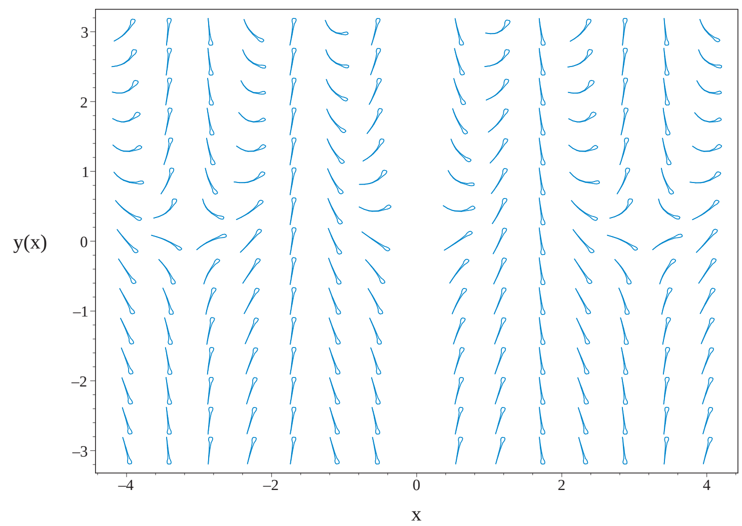


Figure 2.14: Slope field plot
$y' + y\cot(x) = \tan(x)$

Summary of solutions found

$$y = (-\sin(x) + \ln(\sec(x) + \tan(x)) + c_1)\csc(x)$$

Solved as first order Exact ode

Time used: 0.118 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y\cot(x) + \tan(x))\,\mathrm{d}x$$
$$(y\cot(x) - \tan(x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= y\cot(x) - \tan(x) \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\cot(x) - \tan(x)) \\ &= \cot(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cot(x)) - (0)) \\ &= \cot(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cot(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\sin(x))} \\ &= \sin(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sin(x)(y\cot(x) - \tan(x)) \\ &= \cos(x)\, y - \tan(x)\sin(x) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sin(x)(1) \\ &= \sin(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\cos(x)\, y - \tan(x)\sin(x)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y}\, \mathrm{d}y = \int \overline{N}\, \mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y}\, \mathrm{d}y = \int \sin(x)\, \mathrm{d}y$$

$$\phi = y \sin(x) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \cos(x)\, y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \cos(x)\, y - \tan(x)\sin(x)$. Therefore equation (4) becomes

$$\cos(x)\, y - \tan(x)\sin(x) = \cos(x)\, y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\tan(x)\sin(x)$$

Integrating the above w.r.t x gives

$$\int f'(x)\, \mathrm{d}x = \int (-\tan(x)\sin(x))\, \mathrm{d}x$$

$$f(x) = \sin(x) - \ln(\sec(x) + \tan(x)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\sin(x) + \sin(x) - \ln(\sec(x) + \tan(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\sin(x) + \sin(x) - \ln(\sec(x) + \tan(x))$$

Solving for y gives

$$y = -\frac{\sin(x) - \ln(\sec(x) + \tan(x)) - c_1}{\sin(x)}$$

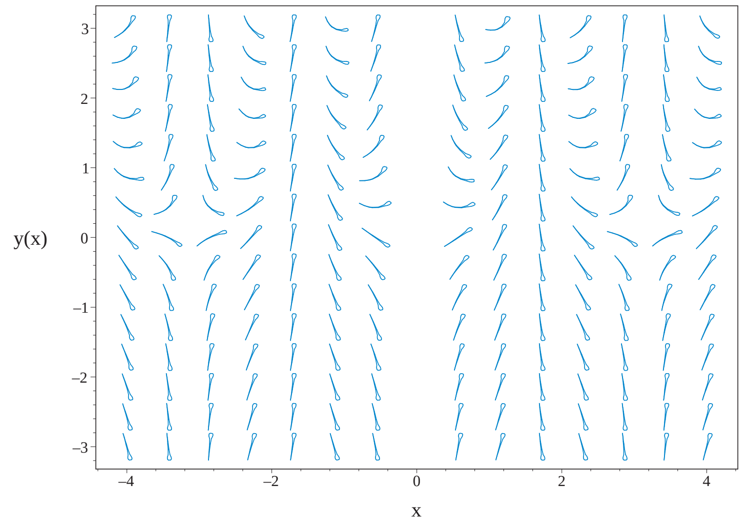


Figure 2.15: Slope field plot
$y' + y\cot(x) = \tan(x)$

Summary of solutions found

$$y = -\frac{\sin(x) - \ln(\sec(x) + \tan(x)) - c_1}{\sin(x)}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) + y(x)\cot(x) = \tan(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = -y(x)\cot(x) + \tan(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + y(x)\cot(x) = \tan(x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\cot(x)\right) = \mu(x)\tan(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\cot(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \mu(x)\cot(x)$
- Solve to find the integrating factor
  $\mu(x) = \sin(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y(x)\mu(x))\right)dx = \int \mu(x)\tan(x)\,dx + C1$
- Evaluate the integral on the lhs
  $y(x)\mu(x) = \int \mu(x)\tan(x)\,dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int \mu(x)\tan(x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \sin(x)$
  $y(x) = \frac{\int \sin(x)\tan(x)dx + C1}{\sin(x)}$
- Evaluate the integrals on the rhs
  $y(x) = \frac{-\sin(x) + \ln(\sec(x) + \tan(x)) + C1}{\sin(x)}$

- Simplify
  $y(x) = (-\sin(x) + \ln(\sec(x) + \tan(x)) + C1)\csc(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 19

```
dsolve(diff(y(x),x)+y(x)*cot(x) = tan(x),y(x),singsol=all)
```

$$y(x) = \csc(x)(-\sin(x) + \ln(\sec(x) + \tan(x)) + c_1)$$

Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 18

```
DSolve[{D[y[x],x]+y[x]*Cot[x]==Tan[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \csc(x)\operatorname{arctanh}(\sin(x)) + c_1\csc(x) - 1$$

2.1.5 Problem 1(e)

Internal problem ID [4194]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 1(e)
Date solved : `Monday, January 27, 2025 at 08:41:45 AM`
CAS classification : `[_linear]`

Solve

$$y' + y\tan(x) = \cot(x)$$

Solved as first order linear ode

Time used: 0.114 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \tan(x) \\ p(x) &= \cot(x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int \tan(x)dx} \\ &= \sec(x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)(\cot(x)) \\ \frac{d}{dx}(y\sec(x)) &= (\sec(x))(\cot(x)) \\ d(y\sec(x)) &= (\cot(x)\sec(x))\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\sec(x) &= \int \cot(x)\sec(x)\,dx \\ &= \ln(\csc(x) - \cot(x)) + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\sec(x)$ gives the final solution

$$y = \cos(x)(\ln(\csc(x) - \cot(x)) + c_1)$$

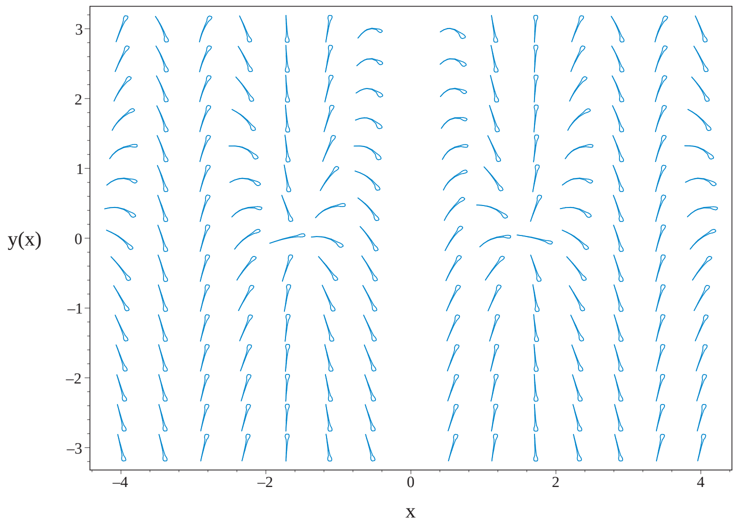


Figure 2.16: Slope field plot
$y' + y\tan(x) = \cot(x)$

Summary of solutions found

$$y = \cos(x)(\ln(\csc(x) - \cot(x)) + c_1)$$

Solved as first order Exact ode

Time used: 0.103 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y\tan(x) + \cot(x))\,\mathrm{d}x$$
$$(y\tan(x) - \cot(x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= y\tan(x) - \cot(x) \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\tan(x) - \cot(x)) \\ &= \tan(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\tan(x)) - (0)) \\ &= \tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int \tan(x) \,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(\cos(x))} \\ &= \sec(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sec(x)(y\tan(x) - \cot(x)) \\ &= y\tan(x)\sec(x) - \csc(x) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sec(x)(1) \\ &= \sec(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (y\tan(x)\sec(x) - \csc(x)) + (\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \overline{N} \, \mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \sec(x) \, \mathrm{d}y$$

$$\phi = y \sec(x) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = y \tan(x) \sec(x) + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = y \tan(x) \sec(x) - \csc(x)$. Therefore equation (4) becomes

$$y \tan(x) \sec(x) - \csc(x) = y \tan(x) \sec(x) + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\csc(x)$$

Integrating the above w.r.t x gives

$$\int f'(x) \, \mathrm{d}x = \int (-\csc(x)) \, \mathrm{d}x$$

$$f(x) = \ln(\csc(x) + \cot(x)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y \sec(x) + \ln(\csc(x) + \cot(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y \sec(x) + \ln(\csc(x) + \cot(x))$$

Solving for y gives

$$y = -\frac{\ln(\csc(x) + \cot(x)) - c_1}{\sec(x)}$$

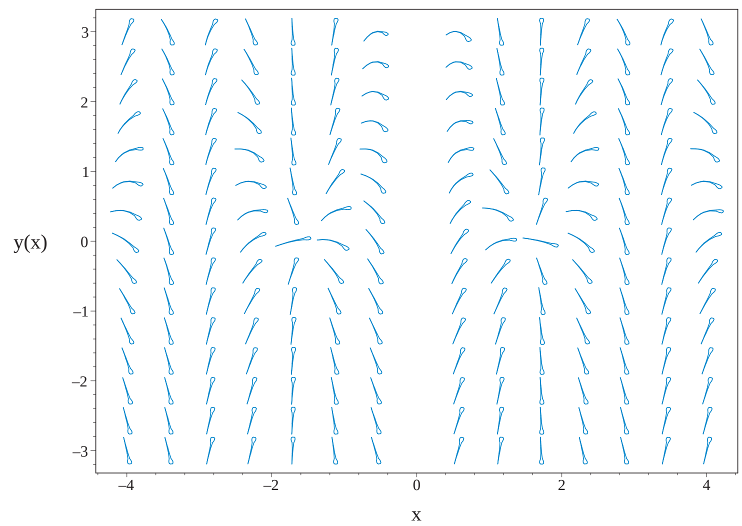


Figure 2.17: Slope field plot
$y' + y\tan(x) = \cot(x)$

Summary of solutions found

$$y = -\frac{\ln(\csc(x) + \cot(x)) - c_1}{\sec(x)}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) + y(x)\tan(x) = \cot(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = -y(x)\tan(x) + \cot(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + y(x)\tan(x) = \cot(x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\tan(x)\right) = \mu(x)\cot(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\tan(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \mu(x)\tan(x)$
- Solve to find the integrating factor
  $\mu(x) = \frac{1}{\cos(x)}$
- Integrate both sides with respect to x
  $\int\left(\frac{d}{dx}(y(x)\mu(x))\right)dx = \int \mu(x)\cot(x)\,dx + C1$
- Evaluate the integral on the lhs
  $y(x)\mu(x) = \int \mu(x)\cot(x)\,dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int \mu(x)\cot(x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{\cos(x)}$
  $y(x) = \cos(x)\left(\int \frac{\cot(x)}{\cos(x)}dx + C1\right)$
- Evaluate the integrals on the rhs
  $y(x) = \cos(x)(\ln(\csc(x) - \cot(x)) + C1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 17

```
dsolve(diff(y(x),x)+y(x)*tan(x) = cot(x),y(x),singsol=all)
```

$$y(x) = (-\ln(\csc(x) + \cot(x)) + c_1)\cos(x)$$

Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 16

```
DSolve[{D[y[x],x]+y[x]*Tan[x]==Cot[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \cos(x)(-\text{arctanh}(\cos(x)) + c_1)$$

2.1.6 Problem 1(f)

Internal problem ID [4195]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 1(f)
Date solved : `Monday, January 27, 2025 at 08:41:47 AM`
CAS classification : `[_linear]`

Solve

$$y' + y\ln(x) = x^{-x}$$

Solved as first order linear ode

Time used: 0.266 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \ln(x)$$
$$p(x) = x^{-x}$$

The integrating factor μ is

$$\mu = e^{\int \ln(x)dx}$$

Therefore the solution is

$$y = \left(\int x^{-x} e^{\int \ln(x)dx} dx + c_1\right) e^{-\int \ln(x)dx}$$

Summary of solutions found

$$y = \left(\int x^{-x} e^{\int \ln(x)dx} dx + c_1\right) e^{-\int \ln(x)dx}$$

Solved as first order Exact ode

Time used: 0.166 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-y\ln(x) + x^{-x}\right)\mathrm{d}x\\ \left(y\ln(x) - x^{-x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\ln(x) - x^{-x}\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\ln(x) - x^{-x}\right)\\ &= \ln(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((\ln(x)) - (0))\\ &= \ln(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \ln(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x\ln(x)-x} \\ &= x^x \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^x \mathrm{e}^{-x}\left(y\ln(x) - x^{-x}\right) \\ &= \mathrm{e}^{-x}(y\ln(x)\,x^x - 1)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x^x \mathrm{e}^{-x}(1) \\ &= x^x \mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\mathrm{e}^{-x}(y\ln(x)\,x^x - 1)\right) + \left(x^x\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y \\ \int \frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int x^x\mathrm{e}^{-x}\,\mathrm{d}y\end{aligned}$$

$$\phi = x^x\mathrm{e}^{-x}y + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = x^x(\ln(x) + 1)\,\mathrm{e}^{-x}y - x^x\mathrm{e}^{-x}y + f'(x) \tag{4}$$

$$= x^x\mathrm{e}^{-x}y\ln(x) + f'(x)$$

But equation (1) says that $\frac{\partial\phi}{\partial x} = \mathrm{e}^{-x}(y\ln(x)\,x^x - 1)$. Therefore equation (4) becomes

$$\mathrm{e}^{-x}(y\ln(x)\,x^x - 1) = x^x\mathrm{e}^{-x}y\ln(x) + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\mathrm{e}^{-x}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\mathrm{e}^{-x}\right)\mathrm{d}x$$
$$f(x) = \mathrm{e}^{-x} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = x^x \mathrm{e}^{-x} y + \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x^x \mathrm{e}^{-x} y + \mathrm{e}^{-x}$$

Solving for y gives

$$y = -\left(\mathrm{e}^{-x} - c_1\right) x^{-x} \mathrm{e}^{x}$$

Summary of solutions found

$$y = -\left(\mathrm{e}^{-x} - c_1\right) x^{-x} \mathrm{e}^{x}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) + y(x)\ln(x) = x^{-x}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = -y(x)\ln(x) + x^{-x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + y(x)\ln(x) = x^{-x}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\ln(x)\right) = \mu(x)\,x^{-x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\ln(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \mu(x)\ln(x)$
- Solve to find the integrating factor
  $\mu(x) = x^x \mathrm{e}^{-x}$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \mu(x)\,x^{-x}dx + C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x) = \int \mu(x)\,x^{-x}dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int \mu(x)x^{-x}dx + C1}{\mu(x)}$
- Substitute $\mu(x) = x^x \mathrm{e}^{-x}$
  $y(x) = \frac{\int x^x \mathrm{e}^{-x} x^{-x}dx + C1}{x^x \mathrm{e}^{-x}}$

- Evaluate the integrals on the rhs

  $y(x) = \frac{-e^{-x}+C1}{x^x e^{-x}}$
- Simplify

  $y(x) = x^{-x}(C1\, e^x - 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 16

```
dsolve(diff(y(x),x)+y(x)*ln(x) = x^(-x),y(x),singsol=all)
```

