A Solution Manual For

Second order enumerated odes

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Contents

CHAPTER 1

Lookup tables for all problems in current book

1.1 section 1

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE
9072	1	$y'' = 0$
9073	2	$y''^2 = 0$
9074	3	$y''^n = 0$
9075	4	$ay'' = 0$
9076	5	$ay''^2 = 0$
9077	6	$ay''^n = 0$
9078	7	$y'' = 1$
9079	8	$y''^2 = 1$
9080	9	$y'' = x$
9081	10	$y''^2 = x$
9082	11	$y''^3 = 0$
9083	12	$y'' + y' = 0$
9084	13	$y''^2 + y' = 0$
9085	14	$y'' + y'^2 = 0$
9086	15	$y'' + y' = 1$
9087	16	$y''^2 + y' = 1$
9088	17	$y'' + y'^2 = 1$
9089	18	$y'' + y' = x$
9090	19	$y''^2 + y' = x$
9091	20	$y'' + y'^2 = x$
9092	21	$y'' + y' + y = 0$
9093	22	$y''^2 + y' + y = 0$
9094	23	$y'' + y'^2 + y = 0$
9095	24	$y'' + y' + y = 1$
9096	25	$y'' + y' + y = x$
9097	26	$y'' + y' + y = x + 1$
9098	27	$y'' + y' + y = x^2 + x + 1$
9099	28	$y'' + y' + y = x^3 + x^2 + x + 1$
9100	29	$y'' + y' + y = \sin(x)$
9101	30	$y'' + y' + y = \cos(x)$
9102	31	$y'' + y' = 1$
9103	32	$y'' + y' = x$
9104	33	$y'' + y' = x + 1$
9105	34	$y'' + y' = x^2 + x + 1$

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Table 1.1 Lookup table
Continued from previous page

ID	problem	ODE
9106	35	$y'' + y' = x^3 + x^2 + x + 1$
9107	36	$y'' + y' = \sin(x)$
9108	37	$y'' + y' = \cos(x)$
9109	38	$y'' + y = 1$
9110	39	$y'' + y = x$
9111	40	$y'' + y = x + 1$
9112	41	$y'' + y = x^2 + x + 1$
9113	42	$y'' + y = x^3 + x^2 + x + 1$
9114	43	$y'' + y = \sin(x)$
9115	44	$y'' + y = \cos(x)$
9116	45	$yy''^2 + y' = 0$
9117	46	$yy''^2 + y'^3 = 0$
9118	47	$y^2y''^2 + y' = 0$
9119	48	$yy''^4 + y'^2 = 0$
9120	49	$y^3y''^2 + y'y = 0$
9121	50	$yy'' + y'^3 = 0$
9122	51	$yy''^3 + y^3y' = 0$
9123	52	$yy''^3 + y^3y'^5 = 0$

1.2 section 2

Table 1.2: Lookup table for all problems in current section

ID	problem	ODE
9124	1	$y'' + xy' + yy'^2 = 0$
9125	2	$y'' + \sin(x)\, y' + yy'^2 = 0$
9126	3	$y'' + (1 - x)\, y' + y^2y'^2 = 0$
9127	4	$y'' + (\sin(x) + 2x)\, y' + \cos(y)\, yy'^2 = 0$
9128	5	$y''y' + y^2 = 0$
9129	6	$y''y' + y^n = 0$
9130	8	$y' = (x + y)^4$
9131	9	$y'' + (x + 3)\, y' + (y^2 + 3)\, y'^2 = 0$
9132	10	$y'' + xy' + yy'^2 = 0$
9133	11	$y'' + \sin(x)\, y' + y'^2 = 0$
9134	12	$3y'' + \cos(x)\, y' + \sin(y)\, y'^2 = 0$
9135	13	$10y'' + x^2y' + \frac{3y'^2}{y} = 0$

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Table 1.2 Lookup table
Continued from previous page

ID	problem	ODE
9136	14	$10y'' + (e^x + 3x)\, y' + \frac{3\, e^y y'^2}{\sin(y)} = 0$
9137	15	$y'' - \frac{2y}{x^2} = x\, e^{-\sqrt{x}}$
9138	16	$y'' - \frac{y'}{\sqrt{x}} + \frac{(x+\sqrt{x}-8)\, y}{4x^2} = x$
9139	17	$y'' + \frac{2y'}{x} + \frac{a^2 y}{x^4} = 0$
9140	18	$(-x^2 + 1)\, y'' - xy' - c^2 y = 0$
9141	19	$x^6 y'' + 3x^5 y' + a^2 y = \frac{1}{x^2}$
9142	20	$x^2 y'' - 3xy' + 3y = 2x^3 - x^2$
9143	21	$y'' + \cot(x)\, y' + 4y \csc(x)^2 = 0$
9144	22	$(x^2 + 1)\, y'' + (x + 1)\, y' + y = 4 \cos(\ln(x + 1))$
9145	23	$y'' + \tan(x)\, y' + \cos(x)^2\, y = 0$
9146	24	$xy'' - y' + 4x^3 y = 8x^3 \sin(x)^2$
9147	25	$xy'' - y' + 4x^3 y = x^5$
9148	25	$\cos(x)\, y'' + \sin(x)\, y' - 2y \cos(x)^3 = 2 \cos(x)^5$
9149	26	$y'' + \left(1 - \frac{1}{x}\right) y' + 4x^2 y\, e^{-2x} = 4(x^3 + x^2)\, e^{-3x}$
9150	27	$y'' - x^2 y' + xy = x^{m+1}$
9151	28	$y'' - \frac{y'}{\sqrt{x}} + \frac{(x+\sqrt{x}-8)\, y}{4x^2} = 0$
9152	29	$\cos(x)^2\, y'' - 2 \cos(x) \sin(x)\, y' + \cos(x)^2\, y = 0$
9153	30	$y'' - 4xy' + (4x^2 - 1)\, y = -3\, e^{x^2} \sin(x)$
9154	31	$y'' - 2bxy' + b^2 x^2 y = x$
9155	32	$y'' - 4xy' + (4x^2 - 3)\, y = e^{x^2}$
9156	33	$y'' - 2 \tan(x)\, y' + 5y = e^{x^2} \sec(x)$
9157	34	$x^2 y'' - 2xy' + 2(x^2 + 1)\, y = 0$
9158	35	$4x^2 y'' + 4x^5 y' + (x^8 + 6x^4 + 4)\, y = 0$
9159	36	$x^2 y'' + (-y + xy')^2 = 0$
9160	37	$xy'' + 2y' - xy = 0$
9161	38	$xy'' + 2y' + xy = 0$
9162	39	$y' + y \cot(x) = 2 \cos(x)$
9163	40	$2xy^2 - y + (y^2 + x + y)\, y' = 0$
9164	41	$y' = x - y^2$
9165	42	$y'''' - y''' - 3y'' + 5y' - 2y = x\, e^x + 3\, e^{-2x}$
9166	43	$x^2 y'' - x(6 + x)\, y' + 10y = 0$
9167	44	$x^2 y'' + xy' + (x^2 - 5)\, y = 0$
9168	45	$x^2 y'' + xy' + (x^2 - 5)\, y = 0$
9169	46	$x^2 y'' - 4xy' + 6y = 0$

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Table 1.2 Lookup table
Continued from previous page

ID	problem	ODE
9170	47	$y''' - xy = 0$
9171	48	$y' = y^{1/3}$
9172	49	$[x'(t) = 3x(t) + y(t), y'(t) = -x(t) + y(t)]$

CHAPTER 2

Book Solved Problems

2.1 section 1

2.1.1 Problem 1

Internal problem ID [9072]
Book : Second order enumerated odes
Section : section 1
Problem number : 1
Date solved : `Monday, January 27, 2025 at 05:31:55 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = 0$$

Solved as second order ode quadrature

Time used: 0.013 (sec)

Integrating twice gives the solution

$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

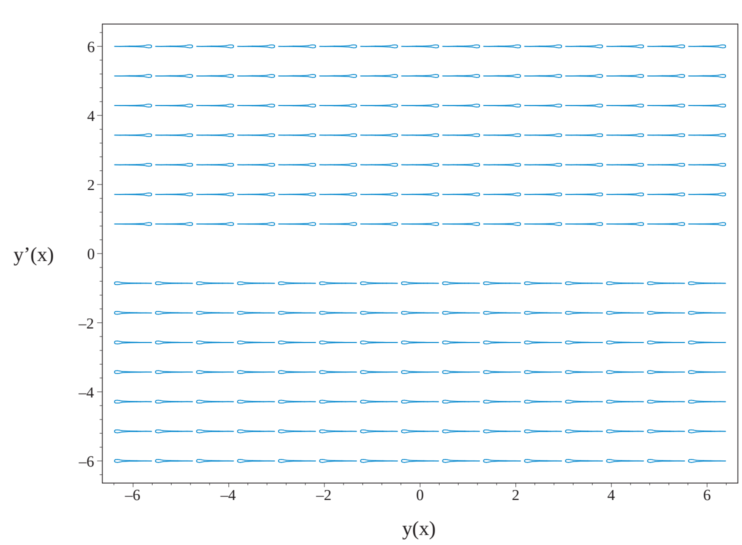


Figure 2.1: Slope field plot
$y'' = 0$

Solved as second order linear constant coeff ode

Time used: 0.042 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(0)^2 - (4)\,(1)\,(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2 x + c_1$$

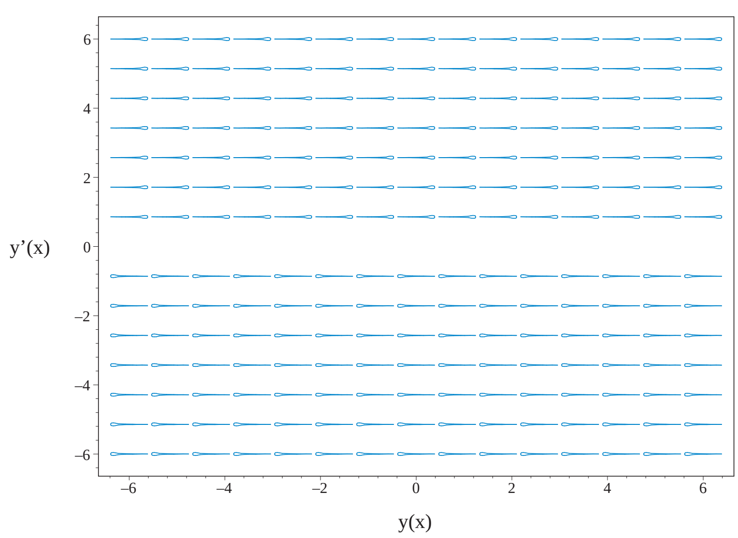


Figure 2.2: Slope field plot
$y'' = 0$

Solved as second order linear exact ode

Time used: 0.055 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y' = c_1$$

We now have a first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dy &= \int c_1\,dx \\ y &= c_1 x + c_2 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

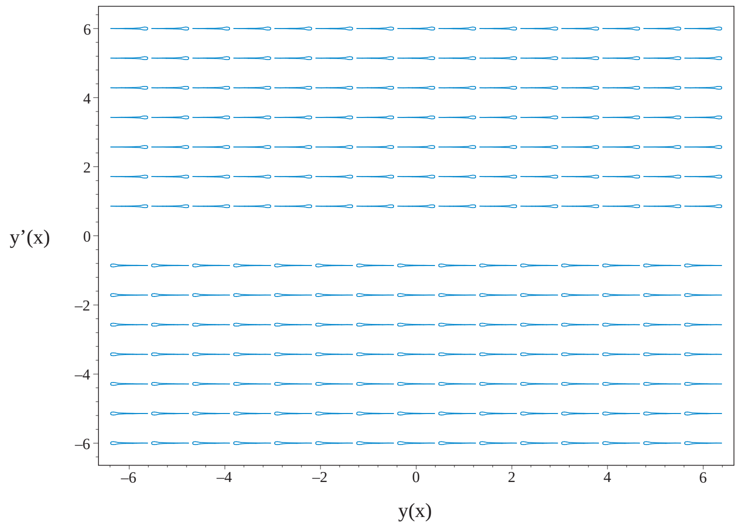


Figure 2.3: Slope field plot
$y'' = 0$

Solved as second order missing y ode

Time used: 0.046 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) = 0$$

Which is now solved for $u(x)$ as first order ode.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\,dx + c_1$$
$$u(x) = c_1$$

In summary, these are the solution found for $u(x)$

$$u(x) = c_1$$

For solution $u(x) = c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\,dx$$
$$y = c_1 x + c_2$$

In summary, these are the solution found for (y)

$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

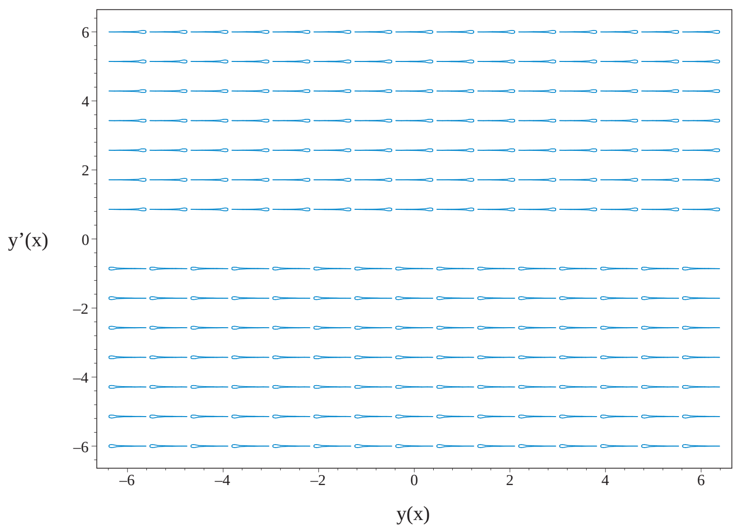


Figure 2.4: Slope field plot
$y'' = 0$

Solved as second order integrable as is ode

Time used: 0.025 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = 0$$
$$y' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1 \, dx$$
$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

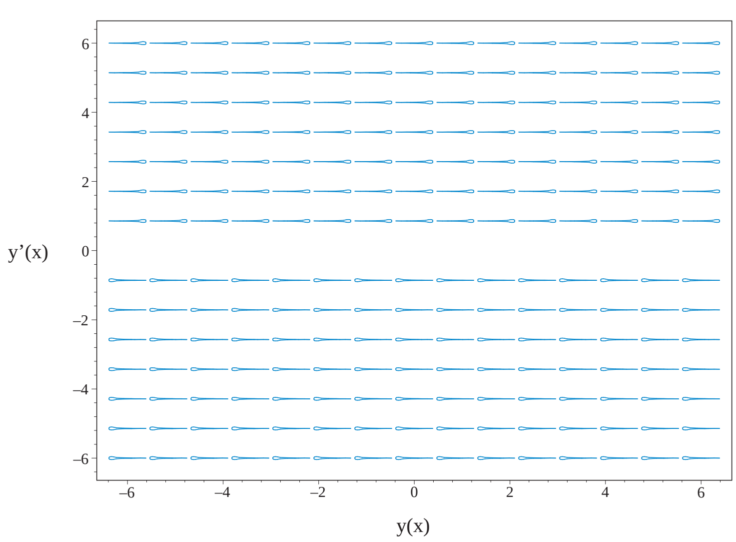


Figure 2.5: Slope field plot
$y'' = 0$

Solved as second order integrable as is ode (ABC method)

Time used: 0.027 (sec)

Writing the ode as

$$y'' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = 0$$
$$y' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1 \, dx$$
$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

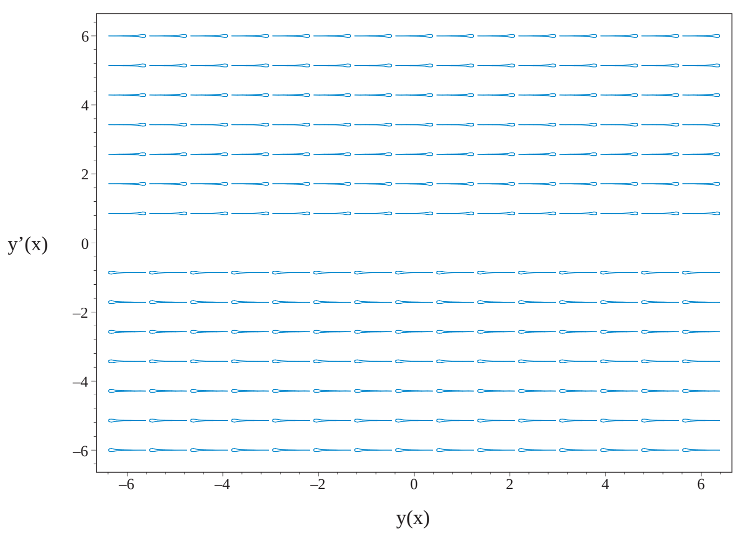


Figure 2.6: Slope field plot
$y'' = 0$

Solved as second order can be made integrable

Time used: 0.299 (sec)

Multiplying the ode by y' gives

$$y'y'' = 0$$

Integrating the above w.r.t x gives

$$\int y'y'' dx = 0$$
$$\frac{y'^2}{2} = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{2}\,\sqrt{c_1} \tag{1}$$
$$y' = -\sqrt{2}\,\sqrt{c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \sqrt{2}\,\sqrt{c_1}\,dx$$
$$y = \sqrt{2}\,\sqrt{c_1}\,x + c_2$$

Solving Eq. (2)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -\sqrt{2}\,\sqrt{c_1}\,dx$$
$$y = -\sqrt{2}\,\sqrt{c_1}\,x + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\sqrt{2}\,\sqrt{c_1}\,x + c_3$$
$$y = \sqrt{2}\,\sqrt{c_1}\,x + c_2$$

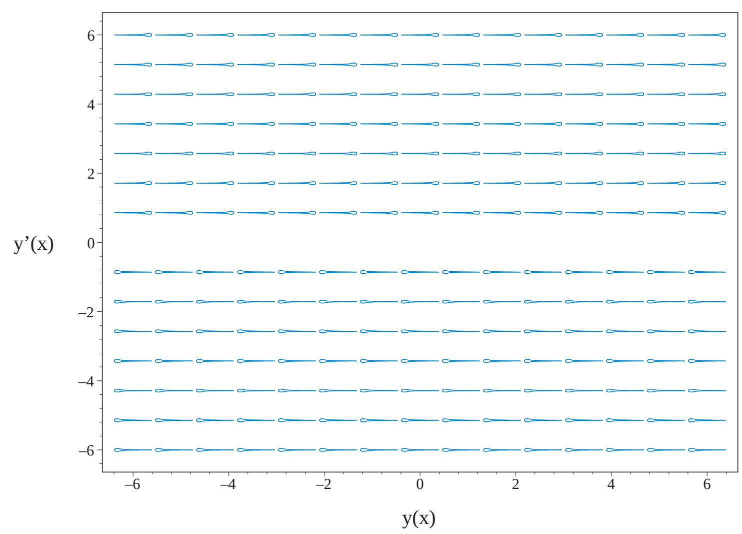


Figure 2.7: Slope field plot
$y'' = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.030 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2 x + c_1$$

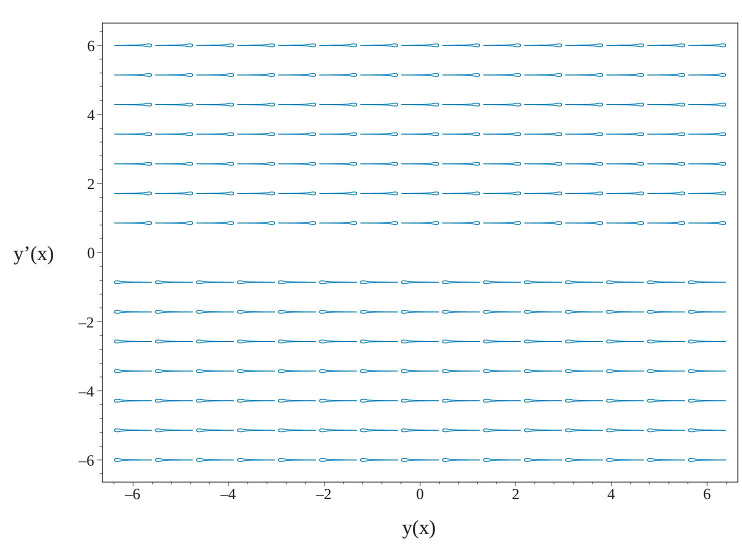


Figure 2.8: Slope field plot
$y'' = 0$

Solved as second order ode adjoint method

Time used: 0.332 (sec)

In normal form the ode

$$y'' = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (0) &= 0 \\ \xi''(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Integrating twice gives the solution

$$\xi = c_1 x + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)} \end{aligned}$$

Or

$$y' - \frac{y c_1}{c_1 x + c_2} = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x) y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{c_1}{c_1 x + c_2} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= e^{\int -\frac{c_1}{c_1 x + c_2} dx} \\ &= \frac{1}{c_1 x + c_2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 x + c_2}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{c_1 x + c_2} = \int 0 \, dx + c_3$$
$$= c_3$$

Dividing throughout by the integrating factor $\frac{1}{c_1 x + c_2}$ gives the final solution

$$y = (c_1 x + c_2) c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = (c_1 x + c_2) c_3$$

The constants can be merged to give

$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

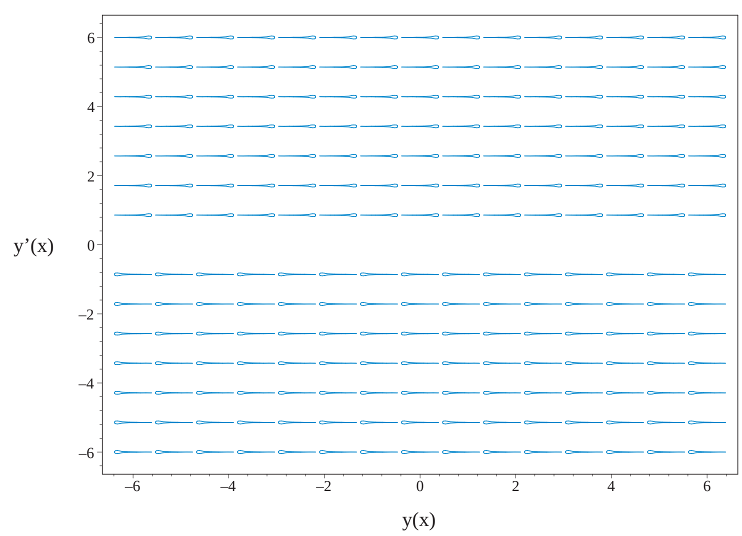


Figure 2.9: Slope field plot
$y'' = 0$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of ODE
  $r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the ODE
  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y(x) = C1y_1(x) + C2y_2(x)$
- Substitute in solutions
  $y(x) = C2x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : `0.000` (sec)
Leaf size : 9

```
dsolve(diff(diff(y(x),x),x) = 0,y(x),singsol=all)
```

$$y = c_1x + c_2$$

Mathematica DSolve solution

Solving time : `0.002` (sec)
Leaf size : 12

```
DSolve[{D[y[x],{x,2}]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2x + c_1$$

2.1.2 Problem 2

Internal problem ID [9073]
Book : Second order enumerated odes
Section : section 1
Problem number : 2
Date solved : `Monday, January 27, 2025 at 05:31:58 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y''^2 = 0$$

Solved as second order missing x ode

Time used: 0.206 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p^2 = 0 \tag{1}$$

$$p'^2 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p^2 = 0$$

Solving gives $p = 0$

Solving equation (2)

Solving for the derivative gives these ODE's to solve

$$p' = 0 \tag{1}$$

$$p' = 0 \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\, dy + c_1$$
$$p = c_1$$

Solving Eq. (2)

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\, dy + c_2$$
$$p = c_2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_3$$
$$y = c_3$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\, dx$$
$$y = c_1 x + c_4$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_2$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_2\, dx$$
$$y = c_2 x + c_5$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_3$$
$$y = c_1 x + c_4$$
$$y = c_2 x + c_5$$

Solved as second order missing y ode

Time used: 0.140 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x)^2 = 0$$

Which is now solved for $u(x)$ as first order ode.

Solving for the derivative gives these ODE's to solve

$$u'(x) = 0 \tag{1}$$
$$u'(x) = 0 \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\, dx + c_1$$
$$u(x) = c_1$$

Solving Eq. (2)

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\, dx + c_2$$
$$u(x) = c_2$$

In summary, these are the solution found for $u(x)$

$$u(x) = c_1$$
$$u(x) = c_2$$

For solution $u(x) = c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\, dx$$
$$y = c_1 x + c_3$$

For solution $u(x) = c_2$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_2$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_2 \, dx$$
$$y = c_2 x + c_4$$

In summary, these are the solution found for (y)

$$y = c_1 x + c_3$$
$$y = c_2 x + c_4$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_3$$
$$y = c_2 x + c_4$$

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2} y(x)\right)^2 = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2} y(x) = 0$
- Characteristic polynomial of ODE
  $r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the ODE
  $y_1(x) = 1$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x)$
- Substitute in solutions
  $y(x) = C2 x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : `0.022` (sec)
Leaf size : 9

```
dsolve(diff(diff(y(x),x),x)^2 = 0,y(x),singsol=all)
```

$$y = c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
DSolve[{(D[y[x],{x,2}])^2==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x + c_1$$

2.1.3 Problem 3

Internal problem ID [9074]
Book : Second order enumerated odes
Section : section 1
Problem number : 3
Date solved : `Monday, January 27, 2025 at 05:31:58 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y''^n = 0$$

Solved as second order missing x ode

Time used: 0.104 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$\left(p(y)\left(\frac{d}{dy}p(y)\right)\right)^n = 0$$

Which is now solved as first order ode for $p(y)$.

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\, dy + c_1$$
$$p = c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\, dx$$
$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

Solved as second order missing y ode

Time used: 0.050 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x)^n = 0$$

Which is now solved for $u(x)$ as first order ode.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\, dx + c_1$$
$$u(x) = c_1$$

In summary, these are the solution found for $u(x)$

$$u(x) = c_1$$

For solution $u(x) = c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\, dx$$
$$y = c_1 x + c_2$$

In summary, these are the solution found for (y)

$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2} y(x)\right)^n = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2} y(x) = 0$
- Characteristic polynomial of ODE
  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial

  $r = 0$
- 1st solution of the ODE

  $y_1(x) = 1$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$
- General solution of the ODE

  $y(x) = C1y_1(x) + C2y_2(x)$
- Substitute in solutions

  $y(x) = C2x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 9

```
dsolve(diff(diff(y(x),x),x)^n = 0,y(x),singsol=all)
```

$$y = c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 24

```
DSolve[{(D[y[x],{x,2}])^n==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}0^{\frac{1}{n}}x^2 + c_2x + c_1$$

2.1.4 Problem 4

Internal problem ID [9075]
Book : Second order enumerated odes
Section : section 1
Problem number : 4
Date solved : `Monday, January 27, 2025 at 05:31:59 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$ay'' = 0$$

Solved as second order ode quadrature

Time used: 0.013 (sec)

The ODE simplifies to

$$y'' = 0$$

Integrating twice gives the solution

$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

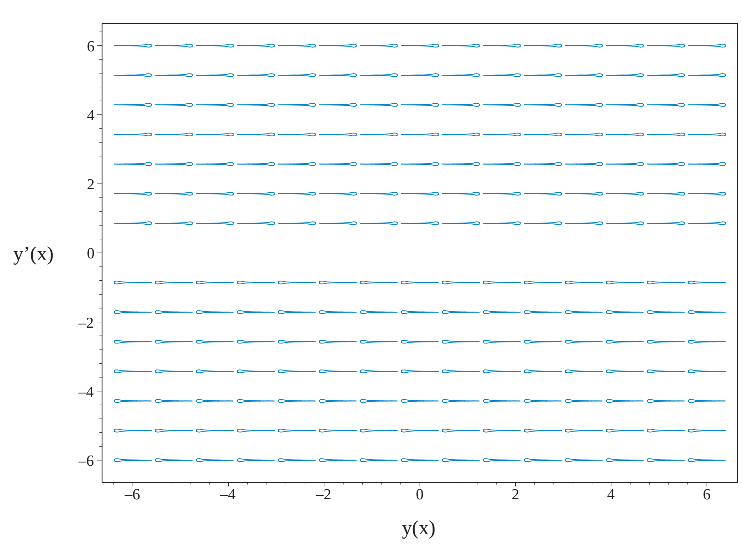


Figure 2.10: Slope field plot
$ay'' = 0$

Solved as second order linear constant coeff ode

Time used: 0.023 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = a, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$a\,\lambda^2 \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a\,\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = a, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(a)} \pm \frac{1}{(2)\,(a)}\sqrt{(0)^2 - (4)\,(a)\,(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2 x + c_1$$

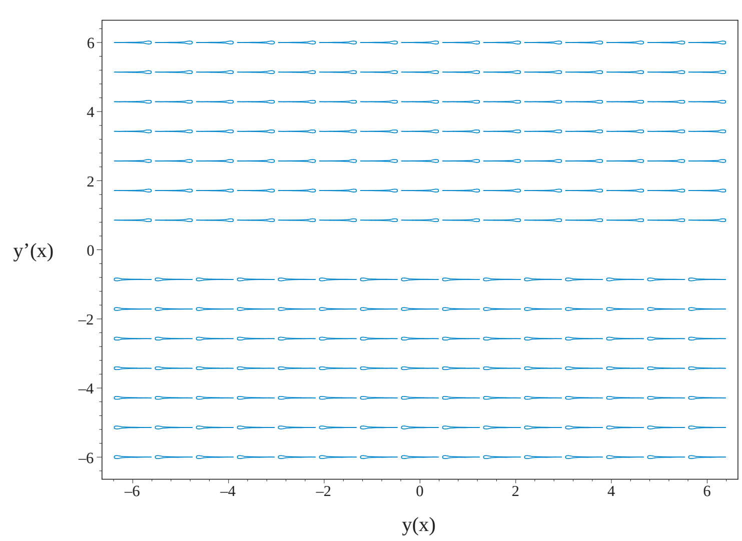


Figure 2.11: Slope field plot
$ay'' = 0$

Solved as second order linear exact ode

Time used: 0.081 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= a \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$ay' = c_1$$

We now have a first order ode to solve which is

$$ay' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dy &= \int \frac{c_1}{a}\,dx \\ y &= \frac{c_1 x}{a} + c_2 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 x}{a} + c_2$$

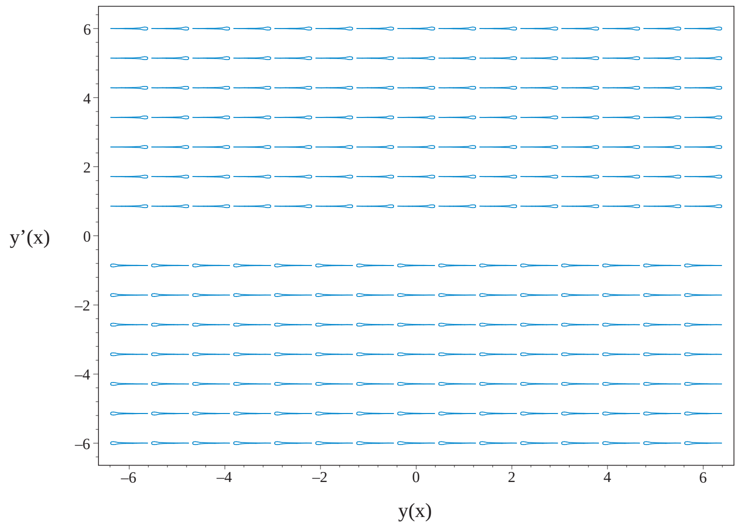


Figure 2.12: Slope field plot
$ay'' = 0$

Solved as second order missing y ode

Time used: 0.047 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$au'(x) = 0$$

Which is now solved for $u(x)$ as first order ode.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\, dx + c_1$$
$$u(x) = c_1$$

In summary, these are the solution found for $u(x)$

$$u(x) = c_1$$

For solution $u(x) = c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\, dx$$
$$y = c_1 x + c_2$$

In summary, these are the solution found for (y)

$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

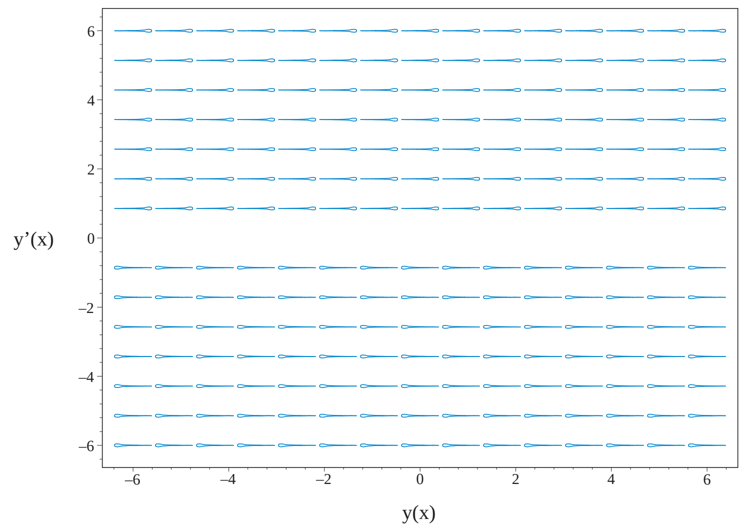


Figure 2.13: Slope field plot
$ay'' = 0$

Solved as second order integrable as is ode

Time used: 0.029 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int a y'' dx = 0$$
$$a y' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{c_1}{a}\, dx$$
$$y = \frac{c_1 x}{a} + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 x}{a} + c_2$$

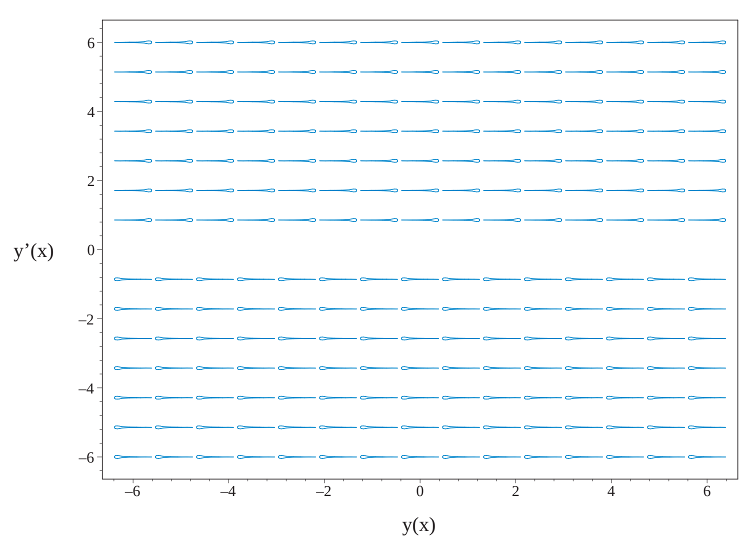


Figure 2.14: Slope field plot
$ay'' = 0$

Solved as second order integrable as is ode (ABC method)

Time used: 0.032 (sec)

Writing the ode as

$$ay'' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int ay''dx = 0$$
$$ay' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{c_1}{a}\,dx$$
$$y = \frac{c_1 x}{a} + c_2$$

Will add steps showing solving for IC soon.

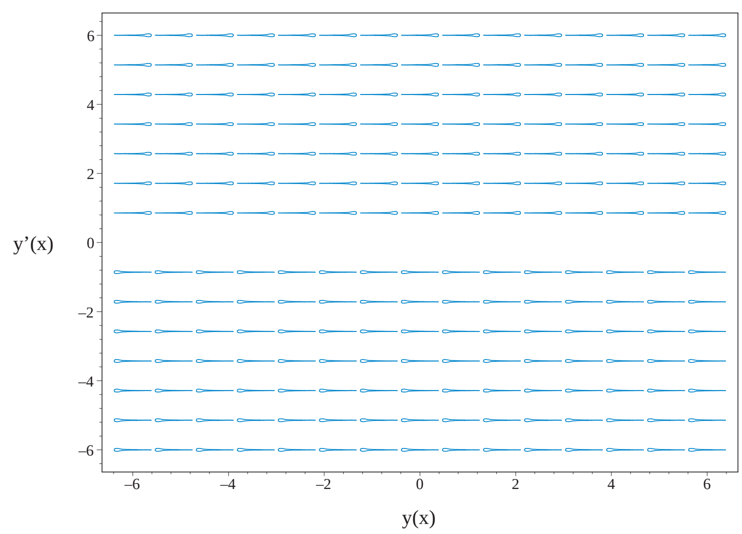


Figure 2.15: Slope field plot
$ay'' = 0$

Solved as second order can be made integrable

Time used: 0.301 (sec)

Multiplying the ode by y' gives

$$ay'y'' = 0$$

Integrating the above w.r.t x gives

$$\int ay'y''dx = 0$$
$$\frac{ay'^2}{2} = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \frac{\sqrt{2}\,\sqrt{ac_1}}{a} \tag{1}$$

$$y' = -\frac{\sqrt{2}\,\sqrt{ac_1}}{a} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{\sqrt{2}\,\sqrt{a c_1}}{a}\,dx$$
$$y = \frac{\sqrt{2}\,\sqrt{a c_1}\,x}{a} + c_2$$

Solving Eq. (2)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -\frac{\sqrt{2}\,\sqrt{a c_1}}{a}\,dx$$
$$y = -\frac{\sqrt{2}\,\sqrt{a c_1}\,x}{a} + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\sqrt{2}\,\sqrt{a c_1}\,x}{a} + c_3$$
$$y = \frac{\sqrt{2}\,\sqrt{a c_1}\,x}{a} + c_2$$

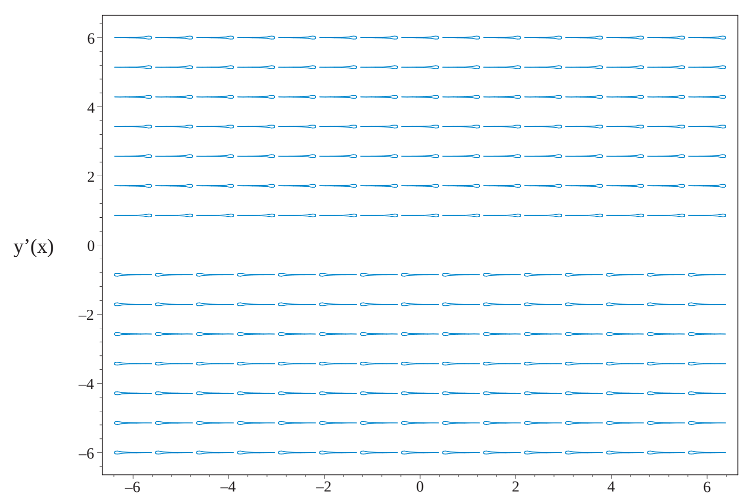


Figure 2.16: Slope field plot
$ay'' = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.028 (sec)

Writing the ode as

$$ay'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= a \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.5: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2 x + c_1$$

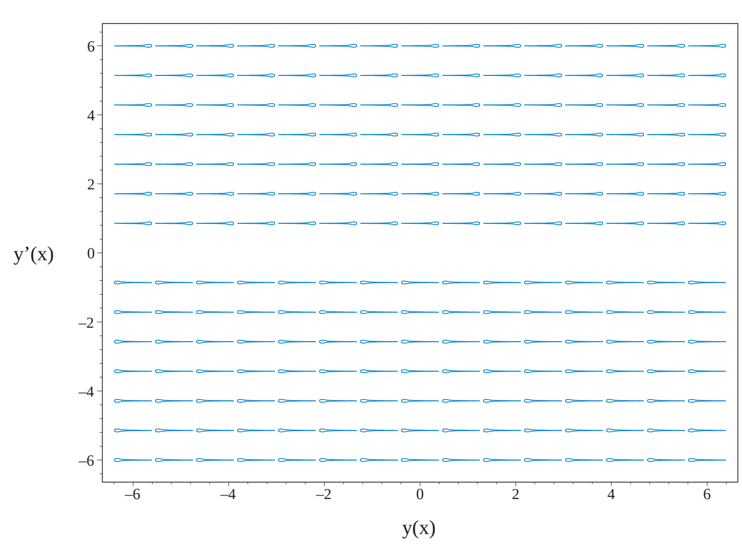


Figure 2.17: Slope field plot
$ay'' = 0$

Solved as second order ode adjoint method

Time used: 0.416 (sec)

In normal form the ode

$$ay'' = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned}p(x) &= 0\\ q(x) &= 0\\ r(x) &= 0\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi'' - (\xi\,p)' + \xi q &= 0\\ \xi'' - (0)' + (0) &= 0\\ \xi''(x) &= 0\end{aligned}$$

Which is solved for $\xi(x)$. Integrating twice gives the solution

$$\xi = c_1 x + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}\end{aligned}$$

Or

$$y' - \frac{y c_1}{c_1 x + c_2} = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= -\frac{c_1}{c_1 x + c_2}\\ p(x) &= 0\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= \mathrm{e}^{\int -\frac{c_1}{c_1 x + c_2}dx}\\ &= \frac{1}{c_1 x + c_2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 x + c_2}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{y}{c_1x+c_2} = \int 0\,dx + c_3$$
$$= c_3$$

Dividing throughout by the integrating factor $\frac{1}{c_1x+c_2}$ gives the final solution

$$y = (c_1x+c_2)\,c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = (c_1x+c_2)\,c_3$$

The constants can be merged to give

$$y = c_1x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1x + c_2$$

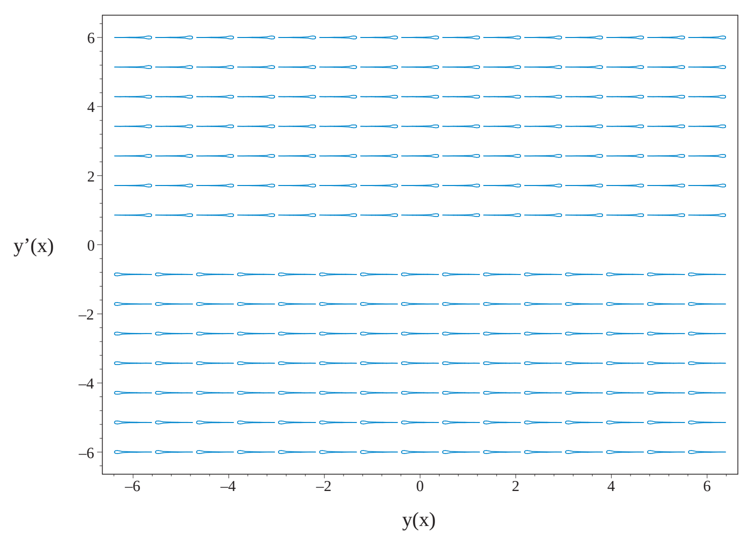


Figure 2.18: Slope field plot
$ay'' = 0$

Maple step by step solution

Let's solve
$a\left(\frac{d^2}{dx^2}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = 0$
- Characteristic polynomial of ODE
  $r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0\pm\left(\sqrt{0}\right)}{2}$
- Roots of the characteristic polynomial
  $r = 0$

- 1st solution of the ODE
  $y_1(x) = 1$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y(x) = C1y_1(x) + C2y_2(x)$
- Substitute in solutions
  $y(x) = C2x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 9

```
dsolve(a*diff(diff(y(x),x),x) = 0,y(x),singsol=all)
```

$$y = c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
DSolve[{a*D[y[x],{x,2}]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2x + c_1$$

2.1.5 Problem 5

Internal problem ID [9076]
Book : Second order enumerated odes
Section : section 1
Problem number : 5
Date solved : `Monday, January 27, 2025 at 05:32:02 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$a{y''}^2 = 0$$

Solved as second order missing x ode

Time used: 0.174 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$a p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p^2 = 0 \tag{1}$$

$$p'^2 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p^2 = 0$$

Solving gives $p = 0$

Solving equation (2)

Solving for the derivative gives these ODE's to solve

$$p' = 0 \tag{1}$$

$$p' = 0 \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\, dy + c_1$$
$$p = c_1$$

Solving Eq. (2)

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\, dy + c_2$$
$$p = c_2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_3$$
$$y = c_3$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\, dx$$
$$y = c_1 x + c_4$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_2$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_2\, dx$$
$$y = c_2 x + c_5$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_3$$
$$y = c_1 x + c_4$$
$$y = c_2 x + c_5$$

Solved as second order missing y ode

Time used: 0.154 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$a u'(x)^2 = 0$$

Which is now solved for $u(x)$ as first order ode.

Solving for the derivative gives these ODE's to solve

$$u'(x) = 0 \tag{1}$$
$$u'(x) = 0 \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\, dx + c_1$$
$$u(x) = c_1$$

Solving Eq. (2)

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\, dx + c_2$$
$$u(x) = c_2$$

In summary, these are the solution found for $u(x)$

$$u(x) = c_1$$
$$u(x) = c_2$$

For solution $u(x) = c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\, dx$$
$$y = c_1 x + c_3$$

For solution $u(x) = c_2$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_2$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_2\, dx$$
$$y = c_2 x + c_4$$

In summary, these are the solution found for (y)

$$y = c_1 x + c_3$$
$$y = c_2 x + c_4$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_3$$
$$y = c_2 x + c_4$$

Maple step by step solution

Let's solve
$a\left(\frac{d^2}{dx^2}y(x)\right)^2 = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = 0$
- Characteristic polynomial of ODE
  $r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the ODE
  $y_1(x) = 1$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x)$
- Substitute in solutions
  $y(x) = C2 x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : `1.697` (sec)
Leaf size : 9

```
dsolve(a*diff(diff(y(x),x),x)^2 = 0,y(x),singsol=all)
```

$$y = c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
DSolve[{a*(D[y[x],{x,2}])^2==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x + c_1$$

2.1.6 Problem 6

Internal problem ID [9077]
Book : Second order enumerated odes
Section : section 1
Problem number : 6
Date solved : `Monday, January 27, 2025 at 05:32:03 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$a{y''}^n = 0$$

Solved as second order missing x ode

Time used: 0.067 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$a\left(p(y)\left(\frac{d}{dy}p(y)\right)\right)^n = 0$$

Which is now solved as first order ode for $p(y)$.