$$y(x) = (e^x c_1 - 1)\, x^{-x}$$

Mathematica DSolve solution

Solving time : 0.083 (sec)
Leaf size : 19

```
DSolve[{D[y[x],x]+y[x]*Log[x]==x^(-x),{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^{-x}(-1 + c_1 e^x)$$

2.1.7 Problem 2(a)

Internal problem ID [4196]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 2(a)
Date solved : `Monday, January 27, 2025 at 08:41:49 AM`
CAS classification : `[_linear]`

Solve

$$xy' + y = x$$

Solved as first order linear ode

Time used: 0.036 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{1}{x} \\ p(x) &= 1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu \\ \frac{\mathrm{d}}{\mathrm{d}x}(yx) &= x \\ \mathrm{d}(yx) &= x\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} yx &= \int x\,dx \\ &= \frac{x^2}{2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor x gives the final solution

$$y = \frac{x^2 + 2c_1}{2x}$$

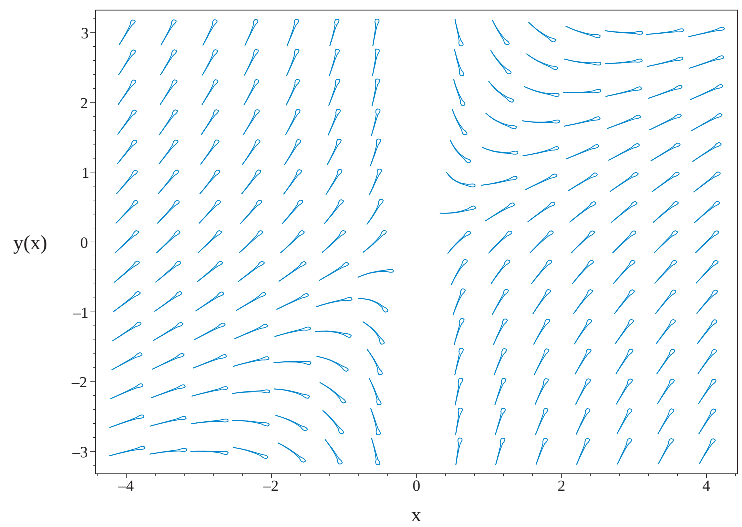


Figure 2.18: Slope field plot
$xy' + y = x$

Summary of solutions found

$$y = \frac{x^2 + 2c_1}{2x}$$

Solved as first order homogeneous class A ode

Time used: 0.245 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y - x}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = -y + x$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = 1 - u$$

$$\frac{du}{dx} = \frac{1 - 2u(x)}{x}$$

Or

$$u'(x) - \frac{1 - 2u(x)}{x} = 0$$

Or

$$u'(x)\, x + 2u(x) - 1 = 0$$

Which is now solved as separable in $u(x)$.

The ode

$$u'(x) = -\frac{2u(x) - 1}{x} \tag{2.6}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{2u(x) - 1}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= -2u + 1 \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$

$$\int \frac{1}{-2u + 1}\, du = \int \frac{1}{x}\, dx$$

$$-\frac{\ln(2u(x) - 1)}{2} = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-2u + 1 = 0$$

for $u(x)$ gives

$$u(x) = \frac{1}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln(2u(x) - 1)}{2} = \ln(x) + c_1$$

$$u(x) = \frac{1}{2}$$

Solving for $u(x)$ gives

$$u(x) = \frac{1}{2}$$

$$u(x) = \frac{x^2 + e^{-2c_1}}{2x^2}$$

Converting $u(x) = \frac{1}{2}$ back to y gives

$$y = \frac{x}{2}$$

Converting $u(x) = \frac{x^2+e^{-2c_1}}{2x^2}$ back to y gives

$$y = \frac{x^2 + e^{-2c_1}}{2x}$$

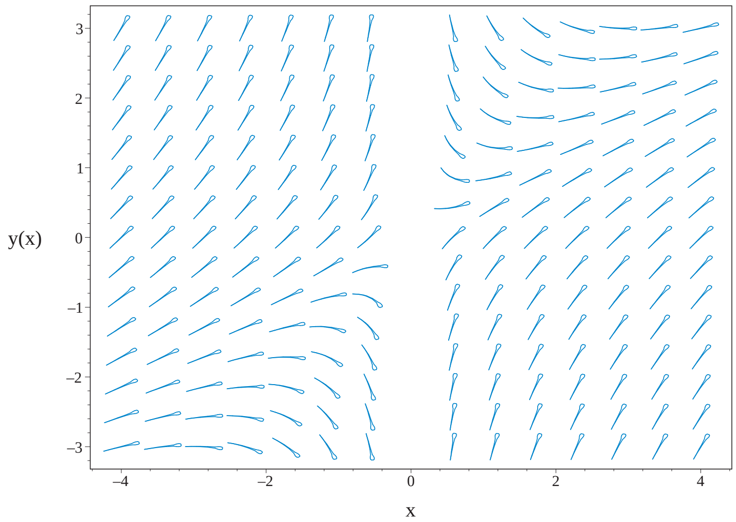


Figure 2.19: Slope field plot
$xy' + y = x$

Summary of solutions found

$$y = \frac{x}{2}$$
$$y = \frac{x^2 + e^{-2c_1}}{2x}$$

Solved as first order homogeneous class D2 ode

Time used: 0.127 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$x(u'(x)\, x + u(x)) + u(x)\, x = x$$

Which is now solved The ode

$$u'(x) = -\frac{2u(x) - 1}{x} \tag{2.7}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{2u(x) - 1}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= -2u + 1 \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$
$$\int \frac{1}{-2u + 1}\, du = \int \frac{1}{x}\, dx$$

$$-\frac{\ln(2u(x) - 1)}{2} = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-2u + 1 = 0$$

for $u(x)$ gives

$$u(x) = \frac{1}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln(2u(x)-1)}{2} = \ln(x) + c_1$$
$$u(x) = \frac{1}{2}$$

Solving for $u(x)$ gives

$$u(x) = \frac{1}{2}$$
$$u(x) = \frac{x^2 + e^{-2c_1}}{2x^2}$$

Converting $u(x) = \frac{1}{2}$ back to y gives

$$y = \frac{x}{2}$$

Converting $u(x) = \frac{x^2+e^{-2c_1}}{2x^2}$ back to y gives

$$y = \frac{x^2 + e^{-2c_1}}{2x}$$

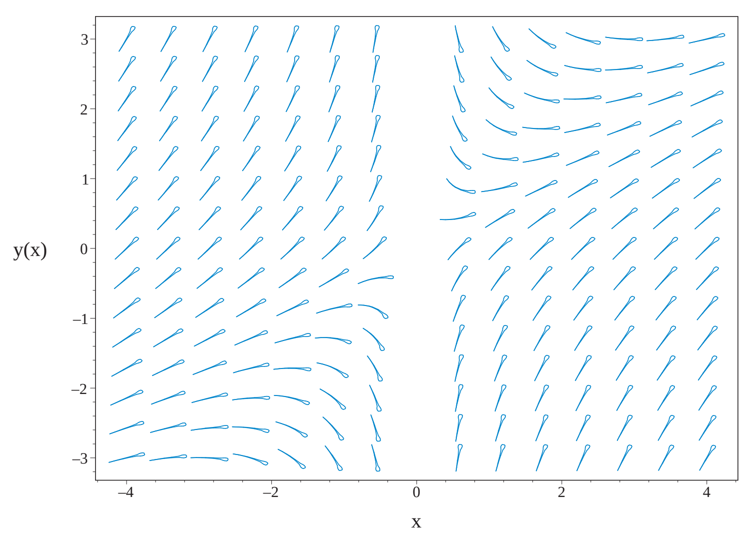


Figure 2.20: Slope field plot
$xy' + y = x$

Summary of solutions found

$$y = \frac{x}{2}$$
$$y = \frac{x^2 + e^{-2c_1}}{2x}$$

Solved as first order homogeneous class Maple C ode

Time used: 0.241 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{Y(X) + y_0 - x_0 - X}{x_0 + X}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{Y(X) - X}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{Y - X}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = X - Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = -u + 1$$
$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{-2u(X) + 1}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{-2u(X) + 1}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 2u(X) - 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{2u(X) - 1}{X} \tag{2.8}$$

is separable as it can be written as

$$\begin{aligned} \frac{d}{dX}u(X) &= -\frac{2u(X) - 1}{X} \\ &= f(X)g(u) \end{aligned}$$

Where

$$f(X) = \frac{1}{X}$$
$$g(u) = -2u + 1$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(X)\, dX$$
$$\int \frac{1}{-2u+1}\, du = \int \frac{1}{X}\, dX$$

$$-\frac{\ln(2u(X)-1)}{2} = \ln(X) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-2u + 1 = 0$$

for $u(X)$ gives

$$u(X) = \frac{1}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln(2u(X)-1)}{2} = \ln(X) + c_1$$
$$u(X) = \frac{1}{2}$$

Solving for $u(X)$ gives

$$u(X) = \frac{1}{2}$$
$$u(X) = \frac{X^2 + e^{-2c_1}}{2X^2}$$

Converting $u(X) = \frac{1}{2}$ back to $Y(X)$ gives

$$Y(X) = \frac{X}{2}$$

Converting $u(X) = \frac{X^2+e^{-2c_1}}{2X^2}$ back to $Y(X)$ gives

$$Y(X) = \frac{X^2 + e^{-2c_1}}{2X}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{X}{2} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = \frac{x}{2}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{X^2 + e^{-2c_1}}{2X} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = \frac{x^2 + e^{-2c_1}}{2x}$$

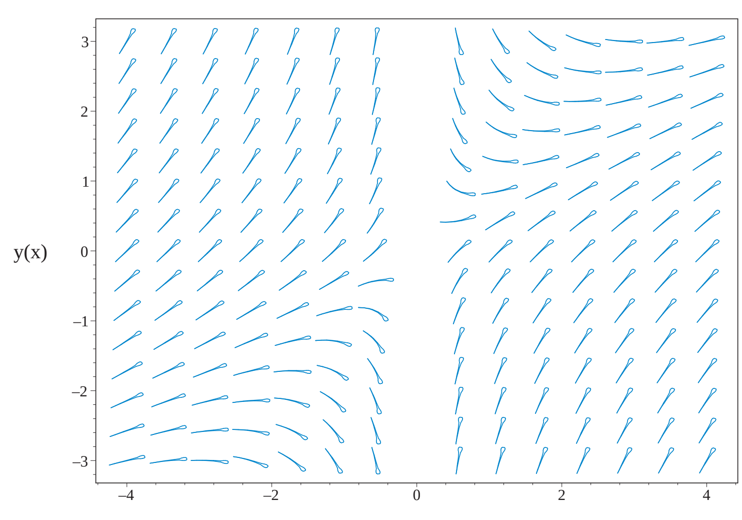


Figure 2.21: Slope field plot
$xy' + y = x$

Solved as first order Exact ode

Time used: 0.117 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = (-y + x)\,\mathrm{d}x$$
$$(y - x)\,\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - x$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y - x)$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x)$$
$$= 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int N\,\mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int x\,\mathrm{d}y$$
$$\phi = yx + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = y - x$. Therefore equation (4) becomes

$$y - x = y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -x$$

Integrating the above w.r.t x gives

$$\int f'(x) \, \mathrm{d}x = \int (-x) \, \mathrm{d}x$$

$$f(x) = -\frac{x^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = yx - \frac{1}{2}x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = yx - \frac{1}{2}x^2$$

Solving for y gives

$$y = \frac{x^2 + 2c_1}{2x}$$

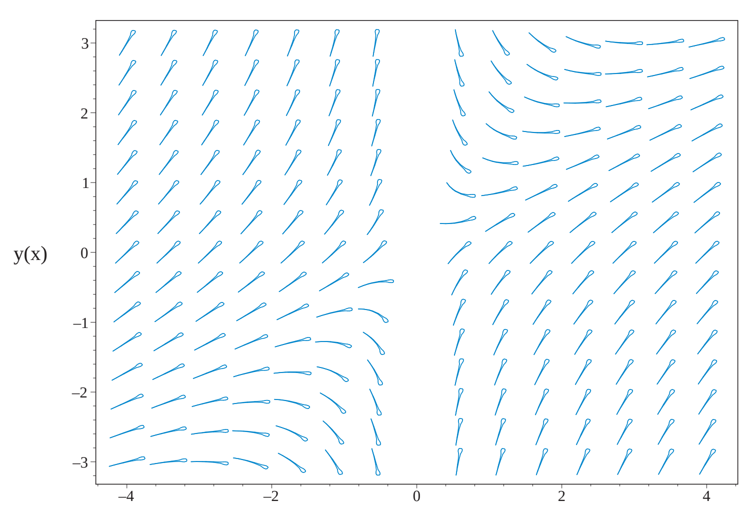


Figure 2.22: Slope field plot
$xy' + y = x$

<u>Summary of solutions found</u>

$$y = \frac{x^2 + 2c_1}{2x}$$

Solved as first order isobaric ode

Time used: 0.104 (sec)

Solving for y' gives

$$y' = -\frac{y-x}{x} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = -\frac{y-x}{x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = -\frac{xu(x) - x}{x}$$

The ode

$$u'(x) = -\frac{2u(x) - 1}{x} \tag{2.9}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{2u(x) - 1}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= -2u + 1 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{1}{-2u+1}\, du &= \int \frac{1}{x}\, dx \end{aligned}$$

$$-\frac{\ln(2u(x) - 1)}{2} = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-2u + 1 = 0$$

for $u(x)$ gives

$$u(x) = \frac{1}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln(2u(x)-1)}{2} = \ln(x) + c_1$$
$$u(x) = \frac{1}{2}$$

Solving for $u(x)$ gives

$$u(x) = \frac{1}{2}$$
$$u(x) = \frac{x^2 + \mathrm{e}^{-2c_1}}{2x^2}$$

Converting $u(x) = \frac{1}{2}$ back to y gives

$$\frac{y}{x} = \frac{1}{2}$$

Converting $u(x) = \frac{x^2+\mathrm{e}^{-2c_1}}{2x^2}$ back to y gives

$$\frac{y}{x} = \frac{x^2 + \mathrm{e}^{-2c_1}}{2x^2}$$

Solving for y gives

$$y = \frac{x}{2}$$
$$y = \frac{x^2 + \mathrm{e}^{-2c_1}}{2x}$$

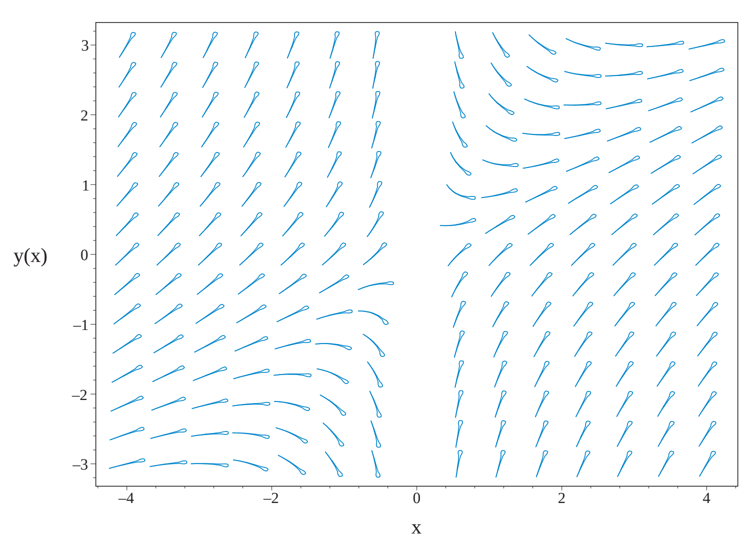


Figure 2.23: Slope field plot
$xy' + y = x$

Summary of solutions found

$$y = \frac{x}{2}$$
$$y = \frac{x^2 + \mathrm{e}^{-2c_1}}{2x}$$

Solved using Lie symmetry for first order ode

Time used: 0.429 (sec)

Writing the ode as

$$y' = -\frac{y-x}{x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(y-x)(b_3-a_2)}{x} - \frac{(y-x)^2 a_3}{x^2} - \left(\frac{1}{x} + \frac{y-x}{x^2}\right)(xa_2 + ya_3 + a_1) + \frac{xb_2 + yb_3 + b_1}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 - 2b_2x^2 - x^2b_3 - 2xya_3 + 2y^2a_3 - xb_1 + ya_1}{x^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_2 - x^2a_3 + 2b_2x^2 + x^2b_3 + 2xya_3 - 2y^2a_3 + xb_1 - ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^2 - a_3v_1^2 + 2a_3v_1v_2 - 2a_3v_2^2 + 2b_2v_1^2 + b_3v_1^2 - a_1v_2 + b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 + 2b_2 + b_3)v_1^2 + 2a_3v_1v_2 + b_1v_1 - 2a_3v_2^2 - a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ -2a_3 &= 0 \\ 2a_3 &= 0 \\ -a_2 - a_3 + 2b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_2 + b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y-x}{x}\right)(x) \\ &= 2y - x \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{2y - x} dy \end{aligned}$$

Which results in

$$S = \frac{\ln (2y - x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y-x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x-4y} \\ S_y &= \frac{1}{2y-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int -\frac{1}{2R}\, dR \\ S(R) &= -\frac{\ln(R)}{2} + c_2 \end{aligned}$$

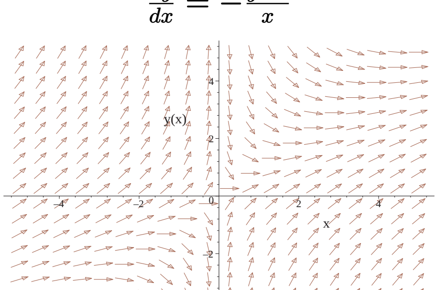
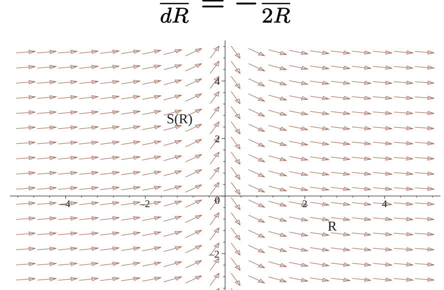
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(2y-x)}{2} = -\frac{\ln(x)}{2} + c_2$$

Which gives

$$y = \frac{x^2 + e^{2c_2}}{2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-x}{x}$ 	$R = x$ $S = \frac{\ln(2y-x)}{2}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

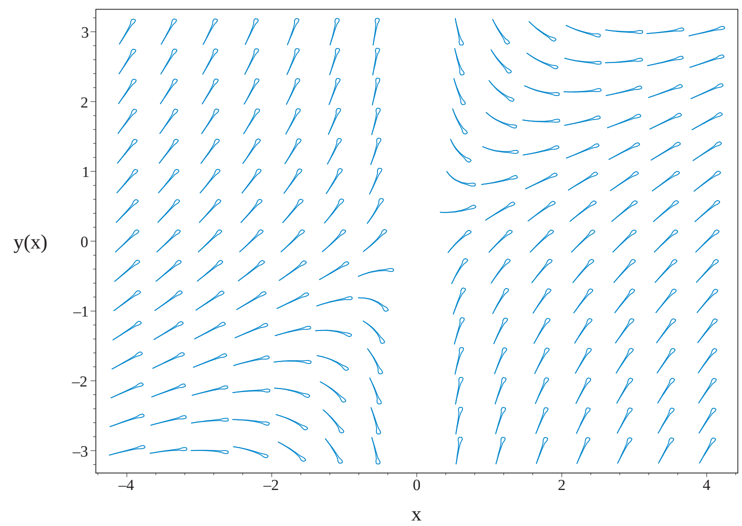


Figure 2.24: Slope field plot
$xy' + y = x$

Summary of solutions found

$$y = \frac{x^2 + e^{2c_2}}{2x}$$

Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y(x)\right) + y(x) = x$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Isolate the derivative
$\frac{d}{dx}y(x) = 1 - \frac{y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
$\frac{d}{dx}y(x) + \frac{y(x)}{x} = 1$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{x}\right) = \mu(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
$\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{x}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
$\frac{d}{dx}\mu(x) = \frac{\mu(x)}{x}$
- Solve to find the integrating factor
$\mu(x) = x$
- Integrate both sides with respect to x
$\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \mu(x)\, dx + C1$
- Evaluate the integral on the lhs
$y(x)\,\mu(x) = \int \mu(x)\, dx + C1$
- Solve for $y(x)$
$y(x) = \frac{\int \mu(x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = x$
$y(x) = \frac{\int x dx + C1}{x}$
- Evaluate the integrals on the rhs
$y(x) = \frac{\frac{x^2}{2} + C1}{x}$

- Simplify

  $y(x) = \frac{x^2+2C1}{2x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 13

```
dsolve(diff(y(x),x)*x+y(x) = x,y(x),singsol=all)
```

$$y(x) = \frac{x}{2} + \frac{c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],x]+y[x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x}{2} + \frac{c_1}{x}$$

2.1.8 Problem 2(b)

Internal problem ID [4197]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 2(b)
Date solved : `Monday, January 27, 2025 at 08:41:51 AM`
CAS classification : `[_linear]`

Solve

$$xy' - y = x^3$$

Solved as first order linear ode

Time used: 0.044 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{1}{x} \\ p(x) &= x^2 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(x^2\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(x^2\right)$$

$$\mathrm{d}\left(\frac{y}{x}\right) = x\,\mathrm{d}x$$

Integrating gives

$$\begin{aligned} \frac{y}{x} &= \int x\,dx \\ &= \frac{x^2}{2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{x}$ gives the final solution

$$y = \frac{x(x^2 + 2c_1)}{2}$$

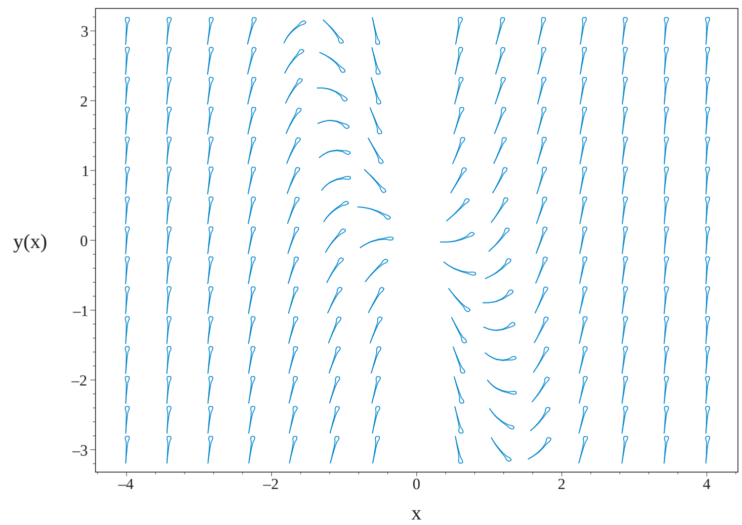


Figure 2.25: Slope field plot
$xy' - y = x^3$

<u>Summary of solutions found</u>

$$y = \frac{x(x^2 + 2c_1)}{2}$$

Solved as first order homogeneous class D2 ode

Time used: 0.028 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$x(u'(x)\,x + u(x)) - u(x)\,x = x^3$$

Which is now solved Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int x\,dx$$
$$u(x) = \frac{x^2}{2} + c_1$$

Converting $u(x) = \frac{x^2}{2} + c_1$ back to y gives

$$y = x\left(\frac{x^2}{2} + c_1\right)$$

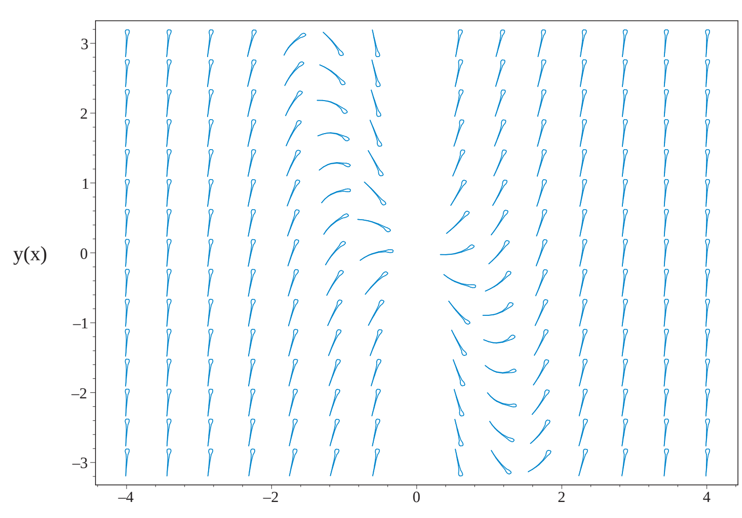


Figure 2.26: Slope field plot
$xy' - y = x^3$

Summary of solutions found

$$y = x\left(\frac{x^2}{2} + c_1\right)$$

Solved as first order Exact ode

Time used: 0.099 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(x^3 + y\right)\mathrm{d}x$$
$$\left(-x^3 - y\right)\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^3 - y$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-x^3 - y\right)$$
$$= -1$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-1) - (1))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}(-x^3 - y)\\ &= \frac{-x^3 - y}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}(x)\\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-x^3 - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y\\ \int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \frac{1}{x}\,\mathrm{d}y\end{aligned}$$

$$\phi = \frac{y}{x} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \frac{-x^3-y}{x^2}$. Therefore equation (4) becomes

$$\frac{-x^3 - y}{x^2} = -\frac{y}{x^2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -x$$

Integrating the above w.r.t x gives

$$\int f'(x)\, dx = \int (-x)\, dx$$

$$f(x) = -\frac{x^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x} - \frac{x^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{y}{x} - \frac{x^2}{2}$$

Solving for y gives

$$y = \frac{x(x^2 + 2c_1)}{2}$$

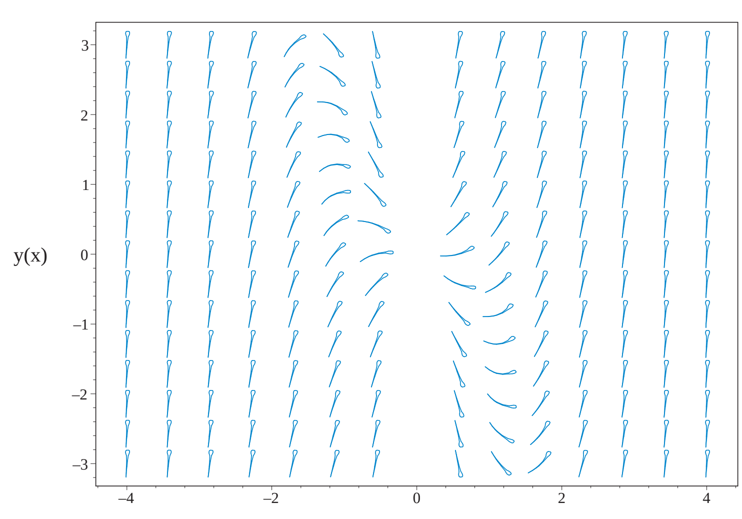


Figure 2.27: Slope field plot
$xy' - y = x^3$

Summary of solutions found

$$y = \frac{x(x^2 + 2c_1)}{2}$$

Solved as first order isobaric ode

Time used: 0.282 (sec)

Solving for y' gives

$$y' = \frac{x^3 + y}{x} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{x^3 + y}{x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 3$$

Since the ode is isobaric of order $m = 3$, then the substitution

$$\begin{aligned} y &= u x^m \\ &= u x^3 \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$3x^2 u(x) + x^3 u'(x) = \frac{x^3 + x^3 u(x)}{x}$$

The ode

$$u'(x) = -\frac{2u(x) - 1}{x} \tag{2.10}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{2u(x) - 1}{x} \\ &= f(x) g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= -2u + 1 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{1}{-2u + 1}\, du &= \int \frac{1}{x}\, dx \end{aligned}$$

$$-\frac{\ln(2u(x) - 1)}{2} = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-2u + 1 = 0$$

for $u(x)$ gives

$$u(x) = \frac{1}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln(2u(x)-1)}{2} = \ln(x) + c_1$$

$$u(x) = \frac{1}{2}$$

Solving for $u(x)$ gives

$$u(x) = \frac{1}{2}$$

$$u(x) = \frac{x^2 + e^{-2c_1}}{2x^2}$$

Converting $u(x) = \frac{1}{2}$ back to y gives

$$\frac{y}{x^3} = \frac{1}{2}$$

Converting $u(x) = \frac{x^2+e^{-2c_1}}{2x^2}$ back to y gives

$$\frac{y}{x^3} = \frac{x^2 + e^{-2c_1}}{2x^2}$$

Solving for y gives

$$y = \frac{x^3}{2}$$

$$y = \frac{x(x^2 + e^{-2c_1})}{2}$$

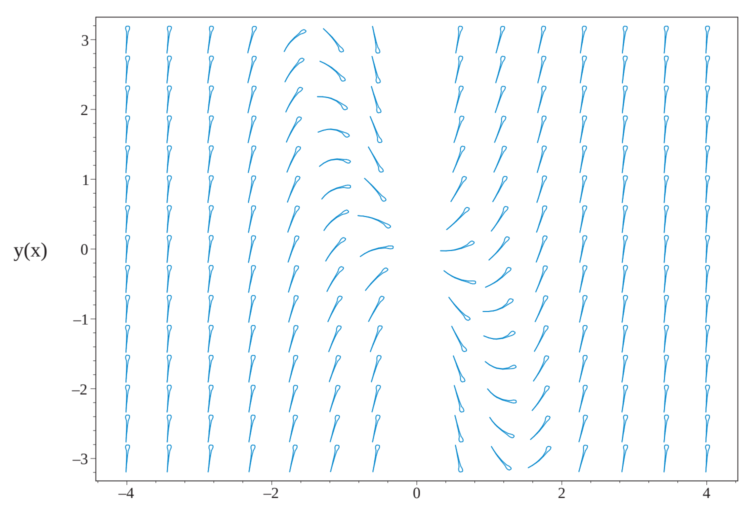


Figure 2.28: Slope field plot
$xy' - y = x^3$

Summary of solutions found

$$y = \frac{x^3}{2}$$

$$y = \frac{x(x^2 + e^{-2c_1})}{2}$$

Solved using Lie symmetry for first order ode

Time used: 0.288 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^3+y}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 + \frac{(x^3+y)(b_3-a_2)}{x} - \frac{(x^3+y)^2 a_3}{x^2} \\
- \left(3x - \frac{x^3+y}{x^2}\right)(x a_2 + y a_3 + a_1) - \frac{x b_2 + y b_3 + b_1}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^6 a_3 + 3x^4 a_2 - x^4 b_3 + 4x^3 y a_3 + 2x^3 a_1 + x b_1 - y a_1}{x^2} = 0
$$

Setting the numerator to zero gives

$$
-x^6 a_3 - 3x^4 a_2 + x^4 b_3 - 4x^3 y a_3 - 2x^3 a_1 - x b_1 + y a_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
-a_3 v_1^6 - 3a_2 v_1^4 - 4a_3 v_1^3 v_2 + b_3 v_1^4 - 2a_1 v_1^3 + a_1 v_2 - b_1 v_1 = 0 \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$-a_3v_1^6 + (-3a_2 + b_3)\, v_1^4 - 4a_3v_1^3v_2 - 2a_1v_1^3 - b_1v_1 + a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -2a_1 &= 0 \\ -4a_3 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -3a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= 3a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= x \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^3 + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int R\, dR \\ S(R) &= \frac{R^2}{2} + c_2 \end{aligned}$$

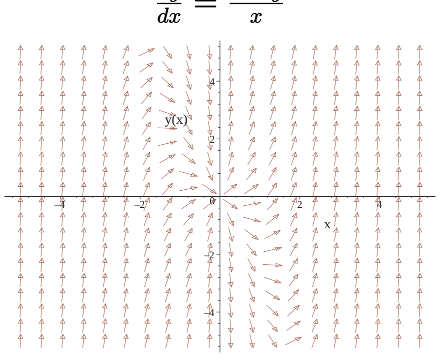
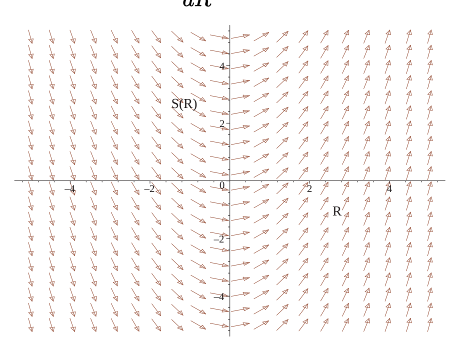
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{y}{x} = \frac{x^2}{2} + c_2$$

Which gives

$$y = \frac{(x^2 + 2c_2)\, x}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = R$ 

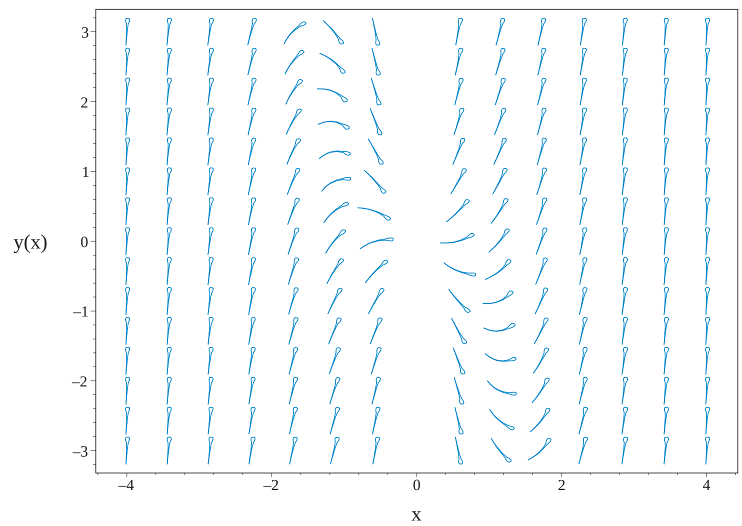


Figure 2.29: Slope field plot
$xy' - y = x^3$

Summary of solutions found

$$y = \frac{(x^2 + 2c_2)\, x}{2}$$

Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y(x)\right) - y(x) = x^3$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = \frac{y(x)+x^3}{x}$
- Collect w.r.t. $y(x)$ and simplify
$\frac{d}{dx}y(x) = \frac{y(x)}{x} + x^2$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
$-\frac{y(x)}{x} + \frac{d}{dx}y(x) = x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$\mu(x)\left(-\frac{y(x)}{x} + \frac{d}{dx}y(x)\right) = \mu(x)\, x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
$\mu(x)\left(-\frac{y(x)}{x} + \frac{d}{dx}y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
$\frac{d}{dx}\mu(x) = -\frac{\mu(x)}{x}$
- Solve to find the integrating factor
$\mu(x) = \frac{1}{x}$
- Integrate both sides with respect to x
$\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \mu(x)\, x^2 dx + C1$
- Evaluate the integral on the lhs
$y(x)\,\mu(x) = \int \mu(x)\, x^2 dx + C1$
- Solve for $y(x)$
$y(x) = \frac{\int \mu(x)x^2 dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x}$
$y(x) = x\left(\int x dx + C1\right)$

- Evaluate the integrals on the rhs

  $y(x) = x\left(\frac{x^2}{2} + C1\right)$

- Simplify

  $y(x) = \frac{x(x^2+2C1)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 14

```
dsolve(diff(y(x),x)*x-y(x) = x^3,y(x),singsol=all)
```

$$y(x) = \frac{(x^2 + 2c_1)\, x}{2}$$

Mathematica DSolve solution

Solving time : `0.028` (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],x]-y[x]==x^3,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^3}{2} + c_1 x$$

2.1.9 Problem 2(c)

Internal problem ID [4198]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 2(c)
Date solved : `Monday, January 27, 2025 at 08:41:53 AM`
CAS classification : `[_linear]`

Solve

$$xy' + ny = x^n$$

Solved as first order linear ode

Time used: 0.086 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{n}{x}$$
$$p(x) = x^{n-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= e^{\int \frac{n}{x}dx}\\ &= x^n\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(x^{n-1}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(y\,x^n) = (x^n)\left(x^{n-1}\right)$$
$$\mathrm{d}(y\,x^n) = \left(x^{n-1}x^n\right)\,\mathrm{d}x$$

Integrating gives

$$\begin{aligned}y\,x^n &= \int x^{n-1}x^n\,dx\\ &= \frac{x^{2n}}{2n} + c_1\end{aligned}$$

Dividing throughout by the integrating factor x^n gives the final solution

$$y = \frac{x^n}{2n} + x^{-n}c_1$$

Summary of solutions found

$$y = \frac{x^n}{2n} + x^{-n}c_1$$

Solved as first order Exact ode

Time used: 0.177 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(-ny+x^n)\,\mathrm{d}x\\ (ny-x^n)\,\mathrm{d}x+(x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=ny-x^n\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(ny-x^n)\\ &=n\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x)\\ &=1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((n) - (1)) \\
&= \frac{n-1}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \frac{n-1}{x} \,\mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{(n-1)\ln(x)} \\
&= x^{n-1}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= x^{n-1}(ny - x^n) \\
&= (ny - x^n)\, x^{n-1}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= x^{n-1}(x) \\
&= x^n
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left((ny - x^n)\, x^{n-1}\right) + (x^n)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y &= \int \overline{N} \,\mathrm{d}y \\
\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y &= \int x^n \,\mathrm{d}y
\end{aligned}$$

$$\phi = y\, x^n + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \frac{x^n n y}{x} + f'(x) \tag{4}$$

$$= ny\, x^{n-1} + f'(x)$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = (ny - x^n)\, x^{n-1}$. Therefore equation (4) becomes

$$(ny - x^n)\, x^{n-1} = ny\, x^{n-1} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$\begin{aligned} f'(x) &= -x^{n-1} x^n \\ &= -x^{2n-1} \end{aligned}$$

Integrating the above w.r.t x results in

$$\begin{aligned} \int f'(x)\, \mathrm{d}x &= \int \left(-x^{2n-1}\right) \mathrm{d}x \\ f(x) &= -\frac{x^{2n}}{2n} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\, x^n - \frac{x^{2n}}{2n} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\, x^n - \frac{x^{2n}}{2n}$$

Solving for y gives

$$y = \frac{(2c_1 n + x^{2n})\, x^{-n}}{2n}$$

Summary of solutions found

$$y = \frac{(2c_1 n + x^{2n})\, x^{-n}}{2n}$$

Solved using Lie symmetry for first order ode

Time used: 0.447 (sec)

Writing the ode as

$$\begin{aligned} y' &= \frac{-ny + x^n}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{(-ny + x^n)(b_3 - a_2)}{x} - \frac{(-ny + x^n)^2 a_3}{x^2} \\
& - \left(\frac{x^n n}{x^2} - \frac{-ny + x^n}{x^2}\right)(x a_2 + y a_3 + a_1) + \frac{n(x b_2 + y b_3 + b_1)}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{n^2 y^2 a_3 + x^n n x a_2 - x^n n y a_3 - n\, x^2 b_2 + n\, y^2 a_3 + x^{2n} a_3 + x^n n a_1 - x^n x b_3 - x^n y a_3 - n x b_1 + n y a_1 - b_2 x^2}{x^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -n^2 y^2 a_3 - x^n n x a_2 + x^n n y a_3 + n\, x^2 b_2 - n\, y^2 a_3 - x^{2n} a_3 \\
& - x^n n a_1 + x^n x b_3 + x^n y a_3 + n x b_1 - n y a_1 + b_2 x^2 + x^n a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y, x^n\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2, x^n = v_3\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -n^2 v_2^2 a_3 - v_3 n v_1 a_2 - n v_2^2 a_3 + v_3 n v_2 a_3 + n v_1^2 b_2 - n v_2 a_1 \\
& - v_3 n a_1 + n v_1 b_1 + v_3 v_2 a_3 - v_3^2 a_3 + b_2 v_1^2 + v_3 v_1 b_3 + v_3 a_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
& (n b_2 + b_2) v_1^2 + (-n a_2 + b_3) v_1 v_3 + n v_1 b_1 + (-n^2 a_3 - n a_3) v_2^2 \\
& + (n a_3 + a_3) v_2 v_3 - n v_2 a_1 - v_3^2 a_3 + (-n a_1 + a_1) v_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
n b_1 &= 0 \\
-a_3 &= 0 \\
-n a_1 &= 0 \\
-n a_1 + a_1 &= 0 \\
n a_3 + a_3 &= 0 \\
-n^2 a_3 - n a_3 &= 0 \\
n b_2 + b_2 &= 0 \\
-n a_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= n a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= n y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= n y - \left(\frac{-n y + x^n}{x}\right)(x) \\
&= 2 n y - x^n \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{2 n y - x^n} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln (2 n y - x^n)}{2 n}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{-n y + x^n}{x}
$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = \frac{x^{n-1}}{-4ny + 2x^n}$$
$$S_y = \frac{1}{2ny - x^n}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -\frac{1}{2R}\, dR$$
$$S(R) = -\frac{\ln(R)}{2} + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(2ny - x^n)}{2n} = -\frac{\ln(x)}{2} + c_2$$

Which gives

$$y = \frac{e^{-n(\ln(x) - 2c_2)} + x^n}{2n}$$

Summary of solutions found

$$y = \frac{e^{-n(\ln(x) - 2c_2)} + x^n}{2n}$$

Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y(x)\right) + ny(x) = x^n$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{-ny(x) + x^n}{x}$

- Collect w.r.t. $y(x)$ and simplify

  $\frac{d}{dx}y(x) = -\frac{ny(x)}{x} + \frac{x^n}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dx}y(x) + \frac{ny(x)}{x} = \frac{x^n}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{ny(x)}{x}\right)=\frac{\mu(x)x^n}{x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$

  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{ny(x)}{x}\right)=\left(\frac{d}{dx}y(x)\right)\mu(x)+y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$

  $\frac{d}{dx}\mu(x)=\frac{\mu(x)n}{x}$
- Solve to find the integrating factor

  $\mu(x)=x^n$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx=\int\frac{\mu(x)x^n}{x}dx+C1$
- Evaluate the integral on the lhs

  $y(x)\,\mu(x)=\int\frac{\mu(x)x^n}{x}dx+C1$
- Solve for $y(x)$

  $y(x)=\frac{\int\frac{\mu(x)x^n}{x}dx+C1}{\mu(x)}$
- Substitute $\mu(x)=x^n$

  $y(x)=\frac{\int\frac{(x^n)^2}{x}dx+C1}{x^n}$
- Evaluate the integrals on the rhs

  $y(x)=\frac{\frac{(x^n)^2}{2n}+C1}{x^n}$
- Simplify

  $y(x)=\frac{x^n}{2n}+x^{-n}C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 20

```
dsolve(diff(y(x),x)*x+n*y(x) = x^n,y(x),singsol=all)
```

$$y(x)=\frac{x^n}{2n}+x^{-n}c_1$$

Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 24

```
DSolve[{x*D[y[x],x]+n*y[x]==x^n,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{x^n}{2n}+c_1x^{-n}$$

2.1.10 Problem 2(d)

Internal problem ID [4199]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 2(d)
Date solved : `Monday, January 27, 2025 at 08:41:55 AM`
CAS classification : `[_linear]`

Solve

$$xy' - ny = x^n$$

Solved as first order linear ode

Time used: 0.090 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{n}{x}$$
$$p(x) = x^{n-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= \mathrm{e}^{\int -\frac{n}{x}dx}\\ &= x^{-n}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x^{n-1}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,x^{-n}\right) &= \left(x^{-n}\right)\left(x^{n-1}\right)\\ \mathrm{d}\left(y\,x^{-n}\right) &= \left(x^{n-1}x^{-n}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^{-n} &= \int x^{n-1}x^{-n}\,dx\\ &= \ln(x) + c_1\end{aligned}$$

Dividing throughout by the integrating factor x^{-n} gives the final solution

$$y = x^n(\ln(x) + c_1)$$

Summary of solutions found

$$y = x^n(\ln(x) + c_1)$$

Solved as first order Exact ode

Time used: 0.132 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(ny+x^n)\,\mathrm{d}x\\ (-ny-x^n)\,\mathrm{d}x+(x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-ny-x^n\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-ny-x^n)\\ &=-n\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x)\\ &=1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-n) - (1)) \\ &= \frac{-n-1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{-n-1}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{(-n-1)\ln(x)} \\ &= x^{-n-1} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x^{-n-1}(-ny - x^n) \\ &= \frac{-1 - x^{-n} n y}{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x^{-n-1}(x) \\ &= x^{-n} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-1 - x^{-n} n y}{x}\right) + \left(x^{-n}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \overline{N} \, \mathrm{d}y \\ \int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int x^{-n} \, \mathrm{d}y \\ \phi &= y\, x^{-n} + f(x) \end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= -\frac{x^{-n} n y}{x} + f'(x) \\ &= -n y\, x^{-n-1} + f'(x)\end{aligned} \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \frac{-1-x^{-n} n y}{x}$. Therefore equation (4) becomes

$$\frac{-1 - x^{-n} n y}{x} = -n y\, x^{-n-1} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$\begin{aligned}f'(x) &= \frac{n y\, x^{-n-1} x - x^{-n} n y - 1}{x} \\ &= -\frac{1}{x}\end{aligned}$$

Integrating the above w.r.t x results in

$$\begin{aligned}\int f'(x)\,\mathrm{d}x &= \int \left(-\frac{1}{x}\right) \mathrm{d}x \\ f(x) &= -\ln(x) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\, x^{-n} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\, x^{-n} - \ln(x)$$

Solving for y gives

$$y = x^n (\ln(x) + c_1)$$

<u>Summary of solutions found</u>

$$y = x^n (\ln(x) + c_1)$$

Solved using Lie symmetry for first order ode

Time used: 0.326 (sec)

Writing the ode as

$$\begin{aligned}y' &= \frac{n y + x^n}{x} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(ny + x^n)(b_3 - a_2)}{x} - \frac{(ny + x^n)^2 a_3}{x^2} - \left(\frac{x^n n}{x^2} - \frac{ny + x^n}{x^2}\right)(xa_2 + ya_3 + a_1) - \frac{n(xb_2 + yb_3 + b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{n^2y^2a_3 + x^n nxa_2 + 3x^n nya_3 + n\,x^2b_2 - n\,y^2a_3 + x^{2n}a_3 + x^n na_1 - x^n xb_3 - x^n ya_3 + nxb_1 - nya_1 - b_2x^2}{x^2} = 0$$

Setting the numerator to zero gives

$$-n^2y^2a_3 - x^n nxa_2 - 3x^n nya_3 - n\,x^2b_2 + n\,y^2a_3 - x^{2n}a_3 - x^n na_1 + x^n xb_3 + x^n ya_3 - nxb_1 + nya_1 + b_2x^2 + x^n a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, x^n\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, x^n = v_3\}$$

The above PDE (6E) now becomes

$$-n^2v_2^2a_3 - v_3nv_1a_2 + nv_2^2a_3 - 3v_3nv_2a_3 - nv_1^2b_2 + nv_2a_1 - v_3na_1 - nv_1b_1 + v_3v_2a_3 - v_3^2a_3 + b_2v_1^2 + v_3v_1b_3 + v_3a_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$(-nb_2 + b_2)\,v_1^2 + (-na_2 + b_3)\,v_1v_3 - nv_1b_1 + \left(-n^2a_3 + na_3\right)v_2^2 + (-3na_3 + a_3)\,v_2v_3 + nv_2a_1 - v_3^2a_3 + (-na_1 + a_1)\,v_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} na_1 &= 0 \\ -a_3 &= 0 \\ -nb_1 &= 0 \\ -na_1 + a_1 &= 0 \\ -3na_3 + a_3 &= 0 \\ -n^2a_3 + na_3 &= 0 \\ -nb_2 + b_2 &= 0 \\ -na_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= na_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= ny \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= ny - \left(\frac{ny + x^n}{x}\right)(x) \\ &= -x^n \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-x^n} dy \end{aligned}$$

Which results in

$$S = -y\,x^{-n}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{ny + x^n}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= ny\, x^{-n-1}\\
S_y &= -x^{-n}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int -\frac{1}{R}\, dR\\
S(R) &= -\ln(R) + c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-yx^{-n} = -\ln(x) + c_2$$

Which gives

$$y = (\ln(x) - c_2)\, x^n$$

Summary of solutions found

$$y = (\ln(x) - c_2)\, x^n$$

Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y(x)\right) - ny(x) = x^n$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

$\frac{d}{dx}y(x) = \frac{ny(x)+x^n}{x}$

- Collect w.r.t. $y(x)$ and simplify

  $\frac{d}{dx}y(x) = \frac{ny(x)}{x} + \frac{x^n}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dx}y(x) - \frac{ny(x)}{x} = \frac{x^n}{x}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{ny(x)}{x}\right) = \frac{\mu(x)x^n}{x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$

  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{ny(x)}{x}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$

  $\frac{d}{dx}\mu(x) = -\frac{\mu(x)n}{x}$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^n}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \frac{\mu(x)x^n}{x} dx + C1$
- Evaluate the integral on the lhs

  $y(x)\,\mu(x) = \int \frac{\mu(x)x^n}{x} dx + C1$
- Solve for $y(x)$

  $y(x) = \frac{\int \frac{\mu(x)x^n}{x} dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x^n}$

  $y(x) = x^n\left(\int \frac{1}{x} dx + C1\right)$
- Evaluate the integrals on the rhs

  $y(x) = x^n(\ln(x) + C1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve(diff(y(x),x)*x-n*y(x) = x^n,y(x),singsol=all)
```

$$y(x) = (\ln(x) + c_1)\,x^n$$

Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 14

```
DSolve[{x*D[y[x],x]-n*y[x]==x^n,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^n(\log(x) + c_1)$$

2.1.11 Problem 2(e)

Internal problem ID [4200]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 2(e)
Date solved : `Monday, January 27, 2025 at 08:41:56 AM`
CAS classification : `[_linear]`

Solve

$$\left(x^{3}+x\right) y'+y = x$$

Solved as first order linear ode

Time used: 0.105 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{1}{x\left(x^{2}+1\right)}$$
$$p(x) = \frac{1}{x^{2}+1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= {\mathrm e}^{\int \frac{1}{x\left(x^{2}+1\right)}d x}\\ &= \frac{x}{\sqrt{x^{2}+1}}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{1}{x^{2}+1}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y x}{\sqrt{x^{2}+1}}\right) = \left(\frac{x}{\sqrt{x^{2}+1}}\right)\left(\frac{1}{x^{2}+1}\right)$$
$$\mathrm{d}\left(\frac{y x}{\sqrt{x^{2}+1}}\right) = \left(\frac{x}{\left(x^{2}+1\right)^{3/2}}\right)\mathrm{d}x$$

Integrating gives

$$\begin{aligned}\frac{y x}{\sqrt{x^{2}+1}} &= \int \frac{x}{\left(x^{2}+1\right)^{3/2}}\,dx\\ &= -\frac{1}{\sqrt{x^{2}+1}} + c_1\end{aligned}$$

Dividing throughout by the integrating factor $\frac{x}{\sqrt{x^2+1}}$ gives the final solution

$$y = \frac{c_1\sqrt{x^2+1}-1}{x}$$

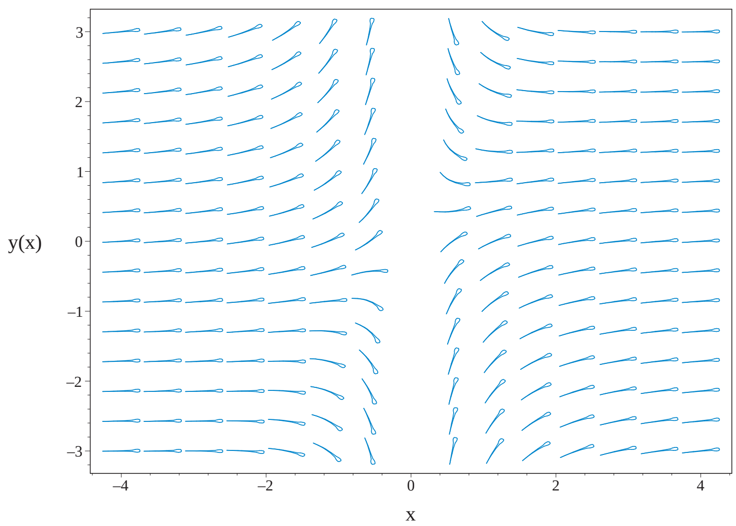


Figure 2.30: Slope field plot
$(x^3+x)\,y'+y=x$

Summary of solutions found

$$y = \frac{c_1\sqrt{x^2+1}-1}{x}$$

Solved as first order Exact ode

Time used: 0.186 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or

might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^3+x)\,\mathrm{d}y &= (-y+x)\,\mathrm{d}x\\ (y-x)\,\mathrm{d}x + (x^3+x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y-x\\ N(x,y) &= x^3+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y-x)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^3+x)\\ &= 3x^2+1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^3+x}\left((1)-(3x^2+1)\right)\\ &= -\frac{3x}{x^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{3x}{x^2+1}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{3\ln(x^2+1)}{2}}\\ &= \frac{1}{(x^2+1)^{3/2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{(x^2+1)^{3/2}}(y-x)\\ &= \frac{y-x}{(x^2+1)^{3/2}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{\left(x^2+1\right)^{3/2}}\left(x^3+x\right) \\ &= \frac{x}{\sqrt{x^2+1}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{gathered}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}=0 \\ \left(\frac{y-x}{\left(x^2+1\right)^{3/2}}\right)+\left(\frac{x}{\sqrt{x^2+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0\end{gathered}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int\overline{N}\,\mathrm{d}y \\ \int\frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int\frac{x}{\sqrt{x^2+1}}\,\mathrm{d}y \\ \phi &= \frac{yx}{\sqrt{x^2+1}}+f(x)\end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{y}{\sqrt{x^2+1}}-\frac{y\,x^2}{\left(x^2+1\right)^{3/2}}+f'(x) \\ &= \frac{y}{\left(x^2+1\right)^{3/2}}+f'(x)\end{aligned} \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial x}=\frac{y-x}{\left(x^2+1\right)^{3/2}}$. Therefore equation (4) becomes

$$\frac{y-x}{\left(x^2+1\right)^{3/2}}=\frac{y}{\left(x^2+1\right)^{3/2}}+f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x)=-\frac{x}{\left(x^2+1\right)^{3/2}}$$

Integrating the above w.r.t x gives

$$\begin{aligned}\int f'(x)\,\mathrm{d}x &= \int\left(-\frac{x}{\left(x^2+1\right)^{3/2}}\right)\mathrm{d}x \\ f(x) &= \frac{1}{\sqrt{x^2+1}}+c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{yx}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{yx}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}$$

Solving for y gives

$$y = \frac{c_1\sqrt{x^2+1}-1}{x}$$

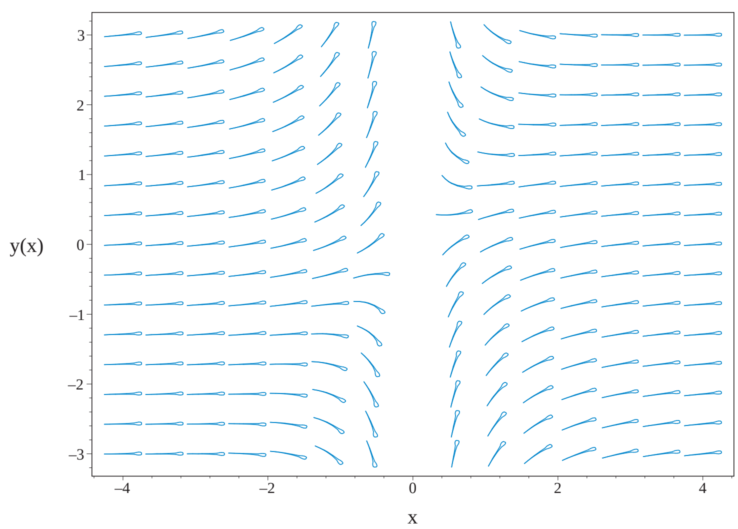


Figure 2.31: Slope field plot
$(x^3+x)\, y' + y = x$

Summary of solutions found

$$y = \frac{c_1\sqrt{x^2+1}-1}{x}$$

Solved using Lie symmetry for first order ode

Time used: 1.940 (sec)