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\, dy + c_1$$
$$p = c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\, dx$$
$$y = c_1x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1x + c_2$$

Solved as second order missing y ode

Time used: 0.059 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$au'(x)^n = 0$$

Which is now solved for $u(x)$ as first order ode.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\,dx + c_1$$
$$u(x) = c_1$$

In summary, these are the solution found for $u(x)$

$$u(x) = c_1$$

For solution $u(x) = c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\,dx$$
$$y = c_1x + c_2$$

In summary, these are the solution found for (y)

$$y = c_1x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1x + c_2$$

Maple step by step solution

Let's solve
$a\left(\frac{d^2}{dx^2}y(x)\right)^n = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = 0$
- Characteristic polynomial of ODE
  $r^2 = 0$

- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the ODE
  $y_1(x) = 1$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y(x) = C1y_1(x) + C2y_2(x)$
- Substitute in solutions
  $y(x) = C2x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 9

```
dsolve(a*diff(diff(y(x),x),x)^n = 0,y(x),singsol=all)
```

$$y = c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.005 (sec)
Leaf size : 24

```
DSolve[{a*(D[y[x],{x,2}])^n==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} 0^{\frac{1}{n}} x^2 + c_2 x + c_1$$

2.1.7 Problem 7

Internal problem ID [9078]
Book : Second order enumerated odes
Section : section 1
Problem number : 7
Date solved : `Monday, January 27, 2025 at 05:32:03 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = 1$$

Solved as second order ode quadrature

Time used: 0.020 (sec)

The ODE can be written as

$$y'' = 1$$

Integrating once gives

$$y' = x + c_1$$

Integrating again gives

$$y = \frac{x^2}{2} + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

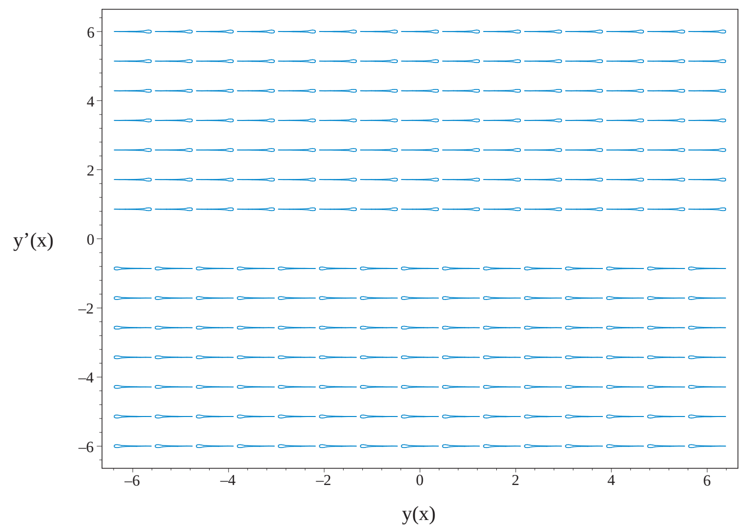


Figure 2.19: Slope field plot
$y'' = 1$

Solved as second order linear constant coeff ode

Time used: 0.089 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_2 x + c_1$$

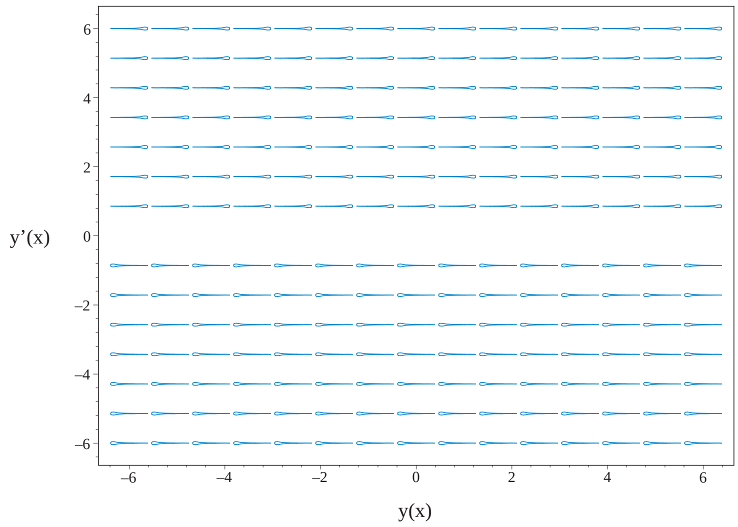


Figure 2.20: Slope field plot
$y'' = 1$

Solved as second order linear exact ode

Time used: 0.059 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1\\ q(x) &= 0\\ r(x) &= 0\\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0\\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int 1\,dx$$

We now have a first order ode to solve which is

$$y' = x + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x + c_1 \, dx$$
$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

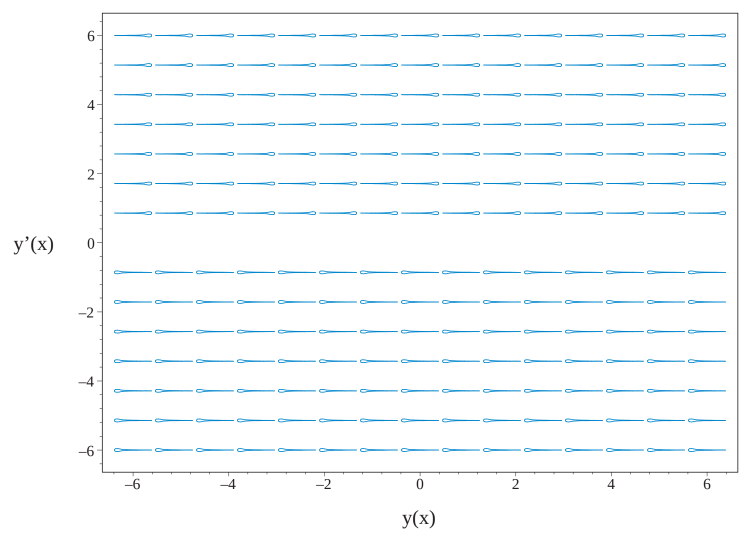


Figure 2.21: Slope field plot
$y'' = 1$

Solved as second order missing y ode

Time used: 0.056 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) - 1 = 0$$

Which is now solved for $u(x)$ as first order ode.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 1 \, dx$$
$$u(x) = x + c_1$$

In summary, these are the solution found for $u(x)$

$$u(x) = x + c_1$$

For solution $u(x) = x + c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = x + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x + c_1 \, dx$$
$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

In summary, these are the solution found for (y)

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

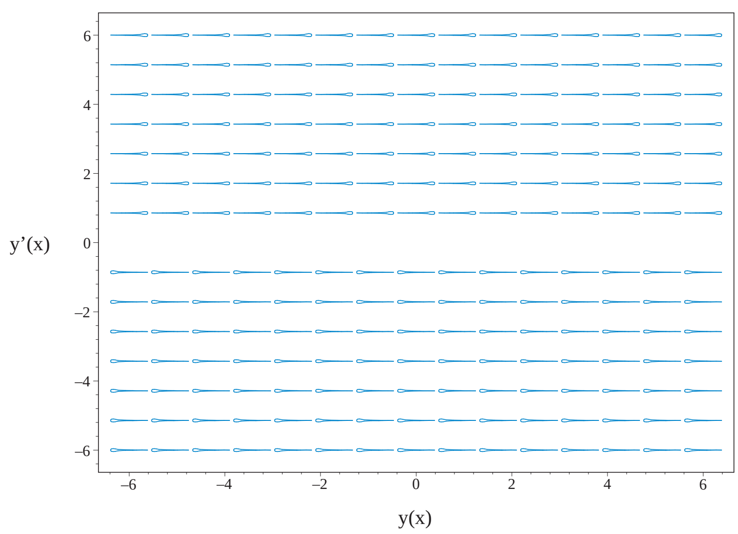


Figure 2.22: Slope field plot
$y'' = 1$

Solved as second order integrable as is ode

Time used: 0.031 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int 1 dx$$
$$y' = x + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x + c_1 \, dx$$
$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

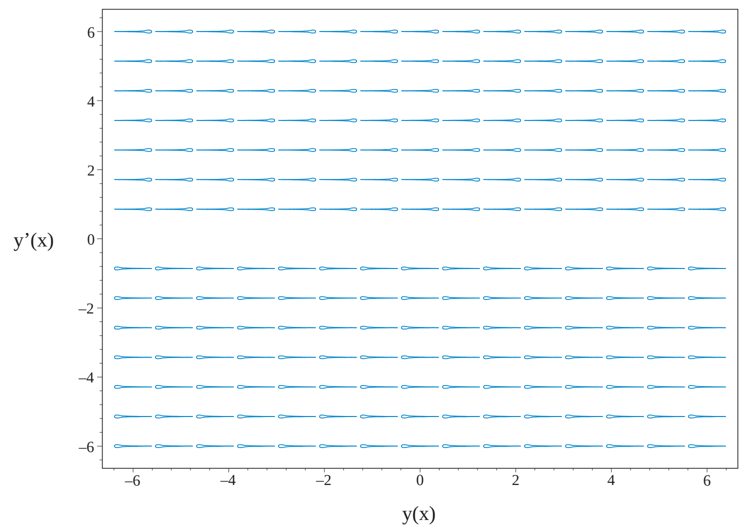


Figure 2.23: Slope field plot
$y'' = 1$

Solved as second order integrable as is ode (ABC method)

Time used: 0.032 (sec)

Writing the ode as

$$y'' = 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int 1 dx$$
$$y' = x + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x + c_1 \, dx$$
$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

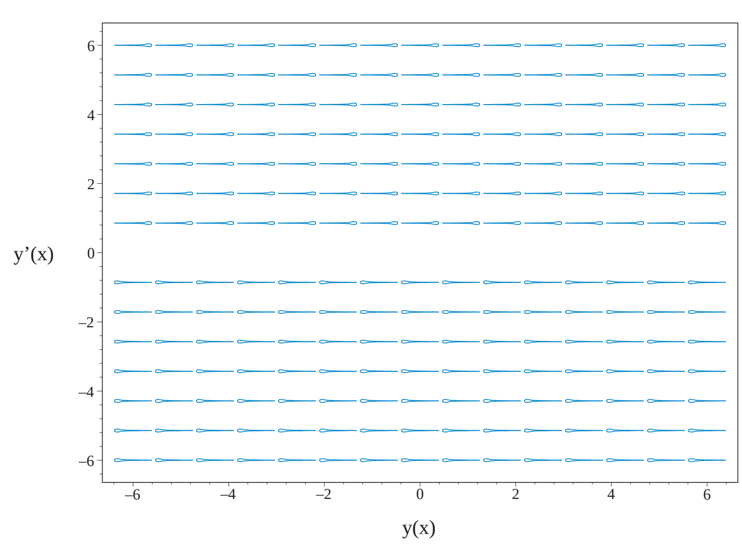


Figure 2.24: Slope field plot
$y'' = 1$

Solved as second order can be made integrable

Time used: 0.467 (sec)

Multiplying the ode by y' gives

$$y'y'' - y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - y')\,dx = 0$$
$$\frac{y'^2}{2} - y = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{2y + 2c_1} \tag{1}$$
$$y' = -\sqrt{2y + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{2y + 2c_1}} dy = dx$$
$$\sqrt{2y + 2c_1} = x + c_2$$

Singular solutions are found by solving

$$\sqrt{2y + 2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -c_1$$

Solving for y gives

$$y = -c_1$$
$$y = \frac{1}{2}c_2^2 + c_2 x + \frac{1}{2}x^2 - c_1$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{2y + 2c_1}} dy = dx$$
$$-\sqrt{2y + 2c_1} = x + c_3$$

Singular solutions are found by solving

$$-\sqrt{2y + 2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -c_1$$

Solving for y gives

$$
\begin{aligned}
y &= -c_1 \\
y &= \frac{1}{2}c_3^2 + c_3 x + \frac{1}{2}x^2 - c_1
\end{aligned}
$$

Will add steps showing solving for IC soon.

The solution

$$y = -c_1$$

was found not to satisfy the ode or the IC. Hence it is removed.

<u>Summary of solutions found</u>

$$
\begin{aligned}
y &= \frac{1}{2}c_2^2 + c_2 x + \frac{1}{2}x^2 - c_1 \\
y &= \frac{1}{2}c_3^2 + c_3 x + \frac{1}{2}x^2 - c_1
\end{aligned}
$$

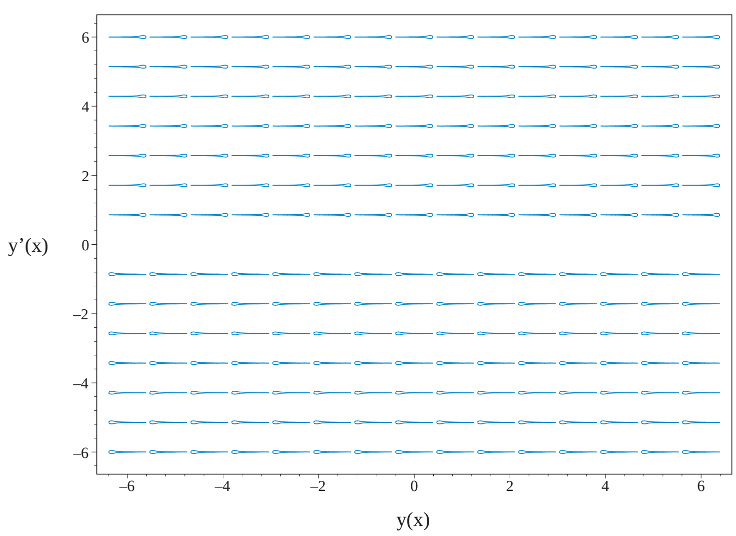


Figure 2.25: Slope field plot
$y'' = 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.047 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.9: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_2 x + c_1$$

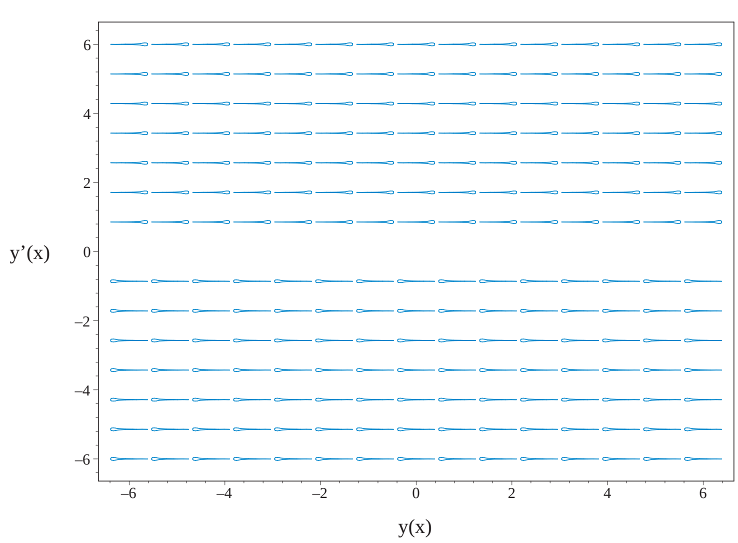


Figure 2.26: Slope field plot
$y'' = 1$

Solved as second order ode adjoint method

Time used: 0.476 (sec)

In normal form the ode

$$y'' = 1 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ r(x) &= 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (0)' + (0) &= 0 \\ \xi''(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Integrating twice gives the solution

$$\xi = c_1 x + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)} \end{aligned}$$

Or

$$y' - \frac{y c_1}{c_1 x + c_2} = \frac{\frac{1}{2} c_1\,x^2 + c_2 x}{c_1 x + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x) y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{c_1}{c_1 x + c_2} \\ p(x) &= \frac{x(c_1 x + 2c_2)}{2c_1 x + 2c_2} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int -\frac{c_1}{c_1 x + c_2} dx} \\ &= \frac{1}{c_1 x + c_2} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x(c_1x+2c_2)}{2c_1x+2c_2}\right)$$

$$\frac{d}{dx}\left(\frac{y}{c_1x+c_2}\right) = \left(\frac{1}{c_1x+c_2}\right)\left(\frac{x(c_1x+2c_2)}{2c_1x+2c_2}\right)$$

$$d\left(\frac{y}{c_1x+c_2}\right) = \left(\frac{x(c_1x+2c_2)}{(2c_1x+2c_2)(c_1x+c_2)}\right) dx$$

Integrating gives

$$\frac{y}{c_1x+c_2} = \int \frac{x(c_1x+2c_2)}{(2c_1x+2c_2)(c_1x+c_2)}\, dx$$

$$= \frac{x}{2c_1} + \frac{c_2^2}{2c_1^2(c_1x+c_2)} + c_3$$

Dividing throughout by the integrating factor $\frac{1}{c_1x+c_2}$ gives the final solution

$$y = \frac{2c_1^3c_3x + (2c_2c_3+x^2)c_1^2 + c_1c_2x + c_2^2}{2c_1^2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{2c_1^3c_3x + (2c_2c_3+x^2)c_1^2 + c_1c_2x + c_2^2}{2c_1^2}$$

The constants can be merged to give

$$y = \frac{2c_1^3x + (x^2+2c_2)c_1^2 + c_1c_2x + c_2^2}{2c_1^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{2c_1^3x + (x^2+2c_2)c_1^2 + c_1c_2x + c_2^2}{2c_1^2}$$

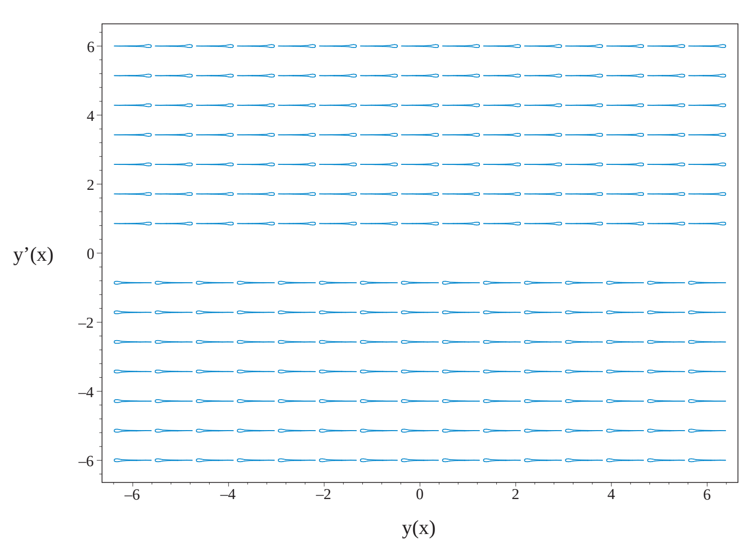


Figure 2.27: Slope field plot
$y'' = 1$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 + C2x + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\left(\int x dx\right) + \left(\int 1 dx\right) x$

  - Compute integrals

    $y_p(x) = \frac{x^2}{2}$

- Substitute particular solution into general solution to ODE

  $y(x) = C1 + C2x + \frac{1}{2}x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x) = 1,y(x),singsol=all)
```

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 19

```
DSolve[{D[y[x],{x,2}]==1,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2}{2} + c_2 x + c_1$$

2.1.8 Problem 8

Internal problem ID [9079]
Book : Second order enumerated odes
Section : section 1
Problem number : 8
Date solved : `Monday, January 27, 2025 at 05:32:06 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y''^2 = 1$$

Factoring the ode gives these factors

$$y'' - 1 = 0 \tag{1}$$

$$y'' + 1 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solved as second order ode quadrature

Time used: 0.019 (sec)

The ODE can be written as

$$y'' = 1$$

Integrating once gives

$$y' = x + c_1$$

Integrating again gives

$$y = \frac{x^2}{2} + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

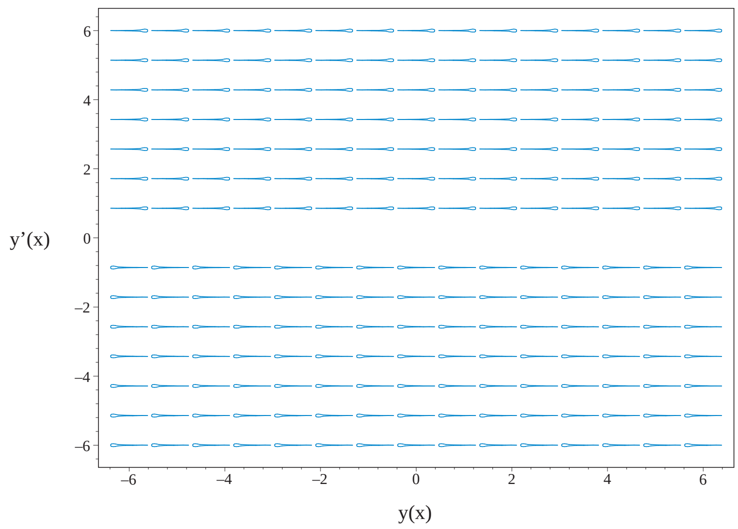


Figure 2.28: Slope field plot
$y'' - 1 = 0$

Solved as second order linear constant coeff ode

Time used: 0.092 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 - 1 = 0$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_2 x + c_1$$

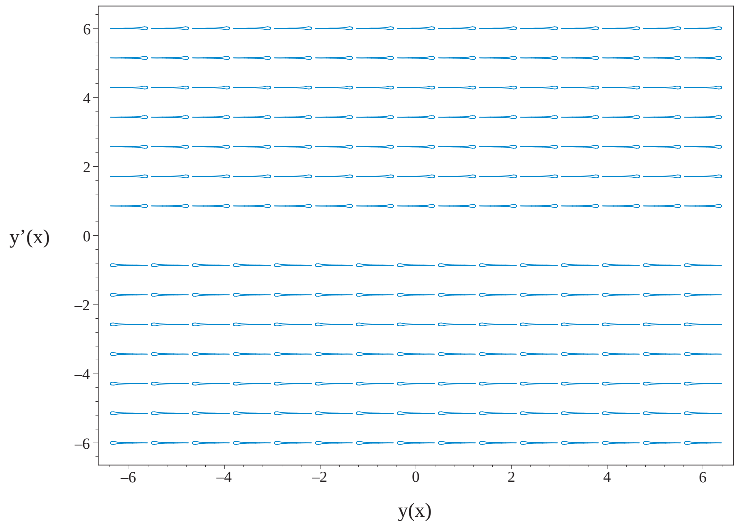


Figure 2.29: Slope field plot
$y'' - 1 = 0$

Solved as second order linear exact ode

Time used: 0.066 (sec)

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int 1\, dx$$

We now have a first order ode to solve which is

$$y' = x + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x + c_1\, dx$$
$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

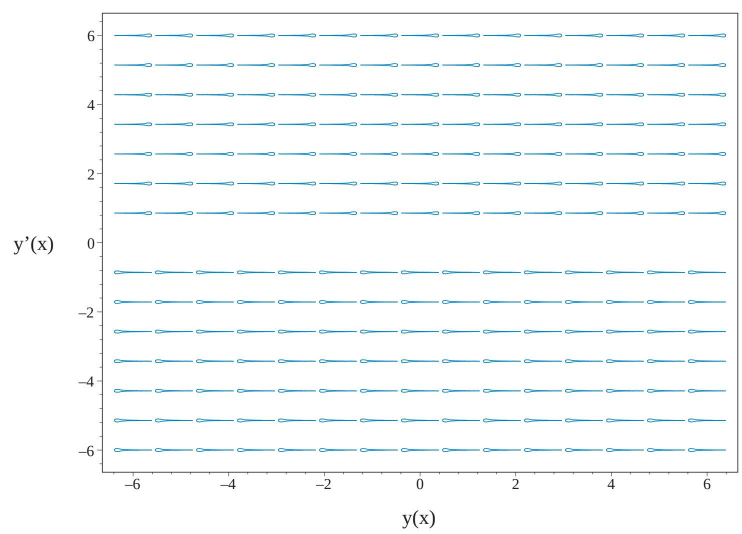


Figure 2.30: Slope field plot
$y'' - 1 = 0$

Solved as second order missing y ode

Time used: 0.059 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) - 1 = 0$$

Which is now solved for $u(x)$ as first order ode.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 1\, dx$$
$$u(x) = x + c_1$$

In summary, these are the solution found for $u(x)$

$$u(x) = x + c_1$$

For solution $u(x) = x + c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = x + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x + c_1\, dx$$
$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

In summary, these are the solution found for (y)

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

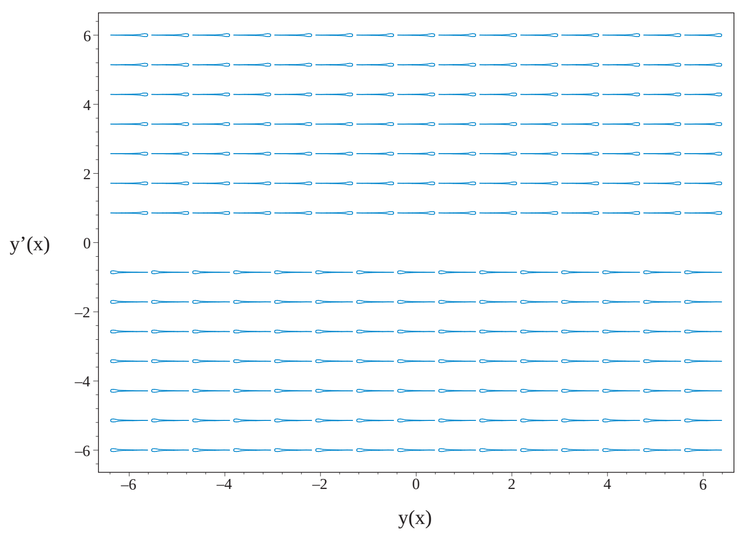


Figure 2.31: Slope field plot
$y'' - 1 = 0$

Solved as second order integrable as is ode

Time used: 0.031 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 1)\, dx = 0$$
$$-x + y' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x + c_1\, dx$$
$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

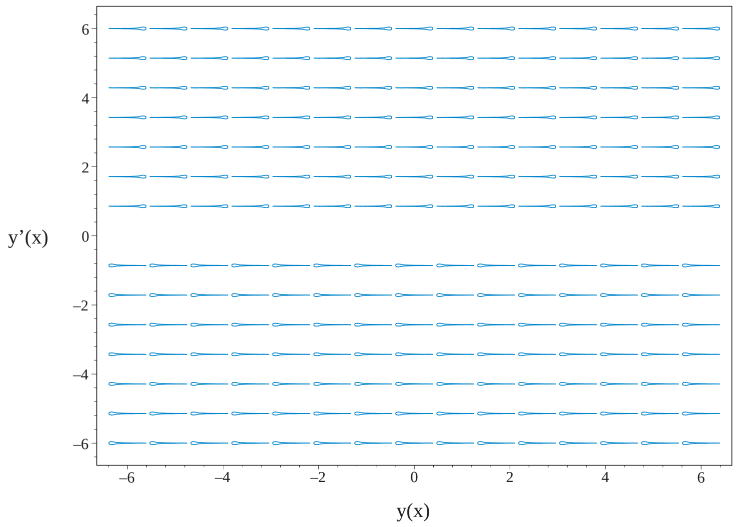


Figure 2.32: Slope field plot
$y'' - 1 = 0$

Solved as second order integrable as is ode (ABC method)

Time used: 0.031 (sec)

Writing the ode as

$$y'' = 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int 1 dx$$
$$y' = x + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x + c_1 \, dx$$
$$y = \frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

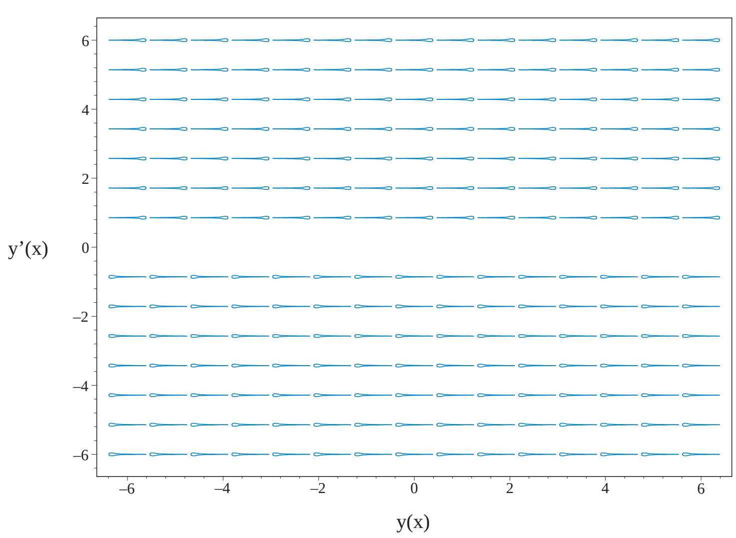


Figure 2.33: Slope field plot
$y'' - 1 = 0$

Solved as second order can be made integrable

Time used: 0.477 (sec)

Multiplying the ode by y' gives

$$y'y'' - y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - y')\, dx = 0$$
$$\frac{y'^2}{2} - y = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{2y + 2c_1} \tag{1}$$
$$y' = -\sqrt{2y + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{2y + 2c_1}} dy = dx$$
$$\sqrt{2y + 2c_1} = x + c_2$$

Singular solutions are found by solving

$$\sqrt{2y + 2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -c_1$$

Solving for y gives

$$y = -c_1$$
$$y = \frac{1}{2}c_2^2 + c_2 x + \frac{1}{2}x^2 - c_1$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{2y + 2c_1}} dy = dx$$
$$-\sqrt{2y + 2c_1} = x + c_3$$

Singular solutions are found by solving

$$-\sqrt{2y + 2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -c_1$$

Solving for y gives

$$y = -c_1$$
$$y = \frac{1}{2}c_3^2 + c_3 x + \frac{1}{2}x^2 - c_1$$

Will add steps showing solving for IC soon.

The solution

$$y = -c_1$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = \frac{1}{2}c_2^2 + c_2 x + \frac{1}{2}x^2 - c_1$$
$$y = \frac{1}{2}c_3^2 + c_3 x + \frac{1}{2}x^2 - c_1$$

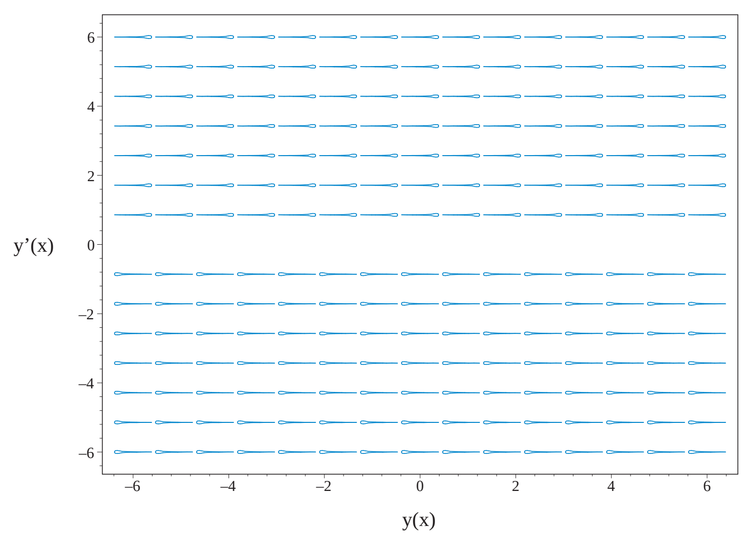


Figure 2.34: Slope field plot
$y'' - 1 = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.047 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.11: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 - 1 = 0$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 + c_2 x + c_1$$

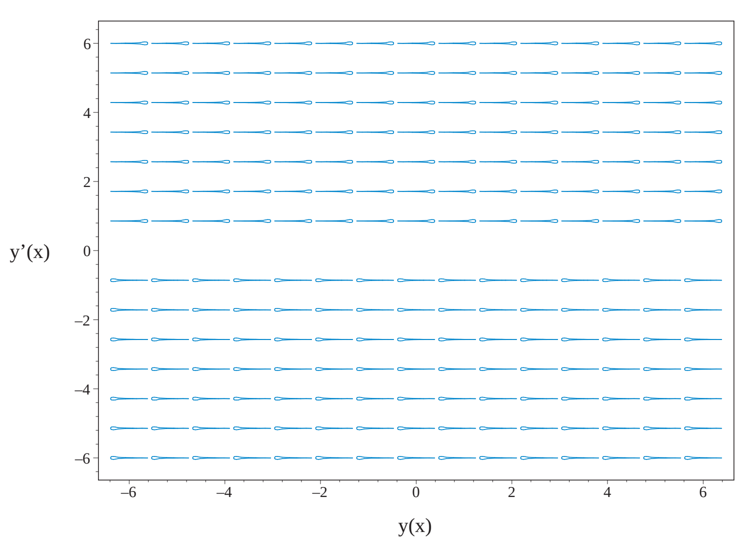


Figure 2.35: Slope field plot
$y'' - 1 = 0$

Solved as second order ode adjoint method

Time used: 0.465 (sec)

In normal form the ode

$$y'' - 1 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ r(x) &= 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (0) &= 0 \\ \xi''(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Integrating twice gives the solution

$$\xi = c_1 x + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)} \end{aligned}$$

Or

$$y' - \frac{y c_1}{c_1 x + c_2} = \frac{\frac{1}{2} c_1\, x^2 + c_2 x}{c_1 x + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x) y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{c_1}{c_1 x + c_2} \\ p(x) &= \frac{x(c_1 x + 2c_2)}{2c_1 x + 2c_2} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= e^{\int -\frac{c_1}{c_1 x + c_2} dx} \\ &= \frac{1}{c_1 x + c_2} \end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x(c_1x+2c_2)}{2c_1x+2c_2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1x+c_2}\right) &= \left(\frac{1}{c_1x+c_2}\right)\left(\frac{x(c_1x+2c_2)}{2c_1x+2c_2}\right) \\
\mathrm{d}\left(\frac{y}{c_1x+c_2}\right) &= \left(\frac{x(c_1x+2c_2)}{(2c_1x+2c_2)(c_1x+c_2)}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{c_1x+c_2} &= \int \frac{x(c_1x+2c_2)}{(2c_1x+2c_2)(c_1x+c_2)}\,dx \\
&= \frac{x}{2c_1} + \frac{c_2^2}{2c_1^2(c_1x+c_2)} + c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_1x+c_2}$ gives the final solution

$$y = \frac{2c_1^3c_3x + (2c_2c_3 + x^2)c_1^2 + c_1c_2x + c_2^2}{2c_1^2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{2c_1^3c_3x + (2c_2c_3 + x^2)c_1^2 + c_1c_2x + c_2^2}{2c_1^2}$$

The constants can be merged to give

$$y = \frac{2c_1^3x + (x^2 + 2c_2)c_1^2 + c_1c_2x + c_2^2}{2c_1^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{2c_1^3x + (x^2 + 2c_2)c_1^2 + c_1c_2x + c_2^2}{2c_1^2}$$

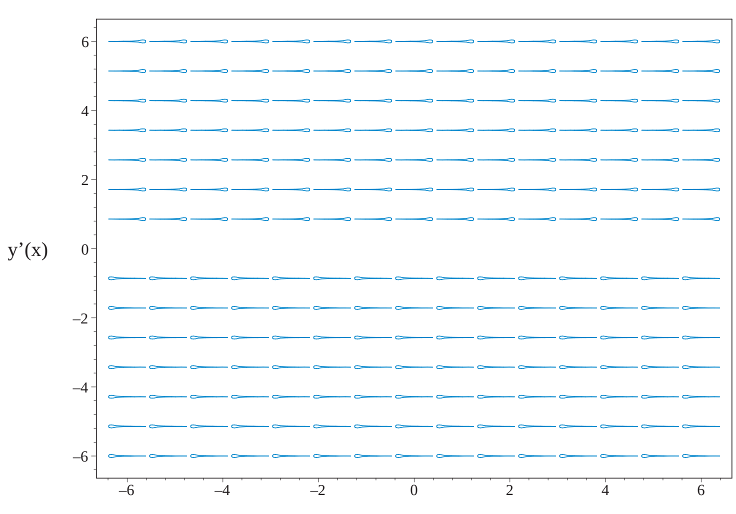


Figure 2.36: Slope field plot
$y'' - 1 = 0$

Solving equation (2)

Solved as second order ode quadrature

Time used: 0.011 (sec)

The ODE can be written as

$$y'' = -1$$

Integrating once gives

$$y' = -x + c_1$$

Integrating again gives

$$y = -\frac{x^2}{2} + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{1}{2}x^2 + c_1 x + c_2$$

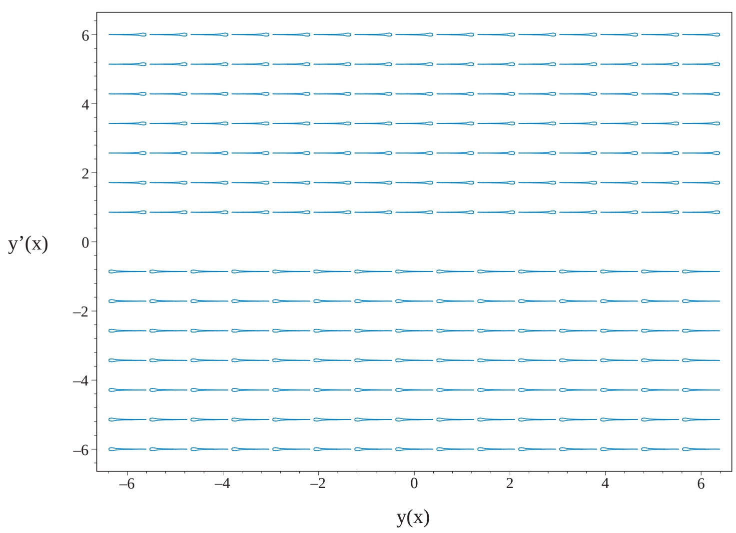


Figure 2.37: Slope field plot
$y'' + 1 = 0$

Solved as second order linear constant coeff ode

Time used: 0.053 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = -1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(0)^2 - (4)\,(1)\,(0)}\\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 + 1 = 0$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(-\frac{x^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{1}{2}x^2 + c_2 x + c_1$$

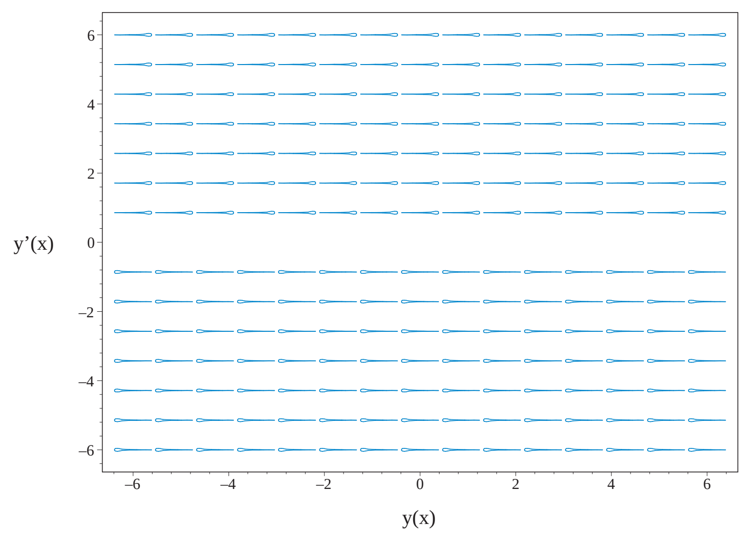


Figure 2.38: Slope field plot
$y'' + 1 = 0$

Solved as second order linear exact ode

Time used: 0.043 (sec)

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= -1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$y'=\int -1\,dx$$

We now have a first order ode to solve which is

$$y'=-x+c_1$$

Since the ode has the form $y'=f(x)$, then we only need to integrate $f(x)$.

$$\int dy=\int -x+c_1\,dx$$
$$y=-\frac{1}{2}x^2+c_1x+c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y=-\frac{1}{2}x^2+c_1x+c_2$$

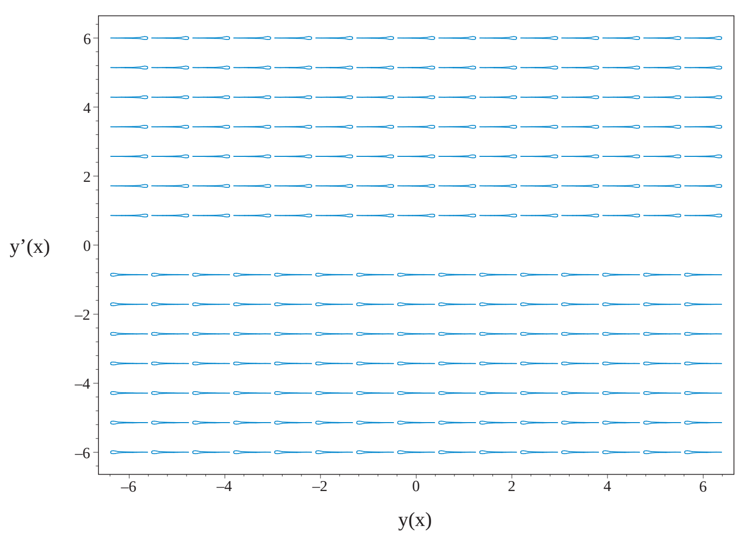


Figure 2.39: Slope field plot
$y''+1=0$

Solved as second order missing y ode

Time used: 0.051 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x)=y'$$

Then

$$u'(x)=y''$$

Hence the ode becomes

$$u'(x) + 1 = 0$$

Which is now solved for $u(x)$ as first order ode.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int -1\, dx$$
$$u(x) = -x + c_1$$

In summary, these are the solution found for $u(x)$

$$u(x) = -x + c_1$$

For solution $u(x) = -x + c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = -x + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -x + c_1\, dx$$
$$y = -\frac{1}{2}x^2 + c_1 x + c_2$$

In summary, these are the solution found for (y)

$$y = -\frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{1}{2}x^2 + c_1 x + c_2$$

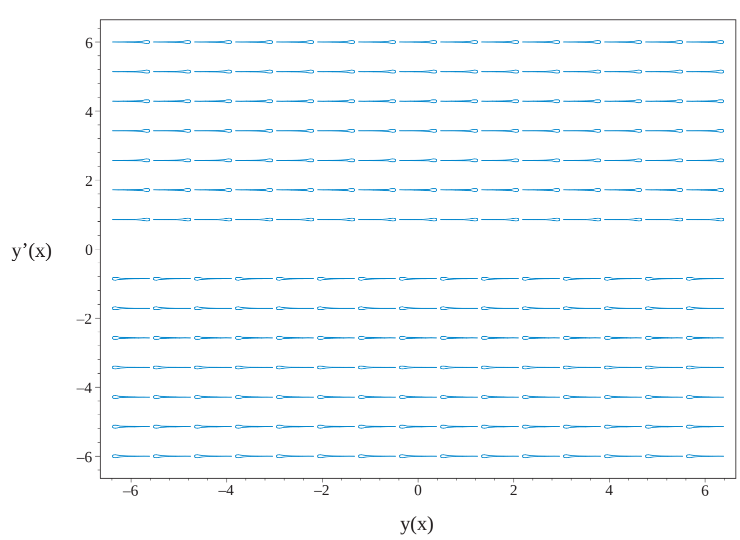


Figure 2.40: Slope field plot
$y'' + 1 = 0$

Solved as second order integrable as is ode

Time used: 0.027 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 1)\, dx = 0$$
$$x + y' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -x + c_1\, dx$$
$$y = -\frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{1}{2}x^2 + c_1 x + c_2$$

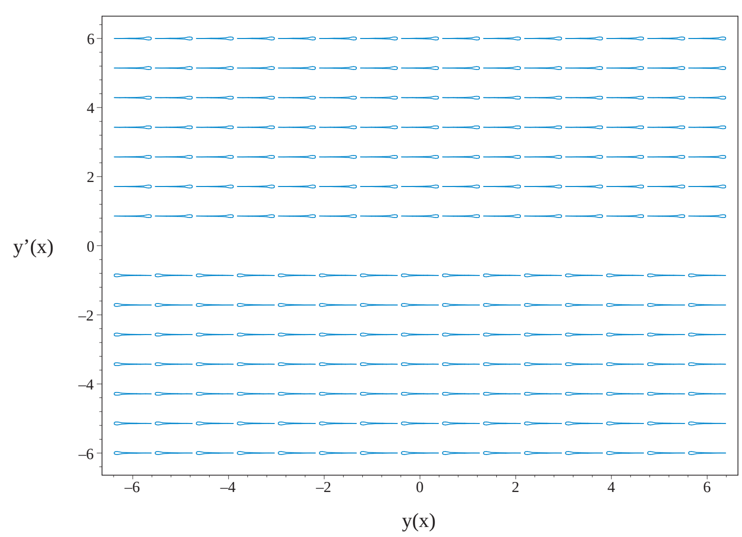


Figure 2.41: Slope field plot
$y'' + 1 = 0$

Solved as second order integrable as is ode (ABC method)

Time used: 0.034 (sec)

Writing the ode as

$$y'' = -1$$

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int (-1)\, dx$$
$$y' = -x + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -x + c_1\, dx$$
$$y = -\frac{1}{2}x^2 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

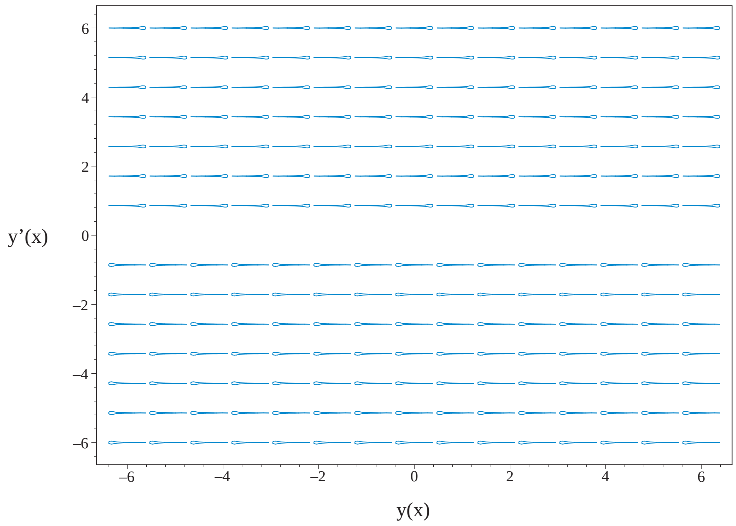


Figure 2.42: Slope field plot
$y'' + 1 = 0$

Solved as second order can be made integrable

Time used: 0.427 (sec)

Multiplying the ode by y' gives

$$y'y'' + y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + y')\, dx = 0$$
$$\frac{y'^2}{2} + y = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{-2y + 2c_1} \tag{1}$$
$$y' = -\sqrt{-2y + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{-2y + 2c_1}} dy = dx$$
$$-\sqrt{-2y + 2c_1} = x + c_2$$

Singular solutions are found by solving

$$\sqrt{-2y + 2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = c_1$$

Solving for y gives

$$y = c_1$$
$$y = -\frac{1}{2}c_2^2 - c_2 x - \frac{1}{2}x^2 + c_1$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{-2y+2c_1}}dy = dx$$
$$\sqrt{-2y+2c_1} = x + c_3$$

Singular solutions are found by solving

$$-\sqrt{-2y+2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = c_1$$

Solving for y gives

$$y = c_1$$
$$y = -\frac{1}{2}c_3^2 - c_3x - \frac{1}{2}x^2 + c_1$$

Will add steps showing solving for IC soon.

The solution

$$y = c_1$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = -\frac{1}{2}c_2^2 - c_2x - \frac{1}{2}x^2 + c_1$$
$$y = -\frac{1}{2}c_3^2 - c_3x - \frac{1}{2}x^2 + c_1$$

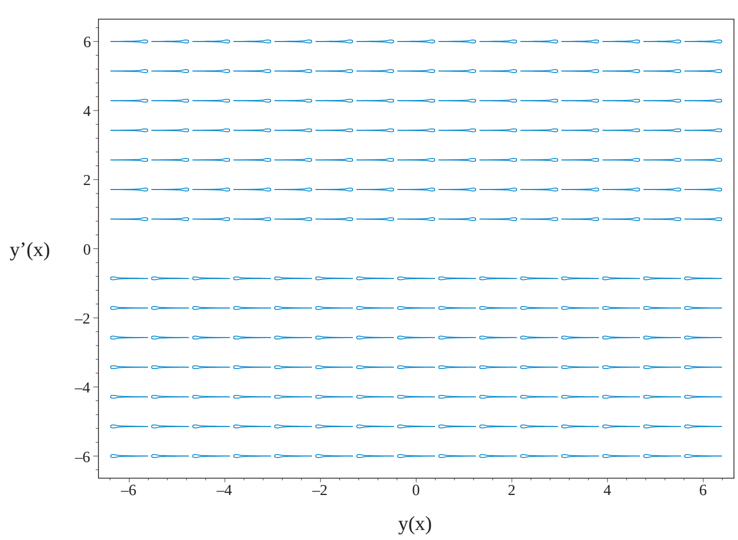


Figure 2.43: Slope field plot
$y'' + 1 = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.048 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.12: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= 1 \int \frac{1}{1}\, dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 + 1 = 0$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(-\frac{x^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{1}{2}x^2 + c_2 x + c_1$$

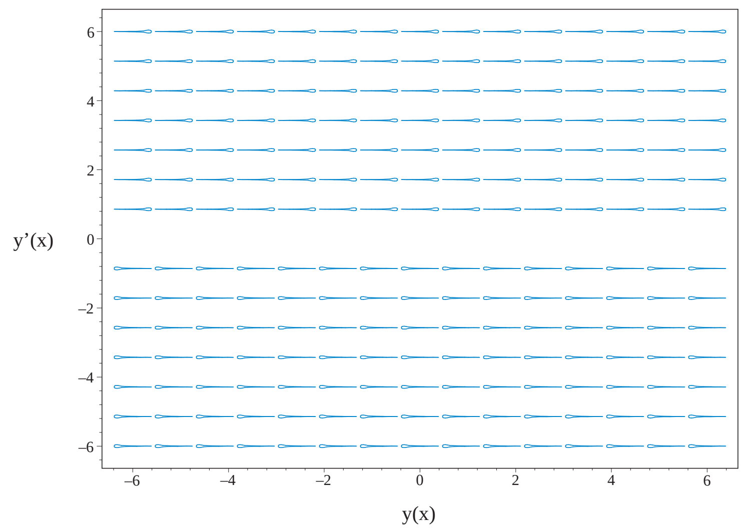


Figure 2.44: Slope field plot
$y'' + 1 = 0$

Solved as second order ode adjoint method

Time used: 0.116 (sec)

In normal form the ode

$$y'' + 1 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ r(x) &= -1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (0) &= 0 \\ \xi''(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Integrating twice gives the solution

$$\xi = c_1 x + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)} \end{aligned}$$

Or

$$y' - \frac{y c_1}{c_1 x + c_2} = \frac{-\frac{1}{2} c_1\, x^2 - c_2 x}{c_1 x + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{c_1}{c_1 x + c_2} \\
p(x) &= -\frac{x(c_1 x + 2c_2)}{2c_1 x + 2c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= \mathrm{e}^{\int -\frac{c_1}{c_1 x + c_2} dx} \\
&= \frac{1}{c_1 x + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{x(c_1 x + 2c_2)}{2c_1 x + 2c_2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 x + c_2}\right) &= \left(\frac{1}{c_1 x + c_2}\right)\left(-\frac{x(c_1 x + 2c_2)}{2c_1 x + 2c_2}\right) \\
\mathrm{d}\left(\frac{y}{c_1 x + c_2}\right) &= \left(-\frac{x(c_1 x + 2c_2)}{(2c_1 x + 2c_2)(c_1 x + c_2)}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 x + c_2} &= \int -\frac{x(c_1 x + 2c_2)}{(2c_1 x + 2c_2)(c_1 x + c_2)} \, dx \\
&= -\frac{x}{2c_1} - \frac{c_2^2}{2c_1^2 (c_1 x + c_2)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 x + c_2}$ gives the final solution

$$
y = (c_1 x + c_2)\left(-\frac{x}{2c_1} - \frac{c_2^2}{2c_1^2 (c_1 x + c_2)} + c_3\right)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = (c_1 x + c_2)\left(-\frac{x}{2c_1} - \frac{c_2^2}{2c_1^2 (c_1 x + c_2)} + c_3\right)
$$

The constants can be merged to give

$$
y = (c_1 x + c_2)\left(-\frac{x}{2c_1} - \frac{c_2^2}{2c_1^2 (c_1 x + c_2)} + 1\right)
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = (c_1 x + c_2)\left(-\frac{x}{2c_1} - \frac{c_2^2}{2c_1^2 (c_1 x + c_2)} + 1\right)
$$

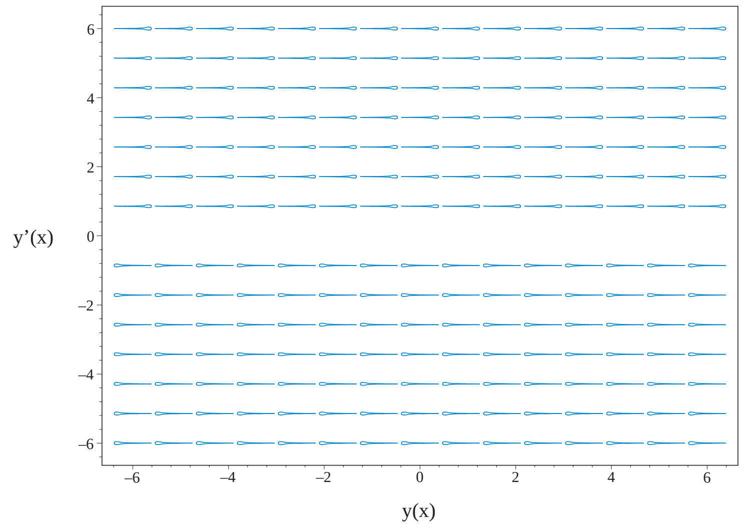


Figure 2.45: Slope field plot
$y'' + 1 = 0$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)^2 = 1,y(x),singsol=all)
```