Writing the ode as

$$y' = -\frac{y-x}{x\,(x^2+1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3a_7 + x^2ya_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^3b_7 + x^2yb_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 \\
&- \frac{(y-x)\left(-3x^2a_7 + x^2b_8 - 2xya_8 + 2xyb_9 - y^2a_9 + 3y^2b_{10} - 2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3\right)}{x(x^2+1)} \\
&- \frac{(y-x)^2\left(x^2a_8 + 2xya_9 + 3y^2a_{10} + xa_5 + 2ya_6 + a_3\right)}{x^2(x^2+1)^2} - \left(\frac{1}{x(x^2+1)} + \frac{y-x}{x^2(x^2+1)}\right. \\
&\left. + \frac{2y-2x}{(x^2+1)^2}\right)\left(x^3a_7 + x^2ya_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1\right) \\
&+ \frac{x^3b_7 + x^2yb_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1}{x(x^2+1)} = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{yb_5x^2 + 2x^3yb_8 - 2x^3ya_9 - 3x^2y^2a_{10} + 6x\,y^3a_{10} + 2x^3y^3a_{10} - 3x^2y^4a_{10} + 4x^4b_7 + 2x^7yb_8 + 4x^5yb_8 + x^6y^2}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&yb_5x^2 + 2x^3yb_8 - 2x^3ya_9 - 3x^2y^2a_{10} + 6x\,y^3a_{10} + 2x^3y^3a_{10} - 3x^2y^4a_{10} + 4x^4b_7 + 2x^7yb_8 + 4x^5yb_8 \\
&+ x^6y^2b_9 + x^4y^2b_9 + 2x^5yb_9 - x^4y^2a_8 + x^4y^2a_9 + 3x^4y^2b_{10} - 2x^3y^3a_9 - 2x^3y^3b_{10} + 2x^3ya_7 \\
&+ 2x^3yb_9 + 3x^2y^2a_9 + 3x^2y^2b_{10} - 2x\,y^3a_9 - 2x\,y^3b_{10} + 2x^7b_4 + 5x^5b_4 + x^5b_5 - 2x^3a_4 - x^3a_5 + x^3b_5 \\
&- 3y^3a_6 + 2b_2x^2 - 2x^3ya_2 + 2x^3ya_3 - 3x^2y^2a_3 - 3x^2ya_1 + 2xya_3 + 3x^8b_7 + 7x^6b_7 - x^6a_7 + x^6b_8 \\
&- 3x^4a_7 + x^4b_8 - x^4a_8 - 4y^4a_{10} + 3x^3b_4 + x^6yb_5 - x^4ya_4 + x^4ya_5 + 2x^4yb_5 + 2x^4yb_6 - 2x^3y^2a_5 \\
&+ 2x^3y^2a_6 - x^3y^2b_6 - 3x^2y^3a_6 + x^2ya_4 + x^2ya_5 - 2x^2ya_6 + 2x^2yb_6 - x\,y^2a_5 + 4x\,y^2a_6 - x\,y^2b_6 \\
&+ x^6b_2 + x^4a_2 + 3x^4b_2 + x^4b_3 + 2x^3a_1 + x^3b_1 - x^2a_2 - x^2a_3 + x^2b_3 - 2y^2a_3 + xb_1 - ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&v_1^4v_2^2b_9 + 2v_1^5v_2b_9 - v_1^4v_2^2a_8 + v_1^4v_2^2a_9 + 3v_1^4v_2^2b_{10} - 2v_1^3v_2^3a_9 - 2v_1^3v_2^3b_{10} \\
&+ 2v_1^3v_2a_7 + 2v_1^3v_2b_9 + 3v_1^2v_2^2a_9 + 3v_1^2v_2^2b_{10} - 2v_1v_2^3a_9 - 2v_1v_2^3b_{10} \\
&- 2v_1^3v_2a_2 + 2v_1^3v_2a_3 - 3v_1^2v_2^2a_3 - 3v_1^2v_2a_1 + 2v_1v_2a_3 + v_1^6v_2b_5 - v_1^4v_2a_4 \\
&+ v_1^4v_2a_5 + 2v_1^4v_2b_5 + 2v_1^4v_2b_6 - 2v_1^3v_2^2a_5 + 2v_1^3v_2^2a_6 - v_1^3v_2^2b_6 - 3v_1^2v_2^3a_6 \\
&+ v_1^2v_2a_4 + v_1^2v_2a_5 - 2v_1^2v_2a_6 + 2v_1^2v_2b_6 - v_1v_2^2a_5 + 4v_1v_2^2a_6 - v_1v_2^2b_6 \\
&+ v_2b_5v_1^2 + 2v_1^3v_2b_8 - 2v_1^3v_2a_9 - 3v_1^2v_2^2a_{10} + 6v_1v_2^3a_{10} + 2v_1^3v_2^3a_{10} \\
&- 3v_1^2v_2^4a_{10} + 2v_1^7v_2b_8 + 4v_1^5v_2b_8 + v_1^6v_2^2b_9 - 2v_2^2a_3 + v_1b_1 - v_2a_1 \\
&+ 4v_1^4b_7 + 2v_1^7b_4 + 5v_1^5b_4 + v_1^5b_5 - 2v_1^3a_4 - v_1^3a_5 + v_1^3b_5 - 3v_2^3a_6 + 2b_2v_1^2 \\
&+ 3v_1^8b_7 + 7v_1^6b_7 - v_1^6a_7 + v_1^6b_8 - 3v_1^4a_7 + v_1^4b_8 - v_1^4a_8 - 4v_2^4a_{10} + 3v_1^3b_4 \\
&+ v_1^6b_2 + v_1^4a_2 + 3v_1^4b_2 + v_1^4b_3 + 2v_1^3a_1 + v_1^3b_1 - v_1^2a_2 - v_1^2a_3 + v_1^2b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(7b_7 - a_7 + b_8 + b_2)\, v_1^6 + (2a_1 - 2a_4 - a_5 + b_1 + 3b_4 + b_5)\, v_1^3 \\
&+ (-a_2 - a_3 + 2b_2 + b_3)\, v_1^2 + (4b_7 - 3a_7 + b_8 - a_8 + a_2 + 3b_2 + b_3)\, v_1^4 \\
&+ (5b_4 + b_5)\, v_1^5 + (b_9 - a_8 + a_9 + 3b_{10})\, v_2^2 v_1^4 + (2b_9 + 4b_8)\, v_2 v_1^5 \\
&+ (-2a_9 - 2b_{10} + 2a_{10})\, v_2^3 v_1^3 + (2a_7 + 2b_9 - 2a_2 + 2a_3 + 2b_8 - 2a_9)\, v_2 v_1^3 \\
&+ (3a_9 + 3b_{10} - 3a_3 - 3a_{10})\, v_2^2 v_1^2 + (-2a_9 - 2b_{10} + 6a_{10})\, v_2^3 v_1 \\
&+ (-3a_1 + a_4 + a_5 - 2a_6 + b_5 + 2b_6)\, v_2 v_1^2 + (-a_4 + a_5 + 2b_5 + 2b_6)\, v_2 v_1^4 \\
&+ (-2a_5 + 2a_6 - b_6)\, v_2^2 v_1^3 + (-a_5 + 4a_6 - b_6)\, v_2^2 v_1 + 2v_1 v_2 a_3 \\
&+ v_1^6 v_2 b_5 - 3v_1^2 v_2^3 a_6 - 3v_1^2 v_2^4 a_{10} + 2v_1^7 v_2 b_8 + v_1^6 v_2^2 b_9 - 2v_2^2 a_3 \\
&+ v_1 b_1 - v_2 a_1 + 2v_1^7 b_4 - 3v_2^3 a_6 + 3v_1^8 b_7 - 4v_2^4 a_{10} = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0 \\
b_5 &= 0 \\
b_9 &= 0 \\
-a_1 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
-3a_6 &= 0 \\
-4a_{10} &= 0 \\
-3a_{10} &= 0 \\
2b_4 &= 0 \\
3b_7 &= 0 \\
2b_8 &= 0 \\
5b_4 + b_5 &= 0 \\
2b_9 + 4b_8 &= 0 \\
-2a_5 + 2a_6 - b_6 &= 0 \\
-a_5 + 4a_6 - b_6 &= 0 \\
-2a_9 - 2b_{10} + 2a_{10} &= 0 \\
-2a_9 - 2b_{10} + 6a_{10} &= 0 \\
-a_2 - a_3 + 2b_2 + b_3 &= 0 \\
-a_4 + a_5 + 2b_5 + 2b_6 &= 0 \\
3a_9 + 3b_{10} - 3a_3 - 3a_{10} &= 0 \\
7b_7 - a_7 + b_8 + b_2 &= 0 \\
b_9 - a_8 + a_9 + 3b_{10} &= 0 \\
-3a_1 + a_4 + a_5 - 2a_6 + b_5 + 2b_6 &= 0 \\
2a_1 - 2a_4 - a_5 + b_1 + 3b_4 + b_5 &= 0 \\
2a_7 + 2b_9 - 2a_2 + 2a_3 + 2b_8 - 2a_9 &= 0 \\
4b_7 - 3a_7 + b_8 - a_8 + a_2 + 3b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 + 2b_{10} \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= -b_3 + b_{10} \\
a_8 &= 2b_{10} \\
a_9 &= -b_{10} \\
a_{10} &= 0 \\
b_1 &= 0 \\
b_2 &= -b_3 + b_{10} \\
b_3 &= b_3 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0 \\
b_7 &= 0 \\
b_8 &= 0 \\
b_9 &= 0 \\
b_{10} &= b_{10}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x^3 + 2x^2y - x\,y^2 + 2x \\
\eta &= y^3 + x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y^3 + x - \left(-\frac{y-x}{x\,(x^2+1)}\right)\left(x^3 + 2x^2y - x\,y^2 + 2x\right) \\
&= \frac{x^3y^3 - x^3y + 3x^2y^2 - x^2 + 2yx}{x^3 + x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^3y^3 - x^3y + 3x^2y^2 - x^2 + 2yx}{x^3+x}} dy \end{aligned}$$

Which results in

$$S = \frac{(x^3 + x)\left(-\frac{\ln(yx+1)}{x^2+1} + \frac{\ln(x\,y^2 - x + 2y)}{2x^2+2}\right)}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y - x}{x\,(x^2 + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{yx + 1} + \frac{y^2 - 1}{2x\,y^2 - 2x + 4y} \\ S_y &= \frac{x^2 + 1}{(yx + 1)\,(x\,y^2 - x + 2y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

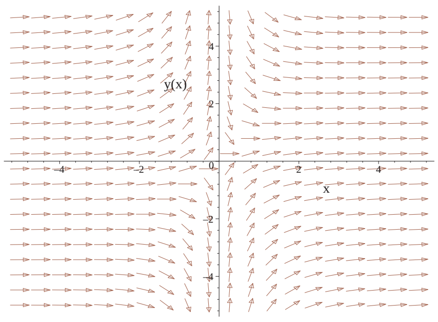
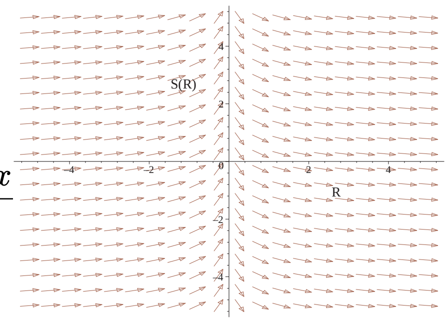
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int -\frac{1}{2R}\, dR \\ S(R) &= -\frac{\ln(R)}{2} + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\ln(yx + 1) + \frac{\ln(xy^2 - x + 2y)}{2} = -\frac{\ln(x)}{2} + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-x}{x(x^2+1)}$ 	$R = x$ $S = -\ln\left(yx+1\right) + \frac{\ln (x}{}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Solving for y gives

$$y = -\frac{e^{2c_2} - 1 - \sqrt{-x^2 e^{2c_2} + x^2 - e^{2c_2} + 1}}{\left(e^{2c_2} - 1\right) x}$$

$$y = -\frac{e^{2c_2} - 1 + \sqrt{-x^2 e^{2c_2} + x^2 - e^{2c_2} + 1}}{\left(e^{2c_2} - 1\right) x}$$

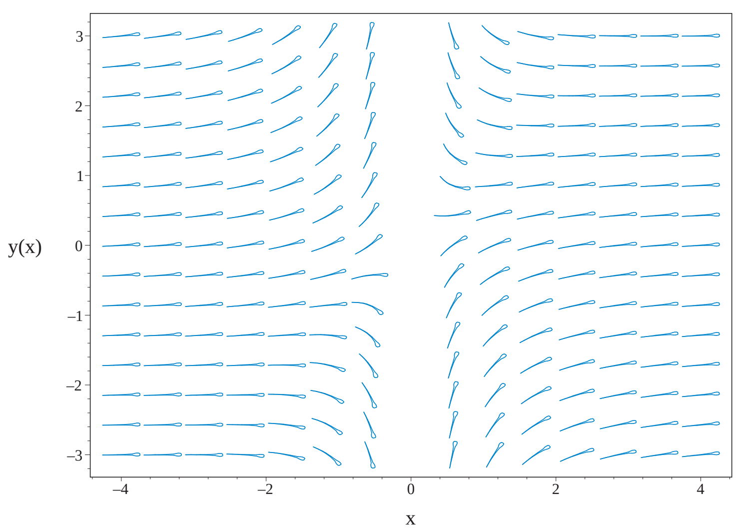


Figure 2.32: Slope field plot
$\left(x^3 + x\right) y' + y = x$

Summary of solutions found

$$y = -\frac{e^{2c_2} - 1 - \sqrt{-x^2 e^{2c_2} + x^2 - e^{2c_2} + 1}}{\left(e^{2c_2} - 1\right) x}$$

$$y = -\frac{e^{2c_2} - 1 + \sqrt{-x^2 e^{2c_2} + x^2 - e^{2c_2} + 1}}{\left(e^{2c_2} - 1\right) x}$$

Maple step by step solution

Let's solve
$(x^3+x)\left(\frac{d}{dx}y(x)\right)+y(x)=x$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x)=\frac{x-y(x)}{x^3+x}$
- Collect w.r.t. $y(x)$ and simplify
  $\frac{d}{dx}y(x)=-\frac{y(x)}{(x^2+1)x}+\frac{1}{x^2+1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x)+\frac{y(x)}{(x^2+1)x}=\frac{1}{x^2+1}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{y(x)}{(x^2+1)x}\right)=\frac{\mu(x)}{x^2+1}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{y(x)}{(x^2+1)x}\right)=\left(\frac{d}{dx}y(x)\right)\mu(x)+y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x)=\frac{\mu(x)}{(x^2+1)x}$
- Solve to find the integrating factor
  $\mu(x)=\frac{x}{\sqrt{x^2+1}}$
- Integrate both sides with respect to x
  $\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx=\int\frac{\mu(x)}{x^2+1}dx+C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x)=\int\frac{\mu(x)}{x^2+1}dx+C1$
- Solve for $y(x)$
  $y(x)=\frac{\int\frac{\mu(x)}{x^2+1}dx+C1}{\mu(x)}$
- Substitute $\mu(x)=\frac{x}{\sqrt{x^2+1}}$
  $y(x)=\frac{\sqrt{x^2+1}\left(\int\frac{x}{(x^2+1)^{3/2}}dx+C1\right)}{x}$
- Evaluate the integrals on the rhs
  $y(x)=\frac{\sqrt{x^2+1}\left(-\frac{1}{\sqrt{x^2+1}}+C1\right)}{x}$
- Simplify
  $y(x)=\frac{C1\sqrt{x^2+1}-1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 19

```
dsolve((x^3+x)*diff(y(x),x)+y(x) = x,y(x),singsol=all)
```

$$y(x) = \frac{\sqrt{x^2+1}\, c_1 - 1}{x}$$

Mathematica DSolve solution

Solving time : 0.039 (sec)
Leaf size : 23

```
DSolve[{(x^3+x)*D[y[x],x]+y[x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-1 + c_1\sqrt{x^2+1}}{x}$$

2.1.12 Problem 3(a)

Internal problem ID [4201]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 3(a)
Date solved : `Monday, January 27, 2025 at 08:41:59 AM`
CAS classification : `[_linear]`

Solve

$$\cot(x)\, y' + y = x$$

Solved as first order linear ode

Time used: 0.112 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \tan(x) \\ p(x) &= x\tan(x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \tan(x)dx} \\ &= \sec(x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x\tan(x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\sec(x)) &= (\sec(x))(x\tan(x)) \\ \mathrm{d}(y\sec(x)) &= (x\tan(x)\sec(x))\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\sec(x) &= \int x\tan(x)\sec(x)\,dx \\ &= \frac{x}{\cos(x)} - \ln(\sec(x)+\tan(x)) + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\sec(x)$ gives the final solution

$$y = -\ln(\sec(x)+\tan(x))\cos(x) + c_1\cos(x) + x$$

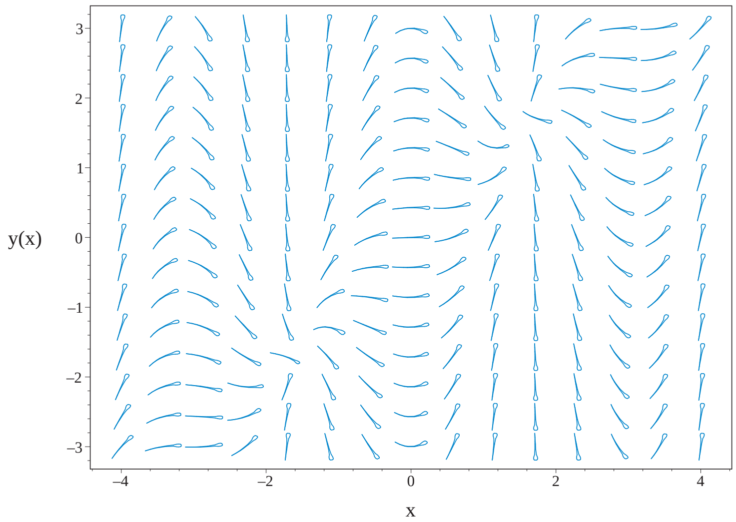


Figure 2.33: Slope field plot
$\cot(x)\, y' + y = x$

Summary of solutions found

$$y = -\ln(\sec(x) + \tan(x))\cos(x) + c_1 \cos(x) + x$$

Solved as first order Exact ode

Time used: 0.164 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\cot(x))\,\mathrm{d}y = (-y + x)\,\mathrm{d}x$$
$$(y - x)\,\mathrm{d}x + (\cot(x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= y-x \\ N(x,y) &= \cot(x) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y-x) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cot(x)) \\ &= -\csc(x)^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \tan(x)\left((1) - \left(-1 - \cot(x)^2\right)\right) \\ &= 2\tan(x) + \cot(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 2\tan(x)+\cot(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(\cos(x))+\ln(\sin(x))} \\ &= \frac{\sin(x)}{\cos(x)^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{\sin(x)}{\cos(x)^2}(y-x) \\ &= (y-x)\sec(x)\tan(x) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{\sin(x)}{\cos(x)^2}(\cot(x)) \\ &= \sec(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((y-x)\sec(x)\tan(x)) + (\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \overline{N} \,\mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \sec(x) \,\mathrm{d}y$$
$$\phi = y \sec(x) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = y \sec(x) \tan(x) + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = (y - x) \sec(x) \tan(x)$. Therefore equation (4) becomes

$$(y - x) \sec(x) \tan(x) = y \sec(x) \tan(x) + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -x \tan(x) \sec(x)$$

Integrating the above w.r.t x gives

$$\int f'(x) \,\mathrm{d}x = \int (-x \tan(x) \sec(x)) \,\mathrm{d}x$$
$$f(x) = -\frac{x}{\cos(x)} + \ln(\sec(x) + \tan(x)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y \sec(x) - \frac{x}{\cos(x)} + \ln(\sec(x) + \tan(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y \sec(x) - \frac{x}{\cos(x)} + \ln(\sec(x) + \tan(x))$$

Solving for y gives

$$y = -\frac{\ln(\sec(x) + \tan(x)) \cos(x) - c_1 \cos(x) - x}{\sec(x) \cos(x)}$$

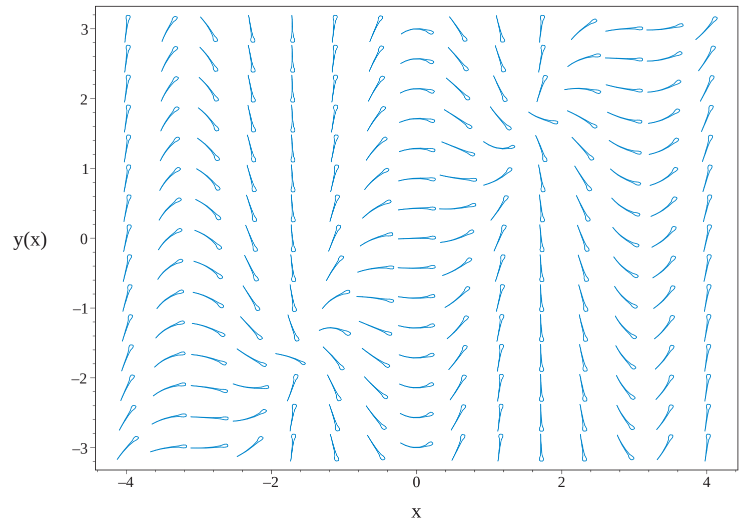


Figure 2.34: Slope field plot
$\cot(x)\, y' + y = x$

Summary of solutions found

$$y = -\frac{\ln(\sec(x) + \tan(x))\cos(x) - c_1 \cos(x) - x}{\sec(x)\cos(x)}$$

Maple step by step solution

Let's solve
$\cot(x)\left(\frac{d}{dx}y(x)\right) + y(x) = x$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{x-y(x)}{\cot(x)}$
- Collect w.r.t. $y(x)$ and simplify
  $\frac{d}{dx}y(x) = -\frac{y(x)}{\cot(x)} + \frac{x}{\cot(x)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + \frac{y(x)}{\cot(x)} = \frac{x}{\cot(x)}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{\cot(x)}\right) = \frac{\mu(x)x}{\cot(x)}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{\cot(x)}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \frac{\mu(x)}{\cot(x)}$
- Solve to find the integrating factor
  $\mu(x) = \frac{1}{\cos(x)}$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \frac{\mu(x)x}{\cot(x)} dx + C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x) = \int \frac{\mu(x)x}{\cot(x)} dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int \frac{\mu(x)x}{\cot(x)} dx + C1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

  $y(x) = \cos(x)\left(\int \frac{x}{\cos(x)\cot(x)}dx + C1\right)$
- Evaluate the integrals on the rhs

  $y(x) = \cos(x)\left(\frac{x}{\cos(x)} - \ln(\sec(x) + \tan(x)) + C1\right)$
- Simplify

  $y(x) = -\ln(\sec(x) + \tan(x))\cos(x) + C1\cos(x) + x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 19

```
dsolve(cot(x)*diff(y(x),x)+y(x) = x,y(x),singsol=all)
```

$$y(x) = x + \cos(x)(-\ln(\sec(x) + \tan(x)) + c_1)$$

Mathematica DSolve solution

Solving time : `0.065` (sec)
Leaf size : 45

```
DSolve[{Cot[x]*D[y[x],x]+y[x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x + \cos(x)\left(\log\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) - \log\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right) + c_1\right)$$

2.1.13 Problem 3(b)

Internal problem ID [4202]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 3(b)
Date solved : `Monday, January 27, 2025 at 08:42:01 AM`
CAS classification : `[_linear]`

Solve

$$\cot(x) y' + y = \tan(x)$$

Solved as first order linear ode

Time used: 0.135 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \tan(x) \\ p(x) &= \tan(x)^2 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int \tan(x) dx} \\ &= \sec(x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(\tan(x)^2\right) \\ \frac{d}{dx}(y \sec(x)) &= (\sec(x))\left(\tan(x)^2\right) \\ d(y\sec(x)) &= \left(\tan(x)^2 \sec(x)\right)\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\sec(x) &= \int \tan(x)^2 \sec(x)\, dx \\ &= \frac{\sin(x)^3}{2\cos(x)^2} + \frac{\sin(x)}{2} - \frac{\ln(\sec(x) + \tan(x))}{2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\sec(x)$ gives the final solution

$$y = \frac{\tan(x)}{2} - \frac{\ln(\sec(x) + \tan(x))\cos(x)}{2} + c_1 \cos(x)$$

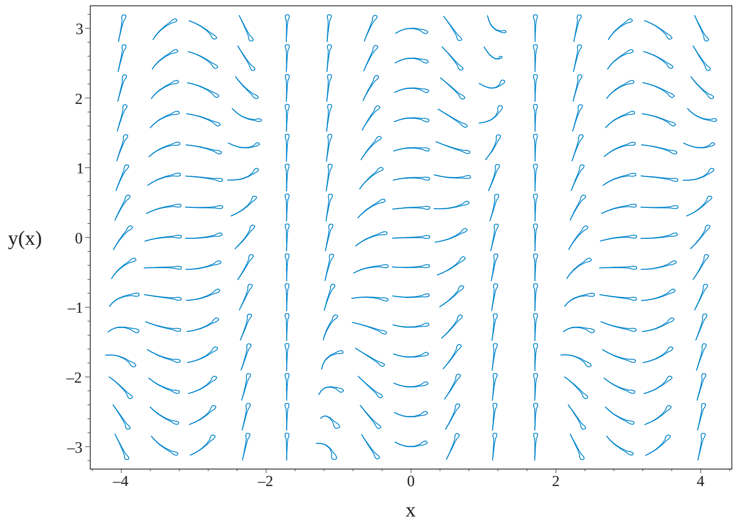


Figure 2.35: Slope field plot
$\cot(x)\,y' + y = \tan(x)$

Summary of solutions found

$$y = \frac{\tan(x)}{2} - \frac{\ln(\sec(x)+\tan(x))\cos(x)}{2} + c_1 \cos(x)$$

Solved as first order Exact ode

Time used: 0.257 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\cot(x))\,\mathrm{d}y = (-y + \tan(x))\,\mathrm{d}x$$
$$(y - \tan(x))\,\mathrm{d}x + (\cot(x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y-\tan(x)\\ N(x,y) &= \cot(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y-\tan(x))\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cot(x))\\ &= -\csc(x)^2\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \tan(x)\left((1)-\left(-1-\cot(x)^2\right)\right)\\ &= 2\tan(x)+\cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int 2\tan(x)+\cot(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(\cos(x))+\ln(\sin(x))}\\ &= \frac{\sin(x)}{\cos(x)^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{\sin(x)}{\cos(x)^2}(y-\tan(x))\\ &= (y-\tan(x))\sec(x)\tan(x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{\sin(x)}{\cos(x)^2}(\cot(x))\\ &= \sec(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ ((y-\tan(x))\sec(x)\tan(x))+(\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \overline{N} \,\mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \sec(x) \,\mathrm{d}y$$

$$\phi = y \sec(x) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = y \sec(x) \tan(x) + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = (y - \tan(x)) \sec(x) \tan(x)$. Therefore equation (4) becomes

$$(y - \tan(x)) \sec(x) \tan(x) = y \sec(x) \tan(x) + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\tan(x)^2 \sec(x)$$

Integrating the above w.r.t x gives

$$\int f'(x) \,\mathrm{d}x = \int \left(-\tan(x)^2 \sec(x)\right) \mathrm{d}x$$

$$f(x) = -\frac{\sin(x)^3}{2\cos(x)^2} - \frac{\sin(x)}{2} + \frac{\ln(\sec(x) + \tan(x))}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y \sec(x) - \frac{\sin(x)^3}{2\cos(x)^2} - \frac{\sin(x)}{2} + \frac{\ln(\sec(x) + \tan(x))}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y \sec(x) - \frac{\sin(x)^3}{2\cos(x)^2} - \frac{\sin(x)}{2} + \frac{\ln(\sec(x) + \tan(x))}{2}$$

Solving for y gives

$$y = \frac{-\ln(\sec(x) + \tan(x)) \cos(x)^2 + \sin(x) \cos(x)^2 + 2c_1 \cos(x)^2 + \sin(x)^3}{2\sec(x) \cos(x)^2}$$

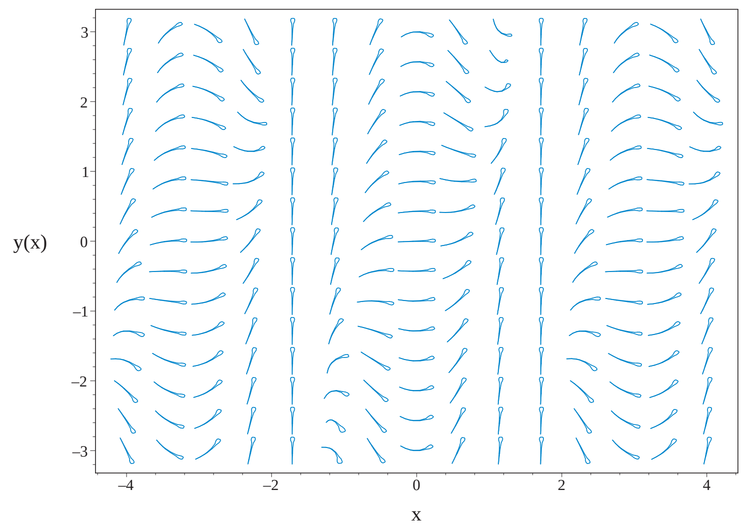


Figure 2.36: Slope field plot
$\cot(x) y' + y = \tan(x)$

Summary of solutions found

$$y = \frac{-\ln(\sec(x) + \tan(x)) \cos(x)^2 + \sin(x) \cos(x)^2 + 2c_1 \cos(x)^2 + \sin(x)^3}{2 \sec(x) \cos(x)^2}$$

Maple step by step solution

Let's solve
$\cot(x) \left(\frac{d}{dx}y(x)\right) + y(x) = \tan(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{-y(x)+\tan(x)}{\cot(x)}$
- Collect w.r.t. $y(x)$ and simplify
  $\frac{d}{dx}y(x) = -\frac{y(x)}{\cot(x)} + \frac{\tan(x)}{\cot(x)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + \frac{y(x)}{\cot(x)} = \frac{\tan(x)}{\cot(x)}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x) \left(\frac{d}{dx}y(x) + \frac{y(x)}{\cot(x)}\right) = \frac{\mu(x)\tan(x)}{\cot(x)}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x) \left(\frac{d}{dx}y(x) + \frac{y(x)}{\cot(x)}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \frac{\mu(x)}{\cot(x)}$
- Solve to find the integrating factor
  $\mu(x) = \frac{1}{\cos(x)}$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \frac{\mu(x)\tan(x)}{\cot(x)} dx + C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x) = \int \frac{\mu(x)\tan(x)}{\cot(x)} dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int \frac{\mu(x)\tan(x)}{\cot(x)} dx + C1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

  $y(x) = \cos(x)\left(\int \frac{\tan(x)}{\cos(x)\cot(x)}dx + C1\right)$

- Evaluate the integrals on the rhs

  $y(x) = \cos(x)\left(\frac{\sin(x)^3}{2\cos(x)^2} + \frac{\sin(x)}{2} - \frac{\ln(\sec(x)+\tan(x))}{2} + C1\right)$

- Simplify

  $y(x) = \frac{\tan(x)}{2} - \frac{\ln(\sec(x)+\tan(x))\cos(x)}{2} + C1\cos(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 23

```
dsolve(cot(x)*diff(y(x),x)+y(x) = tan(x),y(x),singsol=all)
```

$$y(x) = \frac{\tan(x)}{2} - \frac{\cos(x)\ln(\sec(x)+\tan(x))}{2} + \cos(x)\,c_1$$

Mathematica DSolve solution

Solving time : `0.058` (sec)
Leaf size : 25

```
DSolve[{Cot[x]*D[y[x],x]+y[x]==Tan[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}(\cos(x)(-\text{arctanh}(\sin(x))) + \tan(x) + 2c_1\cos(x))$$

2.1.14 Problem 3(c)

Internal problem ID [4203]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 3(c)
Date solved : `Monday, January 27, 2025 at 08:42:04 AM`
CAS classification : `[_linear]`

Solve

$$\tan(x)\,y' + y = \cot(x)$$

Solved as first order linear ode

Time used: 0.129 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \cot(x) \\ p(x) &= \cot(x)^2 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int \cot(x)dx} \\ &= \sin(x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(\cot(x)^2\right) \\ \frac{d}{dx}(y\sin(x)) &= (\sin(x))\left(\cot(x)^2\right) \\ d(y\sin(x)) &= \left(\cot(x)^2\sin(x)\right)\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\sin(x) &= \int \cot(x)^2\sin(x)\,dx \\ &= \cos(x) + \ln(\csc(x) - \cot(x)) + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\sin(x)$ gives the final solution

$$y = (\cos(x) + \ln(\csc(x) - \cot(x)) + c_1)\csc(x)$$

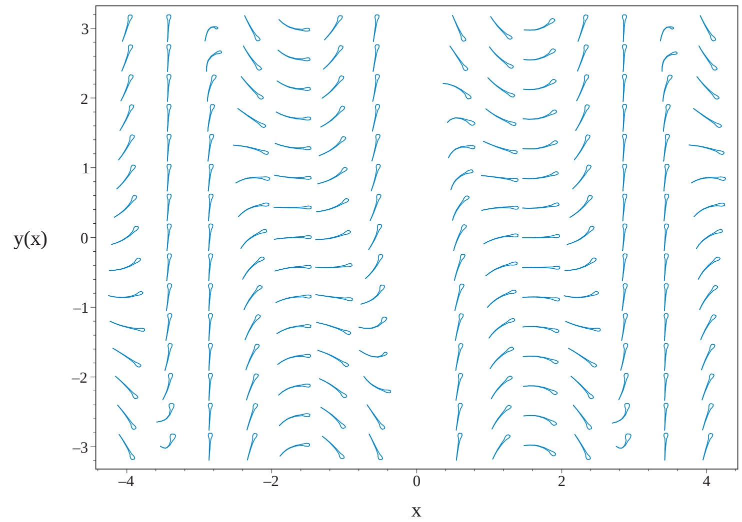


Figure 2.37: Slope field plot
$\tan(x) y' + y = \cot(x)$

Summary of solutions found

$$y = (\cos(x) + \ln(\csc(x) - \cot(x)) + c_1) \csc(x)$$

Solved as first order Exact ode

Time used: 0.195 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\tan(x))\,\mathrm{d}y &= (-y + \cot(x))\,\mathrm{d}x \\ (y - \cot(x))\,\mathrm{d}x + (\tan(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y-\cot(x)\\ N(x,y) &= \tan(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y-\cot(x))\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\tan(x))\\ &= \sec(x)^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \cot(x)\left((1)-\left(\tan(x)^2+1\right)\right)\\ &= -\tan(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\tan(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\cos(x))}\\ &= \cos(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \cos(x)(y-\cot(x))\\ &= (y-\cot(x))\cos(x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \cos(x)(\tan(x))\\ &= \sin(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ ((y-\cot(x))\cos(x))+(\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \, dy = \int \overline{N} \, dy$$

$$\int \frac{\partial \phi}{\partial y} \, dy = \int \sin(x) \, dy$$

$$\phi = y \sin(x) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \cos(x) \, y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = (y - \cot(x)) \cos(x)$. Therefore equation (4) becomes

$$(y - \cot(x)) \cos(x) = \cos(x) \, y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\cos(x) \cot(x)$$

Integrating the above w.r.t x gives

$$\int f'(x) \, dx = \int (-\cos(x) \cot(x)) \, dx$$

$$f(x) = -\cos(x) - \ln(\csc(x) - \cot(x)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y \sin(x) - \cos(x) - \ln(\csc(x) - \cot(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y \sin(x) - \cos(x) - \ln(\csc(x) - \cot(x))$$

Solving for y gives

$$y = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + c_1}{\sin(x)}$$

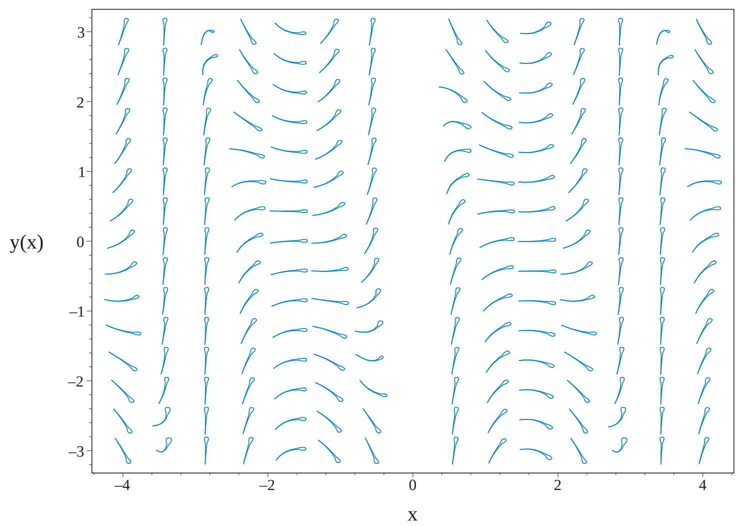


Figure 2.38: Slope field plot
$\tan(x)\, y' + y = \cot(x)$

Summary of solutions found

$$y = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + c_1}{\sin(x)}$$

Maple step by step solution

Let's solve
$\tan(x)\left(\frac{d}{dx}y(x)\right) + y(x) = \cot(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{-y(x)+\cot(x)}{\tan(x)}$
- Collect w.r.t. $y(x)$ and simplify
  $\frac{d}{dx}y(x) = -\frac{y(x)}{\tan(x)} + \frac{\cot(x)}{\tan(x)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + \frac{y(x)}{\tan(x)} = \frac{\cot(x)}{\tan(x)}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{\tan(x)}\right) = \frac{\mu(x)\cot(x)}{\tan(x)}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{\tan(x)}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \frac{\mu(x)}{\tan(x)}$
- Solve to find the integrating factor
  $\mu(x) = \sin(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \frac{\mu(x)\cot(x)}{\tan(x)}dx + C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x) = \int \frac{\mu(x)\cot(x)}{\tan(x)}dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int \frac{\mu(x)\cot(x)}{\tan(x)}dx + C1}{\mu(x)}$

- Substitute $\mu(x) = \sin(x)$

  $y(x) = \frac{\int \frac{\sin(x)\cot(x)}{\tan(x)}dx + C1}{\sin(x)}$
- Evaluate the integrals on the rhs

  $y(x) = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + C1}{\sin(x)}$
- Simplify

  $y(x) = (\cos(x) + \ln(\csc(x) - \cot(x)) + C1)\csc(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 19

```
dsolve(tan(x)*diff(y(x),x)+y(x) = cot(x),y(x),singsol=all)
```

$$y(x) = \csc(x)(\cos(x) + \ln(\csc(x) - \cot(x)) + c_1)$$

Mathematica DSolve solution

Solving time : 0.066 (sec)
Leaf size : 29

```
DSolve[{Tan[x]*D[y[x],x]+y[x]==Cot[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \csc(x)\left(\cos(x) + \log\left(\sin\left(\frac{x}{2}\right)\right) - \log\left(\cos\left(\frac{x}{2}\right)\right) + c_1\right)$$

2.1.15 Problem 3(a)

Internal problem ID [4204]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 3(a)
Date solved : `Monday, January 27, 2025 at 08:42:06 AM`
CAS classification : `[_linear]`

Solve

$$\tan (x) y' = y - \cos (x)$$

Solved as first order linear ode

Time used: 0.120 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\cot (x) \\ p(x) &= -\cos (x) \cot (x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= e^{\int -\cot(x) dx} \\ &= \csc (x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) (-\cos (x) \cot (x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y \csc (x)) &= (\csc (x)) (-\cos (x) \cot (x)) \\ \mathrm{d}(y \csc (x)) &= (-\cos (x) \cot (x) \csc (x))\, \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y \csc (x) &= \int -\cos (x) \cot (x) \csc (x)\, dx \\ &= \cot (x) + x + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\csc (x)$ gives the final solution

$$y = (\cot (x) + x + c_1) \sin (x)$$

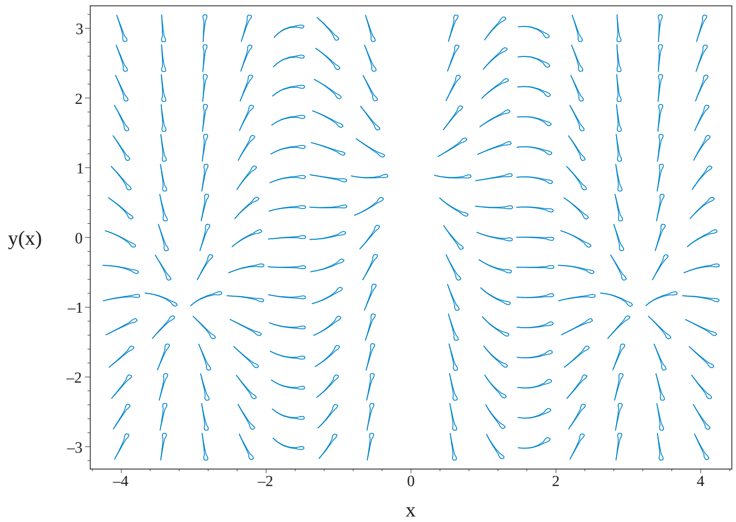


Figure 2.39: Slope field plot
$\tan(x) y' = y - \cos(x)$

Summary of solutions found

$$y = (\cot(x) + x + c_1) \sin(x)$$

Solved as first order Exact ode

Time used: 0.166 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\tan(x))\, \mathrm{d}y = (y - \cos(x))\, \mathrm{d}x$$
$$(-y + \cos(x))\, \mathrm{d}x + (\tan(x))\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -y+\cos(x) \\ N(x,y) &= \tan(x) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y+\cos(x)) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\tan(x)) \\ &= \sec(x)^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \cot(x)\left((-1) - \left(\tan(x)^2+1\right)\right) \\ &= -2\cot(x) - \tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\, dx} \\ &= e^{\int -2\cot(x)-\tan(x)\, dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(\sin(x))+\ln(\cos(x))} \\ &= \frac{\cos(x)}{\sin(x)^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{\cos(x)}{\sin(x)^2}(-y+\cos(x)) \\ &= \cot(x)(-y\csc(x)+\cot(x)) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{\cos(x)}{\sin(x)^2}(\tan(x)) \\ &= \csc(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\ (\cot(x)(-y\csc(x)+\cot(x))) + (\csc(x))\frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \overline{N} \, \mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \csc(x) \, \mathrm{d}y$$
$$\phi = y \csc(x) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -y \csc(x) \cot(x) + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \cot(x)(-y \csc(x) + \cot(x))$. Therefore equation (4) becomes

$$\cot(x)(-y \csc(x) + \cot(x)) = -y \csc(x) \cot(x) + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \cot(x)^2$$

Integrating the above w.r.t x gives

$$\int f'(x) \, \mathrm{d}x = \int \left(\cot(x)^2\right) \mathrm{d}x$$
$$f(x) = -\cot(x) + \frac{\pi}{2} - x + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y \csc(x) - \cot(x) + \frac{\pi}{2} - x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y \csc(x) - \cot(x) + \frac{\pi}{2} - x$$

Solving for y gives

$$y = -\frac{\pi - 2\cot(x) - 2c_1 - 2x}{2\csc(x)}$$

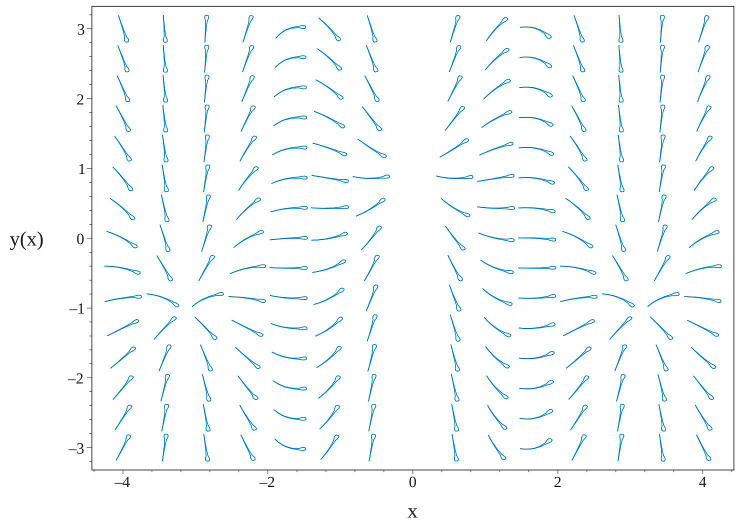


Figure 2.40: Slope field plot
$\tan(x)\, y' = y - \cos(x)$

Summary of solutions found

$$y = -\frac{\pi - 2\cot(x) - 2c_1 - 2x}{2\csc(x)}$$

Maple step by step solution

Let's solve
$\tan(x)\left(\frac{d}{dx}y(x)\right) = y(x) - \cos(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{y(x)-\cos(x)}{\tan(x)}$
- Collect w.r.t. $y(x)$ and simplify
  $\frac{d}{dx}y(x) = \frac{y(x)}{\tan(x)} - \frac{\cos(x)}{\tan(x)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) - \frac{y(x)}{\tan(x)} = -\frac{\cos(x)}{\tan(x)}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{y(x)}{\tan(x)}\right) = -\frac{\mu(x)\cos(x)}{\tan(x)}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{y(x)}{\tan(x)}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = -\frac{\mu(x)}{\tan(x)}$
- Solve to find the integrating factor
  $\mu(x) = \frac{1}{\sin(x)}$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int -\frac{\mu(x)\cos(x)}{\tan(x)} dx + C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x) = \int -\frac{\mu(x)\cos(x)}{\tan(x)} dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int -\frac{\mu(x)\cos(x)}{\tan(x)} dx + C1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\sin(x)}$

  $y(x) = \sin(x)\left(\int -\frac{\cos(x)}{\sin(x)\tan(x)}dx + C1\right)$
- Evaluate the integrals on the rhs