$$y = \frac{1}{2}x^2 + c_1 x + c_2$$
$$y = -\frac{1}{2}x^2 + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 37

```
DSolve[{(D[y[x],{x,2}])^2==1,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{x^2}{2} + c_2 x + c_1$$
$$y(x) \to \frac{x^2}{2} + c_2 x + c_1$$

2.1.9 Problem 9

Internal problem ID [9080]
Book : Second order enumerated odes
Section : section 1
Problem number : 9
Date solved : `Monday, January 27, 2025 at 05:32:11 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = x$$

Solved as second order ode quadrature

Time used: 0.017 (sec)

Integrating once gives

$$y' = \frac{x^2}{2} + c_1$$

Integrating again gives

$$y = \frac{x^3}{6} + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

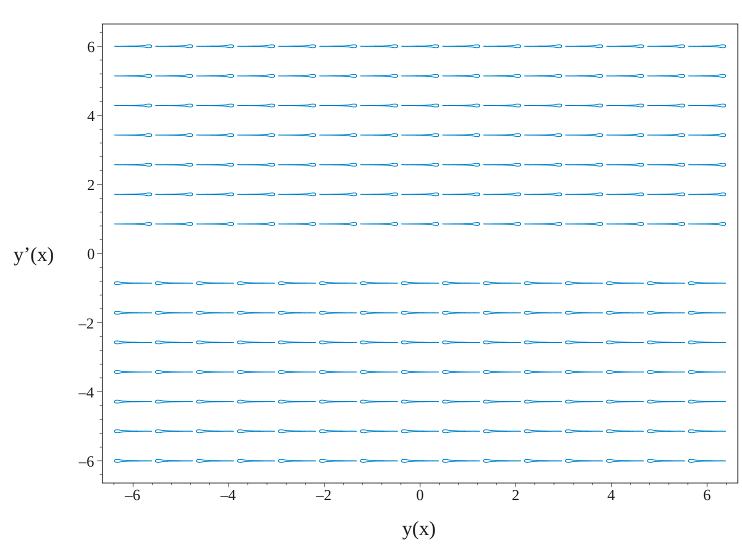


Figure 2.46: Slope field plot
$y'' = x$

Solved as second order linear constant coeff ode

Time used: 0.096 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x^3 + A_1x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_2 + 2A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x + c_1) + \left(\frac{x^3}{6}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{6}x^3 + c_2x + c_1$$

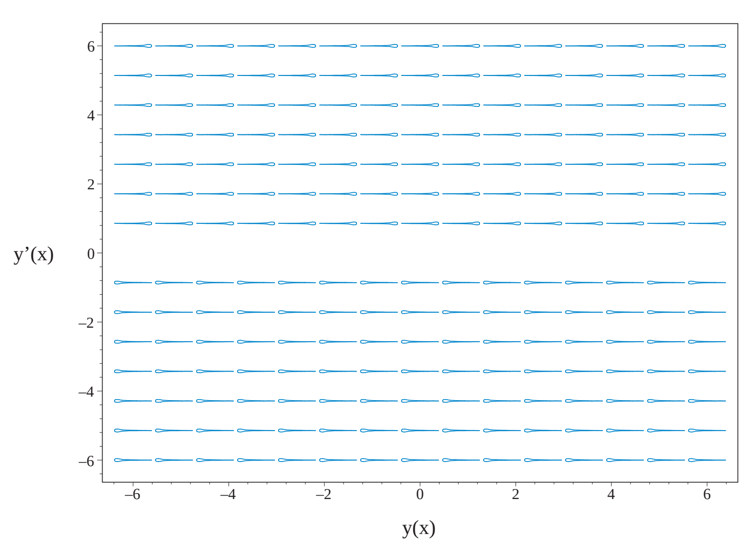


Figure 2.47: Slope field plot
$y'' = x$

Solved as second order linear exact ode

Time used: 0.062 (sec)

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int x\, dx$$

We now have a first order ode to solve which is

$$y' = \frac{x^2}{2} + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dy &= \int \frac{x^2}{2} + c_1\, dx \\ y &= \frac{1}{6}x^3 + c_1 x + c_2 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

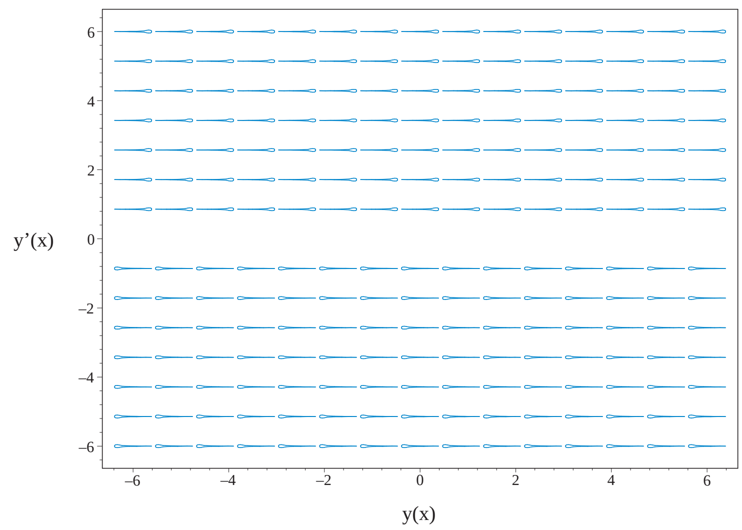


Figure 2.48: Slope field plot
$y'' = x$

Solved as second order missing y ode

Time used: 0.059 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) - x = 0$$

Which is now solved for $u(x)$ as first order ode.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int x\,dx$$
$$u(x) = \frac{x^2}{2} + c_1$$

In summary, these are the solution found for $u(x)$

$$u(x) = \frac{x^2}{2} + c_1$$

For solution $u(x) = \frac{x^2}{2} + c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{x^2}{2} + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{x^2}{2} + c_1\,dx$$
$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

In summary, these are the solution found for (y)

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

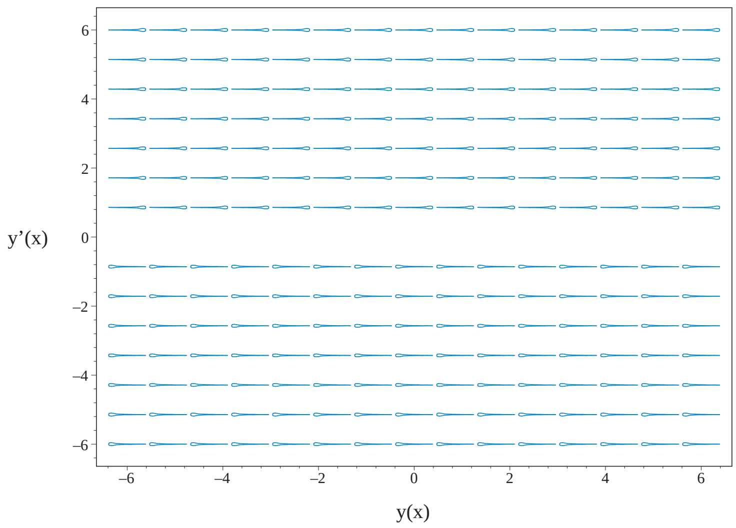


Figure 2.49: Slope field plot
$y'' = x$

Solved as second order integrable as is ode

Time used: 0.033 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int x dx$$
$$y' = \frac{x^2}{2} + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{x^2}{2} + c_1 \, dx$$
$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

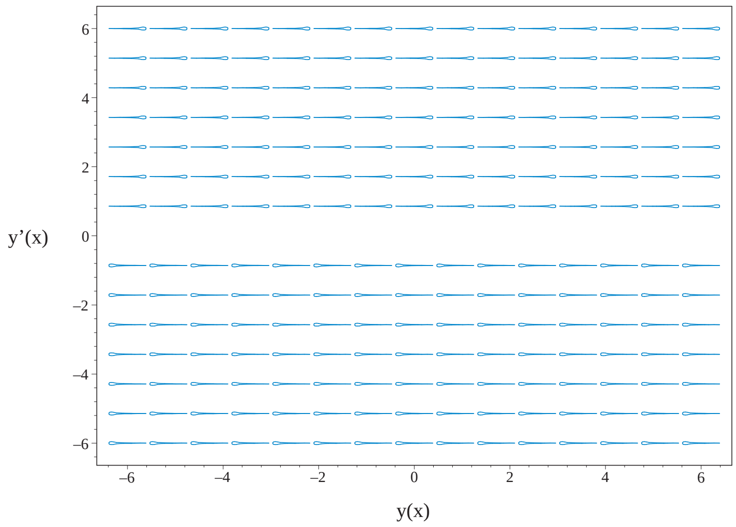


Figure 2.50: Slope field plot
$y'' = x$

Solved as second order integrable as is ode (ABC method)

Time used: 0.033 (sec)

Writing the ode as

$$y'' = x$$

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int x dx$$
$$y' = \frac{x^2}{2} + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{x^2}{2} + c_1 \, dx$$
$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

Will add steps showing solving for IC soon.

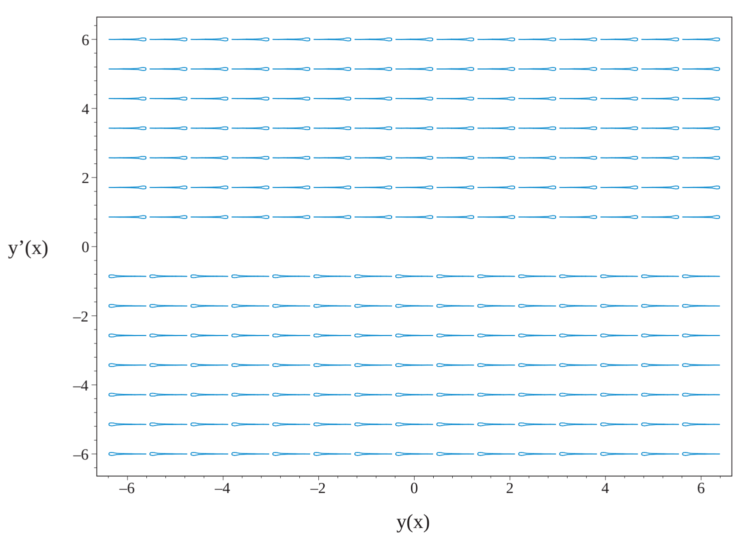


Figure 2.51: Slope field plot
$y'' = x$

Solved as second order ode using Kovacic algorithm

Time used: 0.136 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.13: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x^3 + A_1x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_2 + 2A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x + c_1) + \left(\frac{x^3}{6}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{6}x^3 + c_2x + c_1$$

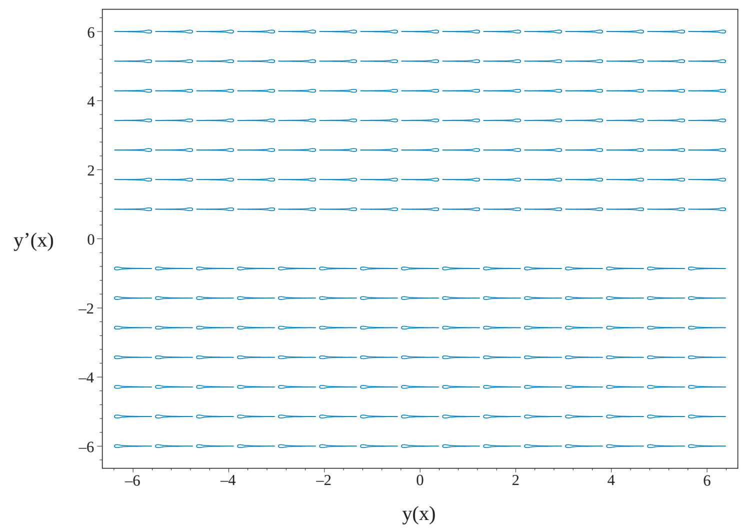


Figure 2.52: Slope field plot
$y'' = x$

Solved as second order ode adjoint method

Time used: 0.383 (sec)

In normal form the ode

$$y'' = x \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ r(x) &= x \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (0)' + (0) &= 0 \\ \xi''(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Integrating twice gives the solution

$$\xi = c_1 x + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)} \end{aligned}$$

Or

$$y' - \frac{y c_1}{c_1 x + c_2} = \frac{\frac{1}{3} c_1\,x^3 + \frac{1}{2} c_2\,x^2}{c_1 x + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x) y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{c_1}{c_1 x + c_2} \\
p(x) &= \frac{2c_1 x^3 + 3c_2 x^2}{6c_1 x + 6c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= \mathrm{e}^{\int -\frac{c_1}{c_1 x + c_2} dx} \\
&= \frac{1}{c_1 x + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{2c_1 x^3 + 3c_2 x^2}{6c_1 x + 6c_2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 x + c_2}\right) &= \left(\frac{1}{c_1 x + c_2}\right)\left(\frac{2c_1 x^3 + 3c_2 x^2}{6c_1 x + 6c_2}\right) \\
\mathrm{d}\left(\frac{y}{c_1 x + c_2}\right) &= \left(\frac{2c_1 x^3 + 3c_2 x^2}{(6c_1 x + 6c_2)(c_1 x + c_2)}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 x + c_2} &= \int \frac{2c_1 x^3 + 3c_2 x^2}{(6c_1 x + 6c_2)(c_1 x + c_2)} \, dx \\
&= \frac{c_1 x^2 - c_2 x}{6c_1^2} - \frac{c_2^3}{6c_1^3 (c_1 x + c_2)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 x + c_2}$ gives the final solution

$$
y = \frac{6c_1^4 c_3 x + (x^3 + 6c_2 c_3) c_1^3 - c_1 c_2^2 x - c_2^3}{6c_1^3}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{6c_1^4 c_3 x + (x^3 + 6c_2 c_3) c_1^3 - c_1 c_2^2 x - c_2^3}{6c_1^3}
$$

The constants can be merged to give

$$
y = \frac{6c_1^4 x + (x^3 + 6c_2) c_1^3 - c_1 c_2^2 x - c_2^3}{6c_1^3}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{6c_1^4 x + (x^3 + 6c_2) c_1^3 - c_1 c_2^2 x - c_2^3}{6c_1^3}
$$

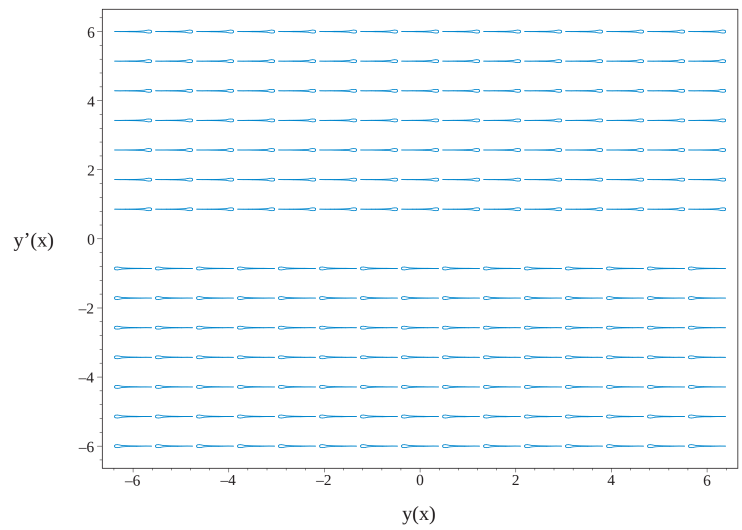


Figure 2.53: Slope field plot
$y'' = x$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = x$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y(x) = C1y_1(x) + C2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 + C2x + y_p(x)$

- □ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\left(\int x^2 dx\right) + \left(\int x dx\right) x$

  - Compute integrals

    $y_p(x) = \frac{x^3}{6}$

- Substitute particular solution into general solution to ODE

$$y(x) = \mathit{C1} + \mathit{C2}x + \tfrac{1}{6}x^3$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x) = x,y(x),singsol=all)
```

$$y = \frac{1}{6}x^3 + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : `0.003` (sec)
Leaf size : 19

```
DSolve[{D[y[x],{x,2}]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^3}{6} + c_2 x + c_1$$

2.1.10 Problem 10

Internal problem ID [9081]
Book : Second order enumerated odes
Section : section 1
Problem number : 10
Date solved : `Monday, January 27, 2025 at 05:32:13 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y''^2 = x$$

Solved as second order missing y ode

Time used: 0.244 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x)^2 - x = 0$$

Which is now solved for $u(x)$ as first order ode.

Solving for the derivative gives these ODE's to solve

$$u'(x) = \sqrt{x} \tag{1}$$
$$u'(x) = -\sqrt{x} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int \sqrt{x}\, dx$$
$$u(x) = \frac{2x^{3/2}}{3} + c_1$$

Solving Eq. (2)

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int -\sqrt{x}\, dx$$
$$u(x) = -\frac{2x^{3/2}}{3} + c_2$$

In summary, these are the solution found for $u(x)$

$$u(x) = -\frac{2x^{3/2}}{3} + c_2$$
$$u(x) = \frac{2x^{3/2}}{3} + c_1$$

For solution $u(x) = -\frac{2x^{3/2}}{3} + c_2$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{2x^{3/2}}{3} + c_2$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -\frac{2x^{3/2}}{3} + c_2\, dx$$
$$y = c_2 x - \frac{4x^{5/2}}{15} + c_3$$

For solution $u(x) = \frac{2x^{3/2}}{3} + c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{2x^{3/2}}{3} + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{2x^{3/2}}{3} + c_1\, dx$$
$$y = c_1 x + \frac{4x^{5/2}}{15} + c_4$$

In summary, these are the solution found for (y)

$$y = c_2 x - \frac{4x^{5/2}}{15} + c_3$$
$$y = c_1 x + \frac{4x^{5/2}}{15} + c_4$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + \frac{4x^{5/2}}{15} + c_4$$
$$y = c_2 x - \frac{4x^{5/2}}{15} + c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resultin
   *** Sublevel 2 ***
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   <- quadrature successful
```

```
--------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   <- quadrature successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)^2 = x,y(x),singsol=all)
```

$$y = \frac{4x^{5/2}}{15} + c_1 x + c_2$$
$$y = -\frac{4x^{5/2}}{15} + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.007 (sec)
Leaf size : 41

```
DSolve[{(D[y[x],{x,2}])^2==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{4x^{5/2}}{15} + c_2 x + c_1$$
$$y(x) \to \frac{4x^{5/2}}{15} + c_2 x + c_1$$

2.1.11 Problem 11

Internal problem ID [9082]
Book : Second order enumerated odes
Section : section 1
Problem number : 11
Date solved : `Monday, January 27, 2025 at 05:32:14 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y''^3 = 0$$

Solved as second order missing x ode

Time used: 0.229 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^3 \left(\frac{d}{dy}p(y)\right)^3 = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p^3 = 0 \tag{1}$$

$$p'^3 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p^3 = 0$$

Solving gives $p = 0$

Solving equation (2)

Solving for the derivative gives these ODE's to solve

$$p' = 0 \tag{1}$$
$$p' = 0 \tag{2}$$
$$p' = 0 \tag{3}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\,dy + c_1$$
$$p = c_1$$

Solving Eq. (2)

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\,dy + c_2$$
$$p = c_2$$

Solving Eq. (3)

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\,dy + c_3$$
$$p = c_3$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_4$$
$$y = c_4$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\,dx$$
$$y = c_1 x + c_5$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_2$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_2 \, dx$$
$$y = c_2 x + c_6$$

For solution (4) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_3$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_3 \, dx$$
$$y = c_3 x + c_7$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_4$$
$$y = c_1 x + c_5$$
$$y = c_2 x + c_6$$
$$y = c_3 x + c_7$$

Solved as second order missing y ode

Time used: 0.229 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x)^3 = 0$$

Which is now solved for $u(x)$ as first order ode.

Solving for the derivative gives these ODE's to solve

$$u'(x) = 0 \tag{1}$$
$$u'(x) = 0 \tag{2}$$
$$u'(x) = 0 \tag{3}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0 \, dx + c_1$$
$$u(x) = c_1$$

Solving Eq. (2)

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\, dx + c_2$$
$$u(x) = c_2$$

Solving Eq. (3)

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int 0\, dx + c_3$$
$$u(x) = c_3$$

In summary, these are the solution found for $u(x)$

$$u(x) = c_1$$
$$u(x) = c_2$$
$$u(x) = c_3$$

For solution $u(x) = c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\, dx$$
$$y = c_1 x + c_4$$

For solution $u(x) = c_2$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_2$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_2\, dx$$
$$y = c_2 x + c_5$$

For solution $u(x) = c_3$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = c_3$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_3\, dx$$
$$y = c_3 x + c_6$$

In summary, these are the solution found for (y)

$$y = c_1 x + c_4$$
$$y = c_2 x + c_5$$
$$y = c_3 x + c_6$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_4$$
$$y = c_2 x + c_5$$
$$y = c_3 x + c_6$$

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)^3 = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y(x) = C1y_1(x) + C2y_2(x)$

- Substitute in solutions

  $y(x) = C2x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 9

```
dsolve(diff(diff(y(x),x),x)^3 = 0,y(x),singsol=all)
```

$$y = c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
DSolve[{(D[y[x],{x,2}])^3==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2x + c_1$$

2.1.12 Problem 12

Internal problem ID [9083]
Book : Second order enumerated odes
Section : section 1
Problem number : 12
Date solved : `Monday, January 27, 2025 at 05:32:15 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y' = 0$$

Solved as second order linear constant coeff ode

Time used: 0.066 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \lambda \,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(0)}\\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 + c_2 \,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 + c_2 \,\mathrm{e}^{-x}$$

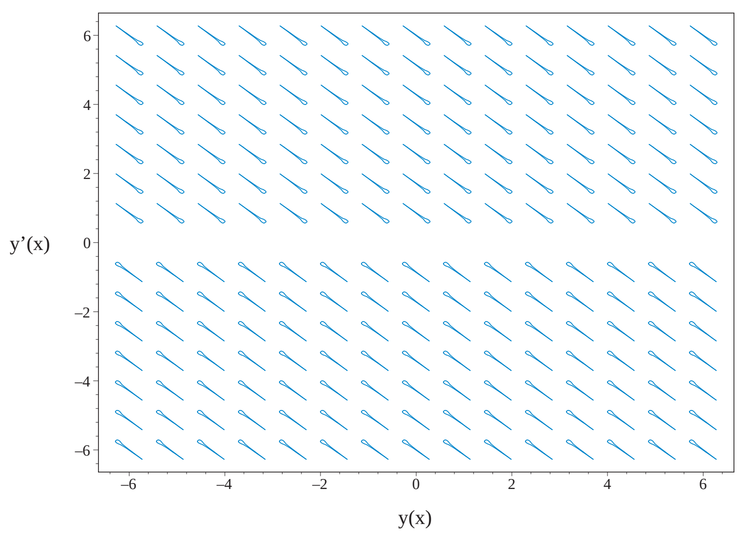


Figure 2.54: Slope field plot
$y'' + y' = 0$

Solved as second order linear exact ode

Time used: 0.142 (sec)

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$p(x) = 1$$
$$q(x) = 1$$
$$r(x) = 0$$
$$s(x) = 0$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\,dx$$

Substituting the above values for p,q,r,s gives

$$y'+y=c_1$$

We now have a first order ode to solve which is

$$y'+y=c_1$$

Integrating gives

$$\int \frac{1}{-y+c_1}dy=dx$$
$$-\ln(-y+c_1)=x+c_2$$

Singular solutions are found by solving

$$-y+c_1=0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y=c_1$$

Solving for y gives

$$y=c_1$$
$$y=-\mathrm{e}^{-x-c_2}+c_1$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y=c_1$$
$$y=-\mathrm{e}^{-x-c_2}+c_1$$

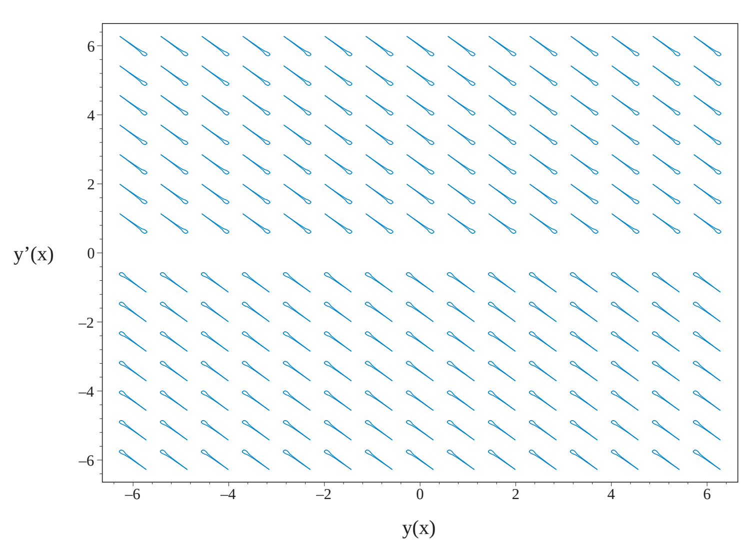


Figure 2.55: Slope field plot
$y''+y'=0$

Solved as second order missing y ode

Time used: 0.113 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) = 0$$

Which is now solved for $u(x)$ as first order ode.

Integrating gives

$$\int -\frac{1}{u} du = dx$$
$$-\ln(u) = x + c_1$$

Singular solutions are found by solving

$$-u = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = e^{-x-c_1}$$

In summary, these are the solution found for $u(x)$

$$u(x) = 0$$
$$u(x) = e^{-x-c_1}$$

For solution $u(x) = 0$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_2$$
$$y = c_2$$

For solution $u(x) = e^{-x-c_1}$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = e^{-x-c_1}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int e^{-x-c_1}\, dx$$
$$y = -e^{-x-c_1} + c_3$$

In summary, these are the solution found for (y)

$$y = c_2$$
$$y = -e^{-x-c_1} + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2$$
$$y = -e^{-x-c_1} + c_3$$

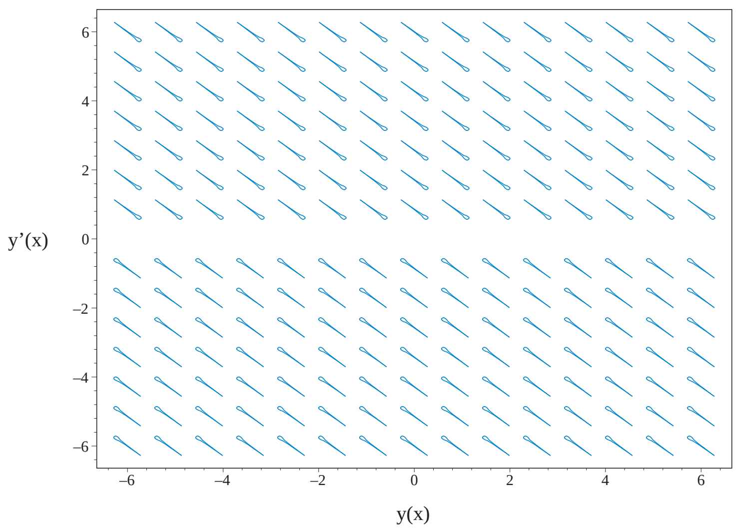


Figure 2.56: Slope field plot
$y'' + y' = 0$

Solved as second order integrable as is ode

Time used: 0.056 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = 0$$
$$y' + y = c_1$$

Which is now solved for y. Integrating gives

$$\int \frac{1}{-y + c_1} dy = dx$$
$$-\ln(-y + c_1) = x + c_2$$

Singular solutions are found by solving

$$-y + c_1 = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = c_1$$

Solving for y gives

$$y = c_1$$
$$y = -e^{-x-c_2} + c_1$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1$$

$$y = -\mathrm{e}^{-x-c_2} + c_1$$

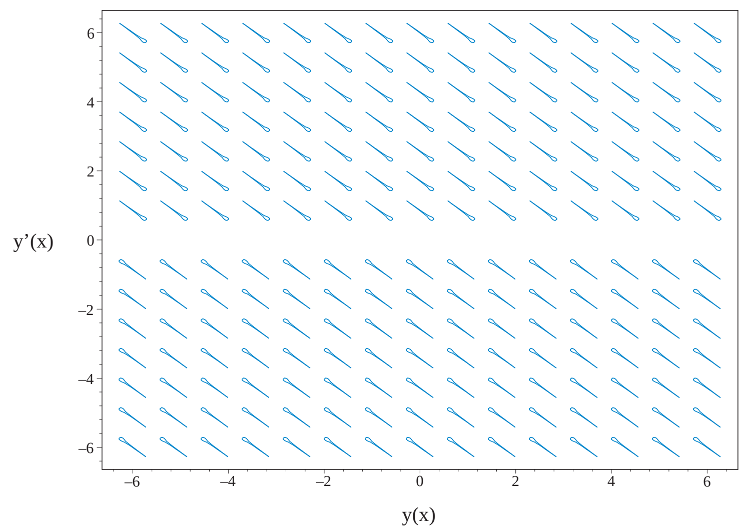


Figure 2.57: Slope field plot
$y'' + y' = 0$

Solved as second order integrable as is ode (ABC method)

Time used: 0.125 (sec)

Writing the ode as

$$y'' + y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = 0$$

$$y' + y = c_1$$

Which is now solved for y. Integrating gives

$$\int \frac{1}{-y + c_1} dy = dx$$

$$-\ln(-y + c_1) = x + c_2$$

Singular solutions are found by solving

$$-y + c_1 = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = c_1$$

Solving for y gives

$$y = c_1$$

$$y = -\mathrm{e}^{-x-c_2} + c_1$$

Will add steps showing solving for IC soon.

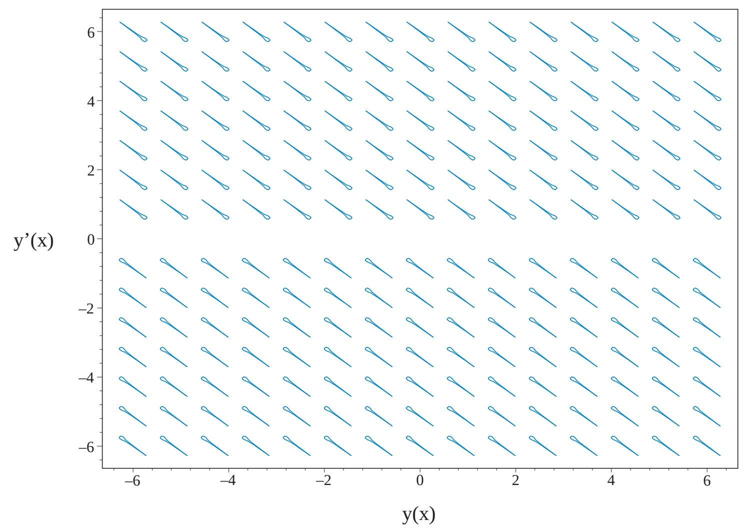


Figure 2.58: Slope field plot
$y'' + y' = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.040 (sec)

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.16: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{e}^{-x} + c_2$$

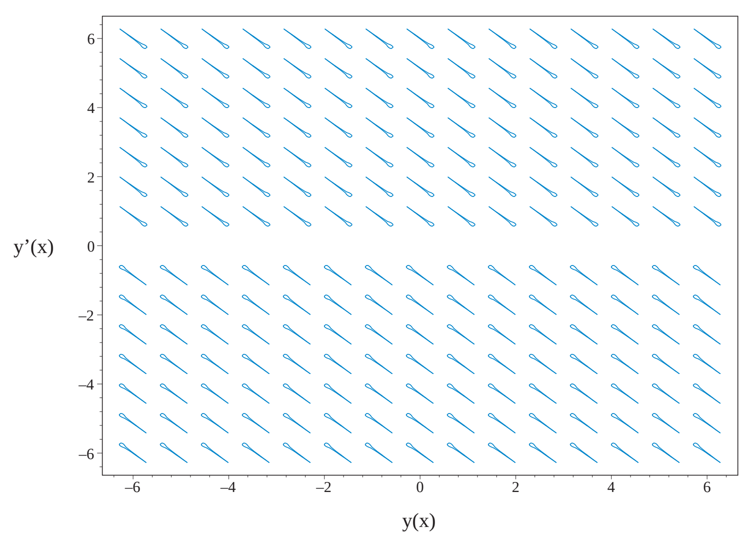


Figure 2.59: Slope field plot
$y'' + y' = 0$

Solved as second order ode adjoint method

Time used: 0.400 (sec)

In normal form the ode

$$y'' + y' = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (0) &= 0 \\ \xi''(x) - \xi'(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} - \lambda e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(0)} \\ &= \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$\xi = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$\xi = c_1 e^{(1)x} + c_2 e^{(0)x}$$

Or

$$\xi = c_1 e^x + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' + y\left(1 - \frac{c_1 e^x}{c_1 e^x + c_2}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{c_2}{c_1\,\mathrm{e}^{x}+c_2} \\
p(x) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int \frac{c_2}{c_1\,\mathrm{e}^{x}+c_2}dx} \\
&= \frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x}+c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x}+c_2}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x}+c_2} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x}+c_2}$ gives the final solution

$$
y = c_3\left(c_1 + c_2\,\mathrm{e}^{-x}\right)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = c_3\left(c_1 + c_2\,\mathrm{e}^{-x}\right)
$$

The constants can be merged to give

$$
y = c_1 + c_2\,\mathrm{e}^{-x}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1 + c_2\,\mathrm{e}^{-x}
$$

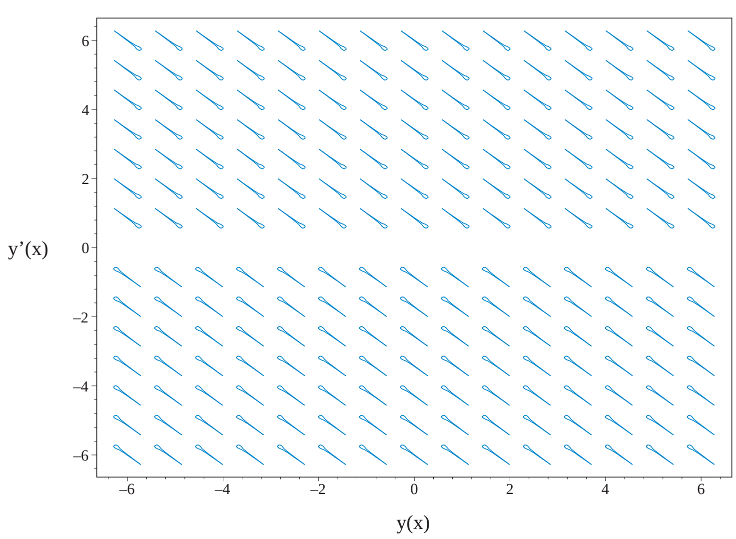


Figure 2.60: Slope field plot
$y'' + y' = 0$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of ODE

  $r^2 + r = 0$
- Factor the characteristic polynomial

  $r(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 0)$
- 1st solution of the ODE

  $y_1(x) = e^{-x}$
- 2nd solution of the ODE

  $y_2(x) = 1$
- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x)$
- Substitute in solutions

  $y(x) = C1\, e^{-x} + C2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 12

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = 0,y(x),singsol=all)
```

$$y = c_1 + c_2 e^{-x}$$

Mathematica DSolve solution

Solving time : `0.01` (sec)
Leaf size : 17

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 - c_1 e^{-x}$$

2.1.13 Problem 13

Internal problem ID [9084]
Book : Second order enumerated odes
Section : section 1
Problem number : 13
Date solved : `Monday, January 27, 2025 at 05:32:17 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y''^2 + y' = 0$$

Solved as second order missing x ode

Time used: 2.523 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y) = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p = 0 \tag{1}$$

$$p'^2 p + 1 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p = 0$$

Solving gives $p = 0$

Solving equation (2)

Let $p = p'$ the ode becomes

$$p^2 p + 1 = 0$$

Solving for p from the above results in

$$p = -\frac{1}{p^2} \tag{1}$$

This has the form

$$p = yf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = p'(y)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. y gives

$$\begin{aligned} p &= f + (yf' + g')\frac{dp}{dy} \\ p - f &= (yf' + g')\frac{dp}{dy} \end{aligned} \tag{2}$$

Comparing the form $p = yf + g$ to (1A) shows that

$$\begin{aligned} f &= 0 \\ g &= -\frac{1}{p^2} \end{aligned}$$

Hence (2) becomes

$$p = \frac{2p'(y)}{p^3} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dy} = 0$ in the above which gives

$$p = 0$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}y} \neq 0$. From eq. (2A). This results in

$$p'(y) = \frac{p(y)^4}{2} \tag{3}$$

This ODE is now solved for $p(y)$. No inversion is needed.

Integrating gives

$$\begin{aligned} \int \frac{2}{p^4} dp &= dy \\ -\frac{2}{3p^3} &= y + c_1 \end{aligned}$$

Singular solutions are found by solving

$$\frac{p^4}{2} = 0$$

for $p(y)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(y) = 0$$

Solving for $p(y)$ gives

$$
\begin{aligned}
p(y) &= 0 \\
p(y) &= \frac{\left(-18(y+c_1)^2\right)^{1/3}}{3c_1+3y} \\
p(y) &= -\frac{\left(-18(y+c_1)^2\right)^{1/3}}{6\,(y+c_1)} - \frac{i\sqrt{3}\left(-18(y+c_1)^2\right)^{1/3}}{6\,(y+c_1)} \\
p(y) &= -\frac{\left(-18(y+c_1)^2\right)^{1/3}}{6\,(y+c_1)} + \frac{i\sqrt{3}\left(-18(y+c_1)^2\right)^{1/3}}{6y+6c_1}
\end{aligned}
$$

Substituing the above solution for p in (2A) gives

$$
\begin{aligned}
p &= -\frac{9(y+c_1)^2}{\left(-18\,(y+c_1)^2\right)^{2/3}} \\
p &= -\frac{18(y+c_1)^2}{\left(-18\,(y+c_1)^2\right)^{2/3}\left(i\sqrt{3}-1\right)} \\
p &= \frac{18(y+c_1)^2}{\left(-18\,(y+c_1)^2\right)^{2/3}\left(1+i\sqrt{3}\right)}
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = 0
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int 0\,dx + c_2 \\
y &= c_2
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = -\frac{9(y+c_1)^2}{\left(-18\,(y+c_1)^2\right)^{2/3}}
$$

Integrating gives

$$
\begin{aligned}
\int -\frac{\left(-18(y+c_1)^2\right)^{2/3}}{9\,(y+c_1)^2} dy &= dx \\
-\frac{18^{2/3}\left(-(y+c_1)^2\right)^{2/3}}{3c_1+3y} &= x + c_3
\end{aligned}
$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = \frac{18(y+c_1)^2}{\left(-18\,(y+c_1)^2\right)^{2/3}\left(1+i\sqrt{3}\right)}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{\left(-18(y+c_1)^2\right)^{2/3}\left(1+i\sqrt{3}\right)}{18\,(y+c_1)^2} dy &= dx \\
\frac{18^{2/3}\left(-(y+c_1)^2\right)^{2/3}\left(1+i\sqrt{3}\right)}{6y+6c_1} &= x + c_4
\end{aligned}
$$

Solving for y gives

$$y = -\frac{1}{12}c_4^3 - \frac{1}{4}c_4^2 x - \frac{1}{4}c_4\, x^2 - \frac{1}{12}x^3 - c_1$$

For solution (4) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{18(y+c_1)^2}{\left(-18\,(y+c_1)^2\right)^{2/3}\left(i\sqrt{3}-1\right)}$$

Integrating gives

$$\int -\frac{\left(-18(y+c_1)^2\right)^{2/3}\left(i\sqrt{3}-1\right)}{18\,(y+c_1)^2}dy = dx$$

$$-\frac{18^{2/3}\left(-(y+c_1)^2\right)^{2/3}\left(i\sqrt{3}-1\right)}{6y+6c_1} = x + c_5$$

Will add steps showing solving for IC soon.

Solving for y from the above solution(s) gives (after possible removing of solutions that do not verify)

$$\begin{aligned}
y &= c_2\\
y &= -\frac{1}{12}c_3^3 - \frac{1}{4}x\, c_3^2 - \frac{1}{4}c_3\, x^2 - \frac{1}{12}x^3 - c_1\\
y &= -\frac{1}{12}c_4^3 - \frac{1}{4}c_4^2 x - \frac{1}{4}c_4\, x^2 - \frac{1}{12}x^3 - c_1\\
y &= -\frac{1}{12}c_5^3 - \frac{1}{4}c_5^2 x - \frac{1}{4}c_5\, x^2 - \frac{1}{12}x^3 - c_1
\end{aligned}$$

Summary of solutions found

$$y = c_2$$

$$y = -\frac{1}{12}c_3^3 - \frac{1}{4}x\, c_3^2 - \frac{1}{4}c_3\, x^2 - \frac{1}{12}x^3 - c_1$$

$$y = -\frac{1}{12}c_4^3 - \frac{1}{4}c_4^2 x - \frac{1}{4}c_4\, x^2 - \frac{1}{12}x^3 - c_1$$

$$y = -\frac{1}{12}c_5^3 - \frac{1}{4}c_5^2 x - \frac{1}{4}c_5\, x^2 - \frac{1}{12}x^3 - c_1$$

Solved as second order missing y ode

Time used: 0.494 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x)^2 + u(x) = 0$$

Which is now solved for $u(x)$ as first order ode.

Solving for the derivative gives these ODE's to solve

$$u'(x) = \sqrt{-u(x)} \tag{1}$$

$$u'(x) = -\sqrt{-u(x)} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$
\begin{aligned}
\int \frac{1}{\sqrt{-u}} du &= dx \\
-2\sqrt{-u} &= x + c_1
\end{aligned}
$$

Singular solutions are found by solving

$$
\sqrt{-u} = 0
$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
u(x) = 0
$$

Solving Eq. (2)

Integrating gives

$$
\begin{aligned}
\int -\frac{1}{\sqrt{-u}} du &= dx \\
2\sqrt{-u} &= x + c_2
\end{aligned}
$$

Singular solutions are found by solving

$$
-\sqrt{-u} = 0
$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
u(x) = 0
$$

In summary, these are the solution found for $u(x)$

$$
\begin{aligned}
-2\sqrt{-u(x)} &= x + c_1 \\
2\sqrt{-u(x)} &= x + c_2 \\
u(x) &= 0
\end{aligned}
$$

For solution $-2\sqrt{-u(x)} = x + c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$
-2\sqrt{-y'} = x + c_1
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int -\frac{1}{4}x^2 - \frac{1}{2}c_1 x - \frac{1}{4}c_1^2 \, dx \\
y &= -\frac{(x + c_1)^3}{12} + c_3
\end{aligned}
$$

For solution $2\sqrt{-u(x)} = x + c_2$, since $u = y'$ then we now have a new first order ode to solve which is

$$
2\sqrt{-y'} = x + c_2
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -\frac{1}{4}x^2 - \frac{1}{2}c_2x - \frac{1}{4}c_2^2\,dx$$

$$y = -\frac{(x+c_2)^3}{12} + c_4$$

For solution $u(x) = 0$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_5$$

$$y = c_5$$

In summary, these are the solution found for (y)

$$y = -\frac{(x+c_1)^3}{12} + c_3$$

$$y = -\frac{(x+c_2)^3}{12} + c_4$$

$$y = c_5$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_5$$

$$y = -\frac{(x+c_1)^3}{12} + c_3$$

$$y = -\frac{(x+c_2)^3}{12} + c_4$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resul
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      -> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)+1/2, y(x)`
         Methods for third order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         <- quadrature successful
      <- 2nd order ODE linearizable_by_differentiation successful
```

```
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful`
```

Maple dsolve solution

Solving time : 0.196 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)^2+diff(y(x),x) = 0,y(x),singsol=all)
```

$$
\begin{aligned}
y &= c_1 \\
y &= -\frac{1}{12}x^3 + \frac{1}{2}c_1 x^2 - c_1^2 x + c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 69

```
DSolve[{(D[y[x],{x,2}])^2+D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to -\frac{x^3}{12} - \frac{1}{4}ic_1 x^2 + \frac{c_1{}^2 x}{4} + c_2 \\
y(x) &\to -\frac{x^3}{12} + \frac{1}{4}ic_1 x^2 + \frac{c_1{}^2 x}{4} + c_2
\end{aligned}
$$

2.1.14 Problem 14

Internal problem ID [9085]
Book : Second order enumerated odes
Section : section 1
Problem number : 14
Date solved : `Monday, January 27, 2025 at 05:32:21 PM`
CAS classification :
`[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible, _mu_xy]]`

Solve

$$y'' + y'^2 = 0$$

Solved as second order missing x ode

Time used: 0.197 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p = 0 \tag{1}$$

$$p' + p = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p = 0$$

Solving gives $p = 0$

Solving equation (2)

Integrating gives

$$\int -\frac{1}{p}dp = dy$$
$$-\ln(p) = y + c_1$$

Singular solutions are found by solving

$$-p = 0$$

for p. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p = 0$$

Solving for p gives

$$p = 0$$
$$p = e^{-y-c_1}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_2$$
$$y = c_2$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = e^{-y-c_1}$$

Integrating gives

$$\int e^{y+c_1}dy = dx$$
$$e^{y+c_1} = x + c_3$$

Solving for y gives

$$y = -c_1 + \ln(x + c_3)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2$$
$$y = -c_1 + \ln(x + c_3)$$

Solved as second order missing y ode

Time used: 0.076 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x)^2 = 0$$

Which is now solved for $u(x)$ as first order ode.

Integrating gives

$$\int -\frac{1}{u^2} du = dx$$
$$\frac{1}{u} = x + c_1$$

Singular solutions are found by solving

$$-u^2 = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = \frac{1}{x + c_1}$$

In summary, these are the solution found for $u(x)$

$$u(x) = 0$$
$$u(x) = \frac{1}{x + c_1}$$

For solution $u(x) = 0$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_2$$
$$y = c_2$$

For solution $u(x) = \frac{1}{x+c_1}$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{1}{x + c_1}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{1}{x + c_1}\, dx$$
$$y = \ln(x + c_1) + c_3$$

In summary, these are the solution found for (y)

$$y = c_2$$
$$y = \ln(x + c_1) + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2$$
$$y = \ln(x + c_1) + c_3$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 + \frac{d^2}{dx^2}y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Make substitution $u = \frac{d}{dx}y(x)$ to reduce order of ODE
  $u(x)^2 + \frac{d}{dx}u(x) = 0$
- Solve for the highest derivative
  $\frac{d}{dx}u(x) = -u(x)^2$
- Separate variables
  $\frac{\frac{d}{dx}u(x)}{u(x)^2} = -1$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}u(x)}{u(x)^2} dx = \int (-1)\, dx + C1$
- Evaluate integral
  $-\frac{1}{u(x)} = C1 - x$
- Solve for $u(x)$
  $u(x) = -\frac{1}{C1 - x}$
- Solve 1st ODE for $u(x)$
  $u(x) = -\frac{1}{C1 - x}$
- Make substitution $u = \frac{d}{dx}y(x)$
  $\frac{d}{dx}y(x) = -\frac{1}{C1 - x}$
- Integrate both sides to solve for $y(x)$
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int -\frac{1}{C1 - x} dx + C2$
- Compute integrals
  $y(x) = \ln(C1 - x) + C2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 10

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)^2 = 0,y(x),singsol=all)
```

$$y = \ln(c_1 x + c_2)$$

Mathematica DSolve solution

Solving time : 0.191 (sec)
Leaf size : 15

```
DSolve[{D[y[x],{x,2}]+(D[y[x],x])^2==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \log(x - c_1) + c_2$$

2.1.15 Problem 15

Internal problem ID [9086]
Book : Second order enumerated odes
Section : section 1
Problem number : 15
Date solved : `Monday, January 27, 2025 at 05:32:21 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y' = 1$$

Solved as second order linear constant coeff ode

Time used: 0.108 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\lambda_{1,2} = \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)}$$
$$= -\frac{1}{2} \pm \frac{1}{2}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 + c_2\,\mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2\,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 \,\mathrm{e}^{-x}\right) + (x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x + c_1 + c_2 \,\mathrm{e}^{-x}$$

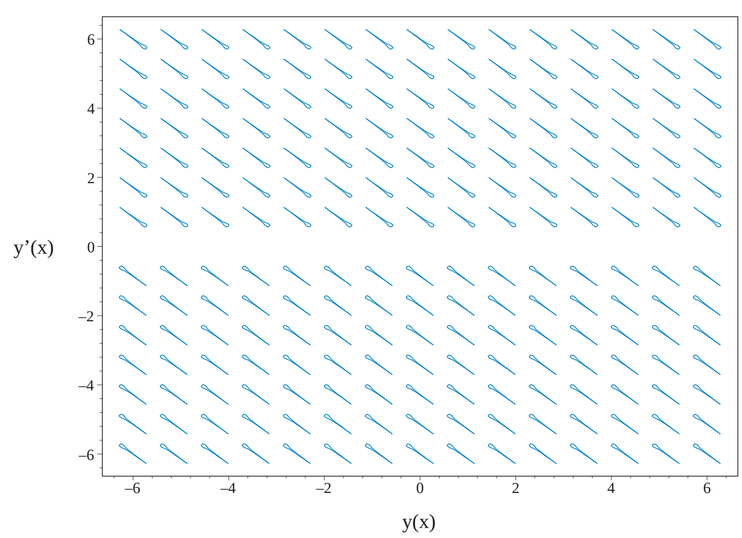


Figure 2.61: Slope field plot
$y'' + y' = 1$

Solved as second order linear exact ode

Time used: 0.109 (sec)

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$y'+y=\int 1\,dx$$

We now have a first order ode to solve which is

$$y'+y=x+c_1$$

In canonical form a linear first order is

$$y'+q(x)y=p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x)&=1\\ p(x)&=x+c_1\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int q\,dx}\\ &=\mathrm{e}^{\int 1dx}\\ &=\mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=\mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\,(x+c_1)\\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x})&=(\mathrm{e}^{x})\,(x+c_1)\\ \mathrm{d}(y\,\mathrm{e}^{x})&=((x+c_1)\,\mathrm{e}^{x})\;\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{x}&=\int(x+c_1)\,\mathrm{e}^{x}\,dx\\ &=(x+c_1-1)\,\mathrm{e}^{x}+c_2\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y=c_1+x-1+c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y=c_1+x-1+c_2\,\mathrm{e}^{-x}$$

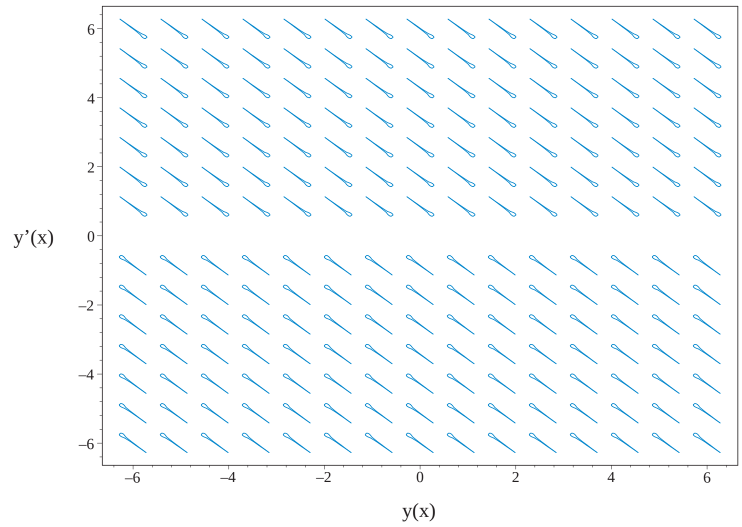


Figure 2.62: Slope field plot
$y'' + y' = 1$

Solved as second order missing y ode

Time used: 0.161 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) - 1 = 0$$

Which is now solved for $u(x)$ as first order ode.

Integrating gives

$$\int \frac{1}{1-u} du = dx$$

$$-\ln(-1+u) = x + c_1$$

Singular solutions are found by solving

$$1 - u = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$u(x) = 1$$

Solving for $u(x)$ gives

$$u(x) = 1$$

$$u(x) = \mathrm{e}^{-x-c_1} + 1$$

In summary, these are the solution found for $u(x)$

$$u(x) = 1$$

$$u(x) = \mathrm{e}^{-x-c_1} + 1$$

For solution $u(x) = 1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = 1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 1\, dx$$
$$y = x + c_2$$

For solution $u(x) = \mathrm{e}^{-x-c_1} + 1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = \mathrm{e}^{-x-c_1} + 1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \mathrm{e}^{-x-c_1} + 1\, dx$$
$$y = x - \mathrm{e}^{-x-c_1} + c_3$$

In summary, these are the solution found for (y)

$$y = x + c_2$$
$$y = x - \mathrm{e}^{-x-c_1} + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x + c_2$$
$$y = x - \mathrm{e}^{-x-c_1} + c_3$$

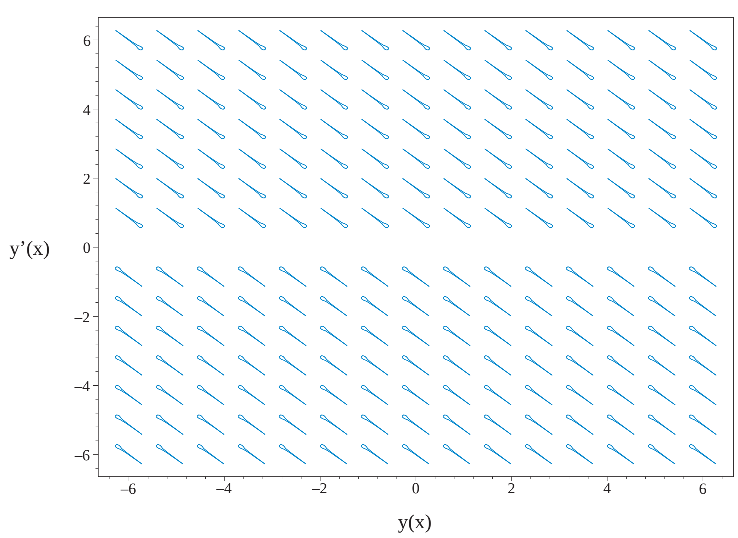


Figure 2.63: Slope field plot
$y'' + y' = 1$

Solved as second order integrable as is ode

Time used: 0.128 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int 1 dx$$
$$y' + y = x + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= \mathrm{e}^{\int 1 dx} \\ &= \mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x + c_1) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})(x + c_1) \\ \mathrm{d}(y\,\mathrm{e}^{x}) &= ((x + c_1)\,\mathrm{e}^{x})\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{x} &= \int (x + c_1)\,\mathrm{e}^{x}\, dx \\ &= (x + c_1 - 1)\,\mathrm{e}^{x} + c_2\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = c_1 + x - 1 + c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 + x - 1 + c_2\,\mathrm{e}^{-x}$$

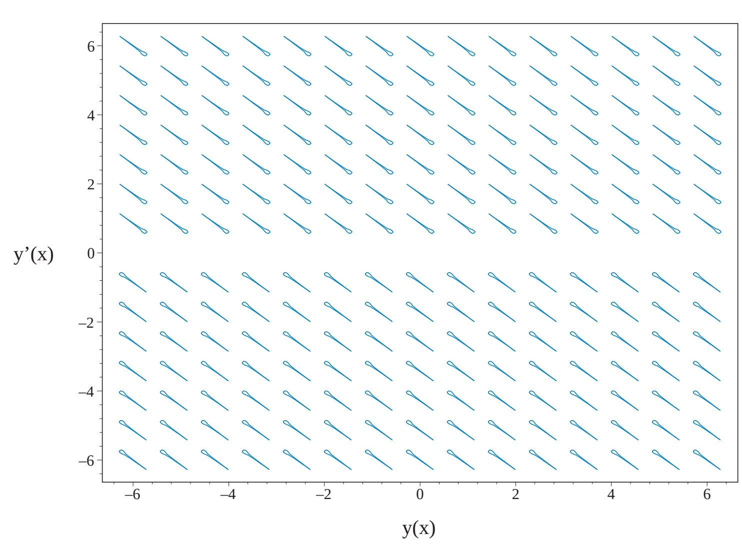


Figure 2.64: Slope field plot
$y'' + y' = 1$

Solved as second order integrable as is ode (ABC method)

Time used: 0.050 (sec)

Writing the ode as

$$y'' + y' = 1$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int (y'' + y')\, dx &= \int 1 dx \\ y' + y &= x + c_1\end{aligned}$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= 1 \\ p(x) &= x + c_1\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= \mathrm{e}^{\int 1 dx} \\ &= \mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(x + c_1) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})\,(x + c_1) \\ \mathrm{d}(y\,\mathrm{e}^{x}) &= ((x + c_1)\,\mathrm{e}^{x})\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{x} &= \int (x + c_1)\,\mathrm{e}^{x}\, dx \\ &= (x + c_1 - 1)\,\mathrm{e}^{x} + c_2\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = c_1 + x - 1 + c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

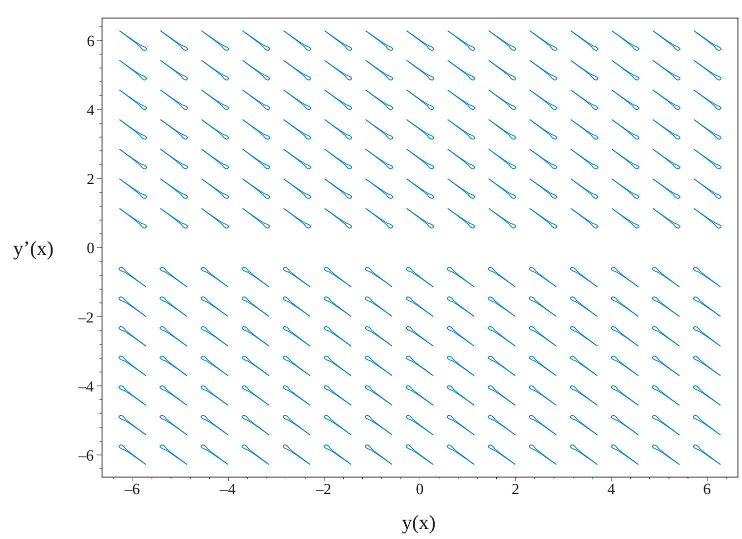


Figure 2.65: Slope field plot
$y'' + y' = 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.062 (sec)

Writing the ode as

$$y'' + y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.19: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2\right) + (x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{-x} + c_2 + x$$

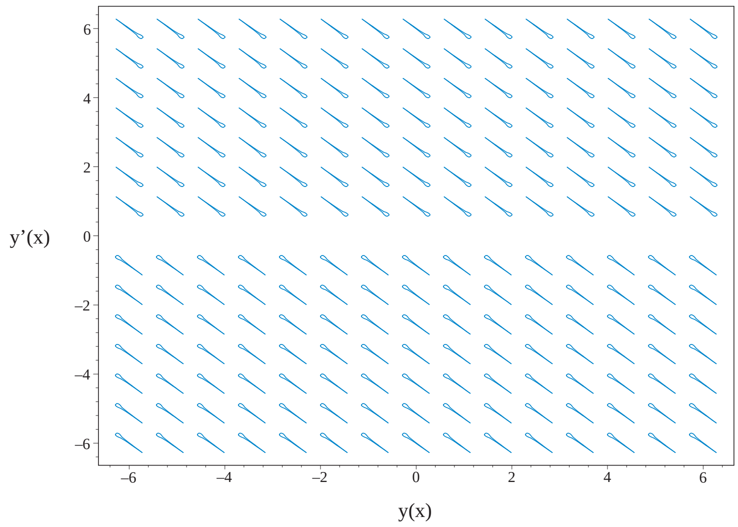


Figure 2.66: Slope field plot
$y'' + y' = 1$

Solved as second order ode adjoint method

Time used: 0.516 (sec)

In normal form the ode

$$y'' + y' = 1 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (0) &= 0 \\ \xi''(x) - \xi'(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(0)}\\ &= \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ \xi &= c_1 e^{(1)x} + c_2 e^{(0)x}\end{aligned}$$

Or

$$\xi = c_1 \, \mathrm{e}^{x} + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' + y\left(1 - \frac{c_1 \, \mathrm{e}^{x}}{c_1 \, \mathrm{e}^{x} + c_2}\right) = \frac{c_2 x + c_1 \, \mathrm{e}^{x}}{c_1 \, \mathrm{e}^{x} + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= \frac{c_2}{c_1 \, \mathrm{e}^{x} + c_2}\\ p(x) &= \frac{c_2 x + c_1 \, \mathrm{e}^{x}}{c_1 \, \mathrm{e}^{x} + c_2}\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int q\, dx}\\ &= \mathrm{e}^{\int \frac{c_2}{c_1 \, \mathrm{e}^{x} + c_2} dx}\\ &= \frac{\mathrm{e}^{x}}{c_1 \, \mathrm{e}^{x} + c_2}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_2 x + c_1\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right) = \left(\frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right)\left(\frac{c_2 x + c_1\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right)$$

$$\mathrm{d}\left(\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right) = \left(\frac{(c_2 x + c_1\,\mathrm{e}^{x})\,\mathrm{e}^{x}}{(c_1\,\mathrm{e}^{x} + c_2)^2}\right)\mathrm{d}x$$

Integrating gives

$$\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2} = \int \frac{(c_2 x + c_1\,\mathrm{e}^{x})\,\mathrm{e}^{x}}{(c_1\,\mathrm{e}^{x} + c_2)^2}\,dx$$
$$= \frac{c_2}{c_1\,(c_1\,\mathrm{e}^{x} + c_2)} + \frac{x\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2} + c_3$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x}+c_2}$ gives the final solution

$$y = \frac{c_2(c_3c_1 + 1)\,\mathrm{e}^{-x} + c_1(c_3c_1 + x)}{c_1}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{c_2(c_3c_1 + 1)\,\mathrm{e}^{-x} + c_1(c_3c_1 + x)}{c_1}$$

The constants can be merged to give

$$y = \frac{c_2(c_1 + 1)\,\mathrm{e}^{-x} + c_1(x + c_1)}{c_1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_2(c_1 + 1)\,\mathrm{e}^{-x} + c_1(x + c_1)}{c_1}$$

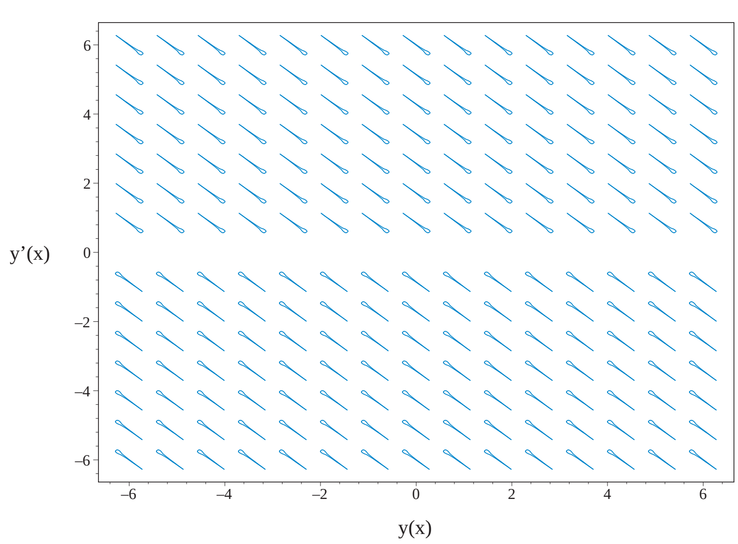


Figure 2.67: Slope field plot
$y'' + y' = 1$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$
- Factor the characteristic polynomial

  $r(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 0)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$
- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\, e^{-x} + C2 + y_p(x)$
- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 1\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{-x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -e^{-x}\left(\int e^{x}dx\right) + \int 1dx$
  - Compute integrals

    $y_p(x) = x - 1$
- Substitute particular solution into general solution to ODE

  $y(x) = C1\, e^{-x} + C2 + x - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = 1,y(x),singsol=all)
```

$$y = -c_1 e^{-x} + x + c_2$$

Mathematica DSolve solution

Solving time : 0.012 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==1,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x - c_1 e^{-x} + c_2$$

2.1.16 Problem 16

Internal problem ID [9087]
Book : Second order enumerated odes
Section : section 1
Problem number : 16
Date solved : `Monday, January 27, 2025 at 05:32:24 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y''^2 + y' = 1$$

Solved as second order missing x ode

Time used: 276.831 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y) = 1$$

Which is now solved as first order ode for $p(y)$.

Let $p = p'$ the ode becomes

$$p^2p^2 + p = 1$$

Solving for p from the above results in

$$p = \frac{-1 + \sqrt{4p^2 + 1}}{2p^2} \tag{1}$$

$$p = -\frac{1 + \sqrt{4p^2 + 1}}{2p^2} \tag{2}$$

This has the form

$$p = yf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = p'(y)$. Each of the above ode's is dAlembert ode which is now solved.

Solving ode 1A

Taking derivative of (*) w.r.t. y gives

$$\begin{aligned} p &= f + (yf' + g')\frac{dp}{dy} \\ p - f &= (yf' + g')\frac{dp}{dy} \end{aligned} \tag{2}$$

Comparing the form $p = yf + g$ to (1A) shows that

$$\begin{aligned} f &= 0 \\ g &= \frac{-1 + \sqrt{4p^2 + 1}}{2p^2} \end{aligned}$$

Hence (2) becomes

$$p = \left(\frac{1}{p^3} - \frac{\sqrt{4p^2 + 1}}{p^3} + \frac{2}{p\sqrt{4p^2 + 1}}\right) p'(y) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dy} = 0$ in the above which gives

$$p = 0$$

Solving the above for p results in

$$p_1 = 0$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$p = 1$$

The general solution is found when $\frac{dp}{dy} \neq 0$. From eq. (2A). This results in

$$p'(y) = \frac{p(y)}{\frac{1}{p(y)^3} - \frac{\sqrt{4p(y)^2+1}}{p(y)^3} + \frac{2}{p(y)\sqrt{4p(y)^2+1}}} \tag{3}$$

This ODE is now solved for $p(y)$. No inversion is needed.

Integrating gives

$$\begin{aligned} \int -\frac{2p^2 - \sqrt{4p^2 + 1} + 1}{p^4\sqrt{4p^2 + 1}} dp &= dy \\ \frac{-2\sqrt{4p^2 + 1}\, p^2 + \sqrt{4p^2 + 1} - 1}{3p^3} &= y + c_1 \end{aligned}$$

Singular solutions are found by solving

$$-\frac{p^4\sqrt{4p^2 + 1}}{2p^2 - \sqrt{4p^2 + 1} + 1} = 0$$

for $p(y)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$\begin{aligned} p(y) &= -\frac{i}{2} \\ p(y) &= \frac{i}{2} \end{aligned}$$

Substituing the above solution for p in (2A) gives

$$p = \frac{-1+\sqrt{4\left(\frac{\left(\left(-3y-3c_1+\sqrt{\frac{81c_1^4+324c_1^3y+486c_1^2y^2+324c_1\,y^3+81y^4-144c_1^2-288c_1y-144y^2+64}{9c_1^2+18c_1y+9y^2-16}}\right)\left(9c_1^2+18c_1y+9y^2-16\right)^2\right)^{1/3}}{9c_1^2+18c_1y+9y^2-16}-\frac{\cdots}{\left(\left(-3y\cdots\right.\right.}\cdots}}{2\left(\frac{\left(\left(-3y-3c_1+\sqrt{\frac{81c_1^4+324c_1^3y+486c_1^2y^2+324c_1\,y^3+81y^4-144c_1^2-288c_1y-144y^2+64}{9c_1^2+18c_1y+9y^2-16}}\right)\left(9c_1^2+18c_1y+9y^2-16\right)^2\right)^{1/3}}{9c_1^2+18c_1y+9y^2-16}-\frac{\cdots}{\left(\left(-3y-3c\cdots\right.\right.}\cdots}$$

Solving ode 2A

Taking derivative of (*) w.r.t. y gives

$$\begin{aligned} p &= f+(yf'+g')\frac{dp}{dy} \\ p-f &= (yf'+g')\frac{dp}{dy} \end{aligned} \tag{2}$$

Comparing the form $p=yf+g$ to (1A) shows that

$$\begin{aligned} f &= 0 \\ g &= \frac{-1-\sqrt{4p^2+1}}{2p^2} \end{aligned}$$

Hence (2) becomes

$$p = \left(-\frac{2}{p\sqrt{4p^2+1}}+\frac{1}{p^3}+\frac{\sqrt{4p^2+1}}{p^3}\right)p'(y) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dy}=0$ in the above which gives

$$p = 0$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}y}\neq 0$. From eq. (2A). This results in

$$p'(y) = \frac{p(y)}{-\frac{2}{p(y)\sqrt{4p(y)^2+1}}+\frac{1}{p(y)^3}+\frac{\sqrt{4p(y)^2+1}}{p(y)^3}} \tag{3}$$

This ODE is now solved for $p(y)$. No inversion is needed.

Integrating gives

$$\begin{aligned} \int \frac{2p^2+\sqrt{4p^2+1}+1}{p^4\sqrt{4p^2+1}}dp &= dy \\ \frac{2\sqrt{4p^2+1}\,p^2-\sqrt{4p^2+1}-1}{3p^3} &= y+c_2 \end{aligned}$$

Singular solutions are found by solving

$$\frac{p^4\sqrt{4p^2+1}}{2p^2+\sqrt{4p^2+1}+1} = 0$$

for $p(y)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
\begin{aligned}
p(y) &= 0 \\
p(y) &= -\frac{i}{2} \\
p(y) &= \frac{i}{2}
\end{aligned}
$$

Substituing the above solution for p in (2A) gives

$$
p = \frac{-1-\sqrt{4\left(\frac{\left(\left(-3y-3c_2+\sqrt{\frac{81c_2^4+324c_2^3y+486c_2^2y^2+324c_2\,y^3+81y^4-144c_2^2-288c_2y-144y^2+64}{9c_2^2+18c_2y+9y^2-16}}\right)(9c_2^2+18c_2y+9y^2-16)^2\right)^{1/3}}{9c_2^2+18c_2y+9y^2-16}-\frac{\cdots}{\left(\left(-3y-3c_2\cdots\right.\right.}\cdots\right.}}{2\left(\frac{\left(\left(-3y-3c_2+\sqrt{\frac{81c_2^4+324c_2^3y+486c_2^2y^2+324c_2\,y^3+81y^4-144c_2^2-288c_2y-144y^2+64}{9c_2^2+18c_2y+9y^2-16}}\right)(9c_2^2+18c_2y+9y^2-16)^2\right)^{1/3}}{9c_2^2+18c_2y+9y^2-16}-\frac{\cdots}{\left(\left(-3y-3c_2+\cdots\right.\right.}\cdots\right.}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = 1
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int 1\, dx \\
y &= x + c_3
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = 2
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int 2\, dx \\
y &= 2x + c_4
\end{aligned}
$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = \frac{-1-\sqrt{4\left(\frac{\left(\left(-3y-3c_2+\sqrt{\frac{81c_2^4+324c_2^3y+486c_2^2y^2+324c_2y^3+81y^4-144c_2^2-288c_2y-144y^2+64}{9c_2^2+18c_2y+9y^2-16}}\right)(9c_2^2+18c_2y+9y^2-16)^2\right)^{1/3}}{9c_2^2+18c_2y+9y^2-16}-\frac{\cdots}{\left(\left(-3y-3c\cdots\right.\right.}\cdots\right.}}{2\left(\frac{\left(\left(-3y-3c_2+\sqrt{\frac{81c_2^4+324c_2^3y+486c_2^2y^2+324c_2y^3+81y^4-144c_2^2-288c_2y-144y^2+64}{9c_2^2+18c_2y+9y^2-16}}\right)(9c_2^2+18c_2y+9y^2-16)^2\right)^{1/3}}{9c_2^2+18c_2y+9y^2-16}-\frac{\cdots}{\left(\left(-3y-3c_2+\cdots\right.\right.}\cdots\right.}
$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y -\frac{2\left(36\tau^2+7\cdots\right.}{\left(1+\sqrt{4\left(\frac{\left(\left(-3\tau-3c_2+\sqrt{\frac{81c_2^4+324c_2^3\tau+486c_2^2\tau^2+324c_2\,\tau^3+81\tau^4-144c_2^2-288c_2\tau-144\tau^2+64}{9c_2^2+18c_2\tau+9\tau^2-16}}\right)(9c_2^2+18c_2\tau+9\tau^2-16)^2\right)^{1/3}}{9c_2^2+18c_2\tau+9\tau^2-16}-\frac{\cdots}{\left(\left(-\cdots\right.\right.}\right.}\right.}$$

For solution (4) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{-1+\sqrt{4\left(\frac{\left(\left(-3y-3c_1+\sqrt{\frac{81c_1^4+324c_1^3y+486c_1^2y^2+324c_1y^3+81y^4-144c_1^2-288c_1y-144y^2+64}{9c_1^2+18c_1y+9y^2-16}}\right)(9c_1^2+18c_1y+9y^2-16)^2\right)^{1/3}}{9c_1^2+18c_1y+9y^2-16}-\frac{\cdots}{\left(\left(-3y\cdots\right.\right.}\right.}}{2\left(\frac{\left(\left(-3y-3c_1+\sqrt{\frac{81c_1^4+324c_1^3y+486c_1^2y^2+324c_1y^3+81y^4-144c_1^2-288c_1y-144y^2+64}{9c_1^2+18c_1y+9y^2-16}}\right)(9c_1^2+18c_1y+9y^2-16)^2\right)^{1/3}}{9c_1^2+18c_1y+9y^2-16}-\frac{\cdots}{\left(\left(-3y-3\cdots\right.\right.}\right.}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{2\left(36\tau^2+72\cdots\right.}{(9c_1^2+18c_1\tau+9\tau^2-16)^2\left(\left(-3\tau-3c_1+\sqrt{\frac{81c_1^4+324c_1^3\tau+486c_1^2\tau^2+324c_1\,\tau^3+81\tau^4-144c_1^2-288c_1\tau-144\tau^2+64}{9c_1^2+18c_1\tau+9\tau^2-16}}\right)(9c_1^2\cdots\right.\right.}$$

Will add steps showing solving for IC soon.

The solution

$$\int^y -\frac{2\left(36\tau^2+72c_2\cdots\right.}{\left(1+\sqrt{4\left(\frac{\left(\left(-3\tau-3c_2+\sqrt{\frac{81c_2^4+324c_2^3\tau+486c_2^2\tau^2+324c_2\,\tau^3+81\tau^4-144c_2^2-288c_2\tau-144\tau^2+64}{9c_2^2+18c_2\tau+9\tau^2-16}}\right)(9c_2^2+18c_2\tau+9\tau^2-16)^2\right)^{1/3}}{9c_2^2+18c_2\tau+9\tau^2-16}-\frac{\cdots}{\left(\left(-3\tau-3\cdots\right.\right.}\right.}\right.} + c_5$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$\int^y \frac{2\left(36\tau^2+72\cdots\right.}{(9c_1^2+18c_1\tau+9\tau^2-16)^2\left(\left(-3\tau-3c_1+\sqrt{\frac{81c_1^4+324c_1^3\tau+486c_1^2\tau^2+324c_1\,\tau^3+81\tau^4-144c_1^2-288c_1\tau-144\tau^2+64}{9c_1^2+18c_1\tau+9\tau^2-16}}\right)(9c_1^2\cdots\right.\right.} + c_6$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = 2x + c_4$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = x + c_3$$

Solved as second order missing y ode

Time used: 3.957 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x)^2 + u(x) - 1 = 0$$

Which is now solved for $u(x)$ as first order ode.

Solving for the derivative gives these ODE's to solve

$$u'(x) = \sqrt{1 - u(x)} \tag{1}$$

$$u'(x) = -\sqrt{1 - u(x)} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{1-u}} du = dx$$

$$-2\sqrt{1-u} = x + c_1$$

Singular solutions are found by solving

$$\sqrt{1-u} = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$u(x) = 1$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{1-u}} du = dx$$

$$2\sqrt{1-u} = x + c_2$$

Singular solutions are found by solving

$$-\sqrt{1-u} = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$u(x) = 1$$

In summary, these are the solution found for $u(x)$

$$
\begin{aligned}
-2\sqrt{1-u(x)} &= x+c_1 \\
2\sqrt{1-u(x)} &= x+c_2 \\
u(x) &= 1
\end{aligned}
$$

For solution $-2\sqrt{1-u(x)} = x+c_1$, since $u = y'$ then we now have a new first order ode to solve which is

$$
-2\sqrt{1-y'} = x+c_1
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int -\frac{1}{4}c_1^2 - \frac{1}{2}c_1 x - \frac{1}{4}x^2 + 1\,dx \\
y &= -\frac{x^3}{12} - \frac{c_1\,x^2}{4} - \frac{(c_1+2)(c_1-2)\,x}{4} + c_3 \\
y &= -\frac{1}{12}x^3 - \frac{1}{4}c_1\,x^2 - \frac{1}{4}c_1^2 x + x + c_3
\end{aligned}
$$

For solution $2\sqrt{1-u(x)} = x+c_2$, since $u = y'$ then we now have a new first order ode to solve which is

$$
2\sqrt{1-y'} = x+c_2
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int -\frac{1}{4}c_2^2 - \frac{1}{2}c_2 x - \frac{1}{4}x^2 + 1\,dx \\
y &= -\frac{x^3}{12} - \frac{c_2\,x^2}{4} - \frac{(c_2+2)(c_2-2)\,x}{4} + c_4 \\
y &= -\frac{1}{12}x^3 - \frac{1}{4}c_2\,x^2 - \frac{1}{4}c_2^2 x + x + c_4
\end{aligned}
$$

For solution $u(x) = 1$, since $u = y'$ then we now have a new first order ode to solve which is

$$
y' = 1
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int 1\,dx \\
y &= x + c_5
\end{aligned}
$$

In summary, these are the solution found for (y)

$$
\begin{aligned}
y &= -\frac{1}{12}x^3 - \frac{1}{4}c_1\,x^2 - \frac{1}{4}c_1^2 x + x + c_3 \\
y &= -\frac{1}{12}x^3 - \frac{1}{4}c_2\,x^2 - \frac{1}{4}c_2^2 x + x + c_4 \\
y &= x + c_5
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= x + c_5 \\
y &= -\frac{1}{12}x^3 - \frac{1}{4}c_1\,x^2 - \frac{1}{4}c_1^2 x + x + c_3 \\
y &= -\frac{1}{12}x^3 - \frac{1}{4}c_2\,x^2 - \frac{1}{4}c_2^2 x + x + c_4
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resul
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
-> Calling odsolve with the ODE`, diff(y(x), x) = 1, y(x), singsol = none`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.195 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x)^2+diff(y(x),x) = 1,y(x),singsol=all)
```

$$y = x + c_1$$
$$y = -\frac{1}{12}x^3 + \frac{1}{2}c_1x^2 - c_1^2x + x + c_2$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 67

```
DSolve[{(D[y[x],{x,2}])^2+D[y[x],x]==1,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{x^3}{12} - \frac{c_1 x^2}{4} + x - \frac{{c_1}^2 x}{4} + c_2$$
$$y(x) \to -\frac{x^3}{12} + \frac{c_1 x^2}{4} + x - \frac{{c_1}^2 x}{4} + c_2$$

2.1.17 Problem 17

Internal problem ID [9088]
Book : Second order enumerated odes
Section : section 1
Problem number : 17
Date solved : `Monday, January 27, 2025 at 05:37:05 PM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]`

Solve

$$y'' + y'^2 = 1$$

Solved as second order missing x ode

Time used: 6.675 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = 1$$

Which is now solved as first order ode for $p(y)$.

Integrating gives

$$\begin{aligned} \int -\frac{p}{p^2-1}dp &= dy \\ -\frac{\ln(p^2-1)}{2} &= y + c_1 \end{aligned}$$

Applying the exponential to both sides gives

$$\begin{aligned} e^{\ln\left(\frac{1}{\sqrt{p^2-1}}\right)} &= e^{y+c_1} \\ \frac{1}{\sqrt{p^2-1}} &= c_1 \,\mathrm{e}^{y} \end{aligned}$$

Singular solutions are found by solving

$$-\frac{p^2-1}{p} = 0$$

for p. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p = -1$$
$$p = 1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{1}{\sqrt{y'^2 - 1}} = c_1 \, e^{y}$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{\sqrt{1 + c_1^2 e^{2y}} \, e^{-y}}{c_1} \tag{1}$$

$$y' = -\frac{\sqrt{1 + c_1^2 e^{2y}} \, e^{-y}}{c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{e^{y} c_1}{\sqrt{1 + e^{2y} c_1^2}} dy = dx$$

$$\frac{c_1 \ln \left(\frac{c_1^2 e^{y}}{\sqrt{c_1^2}} + \sqrt{1 + e^{2y} c_1^2}\right)}{\sqrt{c_1^2}} = x + c_2$$

Singular solutions are found by solving

$$\frac{\sqrt{1 + e^{2y} c_1^2} \, e^{-y}}{c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{\ln \left(-\frac{1}{c_1^2}\right)}{2}$$

Solving for y gives

$$y = \frac{\ln \left(-\frac{1}{c_1^2}\right)}{2}$$

$$y = -x - c_2 + \frac{\ln \left(\frac{\left(e^{2x + 2c_2} - 1\right)^2}{4c_1^2}\right)}{2}$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{e^{y} c_1}{\sqrt{1 + e^{2y} c_1^2}} dy = dx$$

$$-\frac{c_1 \ln \left(\frac{c_1^2 e^{y}}{\sqrt{c_1^2}} + \sqrt{1 + e^{2y} c_1^2}\right)}{\sqrt{c_1^2}} = x + c_3$$

Singular solutions are found by solving

$$-\frac{\sqrt{1+e^{2y}c_1^2}\,e^{-y}}{c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{\ln\left(-\frac{1}{c_1^2}\right)}{2}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -1\, dx$$
$$y = -x + c_4$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 1\, dx$$
$$y = x + c_5$$

Will add steps showing solving for IC soon.

Solving for y from the above solution(s) gives (after possible removing of solutions that do not verify)

$$y = \frac{\ln\left(-\frac{1}{c_1^2}\right)}{2}$$
$$y = -x + c_4$$
$$y = x + c_5$$
$$y = -x - c_2 + \frac{\ln\left(\frac{\left(e^{2x+2c_2}-1\right)^2}{4c_1^2}\right)}{2}$$
$$y = -x - c_3 + \frac{\ln\left(\frac{\left(e^{2x+2c_3}-1\right)^2}{4c_1^2}\right)}{2}$$

The solution

$$y = \frac{\ln\left(-\frac{1}{c_1^2}\right)}{2}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = -x + c_4$$
$$y = x + c_5$$
$$y = -x - c_2 + \frac{\ln\left(\frac{\left(e^{2x+2c_2}-1\right)^2}{4c_1^2}\right)}{2}$$
$$y = -x - c_3 + \frac{\ln\left(\frac{\left(e^{2x+2c_3}-1\right)^2}{4c_1^2}\right)}{2}$$

Solved as second order missing y ode

Time used: 0.259 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x)^2 - 1 = 0$$

Which is now solved for $u(x)$ as first order ode.