  $y(x) = \sin(x)(\cot(x) + x + C1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 15

```
dsolve(tan(x)*diff(y(x),x) = y(x)-cos(x),y(x),singsol=all)
```

$$y(x) = \left(\cot(x) - \frac{\pi}{2} + x + c_1\right)\sin(x)$$

Mathematica DSolve solution

Solving time : 0.054 (sec)
Leaf size : 28

```
DSolve[{Tan[x]*D[y[x],x]==y[x]-Cos[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \cos(x)\,\text{Hypergeometric2F1}\left(-\frac{1}{2}, 1, \frac{1}{2}, -\tan^2(x)\right) + c_1\sin(x)$$

2.1.16 Problem 4(a)

Internal problem ID [4205]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 4(a)
Date solved : `Monday, January 27, 2025 at 08:42:09 AM`
CAS classification : `[_linear]`

Solve

$$y' + y\cos(x) = \sin(2x)$$

Solved as first order linear ode

Time used: 0.131 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \cos(x) \\ p(x) &= \sin(2x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \cos(x)dx} \\ &= \mathrm{e}^{\sin(x)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(\sin(2x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{\sin(x)}\right) &= \left(\mathrm{e}^{\sin(x)}\right)(\sin(2x)) \\ \mathrm{d}\left(y\,\mathrm{e}^{\sin(x)}\right) &= \left(\sin(2x)\,\mathrm{e}^{\sin(x)}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{\sin(x)} &= \int \sin(2x)\,\mathrm{e}^{\sin(x)}\,dx \\ &= 2\sin(x)\,\mathrm{e}^{\sin(x)} - 2\,\mathrm{e}^{\sin(x)} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{\sin(x)}$ gives the final solution

$$y = 2\sin(x) + \mathrm{e}^{-\sin(x)}c_1 - 2$$

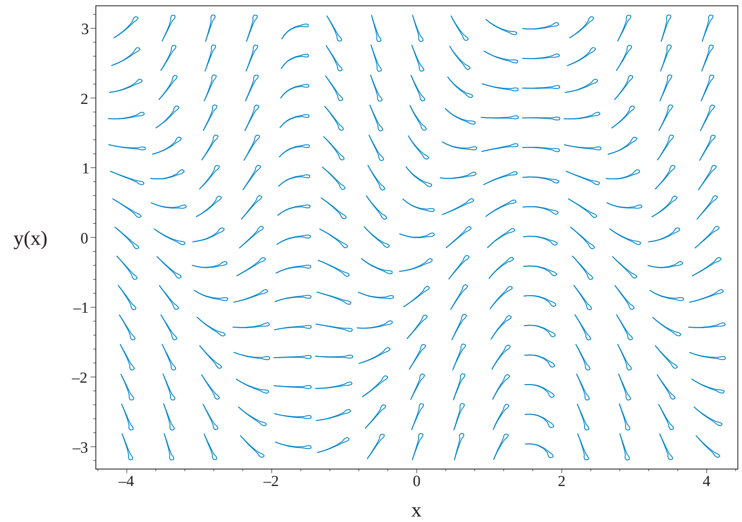


Figure 2.41: Slope field plot
$y' + y\cos(x) = \sin(2x)$

Summary of solutions found

$$y = 2\sin(x) + e^{-\sin(x)}c_1 - 2$$

Solved as first order Exact ode

Time used: 0.122 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y\cos(x) + \sin(2x))\,\mathrm{d}x$$
$$(y\cos(x) - \sin(2x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= y\cos(x) - \sin(2x) \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\cos(x) - \sin(2x)) \\ &= \cos(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cos(x)) - (0)) \\ &= \cos(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \cos(x)\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\sin(x)} \\ &= \mathrm{e}^{\sin(x)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{\sin(x)}(y\cos(x) - \sin(2x)) \\ &= \cos(x)(-2\sin(x) + y)\,\mathrm{e}^{\sin(x)} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{\sin(x)}(1) \\ &= \mathrm{e}^{\sin(x)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\cos(x)(-2\sin(x) + y)\,\mathrm{e}^{\sin(x)}\right) + \left(\mathrm{e}^{\sin(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \overline{N} \, \mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \mathrm{e}^{\sin(x)} \, \mathrm{d}y$$

$$\phi = y \, \mathrm{e}^{\sin(x)} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \mathrm{e}^{\sin(x)} \cos(x) \, y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \cos(x) \left(-2\sin(x) + y\right) \mathrm{e}^{\sin(x)}$. Therefore equation (4) becomes

$$\cos(x) \left(-2\sin(x) + y\right) \mathrm{e}^{\sin(x)} = \mathrm{e}^{\sin(x)} \cos(x) \, y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -2\cos(x) \, \mathrm{e}^{\sin(x)} \sin(x)$$

Integrating the above w.r.t x gives

$$\int f'(x) \, \mathrm{d}x = \int \left(-2\cos(x) \, \mathrm{e}^{\sin(x)} \sin(x)\right) \mathrm{d}x$$

$$f(x) = -2\sin(x) \, \mathrm{e}^{\sin(x)} + 2\,\mathrm{e}^{\sin(x)} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y \, \mathrm{e}^{\sin(x)} - 2\sin(x) \, \mathrm{e}^{\sin(x)} + 2\,\mathrm{e}^{\sin(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y \, \mathrm{e}^{\sin(x)} - 2\sin(x) \, \mathrm{e}^{\sin(x)} + 2\,\mathrm{e}^{\sin(x)}$$

Solving for y gives

$$y = \mathrm{e}^{-\sin(x)} \left(2\sin(x) \, \mathrm{e}^{\sin(x)} - 2\,\mathrm{e}^{\sin(x)} + c_1\right)$$

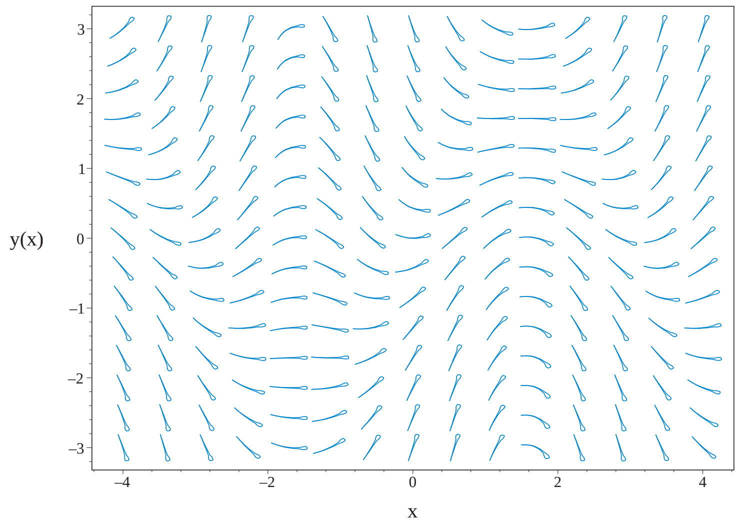


Figure 2.42: Slope field plot
$y' + y\cos(x) = \sin(2x)$

Summary of solutions found

$$y = e^{-\sin(x)}\left(2\sin(x)\, e^{\sin(x)} - 2\, e^{\sin(x)} + c_1\right)$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) + \cos(x)\, y(x) = \sin(2x)$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = -\cos(x)\, y(x) + \sin(2x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
$\frac{d}{dx}y(x) + \cos(x)\, y(x) = \sin(2x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$\mu(x)\left(\frac{d}{dx}y(x) + \cos(x)\, y(x)\right) = \mu(x)\sin(2x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
$\mu(x)\left(\frac{d}{dx}y(x) + \cos(x)\, y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
$\frac{d}{dx}\mu(x) = \mu(x)\cos(x)$
- Solve to find the integrating factor
$\mu(x) = e^{\sin(x)}$
- Integrate both sides with respect to x
$\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \mu(x)\sin(2x)\,dx + C1$
- Evaluate the integral on the lhs
$y(x)\,\mu(x) = \int \mu(x)\sin(2x)\,dx + C1$
- Solve for $y(x)$
$y(x) = \frac{\int \mu(x)\sin(2x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = e^{\sin(x)}$
$y(x) = \frac{\int e^{\sin(x)}\sin(2x)dx + C1}{e^{\sin(x)}}$
- Evaluate the integrals on the rhs
$y(x) = \frac{2\, e^{\sin(x)}\sin(x) - 2\, e^{\sin(x)} + C1}{e^{\sin(x)}}$
- Simplify

$$y(x) = 2\sin(x) + e^{-\sin(x)} C1 - 2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 17

```
dsolve(diff(y(x),x)+cos(x)*y(x) = sin(2*x),y(x),singsol=all)
```

$$y(x) = 2\sin(x) - 2 + e^{-\sin(x)} c_1$$

Mathematica DSolve solution

Solving time : 0.077 (sec)
Leaf size : 20

```
DSolve[{D[y[x],x]+y[x]*Cos[x]==Sin[2*x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 2\sin(x) + c_1 e^{-\sin(x)} - 2$$

2.1.17 Problem 4(b)

Internal problem ID [4206]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 4(b)
Date solved : `Monday, January 27, 2025 at 08:42:11 AM`
CAS classification : `[_linear]`

Solve

$$\cos(x)\,y' + y = \sin(2x)$$

Solved as first order linear ode

Time used: 0.200 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \sec(x) \\ p(x) &= 2\sin(x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int \sec(x)dx} \\ &= \sec(x) + \tan(x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(2\sin(x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y(\sec(x) + \tan(x))) &= (\sec(x) + \tan(x))\,(2\sin(x)) \\ \mathrm{d}(y(\sec(x) + \tan(x))) &= (2\sin(x)\,(\sec(x) + \tan(x)))\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y(\sec(x) + \tan(x)) &= \int 2\sin(x)\,(\sec(x) + \tan(x))\,dx \\ &= -2\sin(x) - 2\ln(\sin(x) - 1) + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\sec(x) + \tan(x)$ gives the final solution

$$y = \frac{(-2\sin(x) - 2\ln(\sin(x) - 1) + c_1)\,(\cos(x) - \sin(x) + 1)}{\cos(x) + 1 + \sin(x)}$$

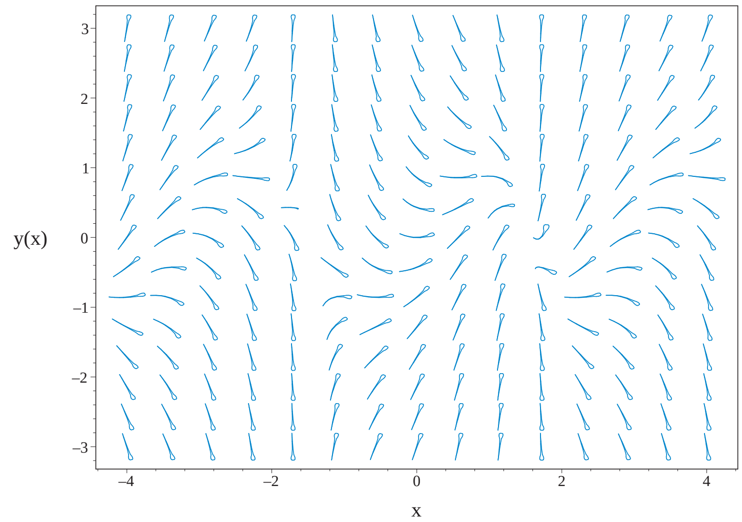


Figure 2.43: Slope field plot
$\cos(x)\, y' + y = \sin(2x)$

Summary of solutions found

$$y = \frac{(-2\sin(x) - 2\ln(\sin(x) - 1) + c_1)(\cos(x) - \sin(x) + 1)}{\cos(x) + 1 + \sin(x)}$$

Solved as first order Exact ode

Time used: 0.362 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\cos(x))\,\mathrm{d}y = (-y + \sin(2x))\,\mathrm{d}x$$
$$(y - \sin(2x))\,\mathrm{d}x + (\cos(x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y - \sin(2x)\\ N(x, y) &= \cos(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \sin(2x))\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(x))\\ &= -\sin(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \sec(x)\left((1) - (-\sin(x))\right)\\ &= \sec(x) + \tan(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int \sec(x)+\tan(x) \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\sec(x)+\tan(x))-\ln(\cos(x))}\\ &= \frac{\sec(x) + \tan(x)}{\cos(x)}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{\sec(x) + \tan(x)}{\cos(x)}(y - \sin(2x))\\ &= \frac{-y + 2\cos(x)\sin(x)}{\sin(x) - 1}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{\sec(x) + \tan(x)}{\cos(x)}(\cos(x))\\ &= \sec(x) + \tan(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-y + 2\cos(x)\sin(x)}{\sin(x) - 1}\right) + (\sec(x) + \tan(x))\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \overline{N}\,\mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \sec(x) + \tan(x)\,\mathrm{d}y$$
$$\phi = y(\sec(x) + \tan(x)) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = y\left(\sec(x)\tan(x) + 1 + \tan(x)^2\right) + f'(x) \tag{4}$$
$$= -\frac{y}{\sin(x) - 1} + f'(x)$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \frac{-y+2\cos(x)\sin(x)}{\sin(x)-1}$. Therefore equation (4) becomes

$$\frac{-y + 2\cos(x)\sin(x)}{\sin(x) - 1} = -\frac{y}{\sin(x) - 1} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \frac{2\cos(x)\sin(x)}{\sin(x) - 1}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(\frac{2\cos(x)\sin(x)}{\sin(x) - 1}\right)\mathrm{d}x$$
$$f(x) = 2\sin(x) + 2\ln(\sin(x) - 1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y(\sec(x) + \tan(x)) + 2\sin(x) + 2\ln(\sin(x) - 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y(\sec(x) + \tan(x)) + 2\sin(x) + 2\ln(\sin(x) - 1)$$

Solving for y gives

$$y = -\frac{2\sin(x) + 2\ln(\sin(x) - 1) - c_1}{\sec(x) + \tan(x)}$$

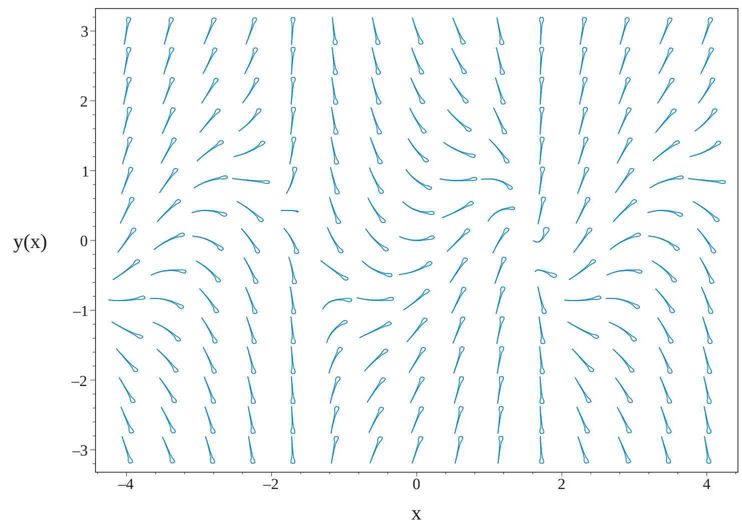


Figure 2.44: Slope field plot
$\cos(x) y' + y = \sin(2x)$

Summary of solutions found

$$y = -\frac{2\sin(x) + 2\ln(\sin(x) - 1) - c_1}{\sec(x) + \tan(x)}$$

Maple step by step solution

Let's solve
$\cos(x)\left(\frac{d}{dx}y(x)\right) + y(x) = \sin(2x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{-y(x)+\sin(2x)}{\cos(x)}$
- Collect w.r.t. $y(x)$ and simplify
  $\frac{d}{dx}y(x) = -\frac{y(x)}{\cos(x)} + \frac{\sin(2x)}{\cos(x)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + \frac{y(x)}{\cos(x)} = \frac{\sin(2x)}{\cos(x)}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{\cos(x)}\right) = \frac{\mu(x)\sin(2x)}{\cos(x)}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{\cos(x)}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \frac{\mu(x)}{\cos(x)}$
- Solve to find the integrating factor
  $\mu(x) = \sec(x) + \tan(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y(x)\mu(x))\right) dx = \int \frac{\mu(x)\sin(2x)}{\cos(x)} dx + C1$
- Evaluate the integral on the lhs

$$y(x)\,\mu(x) = \int \frac{\mu(x)\sin(2x)}{\cos(x)}dx + C1$$

- Solve for $y(x)$

$$y(x) = \frac{\int \frac{\mu(x)\sin(2x)}{\cos(x)}dx + C1}{\mu(x)}$$

- Substitute $\mu(x) = \sec(x) + \tan(x)$

$$y(x) = \frac{\int \frac{(\sec(x)+\tan(x))\sin(2x)}{\cos(x)}dx + C1}{\sec(x)+\tan(x)}$$

- Evaluate the integrals on the rhs

$$y(x) = \frac{-2\sin(x) - 2\ln(\sin(x)-1) + C1}{\sec(x)+\tan(x)}$$

- Simplify

$$y(x) = \frac{(-2\sin(x) - 2\ln(\sin(x)-1) + C1)(\cos(x)-\sin(x)+1)}{\cos(x)+\sin(x)+1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 34

```
dsolve(diff(y(x),x)*cos(x)+y(x) = sin(2*x),y(x),singsol=all)
```

$$y(x) = \frac{(-2\sin(x) - 2\ln(\sin(x)-1) + c_1)(\cos(x)-\sin(x)+1)}{\cos(x)+\sin(x)+1}$$

Mathematica DSolve solution

Solving time : 0.078 (sec)
Leaf size : 42

```
DSolve[{Cos[x]*D[y[x],x]+y[x]==Sin[2*x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-2\operatorname{arctanh}\left(\tan\left(\frac{x}{2}\right)\right)}\left(-2\sin(x) - 4\log\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) + c_1\right)$$

2.1.18 Problem 4(c)

Internal problem ID [4207]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 4(c)
Date solved : `Monday, January 27, 2025 at 08:42:14 AM`
CAS classification : `[_linear]`

Solve

$$y' + y\sin(x) = \sin(2x)$$

Solved as first order linear ode

Time used: 0.121 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \sin(x) \\ p(x) &= \sin(2x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \sin(x)dx} \\ &= \mathrm{e}^{-\cos(x)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(\sin(2x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-\cos(x)}\right) &= \left(\mathrm{e}^{-\cos(x)}\right)(\sin(2x)) \\ \mathrm{d}\left(y\,\mathrm{e}^{-\cos(x)}\right) &= \left(\sin(2x)\,\mathrm{e}^{-\cos(x)}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{-\cos(x)} &= \int \sin(2x)\,\mathrm{e}^{-\cos(x)}\,dx \\ &= 2\cos(x)\,\mathrm{e}^{-\cos(x)} + 2\,\mathrm{e}^{-\cos(x)} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{-\cos(x)}$ gives the final solution

$$y = c_1\,\mathrm{e}^{\cos(x)} + 2\cos(x) + 2$$

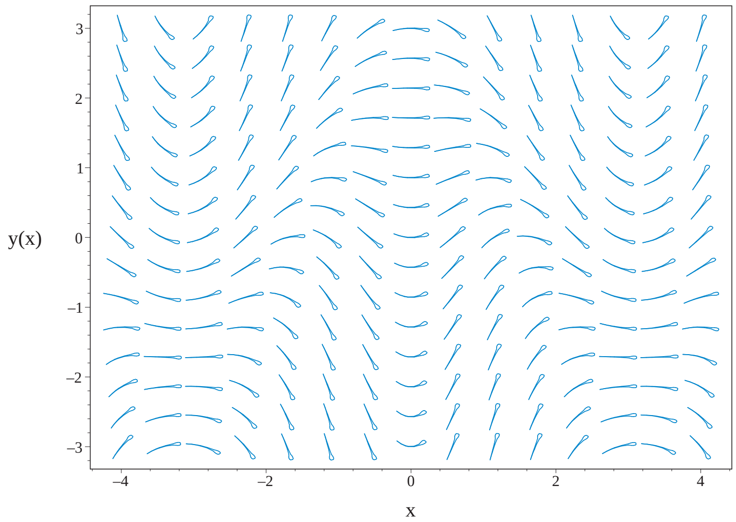


Figure 2.45: Slope field plot
$y' + y\sin(x) = \sin(2x)$

Summary of solutions found

$$y = c_1 e^{\cos(x)} + 2\cos(x) + 2$$

Solved as first order Exact ode

Time used: 0.189 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y\sin(x) + \sin(2x))\,\mathrm{d}x$$
$$(y\sin(x) - \sin(2x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\sin(x) - \sin(2x)\\N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\sin(x) - \sin(2x))\\&= \sin(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\&= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\&= 1((\sin(x)) - (0))\\&= \sin(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\&= e^{\int \sin(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\cos(x)}\\&= \mathrm{e}^{-\cos(x)}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\&= \mathrm{e}^{-\cos(x)}(y\sin(x) - \sin(2x))\\&= \sin(x)(-2\cos(x) + y)\,\mathrm{e}^{-\cos(x)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\&= \mathrm{e}^{-\cos(x)}(1)\\&= \mathrm{e}^{-\cos(x)}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\\left(\sin(x)(-2\cos(x) + y)\,\mathrm{e}^{-\cos(x)}\right) + \left(\mathrm{e}^{-\cos(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \overline{N} \,\mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \mathrm{e}^{-\cos(x)} \,\mathrm{d}y$$
$$\phi = y\,\mathrm{e}^{-\cos(x)} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \sin(x)\,\mathrm{e}^{-\cos(x)} y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \sin(x)\,(-2\cos(x) + y)\,\mathrm{e}^{-\cos(x)}$. Therefore equation (4) becomes

$$\sin(x)\,(-2\cos(x) + y)\,\mathrm{e}^{-\cos(x)} = \sin(x)\,\mathrm{e}^{-\cos(x)} y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -2\sin(x)\,\mathrm{e}^{-\cos(x)}\cos(x)$$

Integrating the above w.r.t x gives

$$\int f'(x) \,\mathrm{d}x = \int \left(-2\sin(x)\,\mathrm{e}^{-\cos(x)}\cos(x)\right) \mathrm{d}x$$
$$f(x) = -2\cos(x)\,\mathrm{e}^{-\cos(x)} - 2\,\mathrm{e}^{-\cos(x)} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\,\mathrm{e}^{-\cos(x)} - 2\cos(x)\,\mathrm{e}^{-\cos(x)} - 2\,\mathrm{e}^{-\cos(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{-\cos(x)} - 2\cos(x)\,\mathrm{e}^{-\cos(x)} - 2\,\mathrm{e}^{-\cos(x)}$$

Solving for y gives

$$y = \mathrm{e}^{\cos(x)}\left(2\cos(x)\,\mathrm{e}^{-\cos(x)} + 2\,\mathrm{e}^{-\cos(x)} + c_1\right)$$

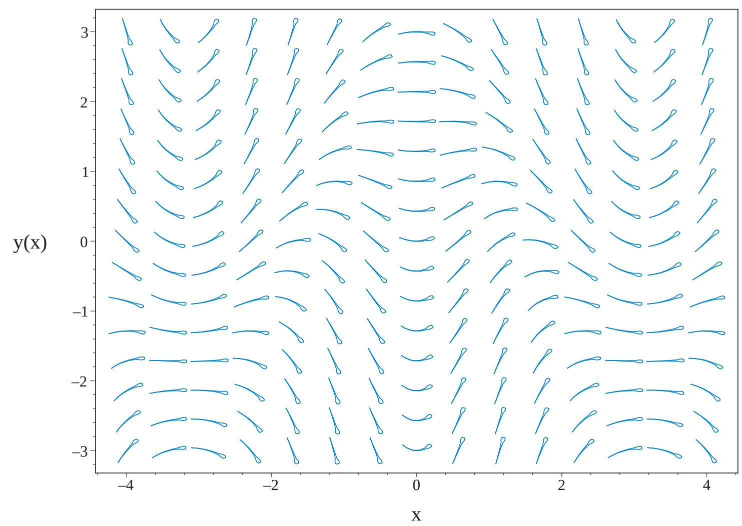


Figure 2.46: Slope field plot
$y' + y\sin(x) = \sin(2x)$

Summary of solutions found

$$y = e^{\cos(x)}\left(2\cos(x)\,e^{-\cos(x)} + 2\,e^{-\cos(x)} + c_1\right)$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) + \sin(x)\,y(x) = \sin(2x)$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = -\sin(x)\,y(x) + \sin(2x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
$\frac{d}{dx}y(x) + \sin(x)\,y(x) = \sin(2x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$\mu(x)\left(\frac{d}{dx}y(x) + \sin(x)\,y(x)\right) = \mu(x)\sin(2x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
$\mu(x)\left(\frac{d}{dx}y(x) + \sin(x)\,y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
$\frac{d}{dx}\mu(x) = \mu(x)\sin(x)$
- Solve to find the integrating factor
$\mu(x) = e^{-\cos(x)}$
- Integrate both sides with respect to x
$\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \mu(x)\sin(2x)\,dx + C1$
- Evaluate the integral on the lhs
$y(x)\,\mu(x) = \int \mu(x)\sin(2x)\,dx + C1$
- Solve for $y(x)$
$y(x) = \frac{\int \mu(x)\sin(2x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = e^{-\cos(x)}$
$y(x) = \frac{\int e^{-\cos(x)}\sin(2x)dx + C1}{e^{-\cos(x)}}$
- Evaluate the integrals on the rhs
$y(x) = \frac{2\,e^{-\cos(x)}\cos(x) + 2\,e^{-\cos(x)} + C1}{e^{-\cos(x)}}$
- Simplify

$$y(x) = C1\,\mathrm{e}^{\cos(x)} + 2\cos(x) + 2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(diff(y(x),x)+y(x)*sin(x) = sin(2*x),y(x),singsol=all)
```