Integrating gives

$$\int \frac{1}{-u^2+1} du = dx$$
$$\operatorname{arctanh}(u) = x + c_1$$

Singular solutions are found by solving

$$-u^2 + 1 = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$u(x) = -1$$
$$u(x) = 1$$

Solving for $u(x)$ gives

$$u(x) = -1$$
$$u(x) = 1$$
$$u(x) = \tanh(x + c_1)$$

In summary, these are the solution found for $u(x)$

$$u(x) = -1$$
$$u(x) = 1$$
$$u(x) = \tanh(x + c_1)$$

For solution $u(x) = -1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = -1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -1\, dx$$
$$y = -x + c_2$$

For solution $u(x) = 1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = 1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 1\, dx$$
$$y = x + c_3$$

For solution $u(x) = \tanh(x + c_1)$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = \tanh(x + c_1)$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \tanh(x + c_1)\; dx$$
$$y = \ln(\cosh(x + c_1)) + c_4$$

In summary, these are the solution found for (y)

$$y = -x + c_2$$
$$y = x + c_3$$
$$y = \ln(\cosh(x + c_1)) + c_4$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -x + c_2$$
$$y = x + c_3$$
$$y = \ln(\cosh(x + c_1)) + c_4$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 + \frac{d^2}{dx^2}y(x) = 1$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Make substitution $u = \frac{d}{dx}y(x)$ to reduce order of ODE
  $u(x)^2 + \frac{d}{dx}u(x) = 1$
- Solve for the highest derivative
  $\frac{d}{dx}u(x) = -u(x)^2 + 1$
- Separate variables
  $\frac{\frac{d}{dx}u(x)}{-u(x)^2+1} = 1$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}u(x)}{-u(x)^2+1}dx = \int 1dx + C1$
- Evaluate integral
  $\mathrm{arctanh}(u(x)) = x + C1$
- Solve for $u(x)$
  $u(x) = \tanh(x + C1)$
- Solve 1st ODE for $u(x)$
  $u(x) = \tanh(x + C1)$

- Make substitution $u = \frac{d}{dx}y(x)$

  $\frac{d}{dx}y(x) = \tanh\left(x + \mathit{C1}\right)$
- Integrate both sides to solve for $y(x)$

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int \tanh\left(x + \mathit{C1}\right) dx + \mathit{C2}$
- Compute integrals

  $y(x) = \ln\left(\cosh\left(x + \mathit{C1}\right)\right) + \mathit{C2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 21

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)^2 = 1,y(x),singsol=all)
```

$$y = x - \ln\left(2\right) + \ln\left(c_1 \mathrm{e}^{-2x} - c_2\right)$$

Mathematica DSolve solution

Solving time : 0.354 (sec)
Leaf size : 48

```
DSolve[{D[y[x],{x,2}]+(D[y[x],x])^2==1,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{2}\log\left(e^{2x}\right) + \log\left(e^{2x} + e^{2c_1}\right) + c_2$$
$$y(x) \to \frac{1}{2}\log\left(e^{2x}\right) + c_2$$

2.1.18 Problem 18

Internal problem ID [9089]
Book : Second order enumerated odes
Section : section 1
Problem number : 18
Date solved : `Monday, January 27, 2025 at 05:37:13 PM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$y'' + y' = x$$

Solved as second order linear constant coeff ode

Time used: 0.123 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)}\\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2\,\mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2\,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2\,\mathrm{e}^{-x}\right) + \left(\frac{1}{2}x^2 - x\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} - x + c_1 + c_2\,\mathrm{e}^{-x}$$

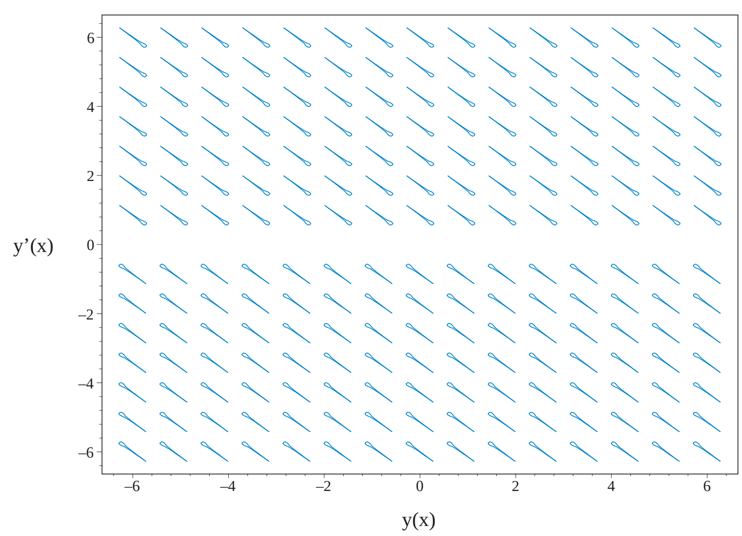


Figure 2.68: Slope field plot
$y'' + y' = x$

Solved as second order linear exact ode

Time used: 0.112 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int x\, dx$$

We now have a first order ode to solve which is

$$y' + y = \frac{x^2}{2} + c_1$$

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= 1 \\ p(x) &= \frac{x^2}{2} + c_1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^2}{2} + c_1\right) \\ \frac{d}{dx}(y\, e^x) &= (e^x)\left(\frac{x^2}{2} + c_1\right) \\ d(y\, e^x) &= \left(\left(\frac{x^2}{2} + c_1\right) e^x\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\, e^x &= \int \left(\frac{x^2}{2} + c_1\right) e^x\, dx \\ &= \frac{(x^2 + 2c_1 - 2x + 2)\, e^x}{2} + c_2 \end{aligned}$$

Dividing throughout by the integrating factor e^x gives the final solution

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 \, e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 \, e^{-x}$$

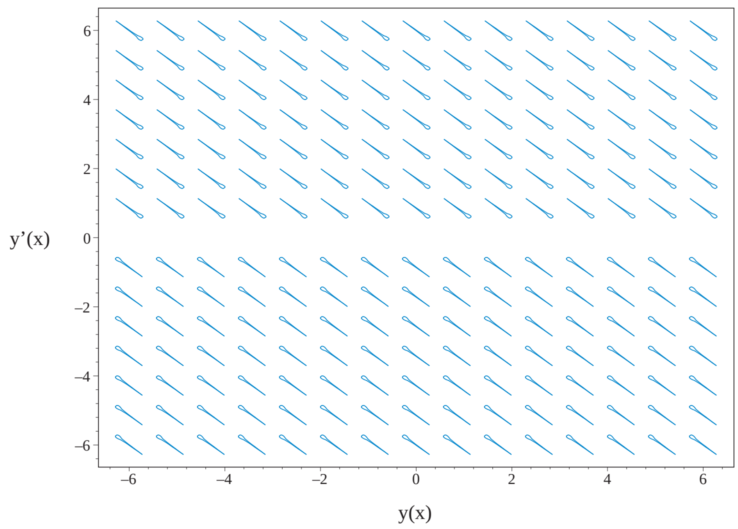


Figure 2.69: Slope field plot
$y'' + y' = x$

Solved as second order missing y ode

Time used: 0.112 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) - x = 0$$

Which is now solved for $u(x)$ as first order ode.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= 1 \\ p(x) &= x \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \, dx} \\ &= e^{\int 1 dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= (\mu)(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}(u\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})(x) \\
\mathrm{d}(u\,\mathrm{e}^{x}) &= (x\,\mathrm{e}^{x})\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
u\,\mathrm{e}^{x} &= \int x\,\mathrm{e}^{x}\,dx \\
&= (x-1)\,\mathrm{e}^{x} + c_1
\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$u(x) = x - 1 + c_1\,\mathrm{e}^{-x}$$

In summary, these are the solution found for $u(x)$

$$u(x) = x - 1 + c_1\,\mathrm{e}^{-x}$$

For solution $u(x) = x - 1 + c_1\,\mathrm{e}^{-x}$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = x - 1 + c_1\,\mathrm{e}^{-x}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned}
\int dy &= \int x - 1 + c_1\,\mathrm{e}^{-x}\,dx \\
y &= -x + \frac{x^2}{2} - c_1\,\mathrm{e}^{-x} + c_2
\end{aligned}$$

In summary, these are the solution found for (y)

$$y = -x + \frac{x^2}{2} - c_1\,\mathrm{e}^{-x} + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -x + \frac{x^2}{2} - c_1\,\mathrm{e}^{-x} + c_2$$

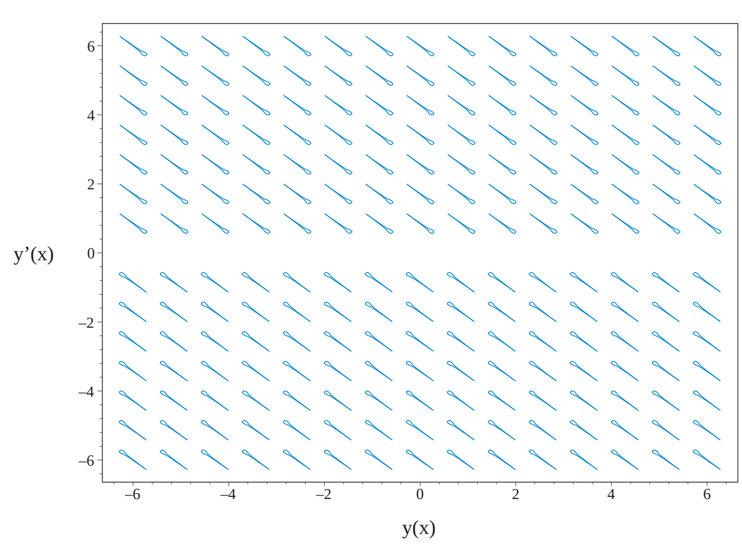


Figure 2.70: Slope field plot
$y'' + y' = x$

Solved as second order integrable as is ode

Time used: 0.147 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int (y''+y')\,dx = \int x dx$$
$$y'+y = \frac{x^2}{2}+c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y'+q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = \frac{x^2}{2}+c_1$$

The integrating factor μ is

$$\mu = e^{\int q\,dx} = e^{\int 1dx} = e^{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2}{2}+c_1\right)$$
$$\frac{d}{dx}(y\,e^{x}) = (e^{x})\left(\frac{x^2}{2}+c_1\right)$$
$$d(y\,e^{x}) = \left(\left(\frac{x^2}{2}+c_1\right)e^{x}\right)\,dx$$

Integrating gives

$$y\,e^{x} = \int \left(\frac{x^2}{2}+c_1\right)e^{x}\,dx = \frac{(x^2+2c_1-2x+2)\,e^{x}}{2}+c_2$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = \frac{x^2}{2}+c_1-x+1+c_2\,e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2}+c_1-x+1+c_2\,e^{-x}$$

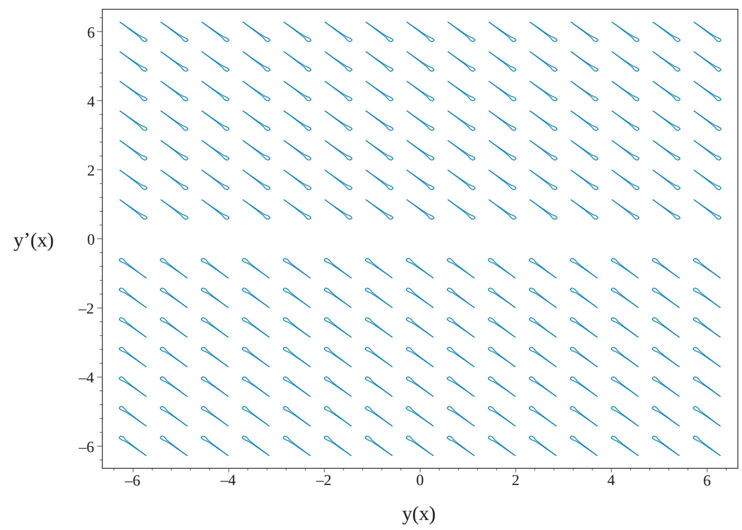


Figure 2.71: Slope field plot
$y'' + y' = x$

Solved as second order integrable as is ode (ABC method)

Time used: 0.051 (sec)

Writing the ode as

$$y'' + y' = x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int x dx$$
$$y' + y = \frac{x^2}{2} + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = \frac{x^2}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= e^{\int 1 dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2}{2} + c_1\right)$$
$$\frac{d}{dx}(y\, e^{x}) = (e^{x})\left(\frac{x^2}{2} + c_1\right)$$
$$d(y\, e^{x}) = \left(\left(\frac{x^2}{2} + c_1\right) e^{x}\right) dx$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{x} &= \int \left(\frac{x^{2}}{2}+c_{1}\right) \mathrm{e}^{x}\,dx \\ &= \frac{(x^{2}+2c_{1}-2x+2)\,\mathrm{e}^{x}}{2} + c_{2} \end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = \frac{x^{2}}{2} + c_{1} - x + 1 + c_{2}\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

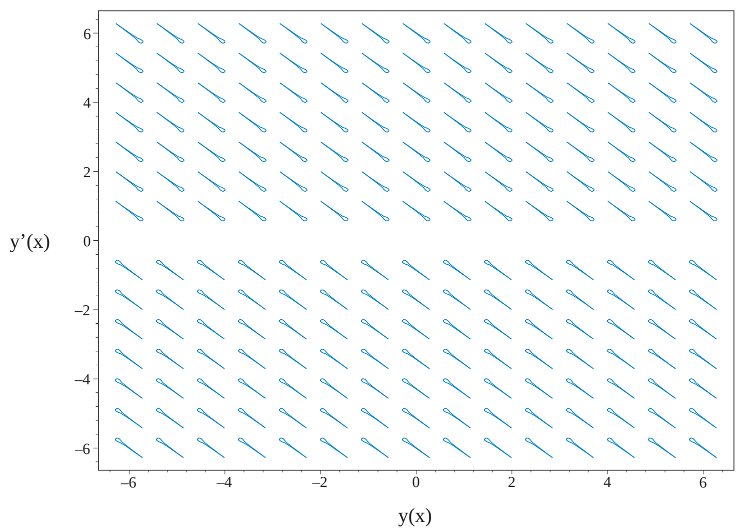


Figure 2.72: Slope field plot
$y'' + y' = x$

Solved as second order ode using Kovacic algorithm

Time used: 0.066 (sec)

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^{2} - 4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.22: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{-x}) + c_2(\mathrm{e}^{-x}(\mathrm{e}^{x})) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\, \mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \,\mathrm{e}^{-x} + c_2\right) + \left(\frac{1}{2}x^2 - x\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{-x} + c_2 + \frac{x^2}{2} - x$$

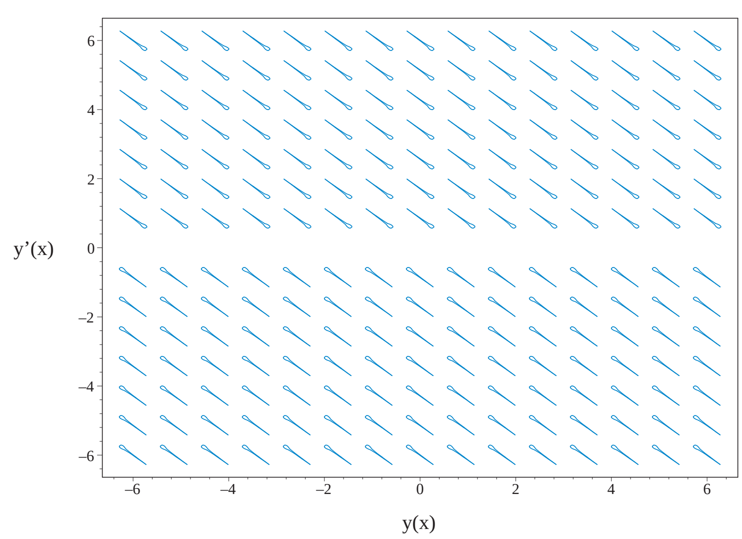


Figure 2.73: Slope field plot
$y'' + y' = x$

Solved as second order ode adjoint method

Time used: 0.546 (sec)

In normal form the ode

$$y'' + y' = x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 0 \\
r(x) &= x
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (\xi(x))' + (0) &= 0 \\
\xi''(x) - \xi'(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
A\xi''(x) + B\xi'(x) + C\xi(x) = 0
$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - \lambda = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)} \\
&= \frac{1}{2} \pm \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{1}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= 0
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
\xi &= c_1 e^{(1)x} + c_2 e^{(0)x}
\end{aligned}
$$

Or

$$
\xi = c_1\, \mathrm{e}^{x} + c_2
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$

$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}$$

Or

$$y' + y\left(1 - \frac{c_1\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right) = \frac{\frac{c_2\,x^2}{2} + c_1(x\,\mathrm{e}^{x} - \mathrm{e}^{x})}{c_1\,\mathrm{e}^{x} + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{c_2}{c_1\,\mathrm{e}^{x} + c_2}$$

$$p(x) = \frac{2c_1(x-1)\,\mathrm{e}^{x} + c_2\,x^2}{2c_1\,\mathrm{e}^{x} + 2c_2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int \frac{c_2}{c_1\,\mathrm{e}^{x}+c_2}dx} \\
&= \frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{2c_1(x-1)\,\mathrm{e}^{x} + c_2\,x^2}{2c_1\,\mathrm{e}^{x} + 2c_2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right) &= \left(\frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right)\left(\frac{2c_1(x-1)\,\mathrm{e}^{x} + c_2\,x^2}{2c_1\,\mathrm{e}^{x} + 2c_2}\right) \\
\mathrm{d}\left(\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right) &= \left(\frac{(2c_1(x-1)\,\mathrm{e}^{x} + c_2\,x^2)\,\mathrm{e}^{x}}{(2c_1\,\mathrm{e}^{x} + 2c_2)\,(c_1\,\mathrm{e}^{x} + c_2)}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2} &= \int \frac{(2c_1(x-1)\,\mathrm{e}^{x} + c_2\,x^2)\,\mathrm{e}^{x}}{(2c_1\,\mathrm{e}^{x} + 2c_2)\,(c_1\,\mathrm{e}^{x} + c_2)}\,dx \\
&= \frac{\mathrm{e}^{x} - x\,\mathrm{e}^{x} + \frac{\mathrm{e}^{x}x^2}{2}}{c_1\,\mathrm{e}^{x} + c_2} + c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x}+c_2}$ gives the final solution

$$y = c_1c_3 + \frac{x^2}{2} - x + c_2c_3\,\mathrm{e}^{-x} + 1$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_1c_3 + \frac{x^2}{2} - x + c_2c_3\,\mathrm{e}^{-x} + 1$$

The constants can be merged to give

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 \,\mathrm{e}^{-x}$$

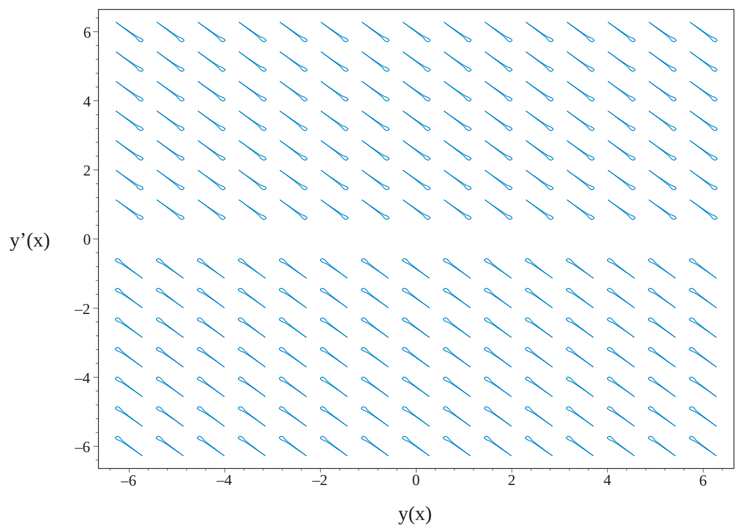


Figure 2.74: Slope field plot
$y'' + y' = x$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = x$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + r = 0$
- Factor the characteristic polynomial
  $r(r+1) = 0$
- Roots of the characteristic polynomial
  $r = (-1, 0)$
- 1st solution of the homogeneous ODE
  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE
  $y_2(x) = 1$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1 \,\mathrm{e}^{-x} + C2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\right]$
  - Wronskian of solutions of the homogeneous equation
    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^{-x}\left(\int \mathrm{e}^{x} x dx\right) + \int x dx$
- Compute integrals

  $y_p(x) = 1 - x + \frac{1}{2}x^2$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\,\mathrm{e}^{-x} + C2 + 1 - x + \frac{x^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+_a, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 21

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = x,y(x),singsol=all)
```

$$y = \frac{x^2}{2} - c_1 \mathrm{e}^{-x} - x + c_2$$

Mathematica DSolve solution

Solving time : 0.043 (sec)
Leaf size : 27

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2}{2} - x - c_1 e^{-x} + c_2$$

2.1.19 Problem 19

Internal problem ID [9090]
Book : Second order enumerated odes
Section : section 1
Problem number : 19
Date solved : `Monday, January 27, 2025 at 05:37:15 PM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$y''^2 + y' = x$$

Solved as second order missing y ode

Time used: 0.581 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x)^2 + u(x) - x = 0$$

Which is now solved for $u(x)$ as first order ode.

Let $p = u'(x)$ the ode becomes

$$p^2 + u - x = 0$$

Solving for $u(x)$ from the above results in

$$u(x) = -p^2 + x \tag{1}$$

This has the form

$$u = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = u'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $u(x) = xf + g$ to (1A) shows that

$$\begin{aligned} f &= 1 \\ g &= -p^2 \end{aligned}$$

Hence (2) becomes

$$p - 1 = -2pp'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 1 = 0$$

Solving the above for p results in

$$p_1 = 1$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$u(x) = x - 1$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x) - 1}{2p(x)} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

Integrating gives

$$\int -\frac{2p}{p-1} dp = dx$$
$$-2p - 2\ln(p-1) = x + c_1$$

Singular solutions are found by solving

$$-\frac{p-1}{2p} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 1$$

Solving for $p(x)$ gives

$$p(x) = 1$$
$$p(x) = \text{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) + 1$$

Substituing the above solution for p in (2A) gives

$$u(x) = x - 1$$
$$u(x) = -\text{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^2 - 2\,\text{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) + x - 1$$

In summary, these are the solution found for $u(x)$

$$u(x) = x - 1$$
$$u(x) = -\text{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^2 - 2\,\text{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) + x - 1$$

For solution $u(x) = x - 1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = x - 1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x - 1\, dx$$
$$y = \frac{1}{2}x^2 - x + c_2$$

For solution $u(x) = -\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^2 - 2\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) + x - 1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = -\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^2 - 2\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) + x - 1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^2 - 2\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) + x - 1\, dx$$
$$y = \frac{2\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^3}{3} + 3\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^2 + 4\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) + \frac{x^2}{2} - x + c_3$$

In summary, these are the solution found for (y)

$$y = \frac{1}{2}x^2 - x + c_2$$
$$y = \frac{2\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^3}{3} + 3\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^2 + 4\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) + \frac{x^2}{2} - x + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}x^2 - x + c_2$$
$$y = \frac{2\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^3}{3} + 3\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right)^2 + 4\operatorname{LambertW}\left(e^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) + \frac{x^2}{2} - x + c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resultin
   *** Sublevel 2 ***
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying 2nd order Liouville
   trying 2nd order WeierstrassP
   trying 2nd order JacobiSN
   differential order: 2; trying a linearization to 3rd order
   trying 2nd order ODE linearizable_by_differentiation
   trying 2nd order, 2 integrating factors of the form mu(x,y)
   trying differential order: 2; missing variables
   `, `-> Computing symmetries using: way = 3
   -> Calling odsolve with the ODE`, diff(_b(_a), _a) = (-_b(_a)+_a)^(1/2), _b(_a), HINT
      symmetry methods on request
   `, `1st order, trying reduction of order with given symmetries:`[1, 1]
```

Maple dsolve solution

Solving time : 0.063 (sec)
Leaf size : 122

```
dsolve(diff(diff(y(x),x),x)^2+diff(y(x),x) = x,y(x),singsol=all)
```

$$y = \int \left(-e^{2 \operatorname{RootOf}\left(\_Z-x-2 e^{\_Z}+2+c_1-\ln\left(e^{\_Z}\left(e^{\_Z}-2\right)^2\right)\right)} + 2 e^{\operatorname{RootOf}\left(\_Z-x-2 e^{\_Z}+2+c_1-\ln\left(e^{\_Z}\left(e^{\_Z}-2\right)^2\right)\right)} + x\right) dx - x + c_2$$

$$y = \frac{2 \operatorname{LambertW}\left(-c_1 e^{-\frac{x}{2}-1}\right)^3}{3} + 3 \operatorname{LambertW}\left(-c_1 e^{-\frac{x}{2}-1}\right)^2 + 4 \operatorname{LambertW}\left(-c_1 e^{-\frac{x}{2}-1}\right) + \frac{x^2}{2} - x + c_2$$

Mathematica DSolve solution

Solving time : 17.332 (sec)
Leaf size : 172

```
DSolve[{(D[y[x],{x,2}])^2+D[y[x],x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2}{3} W\left(e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)^3 + 3W\left(e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)^2 + 4W\left(e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right) + \frac{x^2}{2} - x + c_2$$

$$y(x) \to \frac{2}{3} W\left(-e^{\frac{1}{2}(-x-2+c_1)}\right)^3 + 3W\left(-e^{\frac{1}{2}(-x-2+c_1)}\right)^2 + 4W\left(-e^{\frac{1}{2}(-x-2+c_1)}\right) + \frac{x^2}{2} - x + c_2$$

$$y(x) \to \frac{x^2}{2} - x + c_2$$

2.1.20 Problem 20

Internal problem ID [9091]
Book : Second order enumerated odes
Section : section 1
Problem number : 20
Date solved : `Monday, January 27, 2025 at 05:37:16 PM`
CAS classification :
`[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_xy]]`

Solve

$$y'' + {y'}^2 = x$$

Solved as second order missing y ode

Time used: 0.507 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x)^2 - x = 0$$

Which is now solved for $u(x)$ as first order ode.

This is reduced Riccati ode of the form

$$u'(x) = a\,x^n + bu(x)^2$$

Comparing the given ode to the above shows that

$$\begin{aligned} a &= 1 \\ b &= -1 \\ n &= 1 \end{aligned}$$

Since $n \neq -2$ then the solution of the reduced Riccati ode is given by

$$\begin{aligned} w &= \sqrt{x}\begin{cases} c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) & ab > 0 \\ c_1 \operatorname{BesselI}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) & ab < 0 \end{cases} \qquad (1) \\ u(x) &= -\frac{1}{b}\frac{w'}{w} \\ k &= 1 + \frac{n}{2} \end{aligned}$$

Since $ab < 0$ then EQ(1) gives

$$\begin{aligned} k &= \frac{3}{2} \\ w &= \sqrt{x}\left(c_1 \operatorname{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)\right) \end{aligned}$$

Therefore the solution becomes

$$u(x) = -\frac{1}{b}\frac{w'}{w}$$

Substituting the value of b, w found above and simplyfing gives

$$u(x) = \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_2\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + c_2\,\text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}$$

Letting $c_2 = 1$ the above becomes

$$u(x) = \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}$$

In summary, these are the solution found for $u(x)$

$$u(x) = \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}$$

For solution $u(x) = \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}\, dx$$

$$y = \int \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)} dx + c_3$$

$$y = \int \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)} dx + c_3$$

In summary, these are the solution found for (y)

$$y = \int \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)} dx + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \int \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)} dx + c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 18

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)^2 = x,y(x),singsol=all)
```

$$y = \ln(\pi) + \ln(c_1 \operatorname{AiryAi}(x) - c_2 \operatorname{AiryBi}(x))$$

Mathematica DSolve solution

Solving time : `0.202` (sec)
Leaf size : 15

```
DSolve[{D[y[x],{x,2}]+(D[y[x],x])^2==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \log(x - c_1) + c_2$$

2.1.21 Problem 21

Internal problem ID [9092]
Book : Second order enumerated odes
Section : section 1
Problem number : 21
Date solved : `Monday, January 27, 2025 at 05:37:17 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y' + y = 0$$

Solved as second order linear constant coeff ode

Time used: 0.099 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

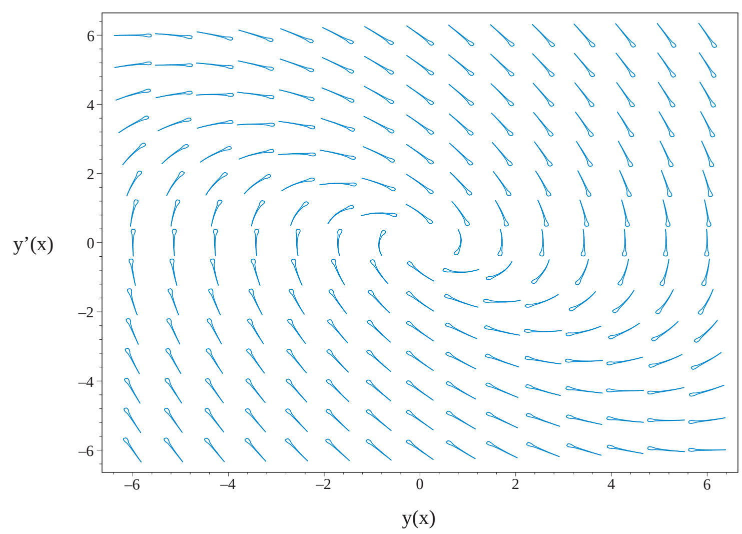


Figure 2.75: Slope field plot
$y'' + y' + y = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.169 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$
$$B = 1 \tag{3}$$
$$C = 1$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.24: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}
$$

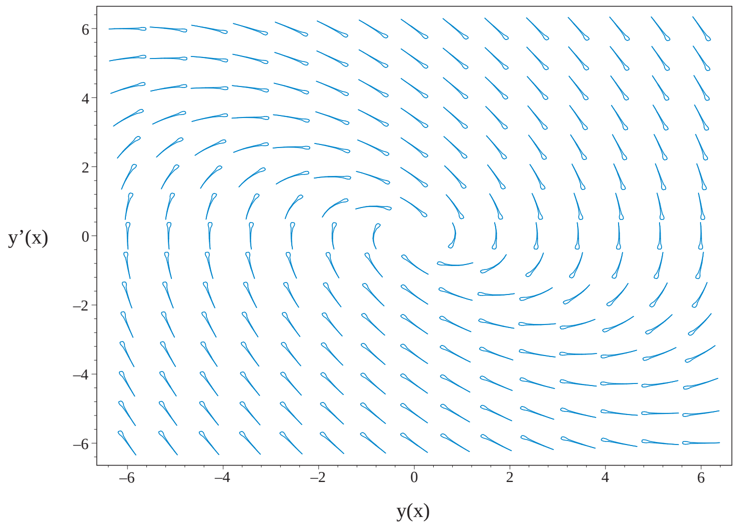


Figure 2.76: Slope field plot
$y'' + y' + y = 0$

Solved as second order ode adjoint method

Time used: 1.662 (sec)

In normal form the ode

$$y'' + y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 1 \\
r(x) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (\xi(x))' + (\xi(x)) &= 0 \\
\xi''(x) - \xi'(x) + \xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(1)} \\ &= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}\end{aligned}$$

Or

$$y' + y\left(1 - \frac{\left(\frac{\mathrm{e}^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + \mathrm{e}^{\frac{x}{2}}\left(-\frac{c_1\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\right)\mathrm{e}^{-\frac{x}{2}}}{c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= -\frac{\left(c_2\sqrt{3} - c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3} + c_2\right)}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)} \\ p(x) &= 0\end{aligned}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1}
\end{aligned}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1}\right) = 0
$$

Integrating gives

$$
\begin{aligned}
\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1}$ gives the final solution

$$
y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}c_3}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}c_3}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}
$$

The constants can be merged to give

$$
y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}
$$

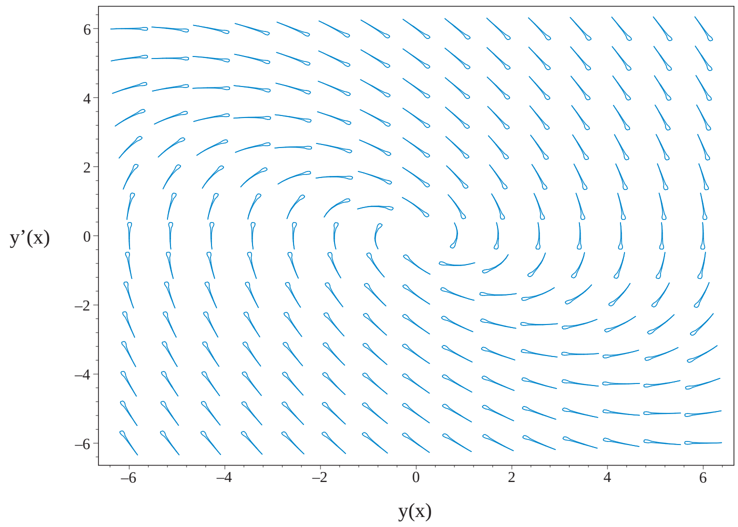


Figure 2.77: Slope field plot
$y'' + y' + y = 0$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x)$

- Substitute in solutions

  $y(x) = C1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = e^{-\frac{x}{2}} \left(c_1 \sin\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 42

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x/2}\left(c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.1.22 Problem 22

Internal problem ID [9093]
Book : Second order enumerated odes
Section : section 1
Problem number : 22
Date solved : Monday, January 27, 2025 at 05:37:20 PM
CAS classification : [[_2nd_order, _missing_x]]

Solve

$$y''^2 + y' + y = 0$$

Does not support ODE with y''^n where $n \neq 1$ unless 1 is missing which is not the case here.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resul
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = exp_sym
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_b(_a)-_a)^(1/2) = 0
         Methods for first order ODEs:
         --- Trying classification methods ---
         trying homogeneous types:
         trying exact
         Looking for potential symmetries
         trying an equivalence to an Abel ODE
         trying 1st order ODE linearizable_by_differentiation
      -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one int
      trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular
      trying differential order: 2; exact nonlinear
      trying 2nd order, integrating factor of the form mu(x,y)
      -> trying 2nd order, the S-function method
         -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods
         -> trying 2nd order, the S-function method
         -> trying 2nd order, No Point Symmetries Class V
```

```
      -> trying 2nd order, No Point Symmetries Class V
      -> trying 2nd order, No Point Symmetries Class V
    trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general
    -> trying 2nd order, dynamical_symmetries, only a reduction of order through one
  solving 2nd order ODE of high degree, Lie methods
`, `2nd order, trying reduction of order with given symmetries:`[1, 0]
```

Maple dsolve solution

Solving time : 0.064 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(diff(y(x),x),x)^2+diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(D[y[x],{x,2}])^2+D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.1.23 Problem 23

Internal problem ID [9094]
Book : Second order enumerated odes
Section : section 1
Problem number : 23
Date solved : `Monday, January 27, 2025 at 05:37:20 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y'^2 + y = 0$$

Solved as second order missing x ode

Time used: 0.869 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 + y = 0$$

Which is now solved as first order ode for $p(y)$.

In canonical form, the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= -\frac{p^2 + y}{p} \end{aligned}$$

This is a Bernoulli ODE.

$$p' = (-1)\,p + (-y)\,\frac{1}{p} \tag{1}$$

The standard Bernoulli ODE has the form

$$p' = f_0(y)p + f_1(y)p^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= -1 \\ f_1 &= -y \end{aligned}$$

The first step is to divide the above equation by p^n which gives

$$\frac{p'}{p^n} = f_0(y)p^{1-n} + f_1(y) \tag{3}$$

The next step is use the substitution $v = p^{1-n}$ in equation (3) which generates a new ODE in $v(y)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $p(y)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(y) &= -1 \\ f_1(y) &= -y \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $p^n = \frac{1}{p}$ gives

$$p'p = -p^2 - y \tag{4}$$

Let

$$\begin{aligned} v &= p^{1-n} \\ &= p^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t y gives

$$v' = 2pp' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{v'(y)}{2} &= -v(y) - y \\ v' &= -2v - 2y \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(y)$ which is now solved.

In canonical form a linear first order is

$$v'(y) + q(y)v(y) = p(y)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(y) &= 2 \\ p(y) &= -2y \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \, dy} \\ &= \mathrm{e}^{\int 2dy} \\ &= \mathrm{e}^{2y} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}(\mu v) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}y}(\mu v) &= (\mu)(-2y) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left(v\,\mathrm{e}^{2y}\right) &= \left(\mathrm{e}^{2y}\right)(-2y) \\ \mathrm{d}\left(v\,\mathrm{e}^{2y}\right) &= \left(-2y\,\mathrm{e}^{2y}\right)\,\mathrm{d}y \end{aligned}$$

Integrating gives

$$\begin{aligned} v\,\mathrm{e}^{2y} &= \int -2y\,\mathrm{e}^{2y}\,dy \\ &= -\frac{(2y-1)\,\mathrm{e}^{2y}}{2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor e^{2y} gives the final solution

$$v(y) = -y + \frac{1}{2} + c_1\,\mathrm{e}^{-2y}$$

The substitution $v = p^{1-n}$ is now used to convert the above solution back to p which results in

$$p^2 = -y + \frac{1}{2} + c_1\,\mathrm{e}^{-2y}$$

Solving for p gives

$$\begin{aligned} p &= -\frac{\sqrt{2 + 4c_1\,\mathrm{e}^{-2y} - 4y}}{2} \\ p &= \frac{\sqrt{2 + 4c_1\,\mathrm{e}^{-2y} - 4y}}{2} \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\sqrt{2 + 4c_1\,\mathrm{e}^{-2y} - 4y}}{2}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} -\frac{2}{\sqrt{2 + 4c_1\,\mathrm{e}^{-2\tau} - 4\tau}} d\tau = x + c_2$$

Singular solutions are found by solving

$$-\frac{\sqrt{2 + 4c_1\,\mathrm{e}^{-2y} - 4y}}{2} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{\mathrm{LambertW}\left(2c_1\,\mathrm{e}^{-1}\right)}{2} + \frac{1}{2}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{2 + 4c_1\,\mathrm{e}^{-2y} - 4y}}{2}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{2}{\sqrt{2 + 4c_1\,\mathrm{e}^{-2\tau} - 4\tau}} d\tau = x + c_3$$

Singular solutions are found by solving

$$\frac{\sqrt{2 + 4c_1\,\mathrm{e}^{-2y} - 4y}}{2} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{\text{LambertW}\left(2c_1\,\mathrm{e}^{-1}\right)}{2} + \frac{1}{2}$$

Will add steps showing solving for IC soon.

The solution

$$y = \frac{\text{LambertW}\left(2c_1\,\mathrm{e}^{-1}\right)}{2} + \frac{1}{2}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\int^{y} -\frac{2}{\sqrt{2+4c_1\,\mathrm{e}^{-2\tau}-4\tau}}d\tau = x + c_2$$
$$\int^{y} \frac{2}{\sqrt{2+4c_1\,\mathrm{e}^{-2\tau}-4\tau}}d\tau = x + c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^2+_a = 0, _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 61

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)^2+y(x) = 0,y(x),singsol=all)
```

$$-2\left(\int^{y}\frac{1}{\sqrt{2+4\,\mathrm{e}^{-2\_a}c_1-4\_a}}d\_a\right)-x-c_2=0$$
$$2\left(\int^{y}\frac{1}{\sqrt{2+4\,\mathrm{e}^{-2\_a}c_1-4\_a}}d\_a\right)-x-c_2=0$$

Mathematica DSolve solution

Solving time : 0.823 (sec)
Leaf size : 272

```
DSolve[{D[y[x],{x,2}]+(D[y[x],x])^2+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} -\frac{\sqrt{2}}{\sqrt{2e^{-2K[1]}c_1 - 2K[1] + 1}}dK[1]\&\right][x + c_2]$$
$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sqrt{2}}{\sqrt{2e^{-2K[2]}c_1 - 2K[2] + 1}}dK[2]\&\right][x + c_2]$$
$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} -\frac{\sqrt{2}}{\sqrt{2e^{-2K[1]}(-c_1) - 2K[1] + 1}}dK[1]\&\right][x + c_2]$$
$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} -\frac{\sqrt{2}}{\sqrt{2e^{-2K[1]}c_1 - 2K[1] + 1}}dK[1]\&\right][x + c_2]$$
$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sqrt{2}}{\sqrt{2e^{-2K[2]}(-c_1) - 2K[2] + 1}}dK[2]\&\right][x + c_2]$$
$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sqrt{2}}{\sqrt{2e^{-2K[2]}c_1 - 2K[2] + 1}}dK[2]\&\right][x + c_2]$$

2.1.24 Problem 24

Internal problem ID [9095]
Book : Second order enumerated odes
Section : section 1
Problem number : 24
Date solved : `Monday, January 27, 2025 at 05:37:22 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y' + y = 1$$

Solved as second order linear constant coeff ode

Time used: 0.149 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + (1) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 1 + e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

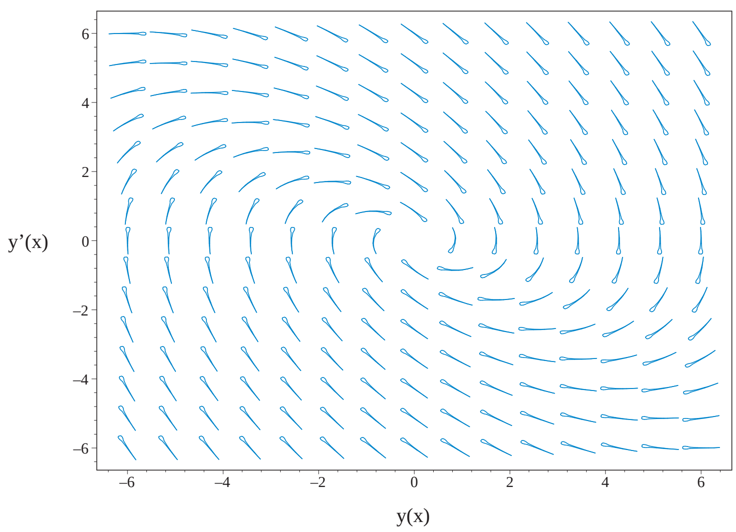


Figure 2.78: Slope field plot
$y'' + y' + y = 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.26: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2 \,\mathrm{e}^{-\frac{x}{2}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \frac{2\,e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \right) + (1) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + 1$$

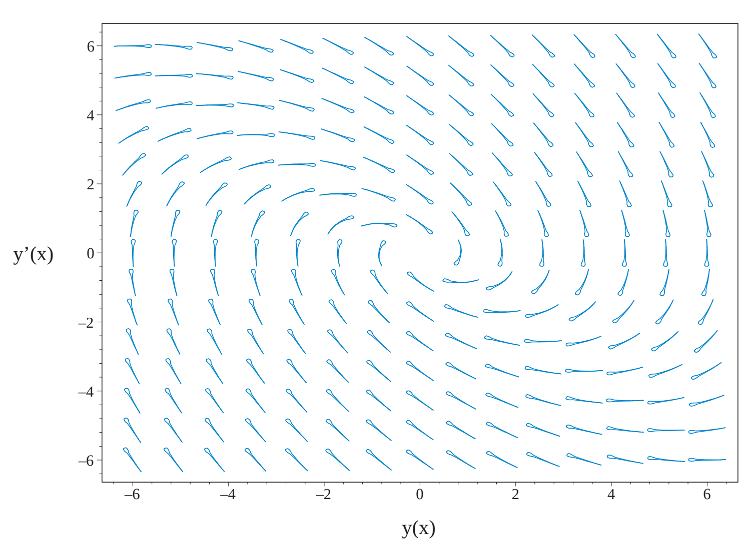


Figure 2.79: Slope field plot
$y'' + y' + y = 1$

Solved as second order ode adjoint method

Time used: 22.033 (sec)

In normal form the ode

$$y'' + y' + y = 1 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (\xi(x)) &= 0 \\ \xi''(x) - \xi'(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(1)} \\ &= \frac{1}{2} \pm \frac{i\sqrt{3}}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1\sqrt{3}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\right)e^{-\frac{x}{2}}}{c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \frac{e^{-\frac{x}{2}}\left(c_1\left(\frac{e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right.}{2}\right.\right.}{}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{(c_2\sqrt{3} - c_1)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)(c_1\sqrt{3} + c_2)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}$$
$$p(x) = \frac{(-c_2\sqrt{3} + c_1)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)(c_1\sqrt{3} + c_2)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= e^{\int -\frac{(c_2\sqrt{3}-c_1)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)(c_1\sqrt{3}+c_2)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\, e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}
\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\left(-c_2\sqrt{3}+c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right)$$

$$\frac{d}{dx}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right)
= \left(\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right)\left(\frac{\left(-c_2\sqrt{3}+c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right)$$

$$d\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right)
= \left(\frac{\left(\left(-c_2\sqrt{3}+c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\right)dx$$

Integrating gives

$$\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)} = \int \frac{\left(\left(-c_2\sqrt{3}+c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\,dx$$

$$= \int \frac{\left(\left(-c_2\sqrt{3}+c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\,dx$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}$ gives the final solution

$$y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(-c_2\sqrt{3}+c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\,dx\right)}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(-c_2\sqrt{3}+c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\,dx\right)}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-c_2\sqrt{3}+c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\cdots\right.\right.}{3}}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)} \cdots\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-c_2\sqrt{3}+c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\cdots\right.\right.}{3}}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)} \cdots\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

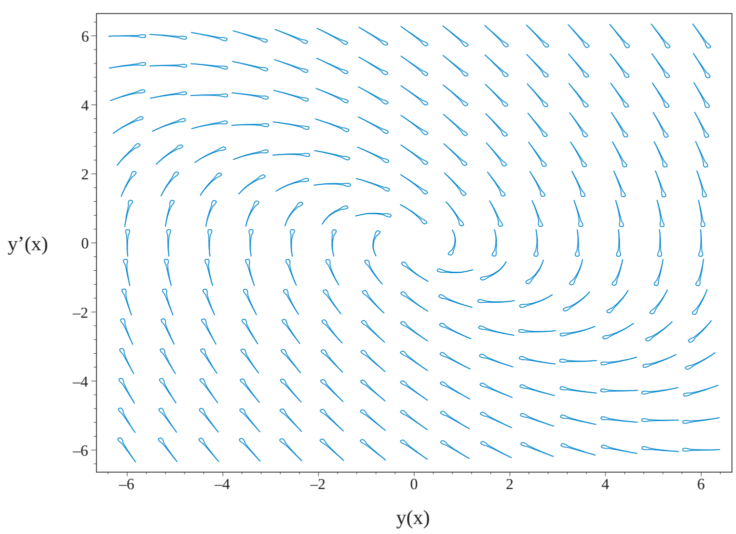


Figure 2.80: Slope field plot
$y'' + y' + y = 1$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

$y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 1\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,\mathrm{e}^{-x}}{2}$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) dx\right)\right)}{3}$

- ◦ Compute integrals

  $y_p(x) = 1$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 32

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = 1,y(x),singsol=all)
```

$$y = c_2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + 1$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 49

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==1,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x/2}\left(e^{x/2} + c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.1.25 Problem 25

Internal problem ID [9096]
Book : Second order enumerated odes
Section : section 1
Problem number : 25
Date solved : `Monday, January 27, 2025 at 05:37:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + y' + y = x$$

Solved as second order linear constant coeff ode

Time used: 0.151 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 + A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + (x - 1) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x - 1 + \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

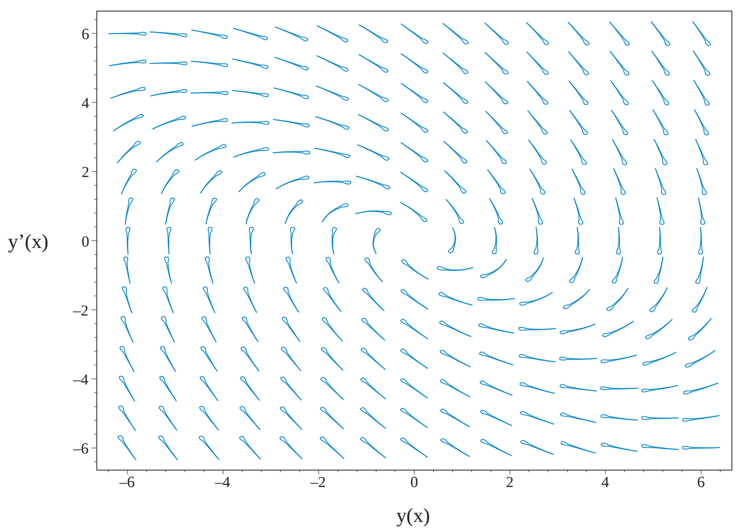


Figure 2.81: Slope field plot
$y'' + y' + y = x$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.28: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 + A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right) + (x - 1) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} + x - 1$$

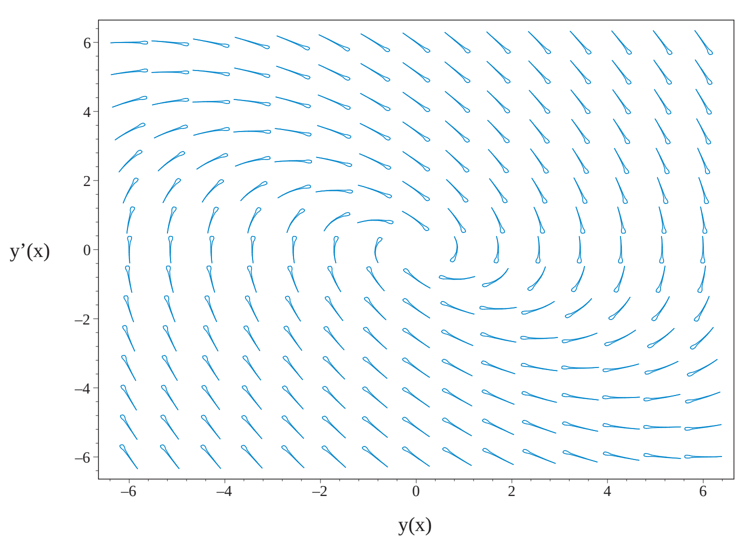


Figure 2.82: Slope field plot
$y'' + y' + y = x$

Solved as second order ode adjoint method

Time used: 32.387 (sec)

In normal form the ode

$$y'' + y' + y = x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= 1\\
q(x) &= 1\\
r(x) &= x
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0\\
\xi'' - (\xi(x))' + (\xi(x)) &= 0\\
\xi''(x) - \xi'(x) + \xi(x) &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(1)}\\
&= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2}\\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2}\\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' + y \left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1 \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} + \frac{c_2 \sqrt{3}\, \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2}\right)\right) e^{-\frac{x}{2}}}{c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)} \right) = \frac{e^{-\frac{x}{2}}\left(c_1\left(\left(\frac{x}{2} + \frac{1}{2}\right) e^{\frac{x}{2}}\right.\right.}{}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{\left(c_2 \sqrt{3} - c_1\right) \cos\left(\frac{\sqrt{3}\, x}{2}\right) - \sin\left(\frac{\sqrt{3}\, x}{2}\right)\left(c_1 \sqrt{3} + c_2\right)}{2c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)} \\
p(x) &= \frac{\left(-c_2(x-1)\sqrt{3} + c_1(x+1)\right) \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \left(c_1(x-1)\sqrt{3} + c_2(x+1)\right) \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)}
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int -\frac{\left(c_2\sqrt{3} - c_1\right)\cos\left(\frac{\sqrt{3}\, x}{2}\right) - \sin\left(\frac{\sqrt{3}\, x}{2}\right)\left(c_1\sqrt{3} + c_2\right)}{2c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)} dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\, x}{2}\right)^2}\, e^{\frac{\sqrt{3}\, \arctan\left(\tan\left(\frac{\sqrt{3}\, x}{2}\right)\right)}{3}}}{c_1 + c_2 \tan\left(\frac{\sqrt{3}\, x}{2}\right)}
\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\left(-c_2(x-1)\sqrt{3}+c_1(x+1)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1(x-1)\sqrt{3}+c_2(x+1)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right)$$

$$\frac{d}{dx}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right)\left(\frac{\left(-c_2(x-1)\sqrt{3}+c_1(x+1)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1(x-1)\sqrt{3}+c_2(x\right.}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right.$$

$$d\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\left(\left(-c_2(x-1)\sqrt{3}+c_1(x+1)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1(x-1)\sqrt{3}+c_2(x+1)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}}{}}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\right.$$

Integrating gives

$$\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)} = \int\frac{\left(\left(-c_2(x-1)\sqrt{3}+c_1(x+1)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1(x-1)\sqrt{3}+c_2(x\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(\right.}$$
$$= \int\frac{\left(\left(-c_2(x-1)\sqrt{3}+c_1(x+1)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1(x-1)\sqrt{3}+c_2(x\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(\right.}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}$ gives the final solution

$$y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int\frac{\left(\left(-c_2(x-1)\sqrt{3}+c_1(x+1)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1(x-1)\sqrt{3}+c_2(x+1)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\right.\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int\frac{\left(\left(-c_2(x-1)\sqrt{3}+c_1(x+1)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1(x-1)\sqrt{3}+c_2(x+1)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\sqrt{s}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}}{2}\right.\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-c_2(x-1)\sqrt{3}+c_1(x+1)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1(x-1)\sqrt{3}+c_2(x+1)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\sqrt{\sec(}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-c_2(x-1)\sqrt{3}+c_1(x+1)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1(x-1)\sqrt{3}+c_2(x+1)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\sqrt{\sec(}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

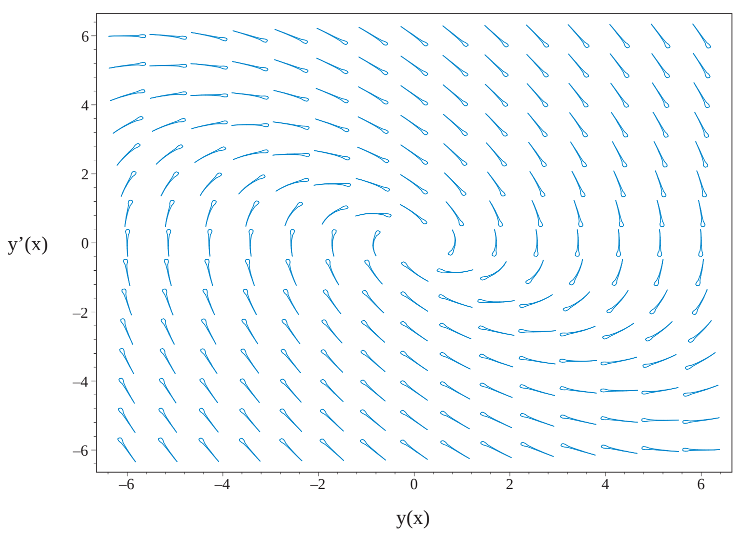


Figure 2.83: Slope field plot
$y'' + y' + y = x$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = x$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

$y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + C2\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\, e^{-x}}{2}$

- ∘ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\sqrt{3}\, e^{-\frac{x}{2}} \left(\cos\left(\frac{\sqrt{3}\, x}{2}\right)\left(\int x\, e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) dx\right) - \sin\left(\frac{\sqrt{3}\, x}{2}\right)\left(\int x\, e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) dx\right)\right)}{3}$

- ∘ Compute integrals

  $y_p(x) = x - 1$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + C2\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) + x - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 33

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = x,y(x),singsol=all)
```

$$y = c_2 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + x - 1$$

Mathematica DSolve solution

Solving time : `0.023` (sec)
Leaf size : 50

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) - 1$$

2.1.26 Problem 26

Internal problem ID [9097]
Book : Second order enumerated odes
Section : section 1
Problem number : 26
Date solved : `Monday, January 27, 2025 at 05:38:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + y' + y = 1 + x$$

Solved as second order linear constant coeff ode

Time used: 0.162 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = 1 + x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 + A_2 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + (x) \end{aligned}$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$y = x + e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

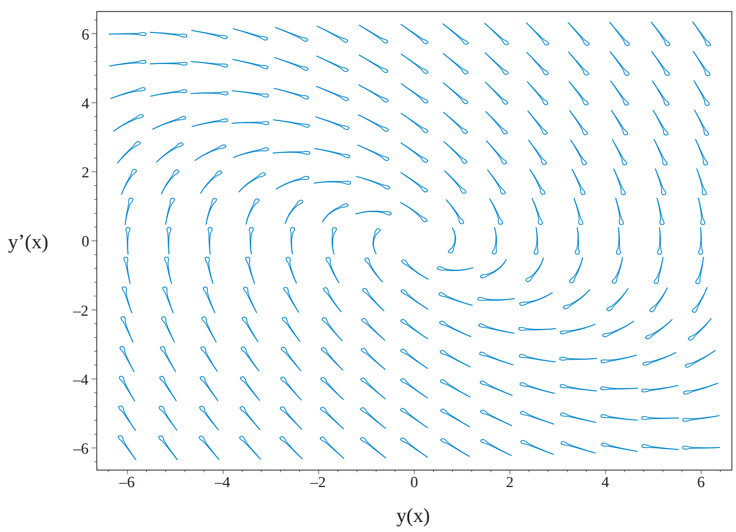


Figure 2.84: Slope field plot
$y'' + y' + y = 1 + x$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.30: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1 + x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \frac{2\,e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 + A_2 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \right) + (x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + x$$

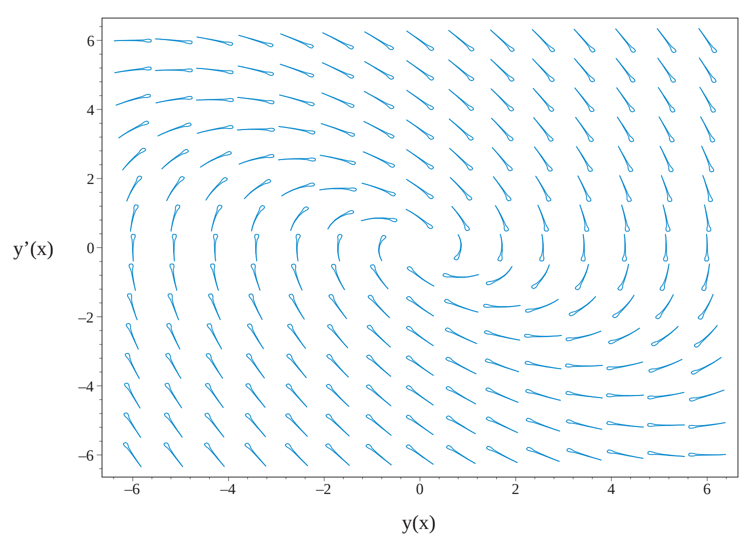


Figure 2.85: Slope field plot
$y'' + y' + y = 1 + x$

Solved as second order ode adjoint method

Time used: 23.880 (sec)

In normal form the ode

$$y'' + y' + y = 1 + x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 1 + x \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (\xi(x)) &= 0 \\ \xi''(x) - \xi'(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(1)} \\ &= \frac{1}{2} \pm \frac{i\sqrt{3}}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y = \int \xi(x)\, r(x)\, dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) = \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}$$

Or

$$y' + y\left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1\sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2\sqrt{3}\, \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\right) e^{-\frac{x}{2}}}{c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \frac{e^{-\frac{x}{2}}\left(c_1\left(\frac{e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right.}{2}\right.\right.}{}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{\left(c_2\sqrt{3} - c_1\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3} + c_2\right)}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}$$
$$p(x) = \frac{\left(-x c_2\sqrt{3} + c_1(x+2)\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(x c_1\sqrt{3} + c_2(x+2)\right)}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int -\frac{\left(c_2\sqrt{3}-c_1\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)} dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\, e^{\frac{\sqrt{3}\, \arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)}
\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{\left(-x c_2\sqrt{3}+c_1(x+2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(x c_1\sqrt{3}+c_2(x+2)\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right)\left(\frac{\left(-x c_2\sqrt{3}+c_1(x+2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(x c_1\sqrt{3}+c_2(x+ \right.}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right.$$

$$\mathrm{d}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\left(\left(-x c_2\sqrt{3}+c_1(x+2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(x c_1\sqrt{3}+c_2(x+2)\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right.\right.}{3}}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\right.$$

Integrating gives

$$\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)} = \int \frac{\left(\left(-x c_2\sqrt{3}+c_1(x+2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(x c_1\sqrt{3}+c_2(x+\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\right.}$$

$$= \int \frac{\left(\left(-x c_2\sqrt{3}+c_1(x+2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(x c_1\sqrt{3}+c_2(x+\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\right.}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}$ gives the final solution

$$y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(-x c_2\sqrt{3}+c_1(x+2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(x c_1\sqrt{3}+c_2(x+2)\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right.}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right.\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(-x c_2\sqrt{3}+c_1(x+2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(x c_1\sqrt{3}+c_2(x+2)\right)\right)\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-x c_2 \sqrt{3}+c_1 (x+2)\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right) \left(x c_1 \sqrt{3}+c_2 (x+2)\right)\right) \sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^{2}}\, \mathrm{e}^{-}}{\left(2 c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right)+2 c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \left(c_1+c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^{2}}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-x c_2 \sqrt{3}+c_1 (x+2)\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right) \left(x c_1 \sqrt{3}+c_2 (x+2)\right)\right) \sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^{2}}\, \mathrm{e}^{-}}{\left(2 c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right)+2 c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \left(c_1+c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^{2}}}$$

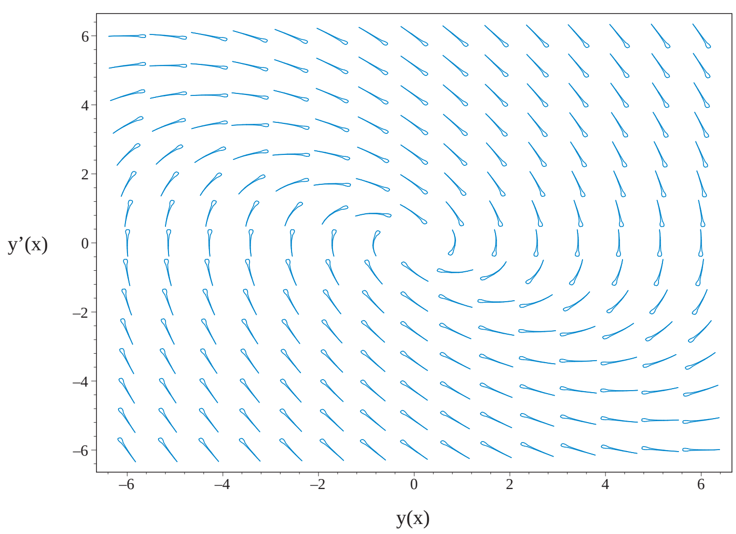


Figure 2.86: Slope field plot
$y'' + y' + y = 1 + x$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = x + 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$$

- General solution of the ODE

$$y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y(x) = C1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x + 1\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\, e^{-x}}{2}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{2\sqrt{3}\, e^{-\frac{x}{2}} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int (x+1) e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int (x+1) e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) dx\right)\right)}{3}$$

- Compute integrals

$$y_p(x) = x$$

- Substitute particular solution into general solution to ODE

$$y(x) = C1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 32

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = x+1,y(x),singsol=all)
```

$$y = c_2 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + x$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 49

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==1+x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

2.1.27 Problem 27

Internal problem ID [9098]
Book : Second order enumerated odes
Section : section 1
Problem number : 27
Date solved : `Monday, January 27, 2025 at 05:38:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + y' + y = x^2 + x + 1$$

Solved as second order linear constant coeff ode

Time used: 0.156 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3 x^2 + A_2 x + 2x A_3 + A_1 + A_2 + 2A_3 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + \left(x^2 - x\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^2 - x + \mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

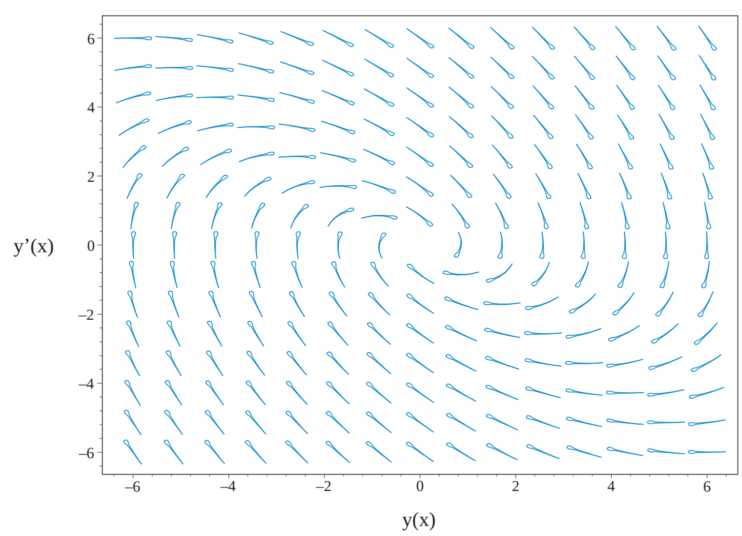


Figure 2.87: Slope field plot
$y'' + y' + y = x^2 + x + 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.273 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.32: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3 x^2 + A_2 x + 2x A_3 + A_1 + A_2 + 2A_3 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \right) + (x^2 - x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} + x^2 - x$$

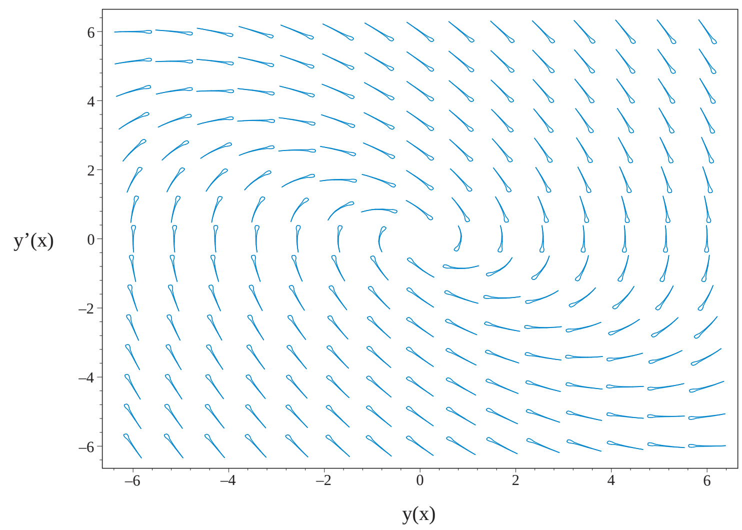


Figure 2.88: Slope field plot
$y'' + y' + y = x^2 + x + 1$

Solved as second order ode adjoint method

Time used: 34.622 (sec)

In normal form the ode

$$y'' + y' + y = x^2 + x + 1 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= 1 \\
r(x) &= x^2 + x + 1
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (\xi(x))' + (\xi(x)) &= 0 \\
\xi''(x) - \xi'(x) + \xi(x) &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} - \lambda\, e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(1)} \\
&= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}
\end{aligned}
$$

Or

$$y' + y\left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\right)e^{-\frac{x}{2}}}{c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \frac{e^{-\frac{x}{2}}\left(c_1\left(\left(\frac{1}{2}x^2 + x\right.\right.\right.}{}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{(c_2\sqrt{3} - c_1)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)(c_1\sqrt{3} + c_2)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)} \\
p(x) &= \frac{\left(-xc_2(x-1)\sqrt{3} + c_1(x^2+3x-2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \left(c_1x(x-1)\sqrt{3} + c_2(x^2+3x-2)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p \\
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) \\
&= (\mu)\left(\frac{\left(-xc_2(x-1)\sqrt{3}+c_1(x^2+3x-2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1x(x-1)\sqrt{3}+c_2(x^2+3x-2)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) \\
&= \left(\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right)\left(\frac{\left(-xc_2(x-1)\sqrt{3}+c_1(x^2+3x-2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1x(x-1)\sqrt{}\right.}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right. \\
&\mathrm{d}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) \\
&= \left(\frac{\left(\left(-xc_2(x-1)\sqrt{3}+c_1(x^2+3x-2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1x(x-1)\sqrt{3}+c_2(x^2+3x-2)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\sqrt{}}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\right.
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)} &= \int \frac{\left(\left(-xc_2(x-1)\sqrt{3}+c_1(x^2+3x-2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1x(x-1)\sqrt{}\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\right.\right.} \\
&= \int \frac{\left(\left(-xc_2(x-1)\sqrt{3}+c_1(x^2+3x-2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1x(x-1)\sqrt{}\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\right.\right.}
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}$ gives the final solution

$$
y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(-xc_2(x-1)\sqrt{3}+c_1(x^2+3x-2)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1x(x-1)\sqrt{3}+c_2(x^2+3x-2)\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-x c_2 (x-1)\sqrt{3}+c_1\left(x^2+3x-2\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1 x(x-1)\sqrt{3}+c_2\left(x^2+3x-2\right)\right)\sin\left(\cdots\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\cdots\right.\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-x c_2 (x-1)\sqrt{3}+c_1\left(x^2+3x-2\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1 x(x-1)\sqrt{3}+c_2\left(x^2+3x-2\right)\right)\sin\left(\cdots\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\cdots\right.\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-x c_2 (x-1)\sqrt{3}+c_1\left(x^2+3x-2\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\left(c_1 x(x-1)\sqrt{3}+c_2\left(x^2+3x-2\right)\right)\sin\left(\cdots\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\cdots\right.\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

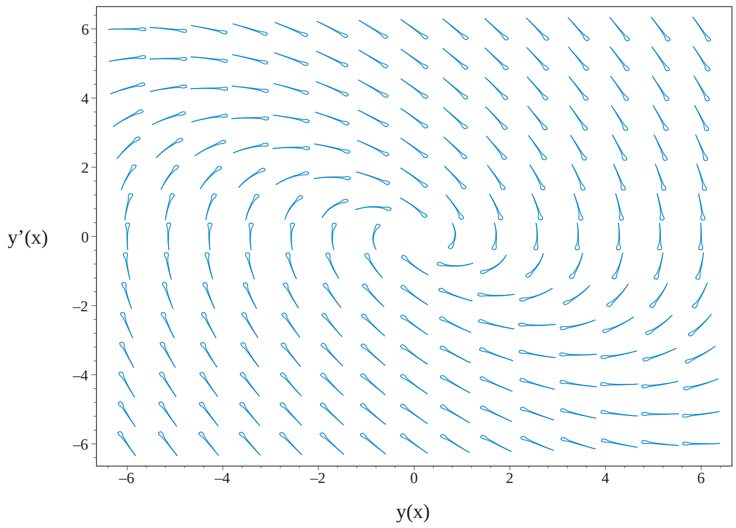


Figure 2.89: Slope field plot
$y'' + y' + y = x^2 + x + 1$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = x^2 + x + 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$
- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$
- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$
- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^2 + x + 1\right]$
- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$
- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\, e^{-x}}{2}$
- ∘ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\sqrt{3}\, e^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}(x^2+x+1)\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}(x^2+x+1)\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$
- ∘ Compute integrals

  $y_p(x) = x(x-1)$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + x(x-1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = x^2+x+1,y(x),singsol=all)
```

$$y = c_2 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + x^2 - x$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 54

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==1+x+x^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x/2}\left(e^{x/2}(x-1)x + c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.1.28 Problem 28

Internal problem ID [9099]
Book : Second order enumerated odes
Section : section 1
Problem number : 28
Date solved : `Monday, January 27, 2025 at 05:39:20 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y' + y = x^3 + x^2 + x + 1$$

Solved as second order linear constant coeff ode

Time used: 0.169 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = x^3 + x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4x^3 + A_3x^2 + 3x^2A_4 + A_2x + 2xA_3 + 6xA_4 + A_1 + A_2 + 2A_3 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 6, A_2 = -1, A_3 = -2, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 - 2x^2 - x + 6$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + \left(x^3 - 2x^2 - x + 6\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^3 - 2x^2 - x + 6 + \mathrm{e}^{-\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

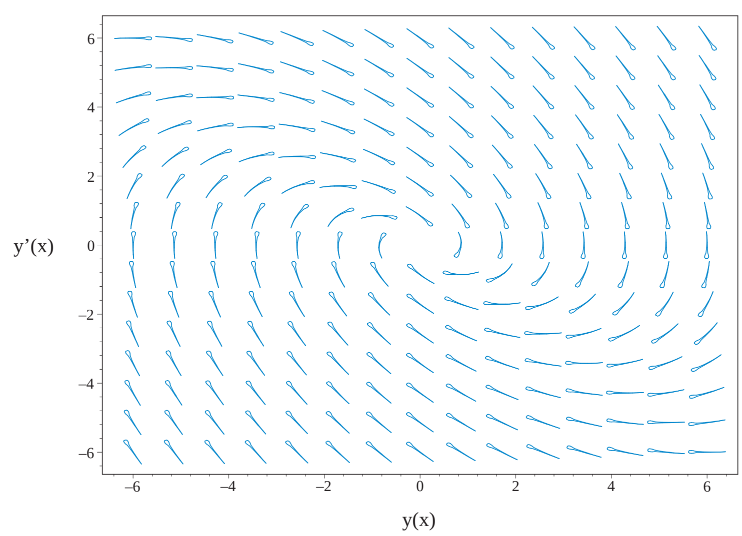


Figure 2.90: Slope field plot
$y'' + y' + y = x^3 + x^2 + x + 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.34: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4 x^3 + A_3 x^2 + 3x^2 A_4 + A_2 x + 2x A_3 + 6x A_4 + A_1 + A_2 + 2A_3 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 6, A_2 = -1, A_3 = -2, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 - 2x^2 - x + 6$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \right) + \left(x^3 - 2x^2 - x + 6\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} + x^3 - 2x^2 - x + 6$$

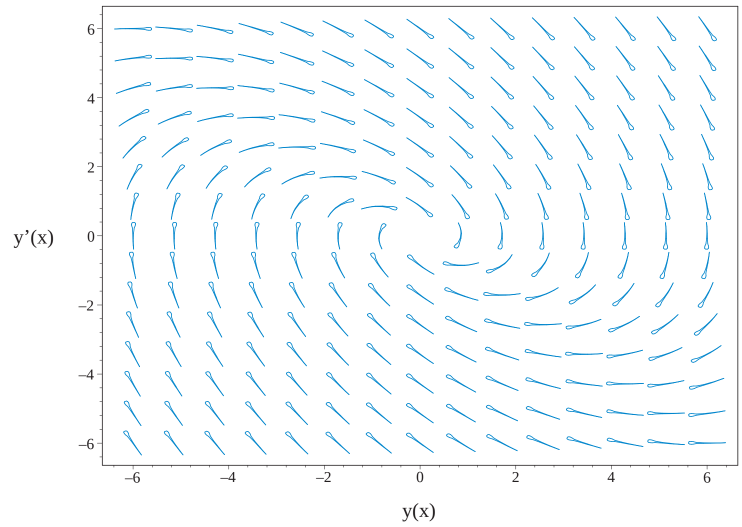


Figure 2.91: Slope field plot
$y'' + y' + y = x^3 + x^2 + x + 1$

Solved as second order ode adjoint method

Time used: 25.213 (sec)

In normal form the ode

$$y'' + y' + y = x^3 + x^2 + x + 1 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned}p(x) &= 1\\ q(x) &= 1\\ r(x) &= x^3 + x^2 + x + 1\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi'' - (\xi\,p)' + \xi q &= 0\\ \xi'' - (\xi(x))' + (\xi(x)) &= 0\\ \xi''(x) - \xi'(x) + \xi(x) &= 0\end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\,\mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=-1, C=1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2-(4)\,(1)\,(1)} \\
&= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$
\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))
$$

Which becomes

$$
\xi = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\right)e^{-\frac{x}{2}}}{c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \frac{e^{-\frac{x}{2}}\left(c_1\left(\left(\frac{1}{2}x^3 + \frac{3}{2}\right.\right.\right.}{}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)} \\
p(x) &= \frac{\left(-c_2(x^3-2x^2-x+6)\sqrt{3} + c_1(x-1)\,(x^2+5x-4)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1(x^3-2x^2-x+6)\right.}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= \mathrm{e}^{\int -\frac{\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}
\end{aligned}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{\left(-c_2(x^3-2x^2-x+6)\sqrt{3}+c_1(x-1)\left(x^2+5x-4\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1(x^3-2x^2-x+\right.}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right.
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right)\left(\frac{\left(-c_2(x^3-2x^2-x+6)\sqrt{3}+c_1(x-1)\left(x^2+5x-4\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right.}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right.}\right.
$$

$$
\mathrm{d}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\left(\left(-c_2(x^3-2x^2-x+6)\sqrt{3}+c_1(x-1)\left(x^2+5x-4\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1(x^3-2x^2-x+6\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1+c_2\tan\right.}\right.
$$

Integrating gives

$$
\begin{aligned}
\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)} &= \int \frac{\left(\left(-c_2(x^3-2x^2-x+6)\sqrt{3}+c_1(x-1)\left(x^2+5x-4\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right.\right.}{\left(2c_1\cos\right.} \\
&= \int \frac{\left(\left(-c_2(x^3-2x^2-x+6)\sqrt{3}+c_1(x-1)\left(x^2+5x-4\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right.\right.}{\left(2c_1\cos\right.}
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}$ gives the final solution

$$
y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(-c_2(x^3-2x^2-x+6)\sqrt{3}+c_1(x-1)\left(x^2+5x-4\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-c_2 (x^3-2x^2-x+6)\sqrt{3}+c_1 (x-1)(x^2+5x-4)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1 (x^3-2x\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right.}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-c_2 (x^3-2x^2-x+6)\sqrt{3}+c_1 (x-1)(x^2+5x-4)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1 (x^3-2x\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right.}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(-c_2 (x^3-2x^2-x+6)\sqrt{3}+c_1 (x-1)(x^2+5x-4)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1 (x^3-2x\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right.}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

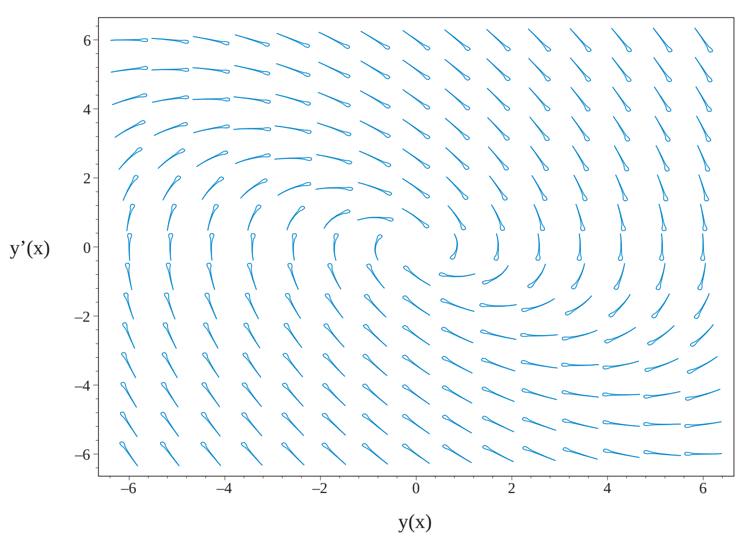


Figure 2.92: Slope field plot
$y'' + y' + y = x^3 + x^2 + x + 1$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = x^3 + x^2 + x + 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^3 + x^2 + x + 1\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,\mathrm{e}^{-x}}{2}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}}(x+1)(x^2+1)\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}}(x+1)(x^2+1)\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$

- ○ Compute integrals

  $y_p(x) = x^3 - 2x^2 - x + 6$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + x^3 - 2x^2 - x + 6$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 43

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = x^3+x^2+x+1,y(x),singsol=all)
```

$$y = c_2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) + \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + x^3 - 2x^2 - x + 6$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 60

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==1+x+x^2+x^3,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^3 - 2x^2 - x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) + 6$$

2.1.29 Problem 29

Internal problem ID [9100]
Book : Second order enumerated odes
Section : section 1
Problem number : 29
Date solved : `Monday, January 27, 2025 at 05:39:46 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y' + y = \sin(x)$$

Solved as second order linear constant coeff ode

Time used: 0.155 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + (-\cos(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\cos(x) + e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

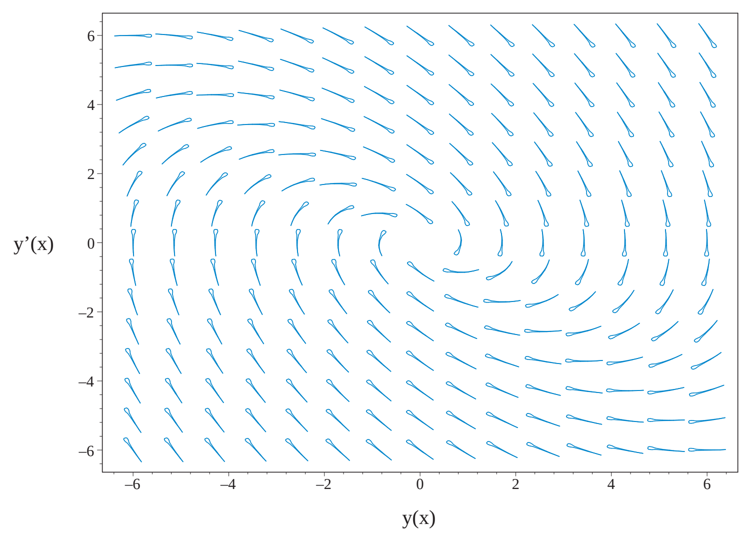


Figure 2.93: Slope field plot
$y'' + y' + y = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.205 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.36: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2 c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \right) + (-\cos(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2 c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \cos(x)$$

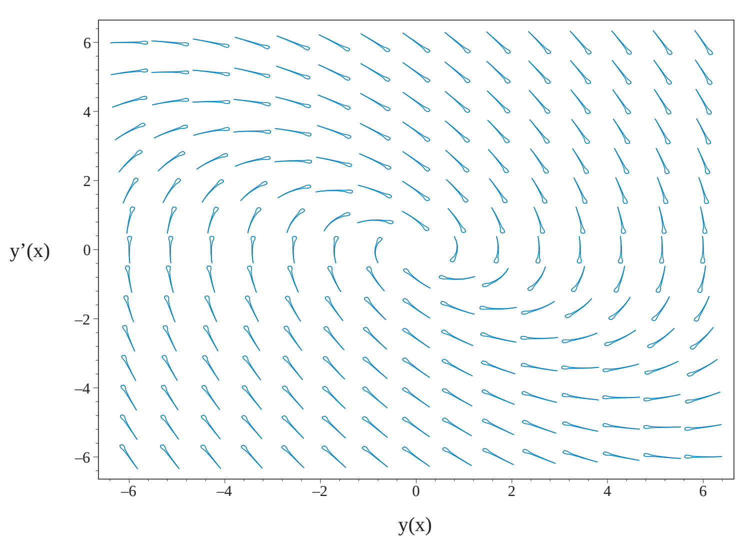


Figure 2.94: Slope field plot
$y'' + y' + y = \sin(x)$

Solved as second order ode adjoint method

Time used: 46.319 (sec)

In normal form the ode

$$y'' + y' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (\xi(x)) &= 0 \\ \xi''(x) - \xi'(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(1)} \\ &= \frac{1}{2} \pm \frac{i\sqrt{3}}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y = \int \xi(x)\, r(x)\, dx$$

$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) = \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}$$

Or

$$y' + y\left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\right)e^{-\frac{x}{2}}}{c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \frac{e^{-\frac{x}{2}}\left(\frac{c_1\left(\frac{\left(-\frac{\sqrt{3}}{2}-1\right)e^{\frac{x}{2}}\cdots}{\frac{1}{4}+(\cdots}\cdots\right)}{\cdots}\right)}{\cdots}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{\left(c_2\sqrt{3} - c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3} + c_2\right)}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}$$

$$p(x) = \frac{\left(\sqrt{3}\,\cos\,(x)\, c_2 - c_1(\cos\,(x) - 2\sin\,(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\,\cos\,(x) + c_2(\cos\,(x) - 2\sin\,(x\cdots\right.}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int -\frac{\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\, e^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)}
\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{\left(\sqrt{3}\,\cos(x)\,c_2 - c_1(\cos(x) - 2\sin(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\,\cos(x) + c_2(\cos(x) - 2\sin\right.}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right)\left(\frac{\left(\sqrt{3}\,\cos(x)\,c_2 - c_1(\cos(x) - 2\sin(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}}{2}\right.}\right.$$

$$\mathrm{d}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\left(\left(\sqrt{3}\,\cos(x)\,c_2 - c_1(\cos(x) - 2\sin(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\,\cos(x) + c_2(\cos(x) - 2\sin(\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\right.$$

Integrating gives

$$\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)} = \int \frac{\left(\left(\sqrt{3}\,\cos(x)\,c_2 - c_1(\cos(x) - 2\sin(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\right.}$$

$$= \int \frac{\left(\left(\sqrt{3}\,\cos(x)\,c_2 - c_1(\cos(x) - 2\sin(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\right.}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}$ gives the final solution

$$y = \frac{\left(c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(\sqrt{3}\,\cos(x)c_2 - c_1(\cos(x) - 2\sin(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\,\cos(x) +\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(\sqrt{3}\,\cos(x)c_2 - c_1(\cos(x) - 2\sin(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\,\cos(x) + c_2(\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1 +\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(\sqrt{3}\cos(x) c_2 - c_1(\cos(x) - 2\sin(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\cos(x) + c_2(\cos(\cdots\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1 + c_2\,\mathrm{t}\cdots\right.}\cdots}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \left(\int \frac{\left(\left(\sqrt{3}\cos(x) c_2 - c_1(\cos(x) - 2\sin(x))\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\cos(x) + c_2(\cos(\cdots\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1 + c_2\,\mathrm{t}\cdots\right.}\cdots}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

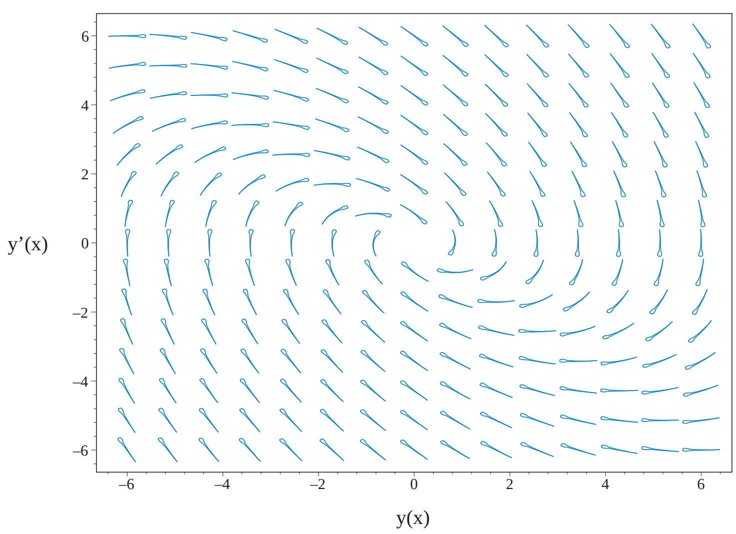


Figure 2.95: Slope field plot
$y'' + y' + y = \sin(x)$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = \sin(x)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + r + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$
- Roots of the characteristic polynomial
  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$
- 1st solution of the homogeneous ODE
  $y_1(x) = \mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)$
- 2nd solution of the homogeneous ODE

$y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$

  ◦ Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{2\sqrt{3}\,e^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\sin(x)\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\sin(x)\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$$

  ◦ Compute integrals

  $y_p(x) = -\cos(x)$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) - \cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 35

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = sin(x),y(x),singsol=all)
```

$$y = c_2 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 - \cos(x)$$

Mathematica DSolve solution

Solving time : `0.309` (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==Sin[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x/2}\left(-e^{x/2}\cos(x) + c_2\cos\left(\frac{\sqrt{3}x}{2}\right) + c_1\sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.1.30 Problem 30

Internal problem ID [9101]
Book : Second order enumerated odes
Section : section 1
Problem number : 30
Date solved : `Monday, January 27, 2025 at 05:40:34 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y' + y = \cos(x)$$

Solved as second order linear constant coeff ode

Time used: 0.168 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + (\sin(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \sin(x) + e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

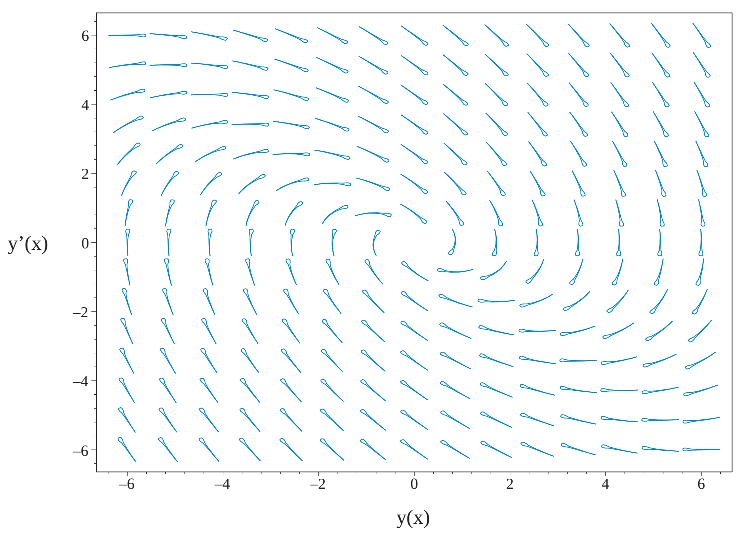


Figure 2.96: Slope field plot
$y'' + y' + y = \cos(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.38: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right) + (\sin(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} + \sin(x)$$

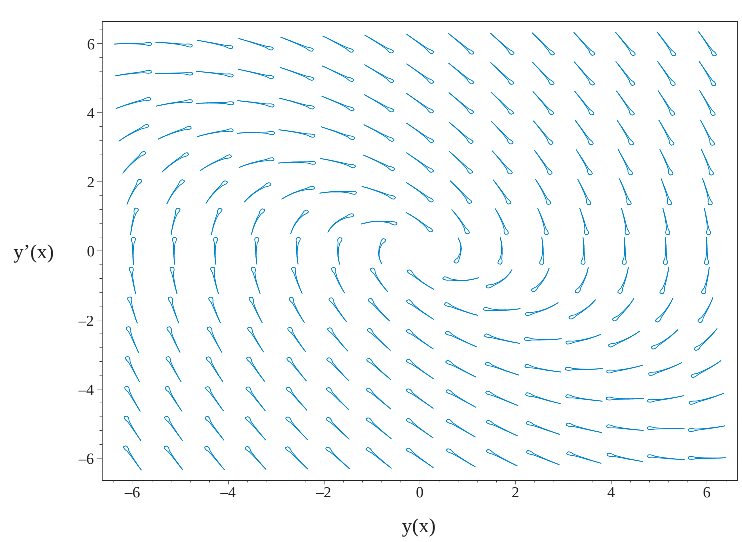


Figure 2.97: Slope field plot
$y'' + y' + y = \cos(x)$

Solved as second order ode adjoint method

Time used: 73.894 (sec)

In normal form the ode

$$y'' + y' + y = \cos(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= \cos(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (\xi(x)) &= 0 \\ \xi''(x) - \xi'(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} - \lambda\, e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(1)} \\ &= \frac{1}{2} \pm \frac{i\sqrt{3}}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)} \right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}$$

Or

$$y' + y \left(1 - \frac{\left(\frac{e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}} \left(-\frac{c_1 \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2 \sqrt{3}\, \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \right) \right) e^{-\frac{x}{2}}}{c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)} \right) = \frac{e^{-\frac{x}{2}} \left(c_1 \left(\frac{e^{\frac{x}{2}} \cos\left(x\left(\frac{\sqrt{3}}{2} \cdots\right.\right.}{\frac{1}{2} + 2\left(\frac{\sqrt{3}}{2} - \cdots\right.} \cdots \right.\right.}{\cdots}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{\left(c_2\sqrt{3} - c_1\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right) \left(c_1\sqrt{3} + c_2\right)}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)} \\
p(x) &= \frac{\left(-c_2\sqrt{3}\, \sin\,(x) + 2c_1\left(\cos\,(x) + \frac{\sin(x)}{2}\right)\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right) \left(c_1\sqrt{3}\, \sin\,(x) + 2\left(\cos\,(x) + \frac{\sin(x)}{2}\right) \cdots\right.}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int -\frac{\left(c_2\sqrt{3} - c_1\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3} + c_2\right)}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)} dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\, e^{\frac{\sqrt{3}\, \arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)}
\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\left(-c_2\sqrt{3}\,\sin(x) + 2c_1\left(\cos(x) + \frac{\sin(x)}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\,\sin(x) + 2\left(\cos(x) + \frac{\sin(x)}{2}\right.\right.}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right.$$

$$\frac{d}{dx}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right)\left(\frac{\left(-c_2\sqrt{3}\,\sin(x) + 2c_1\left(\cos(x) + \frac{\sin(x)}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right.}\right.$$

$$d\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = \left(\frac{\left(\left(-c_2\sqrt{3}\,\sin(x) + 2c_1\left(\cos(x) + \frac{\sin(x)}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\,\sin(x) + 2\left(\cos(x) + \frac{\sin(x)}{2}\right)\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}\right.$$

Integrating gives

$$\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)} = \int \frac{\left(\left(-c_2\sqrt{3}\,\sin(x) + 2c_1\left(\cos(x) + \frac{\sin(x)}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\right.}$$

$$= \int \frac{\left(\left(-c_2\sqrt{3}\,\sin(x) + 2c_1\left(\cos(x) + \frac{\sin(x)}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2\sin\right.}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{c_1+c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)}$ gives the final solution

$$y = \frac{\left(c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(-c_2\sqrt{3}\,\sin(x)+2c_1\left(\cos(x)+\frac{\sin(x)}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\,\sin(x)\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1 + c_2\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}\left(\int \frac{\left(\left(-c_2\sqrt{3}\,\sin(x)+2c_1\left(\cos(x)+\frac{\sin(x)}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}\,\sin(x)+2\right.\right.}{\left(2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(c_1\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \Big(\int \frac{\big(\big(-c_2\sqrt{3}\sin(x)+2c_1\big(\cos(x)+\frac{\sin(x)}{2}\big)\big)\cos\big(\frac{\sqrt{3}\,x}{2}\big)+\sin\big(\frac{\sqrt{3}\,x}{2}\big)\big(c_1\sqrt{3}\sin(x)+2\big(\cos}{\big(2c_1\cos\big(\frac{\sqrt{3}\,x}{2}\big)+2c_2\sin\big(\frac{\sqrt{3}\,x}{2}\big)\big)\big(c_1+c_2}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{3}\,x}{2}\right)\right) e^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} \Big(\int \frac{\big(\big(-c_2\sqrt{3}\sin(x)+2c_1\big(\cos(x)+\frac{\sin(x)}{2}\big)\big)\cos\big(\frac{\sqrt{3}\,x}{2}\big)+\sin\big(\frac{\sqrt{3}\,x}{2}\big)\big(c_1\sqrt{3}\sin(x)+2\big(\cos}{\big(2c_1\cos\big(\frac{\sqrt{3}\,x}{2}\big)+2c_2\sin\big(\frac{\sqrt{3}\,x}{2}\big)\big)\big(c_1+c_2}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