$$y(x) = 2\cos(x) + 2 + \mathrm{e}^{\cos(x)} c_1$$

Mathematica DSolve solution

Solving time : 0.067 (sec)
Leaf size : 18

```
DSolve[{D[y[x],x]+y[x]*Sin[x]==Sin[2*x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 2\cos(x) + c_1 e^{\cos(x)} + 2$$

2.1.19 Problem 4(d)

Internal problem ID [4208]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 4(d)
Date solved : `Monday, January 27, 2025 at 08:42:16 AM`
CAS classification : `[_linear]`

Solve

$$\sin (x) y' + y = \sin (2x)$$

Solved as first order linear ode

Time used: 0.319 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \csc (x)$$
$$p(x) = 2 \cos (x)$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \csc(x)dx}$$

Therefore the solution is

$$y = \left(\int 2 \cos (x) \mathrm{e}^{\int \csc(x)dx} dx + c_1 \right) \mathrm{e}^{-\int \csc(x)dx}$$

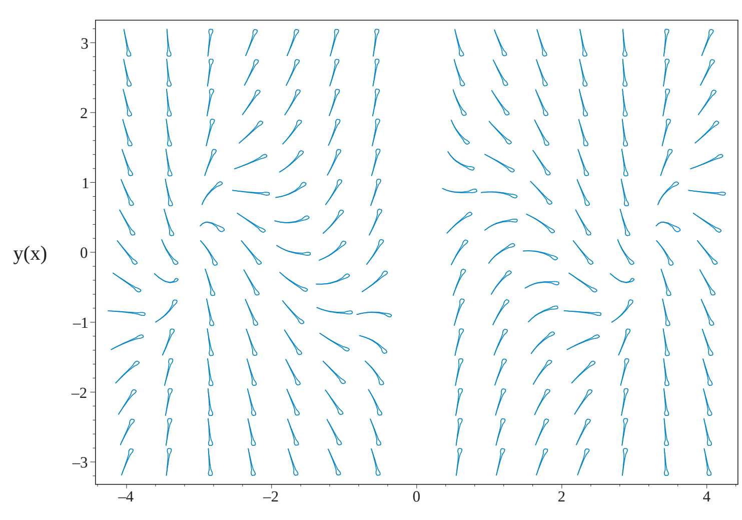


Figure 2.47: Slope field plot
$\sin (x) y' + y = \sin (2x)$

Summary of solutions found

$$y = \left(\int 2 \cos (x) \mathrm{e}^{\int \csc(x)dx} dx + c_1 \right) \mathrm{e}^{-\int \csc(x)dx}$$

Solved as first order Exact ode

Time used: 0.329 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\sin(x))\,\mathrm{d}y&=(-y+\sin(2x))\,\mathrm{d}x\\ (y-\sin(2x))\,\mathrm{d}x+(\sin(x))\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y-\sin(2x)\\ N(x,y)&=\sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(y-\sin(2x))\\ &=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(\sin(x))\\ &=\cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \csc(x)\left((1) - (\cos(x))\right) \\
&= \csc(x) - \cot(x)
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \csc(x) - \cot(x) \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\ln(\sin(x)) - \ln(\csc(x) + \cot(x))} \\
&= \frac{1}{\sin(x)(\csc(x) + \cot(x))}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sin(x)(\csc(x) + \cot(x))}(y - \sin(2x)) \\
&= \frac{y - 2\sin(x)\cos(x)}{\cos(x) + 1}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sin(x)(\csc(x) + \cot(x))}(\sin(x)) \\
&= \frac{1}{\csc(x) + \cot(x)}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y - 2\sin(x)\cos(x)}{\cos(x) + 1}\right) + \left(\frac{1}{\csc(x) + \cot(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \overline{N} \, \mathrm{d}y \\
\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \frac{1}{\csc(x) + \cot(x)} \, \mathrm{d}y
\end{aligned}$$

$$\phi = \frac{y}{\csc(x) + \cot(x)} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= -\frac{y\left(-\csc(x)\cot(x) - 1 - \cot(x)^2\right)}{\left(\csc(x)+\cot(x)\right)^2} + f'(x) \qquad (4)\\ &= \frac{y}{\cos(x)+1} + f'(x)\end{aligned}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \frac{y - 2\sin(x)\cos(x)}{\cos(x)+1}$. Therefore equation (4) becomes

$$\frac{y - 2\sin(x)\cos(x)}{\cos(x)+1} = \frac{y}{\cos(x)+1} + f'(x) \qquad (5)$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{2\sin(x)\cos(x)}{\cos(x)+1}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{2\sin(x)\cos(x)}{\cos(x)+1}\right)\mathrm{d}x$$

$$f(x) = 2\cos(x) - 2\ln(\cos(x)+1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{y}{\csc(x)+\cot(x)} + 2\cos(x) - 2\ln(\cos(x)+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{y}{\csc(x)+\cot(x)} + 2\cos(x) - 2\ln(\cos(x)+1)$$

Solving for y gives

$$\begin{aligned}y = {}& 2\ln(\cos(x)+1)\cot(x) + 2\ln(\cos(x)+1)\csc(x)\\ &- 2\cos(x)\cot(x) - 2\cos(x)\csc(x) + c_1\cot(x) + c_1\csc(x)\end{aligned}$$

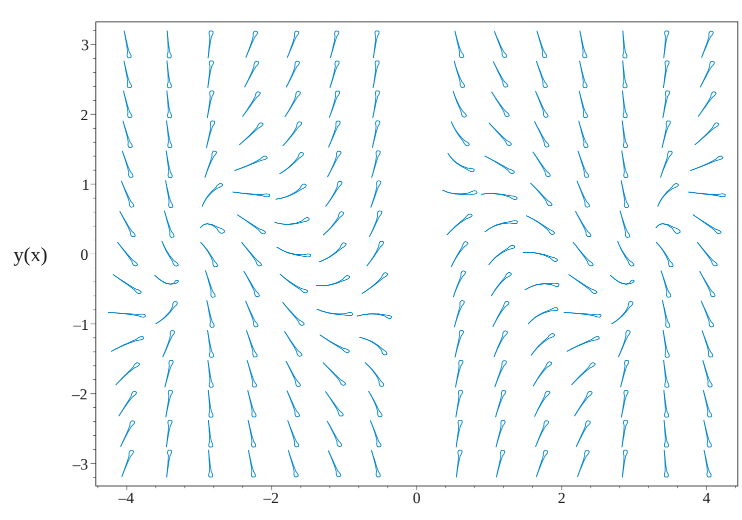


Figure 2.48: Slope field plot
$\sin(x)\,y' + y = \sin(2x)$

<u>Summary of solutions found</u>

$$\begin{aligned}y = {}& 2\ln(\cos(x)+1)\cot(x) + 2\ln(\cos(x)+1)\csc(x)\\ &- 2\cos(x)\cot(x) - 2\cos(x)\csc(x) + c_1\cot(x) + c_1\csc(x)\end{aligned}$$

Maple step by step solution

Let's solve

$\sin(x)\left(\frac{d}{dx}y(x)\right)+y(x)=\sin(2x)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x)=\frac{-y(x)+\sin(2x)}{\sin(x)}$

- Collect w.r.t. $y(x)$ and simplify

  $\frac{d}{dx}y(x)=-\frac{y(x)}{\sin(x)}+\frac{\sin(2x)}{\sin(x)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dx}y(x)+\frac{y(x)}{\sin(x)}=\frac{\sin(2x)}{\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{y(x)}{\sin(x)}\right)=\frac{\mu(x)\sin(2x)}{\sin(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$

  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{y(x)}{\sin(x)}\right)=\left(\frac{d}{dx}y(x)\right)\mu(x)+y(x)\left(\frac{d}{dx}\mu(x)\right)$

- Isolate $\frac{d}{dx}\mu(x)$

  $\frac{d}{dx}\mu(x)=\frac{\mu(x)}{\sin(x)}$

- Solve to find the integrating factor

  $\mu(x)=\csc(x)-\cot(x)$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx=\int\frac{\mu(x)\sin(2x)}{\sin(x)}dx+C1$

- Evaluate the integral on the lhs

  $y(x)\,\mu(x)=\int\frac{\mu(x)\sin(2x)}{\sin(x)}dx+C1$

- Solve for $y(x)$

  $y(x)=\frac{\int\frac{\mu(x)\sin(2x)}{\sin(x)}dx+C1}{\mu(x)}$

- Substitute $\mu(x)=\csc(x)-\cot(x)$

  $y(x)=\frac{\int\frac{(\csc(x)-\cot(x))\sin(2x)}{\sin(x)}dx+C1}{\csc(x)-\cot(x)}$

- Evaluate the integrals on the rhs

  $y(x)=\frac{-2\cos(x)+2\ln(\cos(x)+1)+C1}{\csc(x)-\cot(x)}$

- Simplify

  $y(x)=(-2\cos(x)+2\ln(\cos(x)+1)+C1)(\cos(x)+1)\csc(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 24

```
dsolve(sin(x)*diff(y(x),x)+y(x) = sin(2*x),y(x),singsol=all)
```

$$y(x) = \csc(x)(-2\cos(x) + 2\ln(\cos(x) + 1) + c_1)(\cos(x) + 1)$$

Mathematica DSolve solution

Solving time : 0.268 (sec)
Leaf size : 38

```
DSolve[{Sin[x]*D[y[x],x]+y[x]==Sin[2*x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{\operatorname{arctanh}(\cos(x))}\left(-2\sqrt{\sin^2(x)}\csc(x)\left(\cos(x) + \log\left(\sec^2\left(\frac{x}{2}\right)\right)\right) + c_1\right)$$

2.1.20 Problem 5(a)

Internal problem ID [4209]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 5(a)
Date solved : `Monday, January 27, 2025 at 08:42:20 AM`
CAS classification : `[_linear]`

Solve

$$\sqrt{x^2+1}\, y' + y = 2x$$

Solved as first order linear ode

Time used: 0.101 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{1}{\sqrt{x^2+1}}$$
$$p(x) = \frac{2x}{\sqrt{x^2+1}}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx}\\ &= e^{\int \frac{1}{\sqrt{x^2+1}}dx}\\ &= x + \sqrt{x^2+1}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{2x}{\sqrt{x^2+1}}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\left(x+\sqrt{x^2+1}\right)\right) = \left(x+\sqrt{x^2+1}\right)\left(\frac{2x}{\sqrt{x^2+1}}\right)$$
$$\mathrm{d}\left(y\left(x+\sqrt{x^2+1}\right)\right) = \left(\frac{2x\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}\right)\mathrm{d}x$$

Integrating gives

$$\begin{aligned}y\left(x+\sqrt{x^2+1}\right) &= \int \frac{2x\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}\, dx\\ &= \sqrt{x^2+1}\, x - \operatorname{arcsinh}(x) + x^2 + c_1\end{aligned}$$

Dividing throughout by the integrating factor $x+\sqrt{x^2+1}$ gives the final solution

$$y = \frac{\sqrt{x^2+1}\,x - \operatorname{arcsinh}(x) + x^2 + c_1}{x+\sqrt{x^2+1}}$$

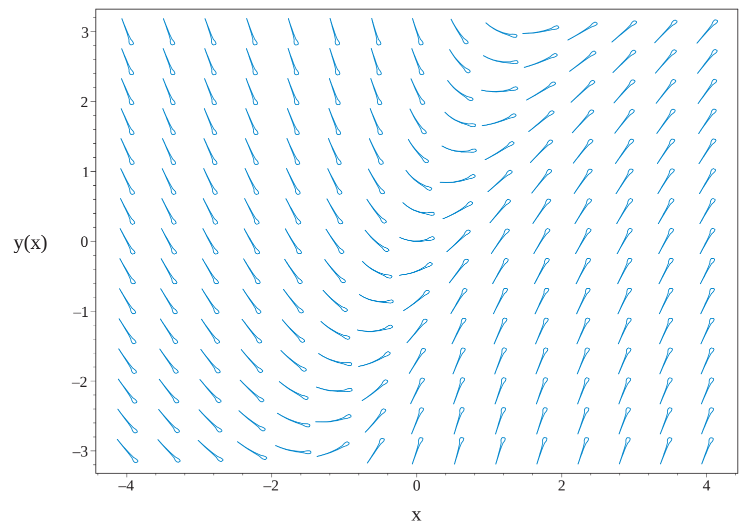


Figure 2.49: Slope field plot
$\sqrt{x^2+1}\,y' + y = 2x$

Summary of solutions found

$$y = \frac{\sqrt{x^2+1}\,x - \operatorname{arcsinh}(x) + x^2 + c_1}{x+\sqrt{x^2+1}}$$

Solved as first order Exact ode

Time used: 0.206 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or

might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\sqrt{x^2+1}\right)\mathrm{d}y &= (-y+2x)\,\mathrm{d}x\\ (y-2x)\,\mathrm{d}x + \left(\sqrt{x^2+1}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y-2x\\ N(x,y) &= \sqrt{x^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y-2x)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{x^2+1}\right)\\ &= \frac{x}{\sqrt{x^2+1}}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{\sqrt{x^2+1}}\left((1) - \left(\frac{x}{\sqrt{x^2+1}}\right)\right)\\ &= \frac{\sqrt{x^2+1}-x}{x^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{\sqrt{x^2+1}-x}{x^2+1}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\operatorname{arcsinh}(x) - \frac{\ln\left(x^2+1\right)}{2}}\\ &= 1 + \frac{x}{\sqrt{x^2+1}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= 1 + \frac{x}{\sqrt{x^2+1}}(y-2x)\\ &= (y-2x)\left(1 + \frac{x}{\sqrt{x^2+1}}\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= 1 + \frac{x}{\sqrt{x^2+1}}\left(\sqrt{x^2+1}\right)\\ &= x + \sqrt{x^2+1}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left((y-2x)\left(1+\frac{x}{\sqrt{x^2+1}}\right)\right) + \left(x+\sqrt{x^2+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y\\ \int \frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int x+\sqrt{x^2+1}\,\mathrm{d}y\\ \phi &= y\left(x+\sqrt{x^2+1}\right) + f(x)\end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = y\left(1+\frac{x}{\sqrt{x^2+1}}\right) + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial x} = (y-2x)\left(1+\frac{x}{\sqrt{x^2+1}}\right)$. Therefore equation (4) becomes

$$(y-2x)\left(1+\frac{x}{\sqrt{x^2+1}}\right) = y\left(1+\frac{x}{\sqrt{x^2+1}}\right) + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{2x\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$$

Integrating the above w.r.t x gives

$$\begin{aligned}\int f'(x)\,\mathrm{d}x &= \int \left(-\frac{2x\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}\right)\mathrm{d}x\\ f(x) &= -\sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2 + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\left(x+\sqrt{x^2+1}\right) - \sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\left(x + \sqrt{x^2+1}\right) - \sqrt{x^2+1}\, x + \operatorname{arcsinh}(x) - x^2$$

Solving for y gives

$$y = \frac{\sqrt{x^2+1}\, x - \operatorname{arcsinh}(x) + x^2 + c_1}{x + \sqrt{x^2+1}}$$

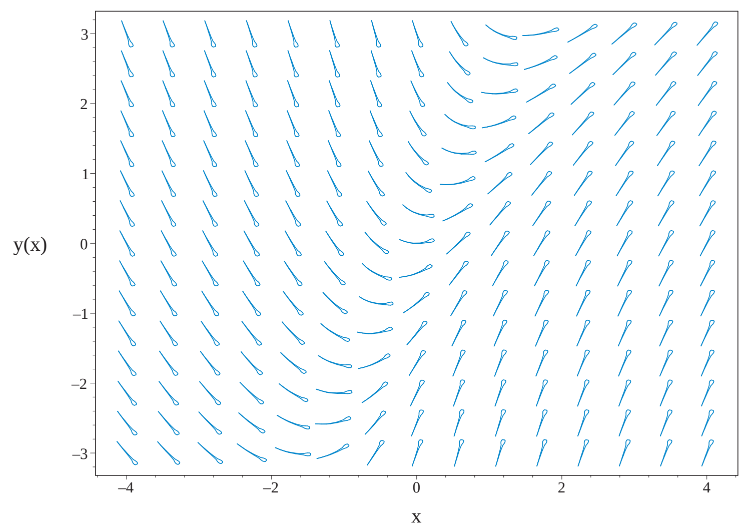


Figure 2.50: Slope field plot
$\sqrt{x^2+1}\, y' + y = 2x$

Summary of solutions found

$$y = \frac{\sqrt{x^2+1}\, x - \operatorname{arcsinh}(x) + x^2 + c_1}{x + \sqrt{x^2+1}}$$

Maple step by step solution

Let's solve
$\sqrt{x^2+1}\left(\frac{d}{dx}y(x)\right) + y(x) = 2x$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{2x - y(x)}{\sqrt{x^2+1}}$
- Collect w.r.t. $y(x)$ and simplify
  $\frac{d}{dx}y(x) = -\frac{y(x)}{\sqrt{x^2+1}} + \frac{2x}{\sqrt{x^2+1}}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + \frac{y(x)}{\sqrt{x^2+1}} = \frac{2x}{\sqrt{x^2+1}}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{\sqrt{x^2+1}}\right) = \frac{2\mu(x)x}{\sqrt{x^2+1}}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + \frac{y(x)}{\sqrt{x^2+1}}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \frac{\mu(x)}{\sqrt{x^2+1}}$

- Solve to find the integrating factor

  $\mu(x) = x + \sqrt{x^2 + 1}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(y(x)\, \mu(x))\right) dx = \int \frac{2\mu(x)x}{\sqrt{x^2+1}} dx + C1$
- Evaluate the integral on the lhs

  $y(x)\, \mu(x) = \int \frac{2\mu(x)x}{\sqrt{x^2+1}} dx + C1$
- Solve for $y(x)$

  $y(x) = \frac{\int \frac{2\mu(x)x}{\sqrt{x^2+1}} dx + C1}{\mu(x)}$
- Substitute $\mu(x) = x + \sqrt{x^2 + 1}$

  $y(x) = \frac{\int \frac{2\left(x+\sqrt{x^2+1}\right)x}{\sqrt{x^2+1}} dx + C1}{x+\sqrt{x^2+1}}$
- Evaluate the integrals on the rhs

  $y(x) = \frac{x^2+x\sqrt{x^2+1}-\operatorname{arcsinh}(x)+C1}{x+\sqrt{x^2+1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 34

```
dsolve((x^2+1)^(1/2)*diff(y(x),x)+y(x) = 2*x,y(x),singsol=all)
```

$$y(x) = \frac{x^2 + x\sqrt{x^2+1} - \operatorname{arcsinh}(x) + c_1}{x + \sqrt{x^2+1}}$$

Mathematica DSolve solution

Solving time : 0.084 (sec)
Leaf size : 33

```
DSolve[{Sqrt[1+x^2]*D[y[x],x]+y[x]==2*x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\operatorname{arcsinh}(x)}\left(-\operatorname{arcsinh}(x) + x^2 + \sqrt{x^2+1}x + c_1\right)$$

2.1.21 Problem 5(b)

Internal problem ID [4210]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 5(b)
Date solved : `Monday, January 27, 2025 at 08:42:22 AM`
CAS classification : `[_linear]`

Solve

$$\sqrt{x^2+1}\, y' - y = 2\sqrt{x^2+1}$$

Solved as first order linear ode

Time used: 0.094 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{1}{\sqrt{x^2+1}} \\ p(x) &= 2 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= \mathrm{e}^{\int -\frac{1}{\sqrt{x^2+1}} dx} \\ &= \frac{1}{x+\sqrt{x^2+1}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(2) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x+\sqrt{x^2+1}}\right) &= \left(\frac{1}{x+\sqrt{x^2+1}}\right)(2) \\ \mathrm{d}\left(\frac{y}{x+\sqrt{x^2+1}}\right) &= \left(\frac{2}{x+\sqrt{x^2+1}}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x+\sqrt{x^2+1}} &= \int \frac{2}{x+\sqrt{x^2+1}}\, dx \\ &= x\sqrt{x^2+1} + \operatorname{arcsinh}(x) - x^2 + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{x+\sqrt{x^2+1}}$ gives the final solution

$$y = \left(x+\sqrt{x^2+1}\right)\left(x\sqrt{x^2+1}+\operatorname{arcsinh}(x)-x^2+c_1\right)$$

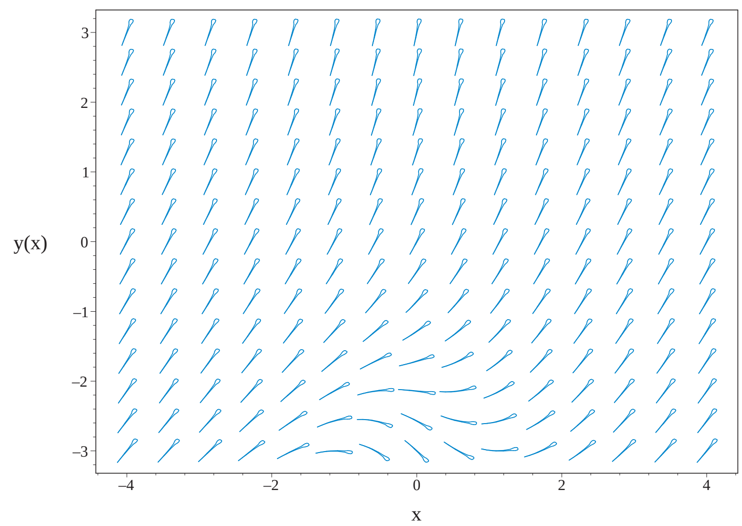


Figure 2.51: Slope field plot
$\sqrt{x^2+1}\,y'-y=2\sqrt{x^2+1}$

Summary of solutions found

$$y = \left(x+\sqrt{x^2+1}\right)\left(x\sqrt{x^2+1}+\operatorname{arcsinh}(x)-x^2+c_1\right)$$

Solved as first order Exact ode

Time used: 0.146 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or

might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\sqrt{x^2+1}\right)\mathrm{d}y &= \left(y+2\sqrt{x^2+1}\right)\mathrm{d}x\\ \left(-y-2\sqrt{x^2+1}\right)\mathrm{d}x + \left(\sqrt{x^2+1}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y-2\sqrt{x^2+1}\\ N(x,y) &= \sqrt{x^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y-2\sqrt{x^2+1}\right)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{x^2+1}\right)\\ &= \frac{x}{\sqrt{x^2+1}}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{\sqrt{x^2+1}}\left((-1) - \left(\frac{x}{\sqrt{x^2+1}}\right)\right)\\ &= \frac{-\sqrt{x^2+1}-x}{x^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{-\sqrt{x^2+1}-x}{x^2+1}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\operatorname{arcsinh}(x) - \frac{\ln\left(x^2+1\right)}{2}}\\ &= \frac{1}{\sqrt{x^2+1}\,\left(x+\sqrt{x^2+1}\right)}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\sqrt{x^2+1}\,\left(x+\sqrt{x^2+1}\right)}\left(-y-2\sqrt{x^2+1}\right)\\ &= -\frac{y+2\sqrt{x^2+1}}{\sqrt{x^2+1}\,\left(x+\sqrt{x^2+1}\right)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{\sqrt{x^2+1}\,\left(x+\sqrt{x^2+1}\right)}\left(\sqrt{x^2+1}\right) \\ &= \frac{1}{x+\sqrt{x^2+1}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\left(-\frac{y+2\sqrt{x^2+1}}{\sqrt{x^2+1}\,\left(x+\sqrt{x^2+1}\right)}\right)+\left(\frac{1}{x+\sqrt{x^2+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int\overline{N}\,\mathrm{d}y \\ \int\frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int\frac{1}{x+\sqrt{x^2+1}}\,\mathrm{d}y \\ \phi &= \left(\sqrt{x^2+1}-x\right)y+f(x)\end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x}=\left(\frac{x}{\sqrt{x^2+1}}-1\right)y+f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial x}=-\frac{y+2\sqrt{x^2+1}}{\sqrt{x^2+1}\,\left(x+\sqrt{x^2+1}\right)}$. Therefore equation (4) becomes

$$-\frac{y+2\sqrt{x^2+1}}{\sqrt{x^2+1}\,\left(x+\sqrt{x^2+1}\right)}=\left(\frac{x}{\sqrt{x^2+1}}-1\right)y+f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x)=-\frac{2}{x+\sqrt{x^2+1}}$$

Integrating the above w.r.t x gives

$$\begin{aligned}\int f'(x)\,\mathrm{d}x &= \int\left(-\frac{2}{x+\sqrt{x^2+1}}\right)\mathrm{d}x \\ f(x) &= x^2-x\sqrt{x^2+1}-\operatorname{arcsinh}(x)+c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi=\left(\sqrt{x^2+1}-x\right)y+x^2-x\sqrt{x^2+1}-\operatorname{arcsinh}(x)+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \left(\sqrt{x^2+1} - x\right) y + x^2 - x\sqrt{x^2+1} - \operatorname{arcsinh}(x)$$

Solving for y gives

$$y = \frac{x\sqrt{x^2+1} + \operatorname{arcsinh}(x) - x^2 + c_1}{\sqrt{x^2+1} - x}$$

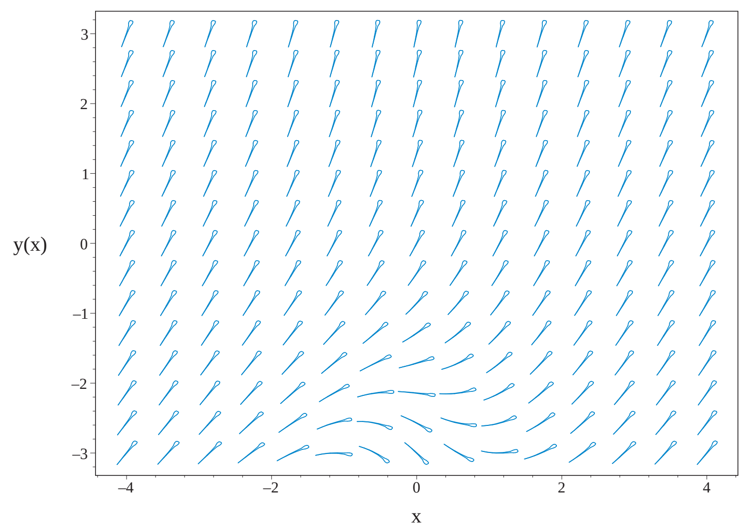


Figure 2.52: Slope field plot
$\sqrt{x^2+1}\, y' - y = 2\sqrt{x^2+1}$

Summary of solutions found

$$y = \frac{x\sqrt{x^2+1} + \operatorname{arcsinh}(x) - x^2 + c_1}{\sqrt{x^2+1} - x}$$

Maple step by step solution

Let's solve
$\sqrt{x^2+1}\left(\frac{d}{dx}y(x)\right) - y(x) = 2\sqrt{x^2+1}$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = \frac{y(x)+2\sqrt{x^2+1}}{\sqrt{x^2+1}}$
- Collect w.r.t. $y(x)$ and simplify
$\frac{d}{dx}y(x) = 2 + \frac{y(x)}{\sqrt{x^2+1}}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
$\frac{d}{dx}y(x) - \frac{y(x)}{\sqrt{x^2+1}} = 2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$\mu(x)\left(\frac{d}{dx}y(x) - \frac{y(x)}{\sqrt{x^2+1}}\right) = 2\mu(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
$\mu(x)\left(\frac{d}{dx}y(x) - \frac{y(x)}{\sqrt{x^2+1}}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
$\frac{d}{dx}\mu(x) = -\frac{\mu(x)}{\sqrt{x^2+1}}$

- Solve to find the integrating factor
  $\mu(x) = \frac{1}{x+\sqrt{x^2+1}}$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int 2\mu(x)\,dx + C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x) = \int 2\mu(x)\,dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int 2\mu(x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x+\sqrt{x^2+1}}$
  $y(x) = \left(x+\sqrt{x^2+1}\right)\left(\int \frac{2}{x+\sqrt{x^2+1}}dx + C1\right)$
- Evaluate the integrals on the rhs
  $y(x) = \left(x+\sqrt{x^2+1}\right)\left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x) - x^2 + C1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 32