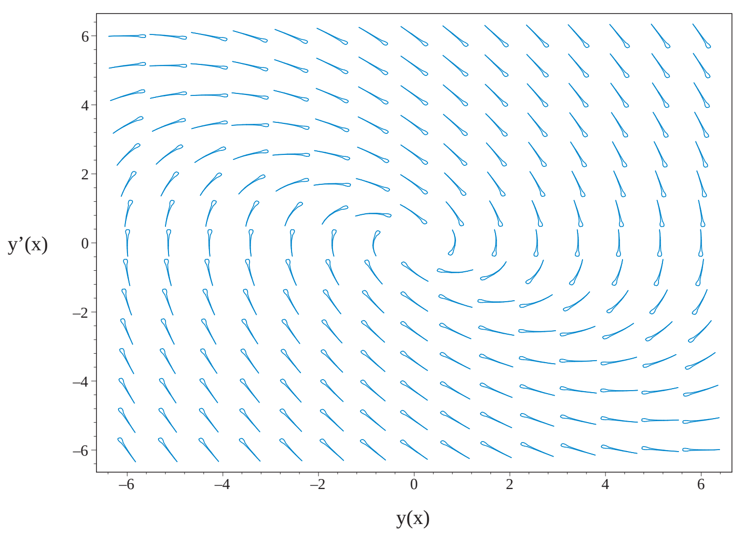


Figure 2.98: Slope field plot
$y'' + y' + y = \cos(x)$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = \cos(x)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + r + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$
- Roots of the characteristic polynomial
  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$
- 1st solution of the homogeneous ODE
  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$
- 2nd solution of the homogeneous ODE

$y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos(x)\right]$
- ○ Wronskian of solutions of the homogeneous equation
  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$$
- ○ Compute Wronskian
  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,\mathrm{e}^{-x}}{2}$
- ○ Substitute functions into equation for $y_p(x)$
  $y_p(x) = -\frac{2\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \cos(x) \sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \cos(x) \cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$
- ○ Compute integrals
  $y_p(x) = \sin(x)$

- Substitute particular solution into general solution to ODE
  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 33

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = cos(x),y(x),singsol=all)
```

$$y = c_2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + \sin(x)$$

Mathematica DSolve solution

Solving time : `1.198` (sec)
Leaf size : 50

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==Cos[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sin(x) + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

2.1.31 Problem 31

Internal problem ID [9102]
Book : Second order enumerated odes
Section : section 1
Problem number : 31
Date solved : `Monday, January 27, 2025 at 05:41:49 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y' = 1$$

Solved as second order linear constant coeff ode

Time used: 0.112 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)}\\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 + c_2 \,\mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 \,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2\,\mathrm{e}^{-x}\right) + (x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x + c_1 + c_2\,\mathrm{e}^{-x}$$

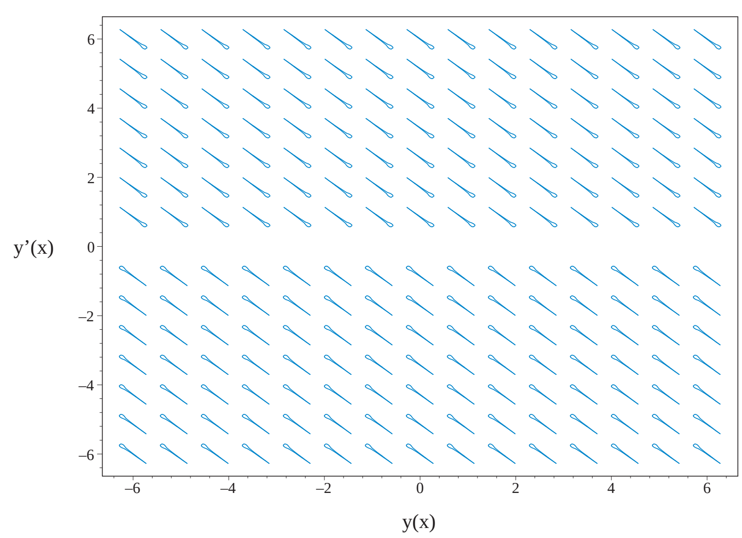


Figure 2.99: Slope field plot
$y'' + y' = 1$

Solved as second order linear exact ode

Time used: 0.110 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$y'+y=\int 1\,dx$$

We now have a first order ode to solve which is

$$y'+y=x+c_1$$

In canonical form a linear first order is

$$y'+q(x)y=p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x)&=1\\ p(x)&=x+c_1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu&=e^{\int q\,dx}\\ &=\mathrm{e}^{\int 1dx}\\ &=\mathrm{e}^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=\mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\,(x+c_1)\\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x})&=(\mathrm{e}^{x})\,(x+c_1)\\ \mathrm{d}(y\,\mathrm{e}^{x})&=((x+c_1)\,\mathrm{e}^{x})\;\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{x}&=\int (x+c_1)\,\mathrm{e}^{x}\,dx\\ &=(x+c_1-1)\,\mathrm{e}^{x}+c_2 \end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y=c_1+x-1+c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y=c_1+x-1+c_2\,\mathrm{e}^{-x}$$

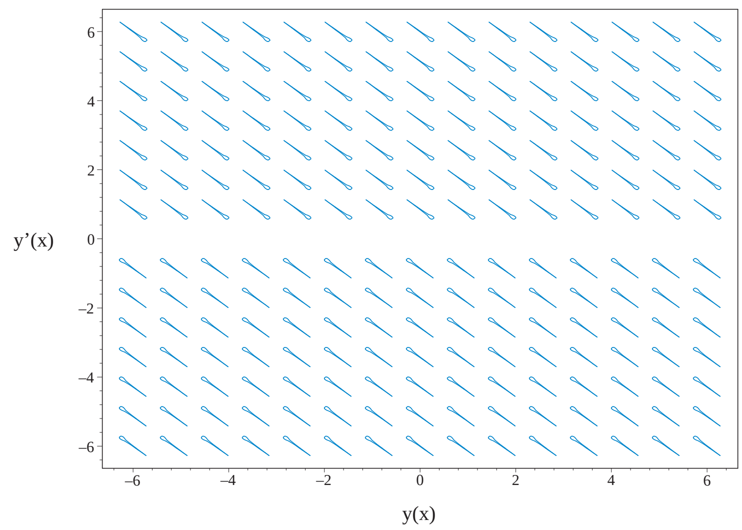


Figure 2.100: Slope field plot
$y'' + y' = 1$

Solved as second order missing y ode

Time used: 0.164 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) - 1 = 0$$

Which is now solved for $u(x)$ as first order ode.

Integrating gives

$$\int \frac{1}{1-u} du = dx$$
$$-\ln(-1+u) = x + c_1$$

Singular solutions are found by solving

$$1 - u = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$u(x) = 1$$

Solving for $u(x)$ gives

$$u(x) = 1$$
$$u(x) = e^{-x-c_1} + 1$$

In summary, these are the solution found for $u(x)$

$$u(x) = 1$$
$$u(x) = e^{-x-c_1} + 1$$

For solution $u(x) = 1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = 1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 1\, dx$$
$$y = x + c_2$$

For solution $u(x) = \mathrm{e}^{-x-c_1} + 1$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = \mathrm{e}^{-x-c_1} + 1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \mathrm{e}^{-x-c_1} + 1\, dx$$
$$y = x - \mathrm{e}^{-x-c_1} + c_3$$

In summary, these are the solution found for (y)

$$y = x + c_2$$
$$y = x - \mathrm{e}^{-x-c_1} + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x + c_2$$
$$y = x - \mathrm{e}^{-x-c_1} + c_3$$

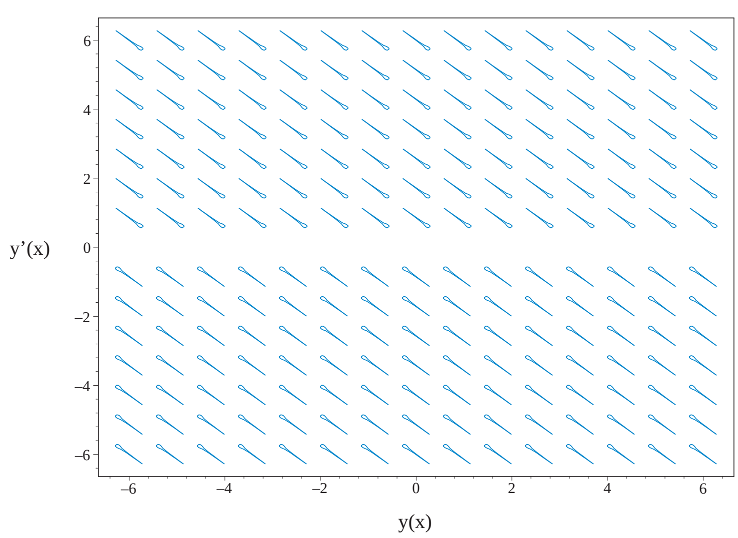


Figure 2.101: Slope field plot
$y'' + y' = 1$

Solved as second order integrable as is ode

Time used: 0.052 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int 1 dx$$
$$y' + y = x + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = x + c_1$$

The integrating factor μ is

$$\mu = e^{\int q\, dx} = e^{\int 1 dx} = e^{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)(x + c_1)$$
$$\frac{d}{dx}(y\,e^{x}) = (e^{x})(x + c_1)$$
$$d(y\,e^{x}) = ((x + c_1)\,e^{x})\, dx$$

Integrating gives

$$y\,e^{x} = \int (x + c_1)\,e^{x}\, dx = (x + c_1 - 1)\,e^{x} + c_2$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = c_1 + x - 1 + c_2\,e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 + x - 1 + c_2\,e^{-x}$$

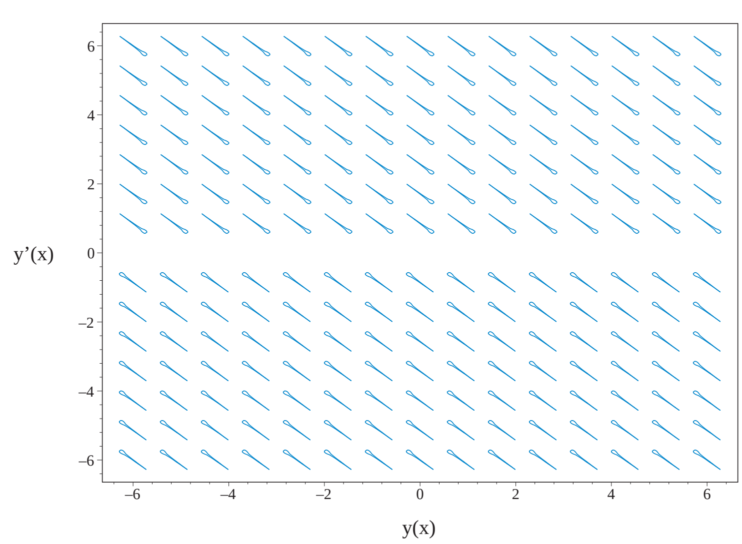


Figure 2.102: Slope field plot
$y'' + y' = 1$

Solved as second order integrable as is ode (ABC method)

Time used: 0.050 (sec)

Writing the ode as

$$y'' + y' = 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int 1 dx$$
$$y' + y = x + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx}\\ &= \mathrm{e}^{\int 1 dx}\\ &= \mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(x + c_1)\\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})\,(x + c_1)\\ \mathrm{d}(y\,\mathrm{e}^{x}) &= ((x + c_1)\,\mathrm{e}^{x})\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{x} &= \int (x + c_1)\,\mathrm{e}^{x}\, dx\\ &= (x + c_1 - 1)\,\mathrm{e}^{x} + c_2\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = c_1 + x - 1 + c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

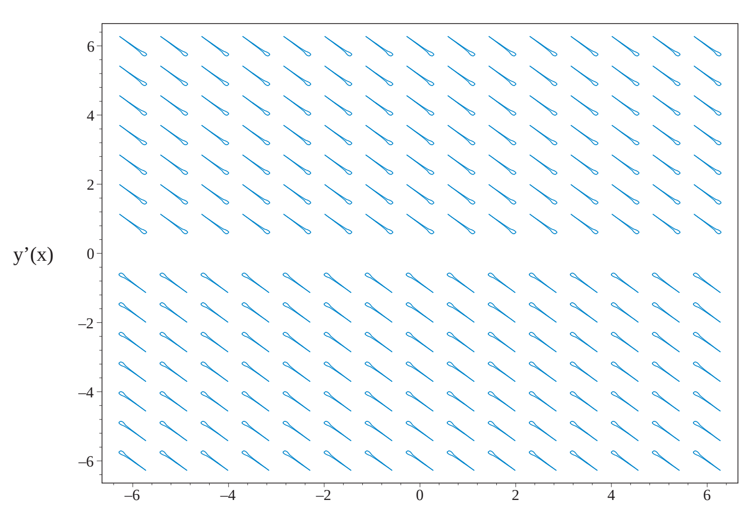


Figure 2.103: Slope field plot
$y'' + y' = 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.060 (sec)

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.40: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx}\\ &= z_1 e^{-\frac{x}{2}}\\ &= z_1\left(e^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx\\ &= y_1(e^{x})\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(e^{-x}\right) + c_2\left(e^{-x}(e^{x})\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \,\mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \,\mathrm{e}^{-x} + c_2\right) + (x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{-x} + c_2 + x$$

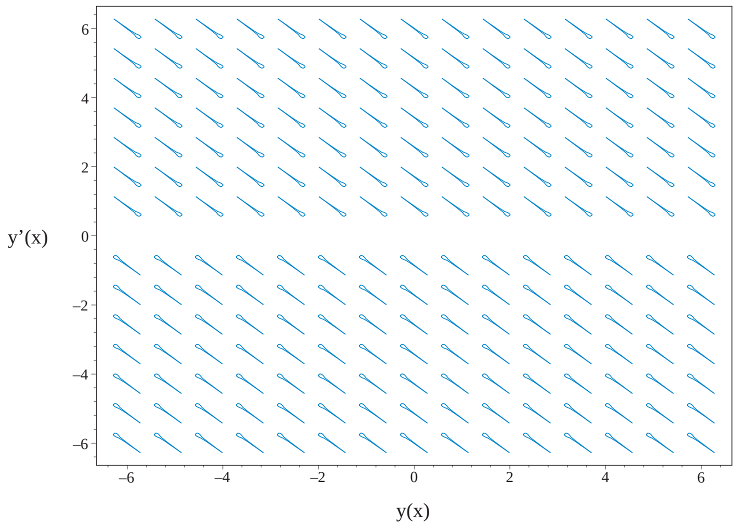


Figure 2.104: Slope field plot
$y'' + y' = 1$

Solved as second order ode adjoint method

Time used: 0.486 (sec)

In normal form the ode

$$y'' + y' = 1 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (0) &= 0 \\ \xi''(x) - \xi'(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(0)}\\ &= \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ \xi &= c_1 e^{(1)x} + c_2 e^{(0)x}\end{aligned}$$

Or

$$\xi = c_1 \, \mathrm{e}^{x} + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' + y\left(1 - \frac{c_1 \, \mathrm{e}^{x}}{c_1 \, \mathrm{e}^{x} + c_2}\right) = \frac{c_2 x + c_1 \, \mathrm{e}^{x}}{c_1 \, \mathrm{e}^{x} + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= \frac{c_2}{c_1 \, \mathrm{e}^{x} + c_2}\\ p(x) &= \frac{c_2 x + c_1 \, \mathrm{e}^{x}}{c_1 \, \mathrm{e}^{x} + c_2}\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx}\\ &= \mathrm{e}^{\int \frac{c_2}{c_1 \, \mathrm{e}^{x} + c_2} dx}\\ &= \frac{\mathrm{e}^{x}}{c_1 \, \mathrm{e}^{x} + c_2}\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_2 x + c_1 e^x}{c_1 e^x + c_2}\right)$$

$$\frac{d}{dx}\left(\frac{y e^x}{c_1 e^x + c_2}\right) = \left(\frac{e^x}{c_1 e^x + c_2}\right)\left(\frac{c_2 x + c_1 e^x}{c_1 e^x + c_2}\right)$$

$$d\left(\frac{y e^x}{c_1 e^x + c_2}\right) = \left(\frac{(c_2 x + c_1 e^x) e^x}{(c_1 e^x + c_2)^2}\right) dx$$

Integrating gives

$$\frac{y e^x}{c_1 e^x + c_2} = \int \frac{(c_2 x + c_1 e^x) e^x}{(c_1 e^x + c_2)^2} dx$$

$$= \frac{c_2}{c_1 (c_1 e^x + c_2)} + \frac{x e^x}{c_1 e^x + c_2} + c_3$$

Dividing throughout by the integrating factor $\frac{e^x}{c_1 e^x + c_2}$ gives the final solution

$$y = \frac{c_2 (c_3 c_1 + 1) e^{-x} + c_1 (c_3 c_1 + x)}{c_1}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{c_2 (c_3 c_1 + 1) e^{-x} + c_1 (c_3 c_1 + x)}{c_1}$$

The constants can be merged to give

$$y = \frac{c_2 (c_1 + 1) e^{-x} + c_1 (x + c_1)}{c_1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_2 (c_1 + 1) e^{-x} + c_1 (x + c_1)}{c_1}$$

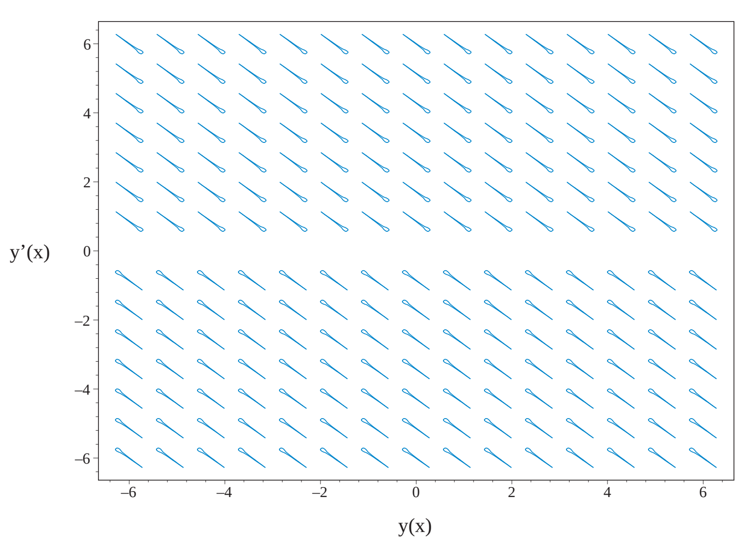


Figure 2.105: Slope field plot
$y'' + y' = 1$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\,\mathrm{e}^{-x} + C2 + y_p(x)$

- □ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x}\left(\int \mathrm{e}^{x}dx\right) + \int 1dx$

  - Compute integrals

    $y_p(x) = x - 1$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\,\mathrm{e}^{-x} + C2 + x - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = 1,y(x),singsol=all)
```

$$y = -c_1\mathrm{e}^{-x} + x + c_2$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==1,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x - c_1 e^{-x} + c_2$$

2.1.32 Problem 32

Internal problem ID [9103]
Book : Second order enumerated odes
Section : section 1
Problem number : 32
Date solved : `Monday, January 27, 2025 at 05:41:51 PM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$y'' + y' = x$$

Solved as second order linear constant coeff ode

Time used: 0.117 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\lambda_{1,2} = \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)}$$
$$= -\frac{1}{2} \pm \frac{1}{2}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 + c_2 \,\mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 \,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2\,\mathrm{e}^{-x}\right) + \left(\frac{1}{2}x^2 - x\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} - x + c_1 + c_2\,\mathrm{e}^{-x}$$

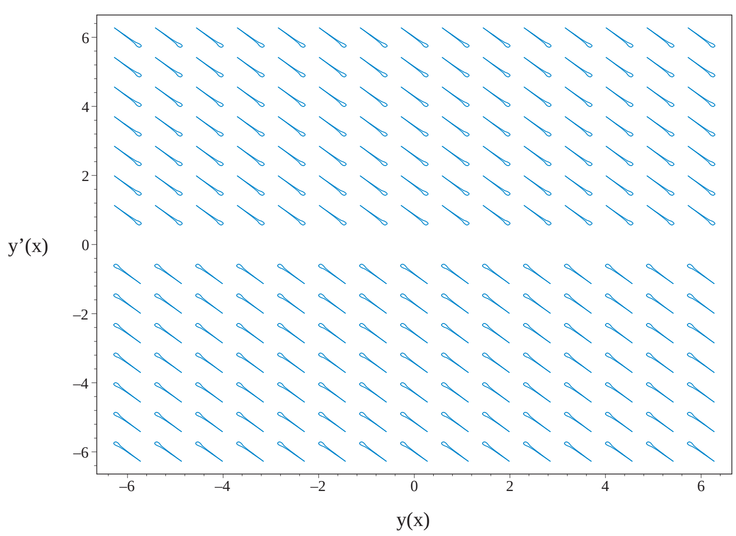


Figure 2.106: Slope field plot
$y'' + y' = x$

Solved as second order linear exact ode

Time used: 0.112 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
y' + y = \int x\, dx
$$

We now have a first order ode to solve which is

$$
y' + y = \frac{x^2}{2} + c_1
$$

In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= 1 \\
p(x) &= \frac{x^2}{2} + c_1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int 1 dx} \\
&= e^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= \mu p \\
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^2}{2} + c_1\right) \\
\frac{d}{dx}(y\, e^{x}) &= (e^{x})\left(\frac{x^2}{2} + c_1\right) \\
d(y\, e^{x}) &= \left(\left(\frac{x^2}{2} + c_1\right) e^{x}\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\, e^{x} &= \int \left(\frac{x^2}{2} + c_1\right) e^{x}\, dx \\
&= \frac{(x^2 + 2c_1 - 2x + 2)\, e^{x}}{2} + c_2
\end{aligned}
$$

Dividing throughout by the integrating factor e^x gives the final solution

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2 e^{-x}$$

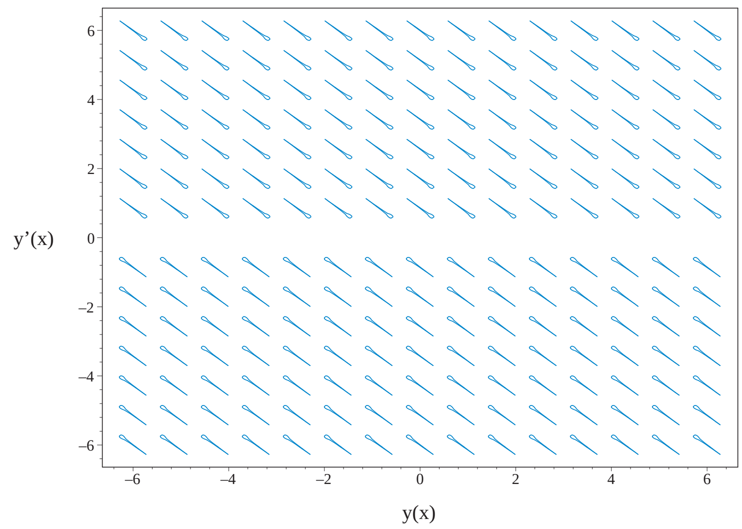


Figure 2.107: Slope field plot
$y'' + y' = x$

Solved as second order missing y ode

Time used: 0.160 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) - x = 0$$

Which is now solved for $u(x)$ as first order ode.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= e^{\int 1dx} \\ &= e^x\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) = (\mu)(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(u\,\mathrm{e}^{x}) = (\mathrm{e}^{x})(x)$$
$$\mathrm{d}(u\,\mathrm{e}^{x}) = (x\,\mathrm{e}^{x})\,\mathrm{d}x$$

Integrating gives

$$u\,\mathrm{e}^{x} = \int x\,\mathrm{e}^{x}\,dx$$
$$= (x-1)\,\mathrm{e}^{x} + c_1$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$u(x) = x - 1 + c_1\,\mathrm{e}^{-x}$$

In summary, these are the solution found for $u(x)$

$$u(x) = x - 1 + c_1\,\mathrm{e}^{-x}$$

For solution $u(x) = x - 1 + c_1\,\mathrm{e}^{-x}$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = x - 1 + c_1\,\mathrm{e}^{-x}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x - 1 + c_1\,\mathrm{e}^{-x}\,dx$$
$$y = -x + \frac{x^2}{2} - c_1\,\mathrm{e}^{-x} + c_2$$

In summary, these are the solution found for (y)

$$y = -x + \frac{x^2}{2} - c_1\,\mathrm{e}^{-x} + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -x + \frac{x^2}{2} - c_1\,\mathrm{e}^{-x} + c_2$$

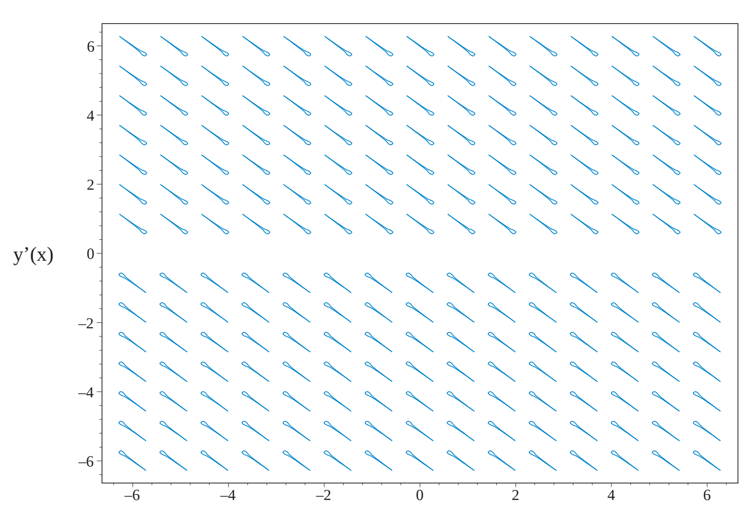


Figure 2.108: Slope field plot
$y'' + y' = x$

Solved as second order integrable as is ode

Time used: 0.053 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int x dx$$
$$y' + y = \frac{x^2}{2} + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = \frac{x^2}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= e^{\int 1 dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^2}{2} + c_1\right) \\ \frac{d}{dx}(y\,e^{x}) &= (e^{x})\left(\frac{x^2}{2} + c_1\right) \\ d(y\,e^{x}) &= \left(\left(\frac{x^2}{2} + c_1\right)e^{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,e^{x} &= \int \left(\frac{x^2}{2} + c_1\right) e^{x}\, dx \\ &= \frac{(x^2 + 2c_1 - 2x + 2)\,e^{x}}{2} + c_2\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2\,e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2\,e^{-x}$$

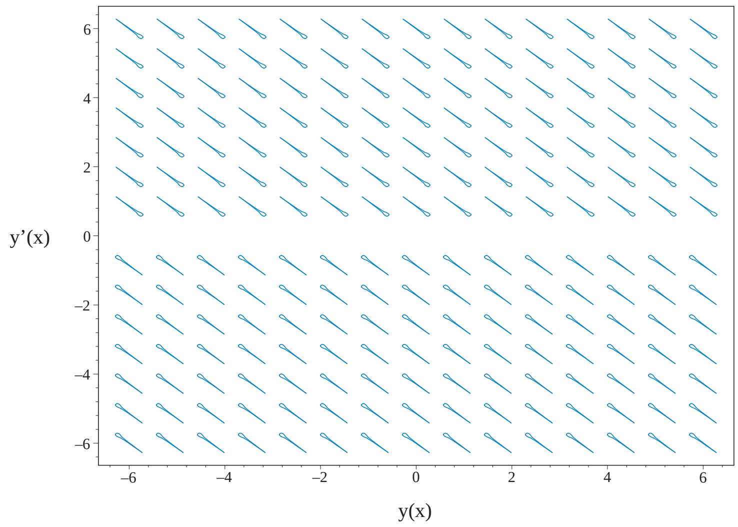


Figure 2.109: Slope field plot
$y'' + y' = x$

Solved as second order integrable as is ode (ABC method)

Time used: 0.053 (sec)

Writing the ode as

$$y'' + y' = x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int x dx$$
$$y' + y = \frac{x^2}{2} + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = \frac{x^2}{2} + c_1$$

The integrating factor μ is

$$\mu = e^{\int q\, dx}$$
$$= e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2}{2} + c_1\right)$$
$$\frac{d}{dx}(y\, e^x) = (e^x)\left(\frac{x^2}{2} + c_1\right)$$
$$d(y\, e^x) = \left(\left(\frac{x^2}{2} + c_1\right) e^x\right) dx$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{x} &= \int \left(\frac{x^{2}}{2}+c_{1}\right) \mathrm{e}^{x}\, dx \\
&= \frac{(x^{2}+2c_{1}-2x+2)\,\mathrm{e}^{x}}{2}+c_{2}
\end{aligned}
$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$
y = \frac{x^{2}}{2}+c_{1}-x+1+c_{2}\,\mathrm{e}^{-x}
$$

Will add steps showing solving for IC soon.

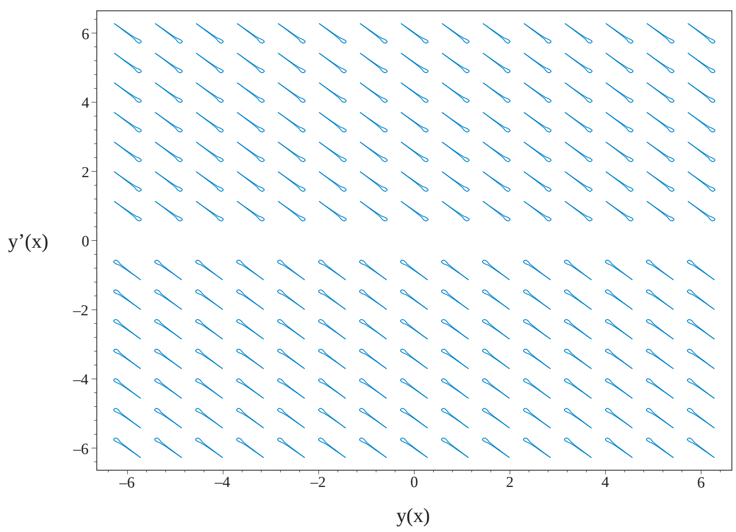


Figure 2.110: Slope field plot
$y''+y'=x$

Solved as second order ode using Kovacic algorithm

Time used: 0.132 (sec)

Writing the ode as

$$
\begin{align}
y''+y' &= 0 \tag{1}\\
Ay''+By'+Cy &= 0 \tag{2}
\end{align}
$$

Comparing (1) and (2) shows that

$$
\begin{align}
A &= 1 \nonumber\\
B &= 1 \tag{3}\\
C &= 0 \nonumber
\end{align}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \nonumber
\end{align}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.42: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-x}\right) + c_2\left(e^{-x}(e^x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \,\mathrm{e}^{-x} + c_2\right) + \left(\frac{1}{2}x^2 - x\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{-x} + c_2 + \frac{x^2}{2} - x$$

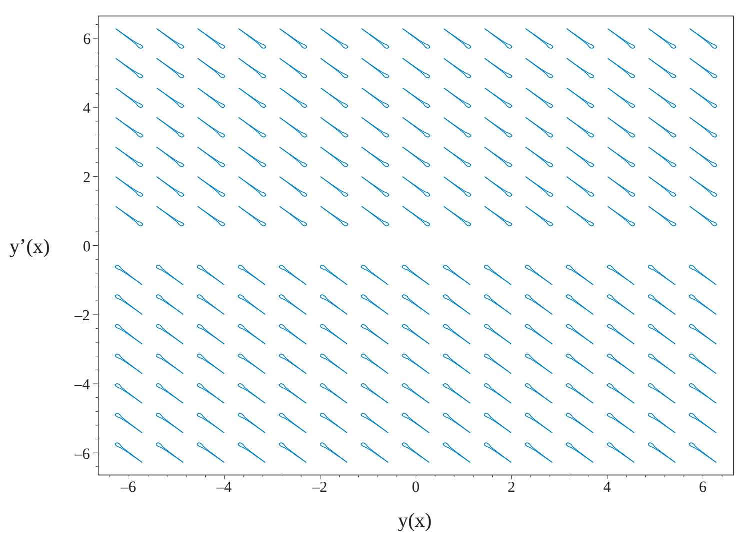


Figure 2.111: Slope field plot
$y'' + y' = x$

Solved as second order ode adjoint method

Time used: 0.464 (sec)

In normal form the ode

$$y'' + y' = x \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 0 \\
r(x) &= x
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (\xi(x))' + (0) &= 0 \\
\xi''(x) - \xi'(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
A\xi''(x) + B\xi'(x) + C\xi(x) = 0
$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - \lambda = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)} \\
&= \frac{1}{2} \pm \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{1}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= 0
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
\xi &= c_1 e^{(1)x} + c_2 e^{(0)x}
\end{aligned}
$$

Or

$$
\xi = c_1\, \mathrm{e}^{x} + c_2
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y' + y\left(1 - \frac{c_1\,e^x}{c_1\,e^x + c_2}\right) = \frac{\frac{c_2\,x^2}{2} + c_1(x\,e^x - e^x)}{c_1\,e^x + c_2}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{c_2}{c_1\,e^x + c_2} \\
p(x) &= \frac{2c_1(x-1)\,e^x + c_2\,x^2}{2c_1\,e^x + 2c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= e^{\int \frac{c_2}{c_1\,e^x + c_2}dx} \\
&= \frac{e^x}{c_1\,e^x + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= \mu p \\
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2c_1(x-1)\,e^x + c_2\,x^2}{2c_1\,e^x + 2c_2}\right) \\
\frac{d}{dx}\left(\frac{y\,e^x}{c_1\,e^x + c_2}\right) &= \left(\frac{e^x}{c_1\,e^x + c_2}\right)\left(\frac{2c_1(x-1)\,e^x + c_2\,x^2}{2c_1\,e^x + 2c_2}\right) \\
d\left(\frac{y\,e^x}{c_1\,e^x + c_2}\right) &= \left(\frac{(2c_1(x-1)\,e^x + c_2\,x^2)\,e^x}{(2c_1\,e^x + 2c_2)\,(c_1\,e^x + c_2)}\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\,e^x}{c_1\,e^x + c_2} &= \int \frac{(2c_1(x-1)\,e^x + c_2\,x^2)\,e^x}{(2c_1\,e^x + 2c_2)\,(c_1\,e^x + c_2)}\,dx \\
&= \frac{e^x - x\,e^x + \frac{e^x x^2}{2}}{c_1\,e^x + c_2} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{e^x}{c_1\,e^x + c_2}$ gives the final solution

$$
y = c_1c_3 + \frac{x^2}{2} - x + c_2c_3\,e^{-x} + 1
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = c_1c_3 + \frac{x^2}{2} - x + c_2c_3\,e^{-x} + 1
$$

The constants can be merged to give

$$
y = \frac{x^2}{2} + c_1 - x + 1 + c_2\,e^{-x}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} + c_1 - x + 1 + c_2\,\mathrm{e}^{-x}$$

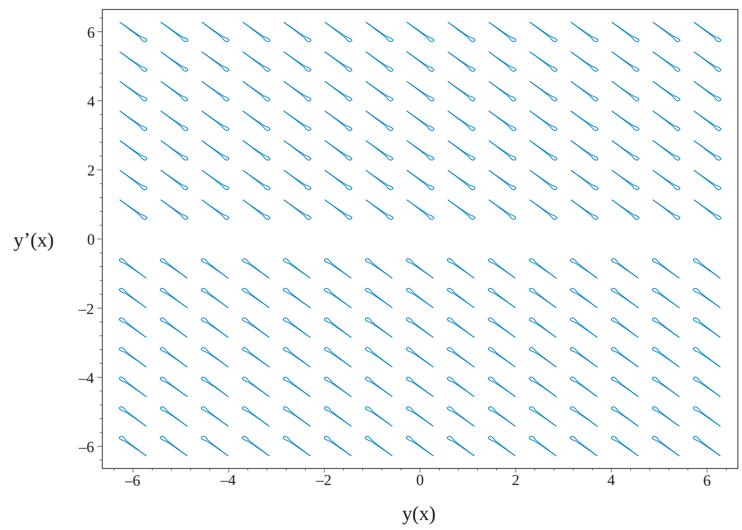


Figure 2.112: Slope field plot
$y'' + y' = x$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = x$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\,\mathrm{e}^{-x} + C2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^{-x}\left(\int \mathrm{e}^{x} x dx\right) + \int x dx$
- Compute integrals

  $y_p(x) = 1 - x + \frac{1}{2}x^2$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\,\mathrm{e}^{-x} + C2 + 1 - x + \frac{x^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 21

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = x,y(x),singsol=all)
```

$$y = \frac{x^2}{2} - c_1 \mathrm{e}^{-x} - x + c_2$$

Mathematica DSolve solution

Solving time : `0.041` (sec)
Leaf size : 27

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{x^2}{2} - x - c_1 e^{-x} + c_2$$

2.1.33 Problem 33

Internal problem ID [9104]
Book : Second order enumerated odes
Section : section 1
Problem number : 33
Date solved : `Monday, January 27, 2025 at 05:41:54 PM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$y'' + y' = 1 + x$$

Solved as second order linear constant coeff ode

Time used: 0.125 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = 1 + x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)}\\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \,\mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 \,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2\,\mathrm{e}^{-x}\right) + \left(\frac{x^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} + c_1 + c_2\,\mathrm{e}^{-x}$$

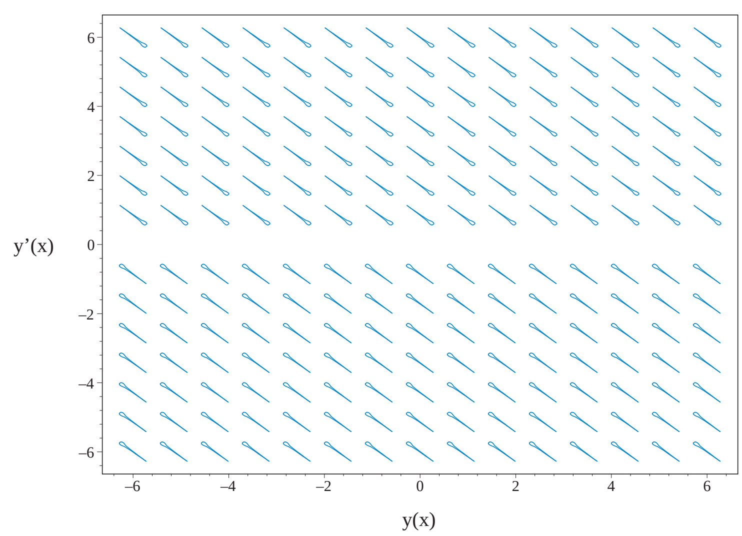


Figure 2.113: Slope field plot
$y'' + y' = 1 + x$

Solved as second order linear exact ode

Time used: 0.117 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 1 + x \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
y' + y = \int 1 + x\, dx
$$

We now have a first order ode to solve which is

$$
y' + y = x + \frac{1}{2}x^2 + c_1
$$

In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= 1 \\
p(x) &= x + \frac{1}{2}x^2 + c_1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x + \frac{1}{2}x^2 + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(y\, \mathrm{e}^{x}) &= (\mathrm{e}^{x})\left(x + \frac{1}{2}x^2 + c_1\right) \\
\mathrm{d}(y\, \mathrm{e}^{x}) &= \left(\left(x + \frac{1}{2}x^2 + c_1\right) \mathrm{e}^{x}\right)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\, \mathrm{e}^{x} &= \int \left(x + \frac{1}{2}x^2 + c_1\right) \mathrm{e}^{x}\, dx \\
&= \frac{(x^2 + 2c_1)\, \mathrm{e}^{x}}{2} + c_2
\end{aligned}
$$

Dividing throughout by the integrating factor e^x gives the final solution

$$y = \frac{x^2}{2} + c_1 + c_2 \,e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} + c_1 + c_2 \,e^{-x}$$

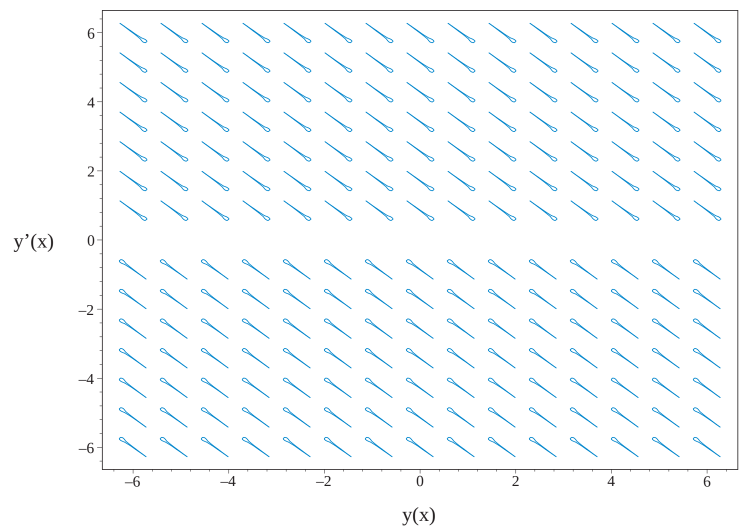


Figure 2.114: Slope field plot
$y'' + y' = 1 + x$

Solved as second order missing y ode

Time used: 0.119 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) - 1 - x = 0$$

Which is now solved for $u(x)$ as first order ode.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= 1 \\ p(x) &= 1 + x \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \,dx} \\ &= e^{\int 1 dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu u) = \mu p$$
$$\frac{d}{dx}(\mu u) = (\mu)(1+x)$$
$$\frac{d}{dx}(u\,e^x) = (e^x)(1+x)$$
$$d(u\,e^x) = ((1+x)\,e^x)\,dx$$

Integrating gives

$$u\,e^x = \int (1+x)\,e^x\,dx$$
$$= x\,e^x + c_1$$

Dividing throughout by the integrating factor e^x gives the final solution

$$u(x) = x + c_1\,e^{-x}$$

In summary, these are the solution found for $u(x)$

$$u(x) = x + c_1\,e^{-x}$$

For solution $u(x) = x + c_1\,e^{-x}$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = x + c_1\,e^{-x}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x + c_1\,e^{-x}\,dx$$
$$y = \frac{x^2}{2} - c_1\,e^{-x} + c_2$$

In summary, these are the solution found for (y)

$$y = \frac{x^2}{2} - c_1\,e^{-x} + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} - c_1\,e^{-x} + c_2$$

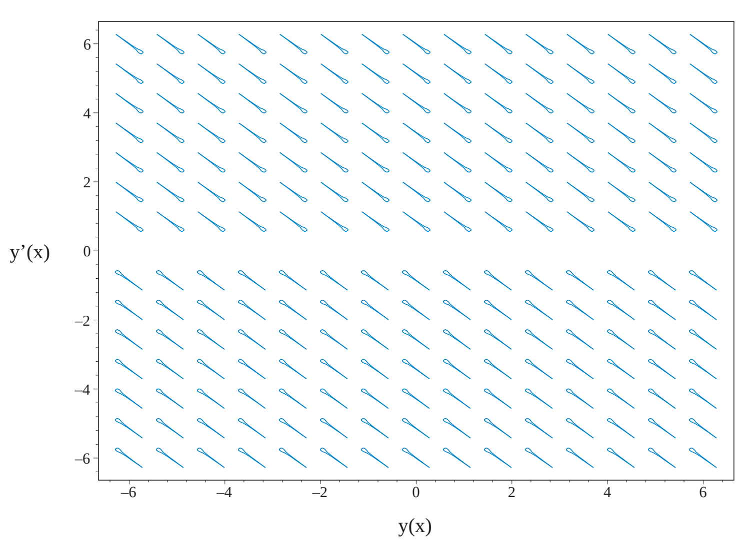


Figure 2.115: Slope field plot
$y'' + y' = 1 + x$

Solved as second order integrable as is ode

Time used: 0.057 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (1 + x)\, dx$$
$$y' + y = x + \frac{1}{2}x^2 + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = x + \frac{1}{2}x^2 + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= e^{\int 1 dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(x + \frac{1}{2}x^2 + c_1\right) \\ \frac{d}{dx}(y\, e^{x}) &= (e^{x})\left(x + \frac{1}{2}x^2 + c_1\right) \\ d(y\, e^{x}) &= \left(\left(x + \frac{1}{2}x^2 + c_1\right) e^{x}\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\, e^{x} &= \int \left(x + \frac{1}{2}x^2 + c_1\right) e^{x}\, dx \\ &= \frac{(x^2 + 2c_1)\, e^{x}}{2} + c_2\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = \frac{x^2}{2} + c_1 + c_2\, e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} + c_1 + c_2\, e^{-x}$$

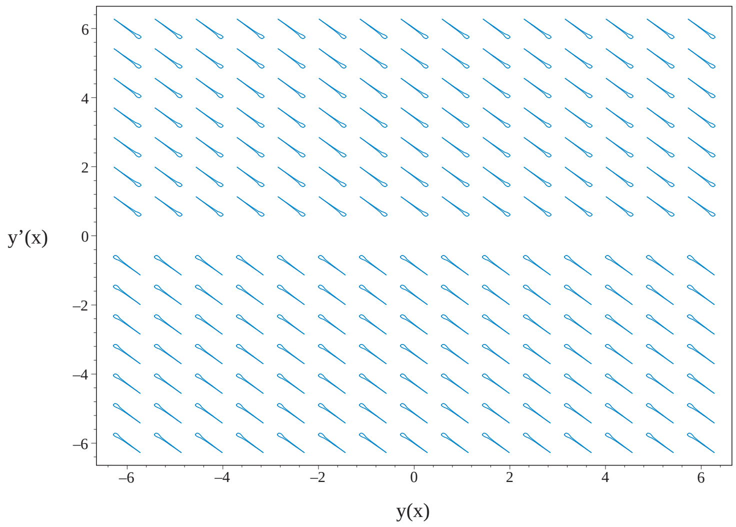


Figure 2.116: Slope field plot
$y'' + y' = 1 + x$

Solved as second order integrable as is ode (ABC method)

Time used: 0.058 (sec)

Writing the ode as

$$y'' + y' = 1 + x$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int (y'' + y')\, dx &= \int (1 + x)\, dx \\ y' + y &= x + \frac{1}{2}x^2 + c_1\end{aligned}$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= 1 \\ p(x) &= x + \frac{1}{2}x^2 + c_1\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= e^{\int 1 dx} \\ &= e^x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(x + \frac{1}{2}x^2 + c_1\right) \\ \frac{d}{dx}(y\,e^x) &= (e^x)\left(x + \frac{1}{2}x^2 + c_1\right) \\ d(y\,e^x) &= \left(\left(x + \frac{1}{2}x^2 + c_1\right)e^x\right)\, dx\end{aligned}$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{x} &= \int \left(x+\frac{1}{2}x^{2}+c_1\right)\mathrm{e}^{x}\,dx \\
&= \frac{\left(x^{2}+2c_1\right)\mathrm{e}^{x}}{2}+c_2
\end{aligned}
$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$
y = \frac{x^{2}}{2}+c_1+c_2\,\mathrm{e}^{-x}
$$

Will add steps showing solving for IC soon.

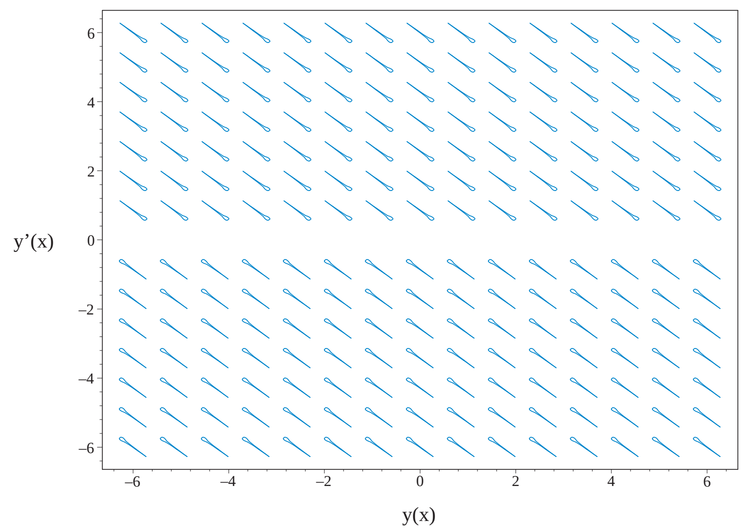


Figure 2.117: Slope field plot
$y''+y'=1+x$

Solved as second order ode using Kovacic algorithm

Time used: 0.066 (sec)

Writing the ode as

$$
\begin{aligned}
y''+y' &= 0 &(1)\\
Ay''+By'+Cy &= 0 &(2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 1 \qquad (3)\\
C &= 0
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1(e^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x}(e^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 2A_2 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \, \mathrm{e}^{-x} + c_2\right) + \left(\frac{x^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \, \mathrm{e}^{-x} + c_2 + \frac{x^2}{2}$$

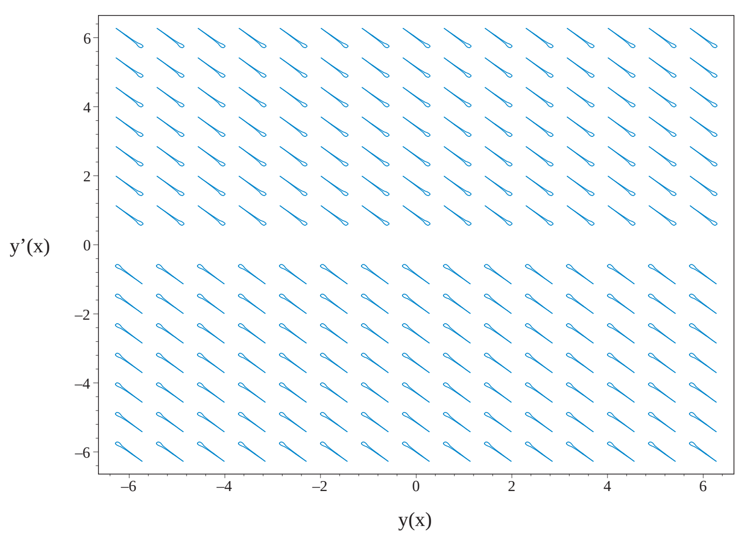


Figure 2.118: Slope field plot
$y'' + y' = 1 + x$

Solved as second order ode adjoint method

Time used: 0.556 (sec)

In normal form the ode

$$y'' + y' = 1 + x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 0 \\
r(x) &= 1 + x
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (\xi(x))' + (0) &= 0 \\
\xi''(x) - \xi'(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
A\xi''(x) + B\xi'(x) + C\xi(x) = 0
$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - \lambda = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)} \\
&= \frac{1}{2} \pm \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{1}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= 0
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
\xi &= c_1 e^{(1)x} + c_2 e^{(0)x}
\end{aligned}
$$

Or

$$
\xi = c_1\, \mathrm{e}^{x} + c_2
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(1 - \frac{c_1\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right) = \frac{c_2 x + c_1\,\mathrm{e}^{x} + \frac{c_2\, x^2}{2} + c_1(x\,\mathrm{e}^{x} - \mathrm{e}^{x})}{c_1\,\mathrm{e}^{x} + c_2}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{c_2}{c_1\,\mathrm{e}^{x} + c_2} \\
p(x) &= \frac{x(2c_1\,\mathrm{e}^{x} + c_2 x + 2c_2)}{2c_1\,\mathrm{e}^{x} + 2c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int \frac{c_2}{c_1\,\mathrm{e}^{x} + c_2} dx} \\
&= \frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x(2c_1\,\mathrm{e}^{x} + c_2 x + 2c_2)}{2c_1\,\mathrm{e}^{x} + 2c_2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right) &= \left(\frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right)\left(\frac{x(2c_1\,\mathrm{e}^{x} + c_2 x + 2c_2)}{2c_1\,\mathrm{e}^{x} + 2c_2}\right) \\
\mathrm{d}\left(\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}\right) &= \left(\frac{x(2c_1\,\mathrm{e}^{x} + c_2 x + 2c_2)\,\mathrm{e}^{x}}{(2c_1\,\mathrm{e}^{x} + 2c_2)\,(c_1\,\mathrm{e}^{x} + c_2)}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2} &= \int \frac{x(2c_1\,\mathrm{e}^{x} + c_2 x + 2c_2)\,\mathrm{e}^{x}}{(2c_1\,\mathrm{e}^{x} + 2c_2)\,(c_1\,\mathrm{e}^{x} + c_2)}\, dx \\
&= \frac{x^2\mathrm{e}^{x}}{2c_1\,\mathrm{e}^{x} + 2c_2} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{x}}{c_1\,\mathrm{e}^{x} + c_2}$ gives the final solution

$$
y = c_1 c_3 + \frac{x^2}{2} + c_2 c_3\,\mathrm{e}^{-x}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = c_1 c_3 + \frac{x^2}{2} + c_2 c_3\,\mathrm{e}^{-x}
$$

The constants can be merged to give

$$
y = \frac{x^2}{2} + c_1 + c_2\,\mathrm{e}^{-x}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{2} + c_1 + c_2\,\mathrm{e}^{-x}$$

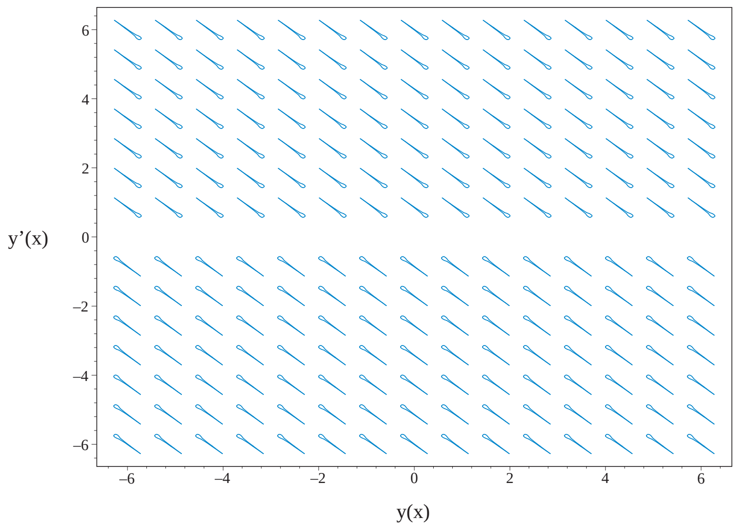


Figure 2.119: Slope field plot
$y'' + y' = 1 + x$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = x + 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$
- Factor the characteristic polynomial

  $r(r + 1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 0)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$
- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\,\mathrm{e}^{-x} + C2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x + 1\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-x}\left(\int e^x(x+1)\,dx\right) + \int (x+1)\,dx$
- Compute integrals

  $y_p(x) = \frac{x^2}{2}$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\,e^{-x} + C2 + \frac{x^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+_a+1, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : 0.000 (sec)
Leaf size : 18

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = x+1,y(x),singsol=all)
```