```
dsolve((x^2+1)^(1/2)*diff(y(x),x)-y(x) = 2*(x^2+1)^(1/2),y(x),singsol=all)
```

$$y(x) = \left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x) - x^2 + c_1\right)\left(x+\sqrt{x^2+1}\right)$$

Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 31

```
DSolve[{Sqrt[1+x^2]*D[y[x],x]-y[x]==2*Sqrt[1+x^2],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{\operatorname{arcsinh}(x)}\left(\operatorname{arcsinh}(x) - x^2 + \sqrt{x^2+1}x + c_1\right)$$

2.1.22 Problem 5(c)

Internal problem ID [4211]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 5(c)
Date solved : `Monday, January 27, 2025 at 08:42:24 AM`
CAS classification : `[_linear]`

Solve

$$\sqrt{(x+a)(x+b)}\,(2y'-3)+y=0$$

Solved as first order linear ode

Time used: 0.310 (sec)

In canonical form a linear first order is

$$y'+q(x)y=p(x)$$

Comparing the above to the given ode shows that

$$q(x)=\frac{1}{2\sqrt{(x+a)(x+b)}}$$
$$p(x)=\frac{3}{2}$$

The integrating factor μ is

$$\mu=e^{\int \frac{1}{2\sqrt{(x+a)(x+b)}}dx}$$

Therefore the solution is

$$y=\left(\int \frac{3\,e^{\int \frac{1}{2\sqrt{(x+a)(x+b)}}dx}}{2}dx+c_1\right)e^{-\int \frac{1}{2\sqrt{(x+a)(x+b)}}dx}$$

Summary of solutions found

$$y=\left(\int \frac{3\,e^{\int \frac{1}{2\sqrt{(x+a)(x+b)}}dx}}{2}dx+c_1\right)e^{-\int \frac{1}{2\sqrt{(x+a)(x+b)}}dx}$$

Solved as first order Exact ode

Time used: 0.321 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2\sqrt{(x+a)\,(x+b)}\right)\mathrm{d}y&=\left(3\sqrt{(x+a)\,(x+b)}-y\right)\mathrm{d}x\\ \left(-3\sqrt{(x+a)\,(x+b)}+y\right)\mathrm{d}x+\left(2\sqrt{(x+a)\,(x+b)}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-3\sqrt{(x+a)\,(x+b)}+y\\ N(x,y)&=2\sqrt{(x+a)\,(x+b)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-3\sqrt{(x+a)\,(x+b)}+y\right)\\ &=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(2\sqrt{(x+a)\,(x+b)}\right)\\ &=\frac{a+b+2x}{\sqrt{(x+a)\,(x+b)}}\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{2\sqrt{(x+a)\,(x+b)}}\left((1)-\left(\frac{a+b+2x}{\sqrt{(x+a)\,(x+b)}}\right)\right)\\ &=\frac{\sqrt{(x+a)\,(x+b)}-a-b-2x}{2\,(x+a)\,(x+b)}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{\sqrt{(x+a)(x+b)}-a-b-2x}{2(x+a)(x+b)} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{\sqrt{(x+b)^2+(a-b)(x+b)}+\frac{(a-b)\ln\left(\frac{a}{2}+\frac{b}{2}+x+\sqrt{(x+b)^2+(a-b)(x+b)}\right)}{2}}{2a-2b} - \frac{\sqrt{(x+a)^2+(b-a)(x+a)}+\frac{(b-a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)^2+(b-a)(x+a)}\right)}{2}}{2(a-b)} - \frac{\ln((x+a)(x+b}{2}} \\
&= \frac{\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2\sqrt{(x+a)(x+b)}}\left(-3\sqrt{(x+a)(x+b)}+y\right) \\
&= \frac{\left(-3\sqrt{(x+a)(x+b)}+y\right)\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2\sqrt{(x+a)(x+b)}}\left(2\sqrt{(x+a)(x+b)}\right) \\
&= \sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\overline{M}
$$

$$
\left(\frac{\left(-3\sqrt{(x+a)(x+b)}+y\right)\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2\sqrt{(x+a)(x+b)}}\right) + \left(\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x}\right.
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (2) w.r.t. y gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \overline{N} \, \mathrm{d}y \\
\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}} \, \mathrm{d}y
\end{aligned}
$$

$$
\phi = \sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\, y + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = \frac{\sqrt{2}\,y\left(2+\frac{a+b+2x}{\sqrt{(x+a)(x+b)}}\right)}{2\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}} + f'(x) \tag{4}$$

$$= \frac{\sqrt{2}\,y\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2\sqrt{(x+a)(x+b)}} + f'(x)$$

But equation (1) says that $\frac{\partial\phi}{\partial x} = \frac{\left(-3\sqrt{(x+a)(x+b)}+y\right)\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2\sqrt{(x+a)(x+b)}}$. Therefore equation (4) becomes

$$\frac{\left(-3\sqrt{(x+a)(x+b)}+y\right)\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2\sqrt{(x+a)(x+b)}} \tag{5}$$
$$= \frac{\sqrt{2}\,y\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2\sqrt{(x+a)(x+b)}} + f'(x)$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{3\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{3\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2}\right) \mathrm{d}x$$

$$f(x) = \int_0^x -\frac{3\sqrt{2}\,\sqrt{a+b+2\tau+2\sqrt{(\tau+a)(\tau+b)}}}{2} d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,y + \int_0^x -\frac{3\sqrt{2}\,\sqrt{a+b+2\tau+2\sqrt{(\tau+a)(\tau+b)}}}{2} d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,y + \int_0^x -\frac{3\sqrt{2}\,\sqrt{a+b+2\tau+2\sqrt{(\tau+a)(\tau+b)}}}{2} d\tau$$

Solving for y gives

$$y = -\frac{\left(\int_0^x -\frac{3\sqrt{2}\,\sqrt{a+b+2\tau+2\sqrt{(\tau+a)(\tau+b)}}}{2} d\tau - c_1\right)\sqrt{2}}{2\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}$$

Summary of solutions found

$$y = -\frac{\left(\int_{0}^{x} -\frac{3\sqrt{2}\,\sqrt{a+b+2\tau+2\sqrt{(\tau+a)(\tau+b)}}}{2}d\tau - c_1\right)\sqrt{2}}{2\sqrt{a+b+2x+2\sqrt{(x+a)\,(x+b)}}}$$

Maple step by step solution

Let's solve

$\sqrt{(x+a)\,(x+b)}\,\left(2\frac{d}{dx}y(x)-3\right)+y(x)=0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x)=-\frac{y(x)-3\sqrt{(x+a)(x+b)}}{2\sqrt{(x+a)(x+b)}}$

- Collect w.r.t. $y(x)$ and simplify

  $\frac{d}{dx}y(x)=\frac{3}{2}-\frac{y(x)}{2\sqrt{(x+a)(x+b)}}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dx}y(x)+\frac{y(x)}{2\sqrt{(x+a)(x+b)}}=\frac{3}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{y(x)}{2\sqrt{(x+a)(x+b)}}\right)=\frac{3\mu(x)}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$

  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{y(x)}{2\sqrt{(x+a)(x+b)}}\right)=\left(\frac{d}{dx}y(x)\right)\mu(x)+y(x)\left(\frac{d}{dx}\mu(x)\right)$

- Isolate $\frac{d}{dx}\mu(x)$

  $\frac{d}{dx}\mu(x)=\frac{\mu(x)}{2\sqrt{(x+a)(x+b)}}$

- Solve to find the integrating factor

  $\mu(x)=\sqrt{2a+2b+4x+4\sqrt{(x+a)\,(x+b)}}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx=\int\frac{3\mu(x)}{2}dx+C1$

- Evaluate the integral on the lhs

  $y(x)\,\mu(x)=\int\frac{3\mu(x)}{2}dx+C1$

- Solve for $y(x)$

  $y(x)=\frac{\int\frac{3\mu(x)}{2}dx+C1}{\mu(x)}$

- Substitute $\mu(x)=\sqrt{2a+2b+4x+4\sqrt{(x+a)\,(x+b)}}$

  $y(x)=\frac{\int\frac{3\sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}}{2}dx+C1}{\sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}}$

- Simplify

  $y(x)=\frac{3\left(\int\sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}dx\right)+2C1}{2\sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 60

```
dsolve(((x+a)*(x+b))^(1/2)*(2*diff(y(x),x)-3)+y(x) = 0,y(x),singsol=all)
```

$$y(x) = \frac{3\left(\int \sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}dx\right)+4c_1}{2\sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}}$$

Mathematica DSolve solution

Solving time : 0.391 (sec)
Leaf size : 115

```
DSolve[{Sqrt[(x+a)*(x+b)]*(2*D[y[x],x]-3)+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(-\frac{\sqrt{a+x}\sqrt{b+x}\operatorname{arctanh}\left(\frac{\sqrt{b+x}}{\sqrt{a+x}}\right)}{\sqrt{(a+x)(b+x)}}\right)\left(\int_1^x \frac{3}{2}\exp\left(\frac{\operatorname{arctanh}\left(\frac{\sqrt{b+K[1]}}{\sqrt{a+K[1]}}\right)\sqrt{a+K[1]}\sqrt{b+K[1]}}{\sqrt{(a+K[1])(b+K[1])}}\right)dK[1]+c_1\right)$$

2.1.23 Problem 5(d)

Internal problem ID [4212]
Book : Elementary Differential equations, Chaundy, 1969
Section : Exercises 3, page 60
Problem number : 5(d)
Date solved : `Monday, January 27, 2025 at 08:42:27 AM`
CAS classification : `[_linear]`

Solve

$$\sqrt{(x+a)\,(x+b)}\,y'+y=\sqrt{x+a}-\sqrt{x+b}$$

Solved as first order linear ode

Time used: 0.317 (sec)

In canonical form a linear first order is

$$y'+q(x)y=p(x)$$

Comparing the above to the given ode shows that

$$q(x)=\frac{1}{\sqrt{(x+a)\,(x+b)}}$$
$$p(x)=\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)\,(x+b)}}$$

The integrating factor μ is

$$\mu=\mathrm{e}^{\int\frac{1}{\sqrt{(x+a)(x+b)}}dx}$$

Therefore the solution is

$$y=\left(\int\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)\mathrm{e}^{\int\frac{1}{\sqrt{(x+a)(x+b)}}dx}}{\sqrt{(x+a)\,(x+b)}}dx+c_1\right)\mathrm{e}^{-\int\frac{1}{\sqrt{(x+a)(x+b)}}dx}$$

Summary of solutions found

$$y=\left(\int\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)\mathrm{e}^{\int\frac{1}{\sqrt{(x+a)(x+b)}}dx}}{\sqrt{(x+a)\,(x+b)}}dx+c_1\right)\mathrm{e}^{-\int\frac{1}{\sqrt{(x+a)(x+b)}}dx}$$

Solved as first order Exact ode

Time used: 0.251 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(\sqrt{(x+a)\,(x+b)}\right)\mathrm{d}y=\left(-y+\sqrt{x+a}-\sqrt{x+b}\right)\mathrm{d}x$$

$$\left(y-\sqrt{x+a}+\sqrt{x+b}\right)\mathrm{d}x+\left(\sqrt{(x+a)\,(x+b)}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y-\sqrt{x+a}+\sqrt{x+b}\\ N(x,y)&=\sqrt{(x+a)\,(x+b)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(y-\sqrt{x+a}+\sqrt{x+b}\right)\\ &=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\sqrt{(x+a)\,(x+b)}\right)\\ &=\frac{a+b+2x}{2\sqrt{(x+a)\,(x+b)}}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{\sqrt{(x+a)(x+b)}}\left((1) - \left(\frac{a+b+2x}{2\sqrt{(x+a)(x+b)}}\right)\right) \\
&= \frac{2\sqrt{(x+a)(x+b)} - a - b - 2x}{2(x+a)(x+b)}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2\sqrt{(x+a)(x+b)}-a-b-2x}{2(x+a)(x+b)} \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{\sqrt{(x+b)^2+(a-b)(x+b)}+\frac{(a-b)\ln\left(\frac{a}{2}+\frac{b}{2}+x+\sqrt{(x+b)^2+(a-b)(x+b)}\right)}{2}}{a-b} - \frac{\sqrt{(x+a)^2+(b-a)(x+a)}+\frac{(b-a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)^2+(b-a)(x+a)}\right)}{2}}{a-b} - \frac{\ln((x+a)(x+b}{2}} \\
&= \frac{a+b+2x+2\sqrt{(x+a)(x+b)}}{2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{a+b+2x+2\sqrt{(x+a)(x+b)}}{2\sqrt{(x+a)(x+b)}}\left(y - \sqrt{x+a} + \sqrt{x+b}\right) \\
&= \frac{\left(y - \sqrt{x+a} + \sqrt{x+b}\right)\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)}{2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{a+b+2x+2\sqrt{(x+a)(x+b)}}{2\sqrt{(x+a)(x+b)}}\left(\sqrt{(x+a)(x+b)}\right) \\
&= \frac{a}{2} + \frac{b}{2} + x + \sqrt{(x+a)(x+b)}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\left(y - \sqrt{x+a} + \sqrt{x+b}\right)\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)}{2\sqrt{(x+a)(x+b)}}\right) + \left(\frac{a}{2} + \frac{b}{2} + x + \sqrt{(x+a)(x+b)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \overline{N}\,\mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \frac{a}{2} + \frac{b}{2} + x + \sqrt{(x+a)(x+b)}\,\mathrm{d}y$$

$$\phi = \frac{\left(a + b + 2x + 2\sqrt{(x+a)(x+b)}\right) y}{2} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \frac{\left(2 + \frac{a+b+2x}{\sqrt{(x+a)(x+b)}}\right) y}{2} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \frac{(y-\sqrt{x+a}+\sqrt{x+b})\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)}{2\sqrt{(x+a)(x+b)}}$. Therefore equation (4) becomes

$$\frac{\left(y - \sqrt{x+a} + \sqrt{x+b}\right)\left(a + b + 2x + 2\sqrt{(x+a)(x+b)}\right)}{2\sqrt{(x+a)(x+b)}} = \frac{\left(2 + \frac{a+b+2x}{\sqrt{(x+a)(x+b)}}\right) y}{2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{\left(a + b + 2x + 2\sqrt{(x+a)(x+b)}\right)\left(\sqrt{x+a} - \sqrt{x+b}\right)}{2\sqrt{(x+a)(x+b)}}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{\left(a + b + 2x + 2\sqrt{(x+a)(x+b)}\right)\left(\sqrt{x+a} - \sqrt{x+b}\right)}{2\sqrt{(x+a)(x+b)}}\right) \mathrm{d}x$$

$$\begin{aligned} f(x) = &-\frac{\sqrt{(x+a)(x+b)}\,(2x - b + 3a)}{3\sqrt{x+a}} \\ &+ \frac{\sqrt{(x+a)(x+b)}\,(2x - a + 3b)}{3\sqrt{x+b}} - \frac{2(x+a)^{3/2}}{3} + \frac{2(x+b)^{3/2}}{3} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\begin{aligned} \phi = &\frac{\left(a + b + 2x + 2\sqrt{(x+a)(x+b)}\right) y}{2} - \frac{\sqrt{(x+a)(x+b)}\,(2x - b + 3a)}{3\sqrt{x+a}} \\ &+ \frac{\sqrt{(x+a)(x+b)}\,(2x - a + 3b)}{3\sqrt{x+b}} - \frac{2(x+a)^{3/2}}{3} + \frac{2(x+b)^{3/2}}{3} + c_1 \end{aligned}$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$\begin{aligned} c_1 = &\frac{\left(a + b + 2x + 2\sqrt{(x+a)(x+b)}\right) y}{2} - \frac{\sqrt{(x+a)(x+b)}\,(2x - b + 3a)}{3\sqrt{x+a}} \\ &+ \frac{\sqrt{(x+a)(x+b)}\,(2x - a + 3b)}{3\sqrt{x+b}} - \frac{2(x+a)^{3/2}}{3} + \frac{2(x+b)^{3/2}}{3} \end{aligned}$$

Summary of solutions found

$$\frac{\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)y}{2}-\frac{\sqrt{(x+a)(x+b)}\,(2x-b+3a)}{3\sqrt{x+a}}+\frac{\sqrt{(x+a)(x+b)}\,(2x-a+3b)}{3\sqrt{x+b}}-\frac{2(x+a)^{3/2}}{3}+\frac{2(x+b)^{3/2}}{3}=c_1$$

Maple step by step solution

Let's solve

$\sqrt{(x+a)(x+b)}\left(\frac{d}{dx}y(x)\right)+y(x)=\sqrt{x+a}-\sqrt{x+b}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x)=\frac{-y(x)+\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)(x+b)}}$

- Collect w.r.t. $y(x)$ and simplify

  $\frac{d}{dx}y(x)=-\frac{y(x)}{\sqrt{(x+a)(x+b)}}+\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)(x+b)}}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dx}y(x)+\frac{y(x)}{\sqrt{(x+a)(x+b)}}=\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)(x+b)}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{y(x)}{\sqrt{(x+a)(x+b)}}\right)=\frac{\mu(x)\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\sqrt{(x+a)(x+b)}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$

  $\mu(x)\left(\frac{d}{dx}y(x)+\frac{y(x)}{\sqrt{(x+a)(x+b)}}\right)=\left(\frac{d}{dx}y(x)\right)\mu(x)+y(x)\left(\frac{d}{dx}\mu(x)\right)$

- Isolate $\frac{d}{dx}\mu(x)$

  $\frac{d}{dx}\mu(x)=\frac{\mu(x)}{\sqrt{(x+a)(x+b)}}$

- Solve to find the integrating factor

  $\mu(x)=a+b+2x+2\sqrt{(x+a)(x+b)}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx=\int\frac{\mu(x)\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\sqrt{(x+a)(x+b)}}dx+C1$

- Evaluate the integral on the lhs

  $y(x)\,\mu(x)=\int\frac{\mu(x)\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\sqrt{(x+a)(x+b)}}dx+C1$

- Solve for $y(x)$

  $y(x)=\frac{\int\frac{\mu(x)\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\sqrt{(x+a)(x+b)}}dx+C1}{\mu(x)}$

- Substitute $\mu(x)=a+b+2x+2\sqrt{(x+a)(x+b)}$

  $y(x)=\frac{\int\frac{\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\sqrt{(x+a)(x+b)}}dx+C1}{a+b+2x+2\sqrt{(x+a)(x+b)}}$

- Evaluate the integrals on the rhs

  $y(x)=\frac{\frac{4(x+a)^{3/2}}{3}-\frac{4(x+b)^{3/2}}{3}+\frac{2\sqrt{x+a}\,(x+b)(2x-b+3a)}{3\sqrt{(x+a)(x+b)}}-\frac{2\sqrt{x+b}\,(x+a)(2x-a+3b)}{3\sqrt{(x+a)(x+b)}}+C1}{a+b+2x+2\sqrt{(x+a)(x+b)}}$

- Simplify

  $y(x)=\frac{2\left(\left((2a+2x)\sqrt{x+a}+(-2b-2x)\sqrt{x+b}+\frac{3C1}{2}\right)\sqrt{(x+a)(x+b)}+3(x+b)\left(-\frac{b}{3}+a+\frac{2x}{3}\right)\sqrt{x+a}+\sqrt{x+b}\,(x+a)(-2x+a-3b)\right)}{\sqrt{(x+a)(x+b)}\left(3a+3b+6x+6\sqrt{(x+a)(x+b)}\right)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 114

```
dsolve(((x+a)*(x+b))^(1/2)*diff(y(x),x)+y(x) = (x+a)^(1/2)-(x+b)^(1/2),y(x),singsol=al
```

$$y(x) = \frac{2\left(\left(2a+2x\right)\sqrt{x+a}+\left(-2b-2x\right)\sqrt{x+b}+3c_1\right)\sqrt{\left(x+a\right)\left(x+b\right)}+6(x+b)\left(-\frac{b}{3}+a+\frac{2x}{3}\right)\sqrt{x+a}+}{\sqrt{\left(x+a\right)\left(x+b\right)}\left(3a+3b+6x+6\sqrt{\left(x+a\right)\left(x+b\right)}\right)}$$

Mathematica DSolve solution

Solving time : 1.687 (sec)
Leaf size : 145

```
DSolve[{Sqrt[(x+a)*(x+b)]*D[y[x],x]+y[x]==Sqrt[x+a]-Sqrt[x+b],{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \exp\left(-\frac{2\sqrt{a+x}\sqrt{b+x}\operatorname{arctanh}\left(\frac{\sqrt{b+x}}{\sqrt{a+x}}\right)}{\sqrt{(a+x)(b+x)}}\right)\left(\int_1^x \frac{\exp\left(\frac{2\operatorname{arctanh}\left(\frac{\sqrt{b+K[1]}}{\sqrt{a+K[1]}}\right)\sqrt{a+K[1]}\sqrt{b+K[1]}}{\sqrt{(a+K[1])(b+K[1])}}\right)\left(\sqrt{a+K[1]}\right.}{\sqrt{(a+K[1])(b+K[1])}} + c_1\right)$$