$$y = \frac{x^2}{2} - c_1 e^{-x} + c_2$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 24

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==1+x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2}{2} - c_1 e^{-x} + c_2$$

2.1.34 Problem 34

Internal problem ID [9105]
Book : Second order enumerated odes
Section : section 1
Problem number : 34
Date solved : `Monday, January 27, 2025 at 05:41:56 PM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$y'' + y' = x^2 + x + 1$$

Solved as second order linear constant coeff ode

Time used: 0.129 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \,\mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 \,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3x^2 A_3 + 2x A_2 + 6x A_3 + A_1 + 2A_2 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 2, A_2 = -\frac{1}{2}, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 \,\mathrm{e}^{-x}\right) + \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^3}{3} - \frac{x^2}{2} + 2x + c_1 + c_2 \,\mathrm{e}^{-x}$$

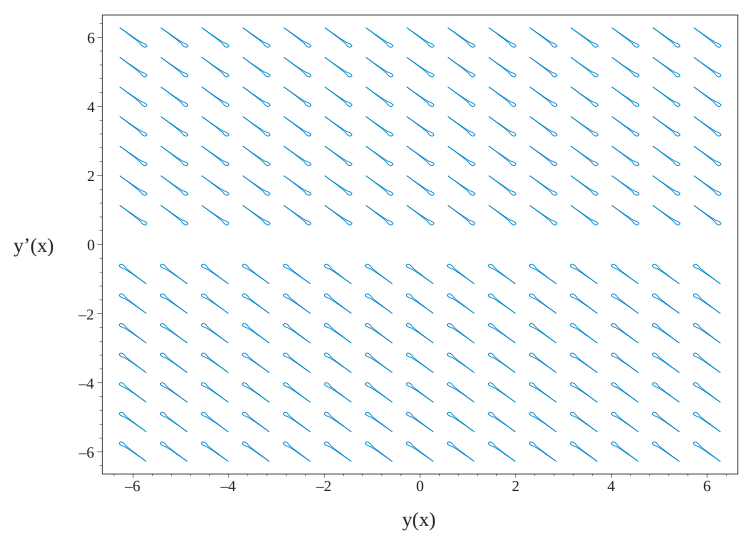


Figure 2.120: Slope field plot
$y'' + y' = x^2 + x + 1$

Solved as second order linear exact ode

Time used: 0.124 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x^2 + x + 1 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)
$$

Integrating gives

$$
p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\; dx
$$

Substituting the above values for p, q, r, s gives

$$
y' + y = \int x^2 + x + 1\, dx
$$

We now have a first order ode to solve which is

$$
y' + y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1
$$

In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= 1 \\
p(x) &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\
\mathrm{d}(y\,\mathrm{e}^{x}) &= \left(\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)\mathrm{e}^{x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{x} &= \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)\mathrm{e}^{x}\, dx \\
&= \frac{(2x^3 - 3x^2 + 6c_1 + 12x - 12)\,\mathrm{e}^{x}}{6} + c_2
\end{aligned}
$$

Dividing throughout by the integrating factor e^x gives the final solution

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 \, e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 \, e^{-x}$$

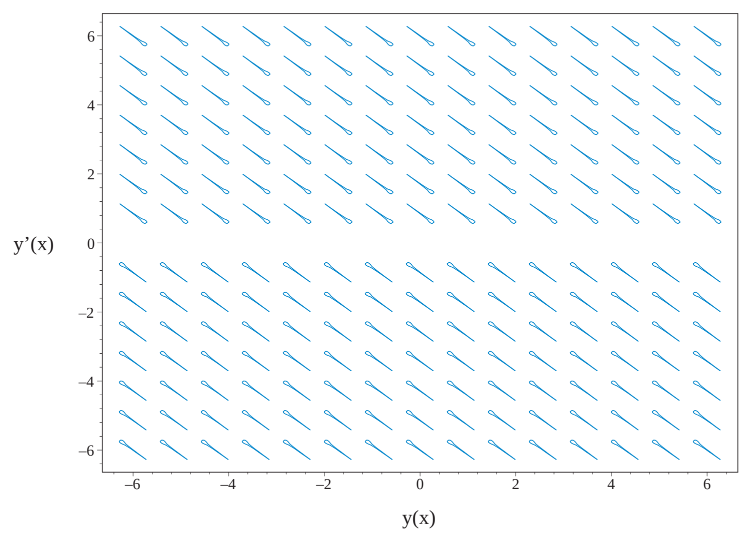


Figure 2.121: Slope field plot
$y'' + y' = x^2 + x + 1$

Solved as second order missing y ode

Time used: 0.186 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) - x^2 - x - 1 = 0$$

Which is now solved for $u(x)$ as first order ode.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= 1 \\ p(x) &= x^2 + x + 1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \, dx} \\ &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= (\mu)\left(x^2+x+1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(u\,\mathrm{e}^x) &= (\mathrm{e}^x)\left(x^2+x+1\right) \\
\mathrm{d}(u\,\mathrm{e}^x) &= \left(\left(x^2+x+1\right)\mathrm{e}^x\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
u\,\mathrm{e}^x &= \int \left(x^2+x+1\right)\mathrm{e}^x\,dx \\
&= \left(x^2-x+2\right)\mathrm{e}^x + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor e^x gives the final solution

$$
u(x) = x^2 - x + 2 + c_1\,\mathrm{e}^{-x}
$$

In summary, these are the solution found for $u(x)$

$$
u(x) = x^2 - x + 2 + c_1\,\mathrm{e}^{-x}
$$

For solution $u(x) = x^2 - x + 2 + c_1\,\mathrm{e}^{-x}$, since $u = y'$ then we now have a new first order ode to solve which is

$$
y' = x^2 - x + 2 + c_1\,\mathrm{e}^{-x}
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int x^2 - x + 2 + c_1\,\mathrm{e}^{-x}\,dx \\
y &= 2x + \frac{x^3}{3} - c_1\,\mathrm{e}^{-x} - \frac{x^2}{2} + c_2
\end{aligned}
$$

In summary, these are the solution found for (y)

$$
y = 2x + \frac{x^3}{3} - c_1\,\mathrm{e}^{-x} - \frac{x^2}{2} + c_2
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = 2x + \frac{x^3}{3} - c_1\,\mathrm{e}^{-x} - \frac{x^2}{2} + c_2
$$

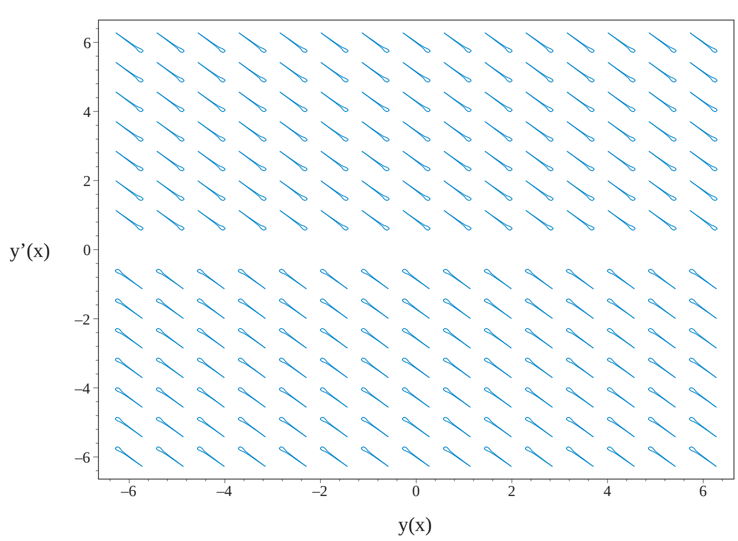


Figure 2.122: Slope field plot
$y'' + y' = x^2 + x + 1$

Solved as second order integrable as is ode

Time used: 0.059 (sec)

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y'' + y')\, dx &= \int (x^2 + x + 1)\, dx \\
y' + y &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1
\end{aligned}
$$

Which is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= 1 \\
p(x) &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int 1 dx} \\
&= e^x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= \mu p \\
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\
\frac{d}{dx}(y\, e^x) &= (e^x)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\
d(y\, e^x) &= \left(\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) e^x\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\, e^x &= \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) e^x\, dx \\
&= \frac{(2x^3 - 3x^2 + 6c_1 + 12x - 12)\, e^x}{6} + c_2
\end{aligned}
$$

Dividing throughout by the integrating factor e^x gives the final solution

$$
y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2\, e^{-x}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2\, e^{-x}
$$

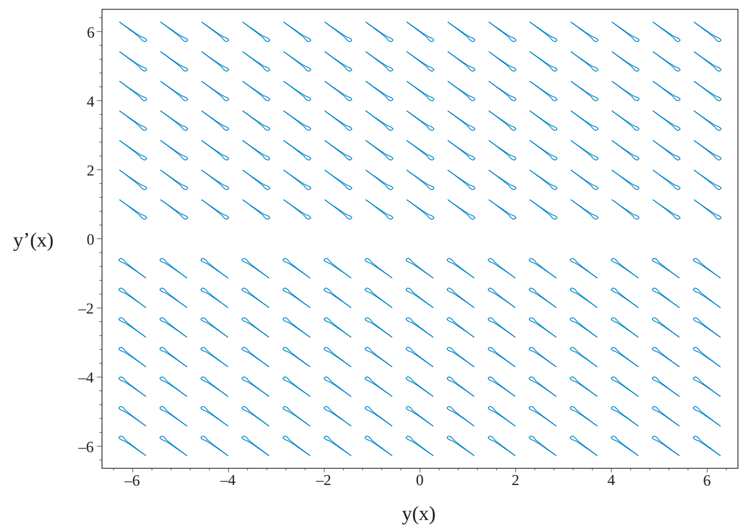


Figure 2.123: Slope field plot
$y'' + y' = x^2 + x + 1$

Solved as second order integrable as is ode (ABC method)

Time used: 0.059 (sec)

Writing the ode as

$$y'' + y' = x^2 + x + 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x^2 + x + 1)\, dx$$
$$y' + y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx}\\ &= \mathrm{e}^{\int 1 dx}\\ &= \mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) = (\mathrm{e}^{x})\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$\mathrm{d}(y\,\mathrm{e}^{x}) = \left(\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)\mathrm{e}^{x}\right)\, \mathrm{d}x$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{x} &= \int \left(\frac{1}{3}x^{3}+\frac{1}{2}x^{2}+x+c_1\right)\mathrm{e}^{x}\,dx \\ &= \frac{(2x^{3}-3x^{2}+6c_1+12x-12)\,\mathrm{e}^{x}}{6}+c_2 \end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = \frac{x^{3}}{3}-\frac{x^{2}}{2}+c_1+2x-2+c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

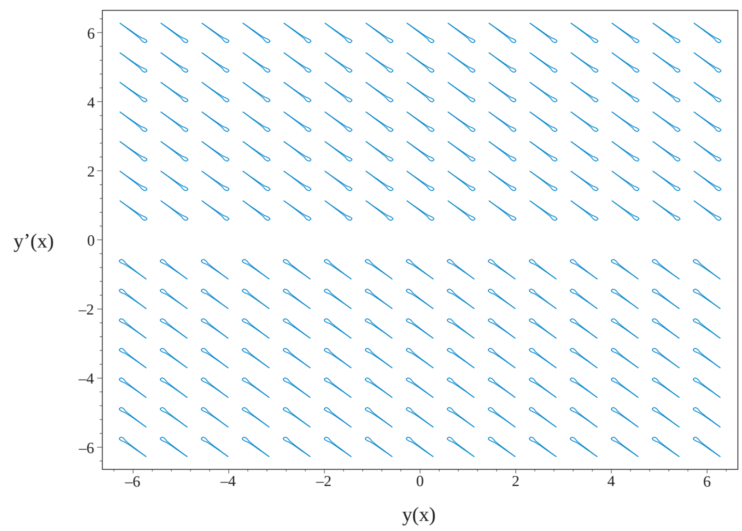


Figure 2.124: Slope field plot
$y''+y'=x^{2}+x+1$

Solved as second order ode using Kovacic algorithm

Time used: 0.071 (sec)

Writing the ode as

$$y''+y'=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=1\\ C&=0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1(e^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-x}\right) + c_2\left(e^{-x}(e^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3x^2A_3 + 2xA_2 + 6xA_3 + A_1 + 2A_2 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 2, A_2 = -\frac{1}{2}, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \,\mathrm{e}^{-x} + c_2\right) + \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{-x} + c_2 + \frac{x^3}{3} - \frac{x^2}{2} + 2x$$

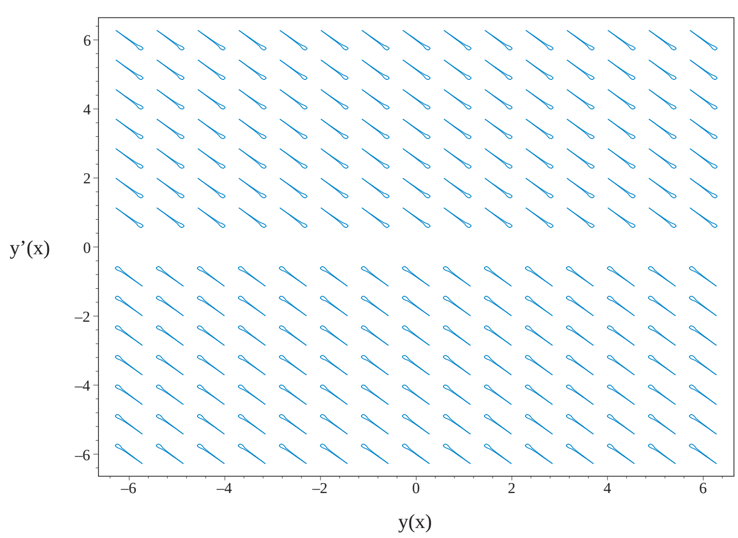


Figure 2.125: Slope field plot
$y'' + y' = x^2 + x + 1$

Solved as second order ode adjoint method

Time used: 0.569 (sec)

In normal form the ode

$$y'' + y' = x^2 + x + 1 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 0 \\
r(x) &= x^2 + x + 1
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (\xi(x))' + (0) &= 0 \\
\xi''(x) - \xi'(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
A\xi''(x) + B\xi'(x) + C\xi(x) = 0
$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - \lambda = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)} \\
&= \frac{1}{2} \pm \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{1}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= 0
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
\xi &= c_1 e^{(1)x} + c_2 e^{(0)x}
\end{aligned}
$$

Or

$$
\xi = c_1\, \mathrm{e}^{x} + c_2
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(1 - \frac{c_1\,e^x}{c_1\,e^x + c_2}\right) = \frac{c_2 x + c_1\,e^x + \frac{c_2 x^2}{2} + \frac{c_2 x^3}{3} + c_1(e^x x - e^x) + c_1(e^x x^2 - 2\,e^x x + 2\,e^x)}{c_1\,e^x + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{c_2}{c_1\,e^x + c_2}$$
$$p(x) = \frac{6c_1(x^2 - x + 2)\,e^x + 2\left(x^2 + \frac{3}{2}x + 3\right)c_2 x}{6c_1\,e^x + 6c_2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= e^{\int \frac{c_2}{c_1\,e^x + c_2}dx}\\ &= \frac{e^x}{c_1\,e^x + c_2}\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{6c_1(x^2 - x + 2)\,e^x + 2\left(x^2 + \frac{3}{2}x + 3\right)c_2 x}{6c_1\,e^x + 6c_2}\right)$$
$$\frac{d}{dx}\left(\frac{y\,e^x}{c_1\,e^x + c_2}\right) = \left(\frac{e^x}{c_1\,e^x + c_2}\right)\left(\frac{6c_1(x^2 - x + 2)\,e^x + 2\left(x^2 + \frac{3}{2}x + 3\right)c_2 x}{6c_1\,e^x + 6c_2}\right)$$
$$d\left(\frac{y\,e^x}{c_1\,e^x + c_2}\right) = \left(\frac{\left(6c_1(x^2 - x + 2)\,e^x + 2\left(x^2 + \frac{3}{2}x + 3\right)c_2 x\right)e^x}{(6c_1\,e^x + 6c_2)\,(c_1\,e^x + c_2)}\right)dx$$

Integrating gives

$$\begin{aligned}\frac{y\,e^x}{c_1\,e^x + c_2} &= \int \frac{\left(6c_1(x^2 - x + 2)\,e^x + 2\left(x^2 + \frac{3}{2}x + 3\right)c_2 x\right)e^x}{(6c_1\,e^x + 6c_2)\,(c_1\,e^x + c_2)}\,dx\\ &= \frac{-2\,e^x + \frac{x^3 e^x}{3} + 2\,e^x x - \frac{e^x x^2}{2}}{c_1\,e^x + c_2} + c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{e^x}{c_1\,e^x + c_2}$ gives the final solution

$$y = \frac{x^3}{3} + c_1 c_3 - \frac{x^2}{2} + 2x + c_2 c_3\,e^{-x} - 2$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{x^3}{3} + c_1 c_3 - \frac{x^2}{2} + 2x + c_2 c_3\,e^{-x} - 2$$

The constants can be merged to give

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 \,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + 2x - 2 + c_2 \,\mathrm{e}^{-x}$$

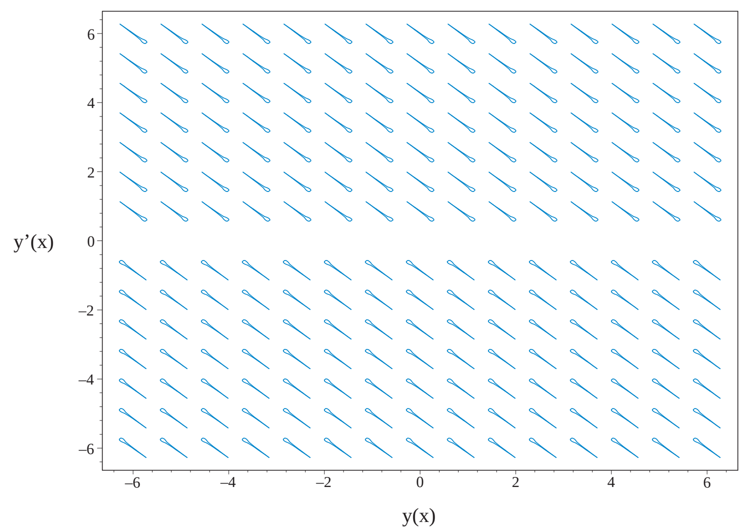


Figure 2.126: Slope field plot
$y'' + y' = x^2 + x + 1$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = x^2 + x + 1$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + r = 0$
- Factor the characteristic polynomial
  $r(r+1) = 0$
- Roots of the characteristic polynomial
  $r = (-1, 0)$
- 1st solution of the homogeneous ODE
  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE
  $y_2(x) = 1$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1 \,\mathrm{e}^{-x} + C2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^2 + x + 1\right]$
  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix}$$

- Compute Wronskian
  $W(y_1(x), y_2(x)) = e^{-x}$
- Substitute functions into equation for $y_p(x)$
  $y_p(x) = -e^{-x}\left(\int (x^2+x+1)\, e^x dx\right) + \int (x^2+x+1)\, dx$
- Compute integrals
  $y_p(x) = -\frac{1}{2}x^2 + 2x - 2 + \frac{1}{3}x^3$

- Substitute particular solution into general solution to ODE
  $y(x) = C1\, e^{-x} + C2 - \frac{x^2}{2} + 2x - 2 + \frac{x^3}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2-_b(_a)+_a+1, _b(_a)`   *** Sub
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 26

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = x^2+x+1,y(x),singsol=all)
```

$$y = \frac{x^3}{3} - c_1 e^{-x} - \frac{x^2}{2} + 2x + c_2$$

Mathematica DSolve solution

Solving time : 0.106 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==1+x+x^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^3}{3} - \frac{x^2}{2} + 2x - c_1 e^{-x} + c_2$$

2.1.35 Problem 35

Internal problem ID [9106]
Book : Second order enumerated odes
Section : section 1
Problem number : 35
Date solved : `Monday, January 27, 2025 at 05:41:59 PM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$y'' + y' = x^3 + x^2 + x + 1$$

Solved as second order linear constant coeff ode

Time used: 0.130 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x^3 + x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \lambda \, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\lambda_{1,2} = \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)}$$
$$= -\frac{1}{2} \pm \frac{1}{2}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 + c_2\,\mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2\,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_4x^4 + A_3x^3 + A_2x^2 + A_1x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4x^3A_4 + 3x^2A_3 + 12x^2A_4 + 2xA_2 + 6xA_3 + A_1 + 2A_2 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -4, A_2 = \frac{5}{2}, A_3 = -\frac{2}{3}, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2\,\mathrm{e}^{-x}\right) + \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x + c_1 + c_2\,\mathrm{e}^{-x}$$

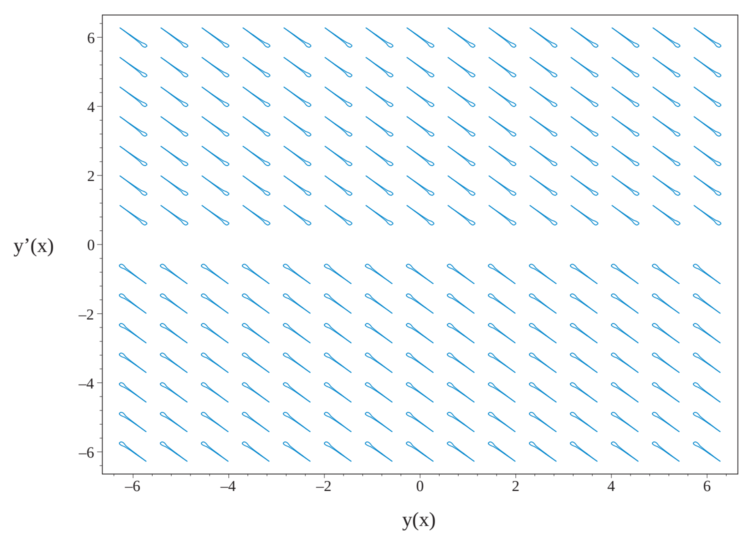


Figure 2.127: Slope field plot
$y'' + y' = x^3 + x^2 + x + 1$

Solved as second order linear exact ode

Time used: 0.138 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x^3 + x^2 + x + 1 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx
$$

Substituting the above values for p, q, r, s gives

$$
y' + y = \int x^3 + x^2 + x + 1\, dx
$$

We now have a first order ode to solve which is

$$
y' + y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1
$$

In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= 1 \\
p(x) &= \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\
\mathrm{d}(y\,\mathrm{e}^{x}) &= \left(\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)\mathrm{e}^{x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{x} &= \int \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)\mathrm{e}^{x}\, dx \\
&= \frac{(3x^4 - 8x^3 + 30x^2 + 12c_1 - 48x + 48)\,\mathrm{e}^{x}}{12} + c_2
\end{aligned}
$$

Dividing throughout by the integrating factor e^x gives the final solution

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2 e^{-x}$$

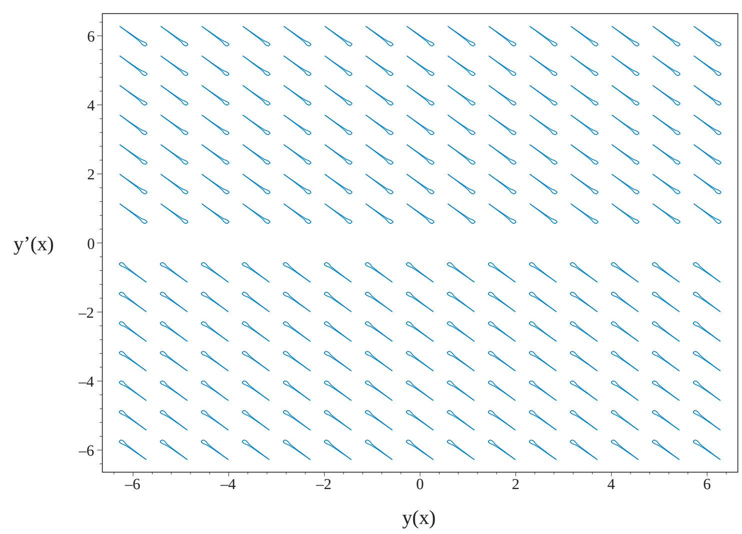


Figure 2.128: Slope field plot
$y'' + y' = x^3 + x^2 + x + 1$

Solved as second order missing y ode

Time used: 0.170 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) - x^3 - x^2 - x - 1 = 0$$

Which is now solved for $u(x)$ as first order ode.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= 1 \\ p(x) &= x^3 + x^2 + x + 1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \, dx} \\ &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu u) &= \mu p \\
\frac{d}{dx}(\mu u) &= (\mu)\left(x^3+x^2+x+1\right) \\
\frac{d}{dx}(u\,e^x) &= (e^x)\left(x^3+x^2+x+1\right) \\
d(u\,e^x) &= \left(\left(x^3+x^2+x+1\right)e^x\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
u\,e^x &= \int \left(x^3+x^2+x+1\right)e^x\,dx \\
&= \left(x^3-2x^2+5x-4\right)e^x + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor e^x gives the final solution

$$
u(x) = x^3-2x^2+5x-4+c_1\,e^{-x}
$$

In summary, these are the solution found for $u(x)$

$$
u(x) = x^3-2x^2+5x-4+c_1\,e^{-x}
$$

For solution $u(x) = x^3-2x^2+5x-4+c_1\,e^{-x}$, since $u = y'$ then we now have a new first order ode to solve which is

$$
y' = x^3-2x^2+5x-4+c_1\,e^{-x}
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int x^3-2x^2+5x-4+c_1\,e^{-x}\,dx \\
y &= -4x+\frac{x^4}{4}-c_1\,e^{-x}+\frac{5x^2}{2}-\frac{2x^3}{3}+c_2
\end{aligned}
$$

In summary, these are the solution found for (y)

$$
y = -4x+\frac{x^4}{4}-c_1\,e^{-x}+\frac{5x^2}{2}-\frac{2x^3}{3}+c_2
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = -4x+\frac{x^4}{4}-c_1\,e^{-x}+\frac{5x^2}{2}-\frac{2x^3}{3}+c_2
$$

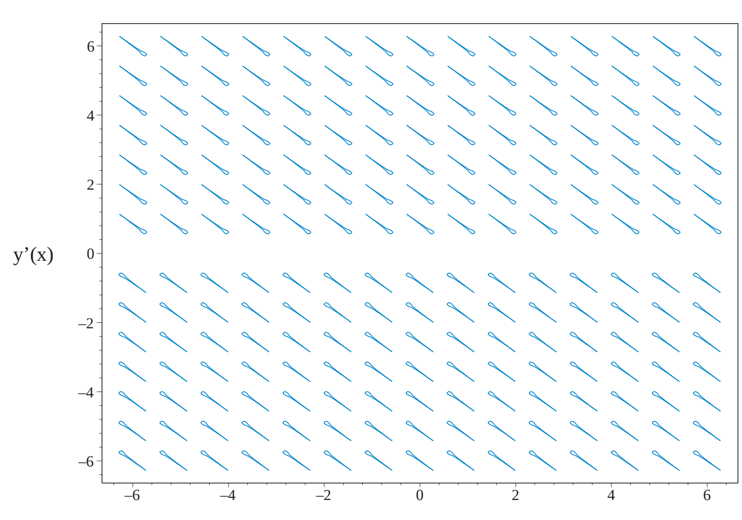


Figure 2.129: Slope field plot
$y'' + y' = x^3 + x^2 + x + 1$

Solved as second order integrable as is ode

Time used: 0.066 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x^3 + x^2 + x + 1)\, dx$$

$$y' + y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= 1 \\ p(x) &= \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\ \frac{d}{dx}(y\, e^x) &= (e^x)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) \\ d(y\, e^x) &= \left(\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) e^x\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\, e^x &= \int \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) e^x\, dx \\ &= \frac{(3x^4 - 8x^3 + 30x^2 + 12c_1 - 48x + 48)\, e^x}{12} + c_2 \end{aligned}$$

Dividing throughout by the integrating factor e^x gives the final solution

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2\, e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2\, e^{-x}$$

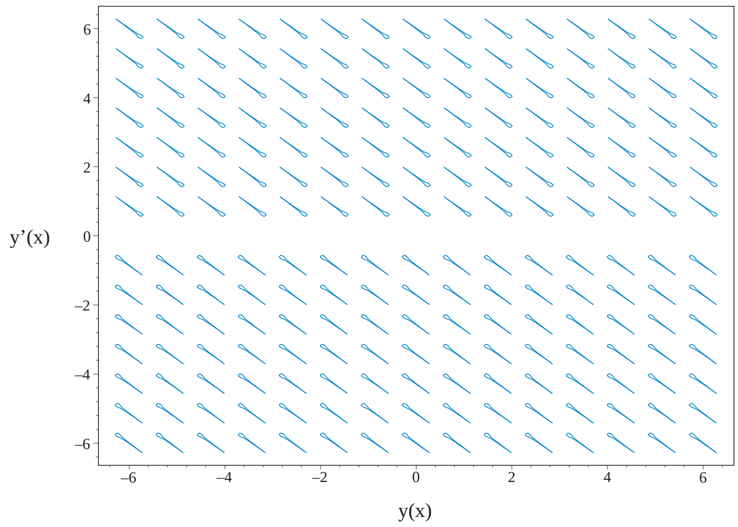


Figure 2.130: Slope field plot
$y'' + y' = x^3 + x^2 + x + 1$

Solved as second order integrable as is ode (ABC method)

Time used: 0.066 (sec)

Writing the ode as

$$y'' + y' = x^3 + x^2 + x + 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (x^3 + x^2 + x + 1)\, dx$$
$$y' + y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= e^{\int 1 dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$\frac{d}{dx}(y\, e^{x}) = (e^{x})\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right)$$
$$d(y\, e^{x}) = \left(\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c_1\right) e^{x}\right)\, dx$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{x} &= \int \left(\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x+c_1\right)\mathrm{e}^{x}\,dx \\
&= \frac{(3x^4-8x^3+30x^2+12c_1-48x+48)\,\mathrm{e}^{x}}{12}+c_2
\end{aligned}
$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$
y=\frac{x^4}{4}-\frac{2x^3}{3}+\frac{5x^2}{2}+c_1-4x+4+c_2\,\mathrm{e}^{-x}
$$

Will add steps showing solving for IC soon.

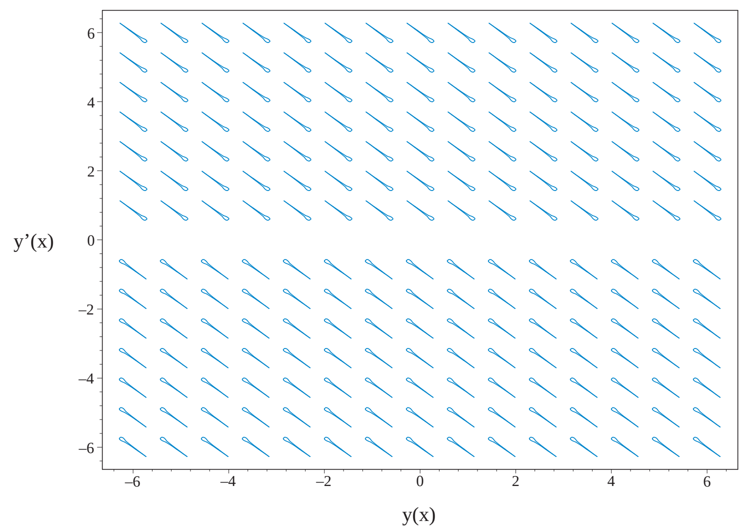


Figure 2.131: Slope field plot
$y''+y'=x^3+x^2+x+1$

Solved as second order ode using Kovacic algorithm

Time used: 0.138 (sec)

Writing the ode as

$$
\begin{aligned}
y''+y' &= 0 \quad (1)\\
Ay''+By'+Cy &= 0 \quad (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 1 \quad (3)\\
C &= 0
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x)=ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x)=rz(x) \quad (4)
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \quad (5)\\
&= \frac{2AB'-2BA'+B^2-4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$
r=\frac{1}{4} \quad (6)
$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.48: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2} dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^{-x}) + c_2(e^{-x}(e^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4x^3A_4 + 3x^2A_3 + 12x^2A_4 + 2xA_2 + 6xA_3 + A_1 + 2A_2 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -4, A_2 = \frac{5}{2}, A_3 = -\frac{2}{3}, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\,\mathrm{e}^{-x} + c_2) + \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{e}^{-x} + c_2 + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x$$

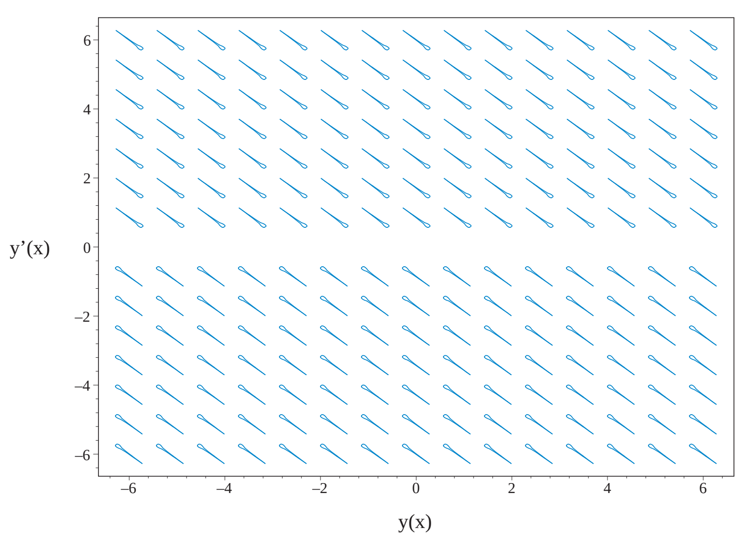


Figure 2.132: Slope field plot
$y'' + y' = x^3 + x^2 + x + 1$

Solved as second order ode adjoint method

Time used: 0.551 (sec)

In normal form the ode

$$y'' + y' = x^3 + x^2 + x + 1 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 0 \\
r(x) &= x^3 + x^2 + x + 1
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (\xi(x))' + (0) &= 0 \\
\xi''(x) - \xi'(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
A\xi''(x) + B\xi'(x) + C\xi(x) = 0
$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - \lambda = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)} \\
&= \frac{1}{2} \pm \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{1}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= 0
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
\xi &= c_1 e^{(1)x} + c_2 e^{(0)x}
\end{aligned}
$$

Or

$$
\xi = c_1\, \mathrm{e}^{x} + c_2
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(1 - \frac{c_1\, \mathrm{e}^{x}}{c_1\, \mathrm{e}^{x} + c_2}\right) = \frac{c_2 x + c_1\, \mathrm{e}^{x} + \frac{c_2\, x^{2}}{2} + \frac{c_2\, x^{3}}{3} + \frac{c_2\, x^{4}}{4} + c_1(\mathrm{e}^{x} x - \mathrm{e}^{x}) + c_1(\mathrm{e}^{x} x^{2} - 2\,\mathrm{e}^{x} x + 2\,\mathrm{e}^{x}) + c_1(\mathrm{e}^{x} x}{c_1\, \mathrm{e}^{x} + c_2}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{c_2}{c_1\, \mathrm{e}^{x} + c_2} \\
p(x) &= \frac{12c_1(x-1)\,(x^{2} - x + 4)\,\mathrm{e}^{x} + 3c_2\left(x^{3} + \frac{4}{3}x^{2} + 2x + 4\right) x}{12c_1\, \mathrm{e}^{x} + 12c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int \frac{c_2}{c_1\, \mathrm{e}^{x} + c_2} dx} \\
&= \frac{\mathrm{e}^{x}}{c_1\, \mathrm{e}^{x} + c_2}
\end{aligned}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{12c_1(x-1)\,(x^{2} - x + 4)\,\mathrm{e}^{x} + 3c_2\left(x^{3} + \frac{4}{3}x^{2} + 2x + 4\right) x}{12c_1\, \mathrm{e}^{x} + 12c_2}\right)
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\, \mathrm{e}^{x}}{c_1\, \mathrm{e}^{x} + c_2}\right) = \left(\frac{\mathrm{e}^{x}}{c_1\, \mathrm{e}^{x} + c_2}\right)\left(\frac{12c_1(x-1)\,(x^{2} - x + 4)\,\mathrm{e}^{x} + 3c_2\left(x^{3} + \frac{4}{3}x^{2} + 2x + 4\right) x}{12c_1\, \mathrm{e}^{x} + 12c_2}\right)
$$

$$
\mathrm{d}\left(\frac{y\, \mathrm{e}^{x}}{c_1\, \mathrm{e}^{x} + c_2}\right) = \left(\frac{\left(12c_1(x-1)\,(x^{2} - x + 4)\,\mathrm{e}^{x} + 3c_2\left(x^{3} + \frac{4}{3}x^{2} + 2x + 4\right) x\right) \mathrm{e}^{x}}{(12c_1\, \mathrm{e}^{x} + 12c_2)\,(c_1\, \mathrm{e}^{x} + c_2)}\right) \mathrm{d}x
$$

Integrating gives

$$
\begin{aligned}
\frac{y\, \mathrm{e}^{x}}{c_1\, \mathrm{e}^{x} + c_2} &= \int \frac{\left(12c_1(x-1)\,(x^{2} - x + 4)\,\mathrm{e}^{x} + 3c_2\left(x^{3} + \frac{4}{3}x^{2} + 2x + 4\right) x\right) \mathrm{e}^{x}}{(12c_1\, \mathrm{e}^{x} + 12c_2)\,(c_1\, \mathrm{e}^{x} + c_2)}\, dx \\
&= \frac{4\,\mathrm{e}^{x} + \frac{x^{4}\mathrm{e}^{x}}{4} - 4\,\mathrm{e}^{x} x + \frac{5\,\mathrm{e}^{x} x^{2}}{2} - \frac{2\,\mathrm{e}^{x} x^{3}}{3}}{c_1\, \mathrm{e}^{x} + c_2} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{x}}{c_1\, \mathrm{e}^{x} + c_2}$ gives the final solution

$$
y = \frac{x^{4}}{4} - \frac{2x^{3}}{3} + c_1 c_3 + \frac{5x^{2}}{2} - 4x + c_2 c_3\, \mathrm{e}^{-x} + 4
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{x^{4}}{4} - \frac{2x^{3}}{3} + c_1 c_3 + \frac{5x^{2}}{2} - 4x + c_2 c_3\, \mathrm{e}^{-x} + 4
$$

The constants can be merged to give

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + c_1 - 4x + 4 + c_2\,\mathrm{e}^{-x}$$

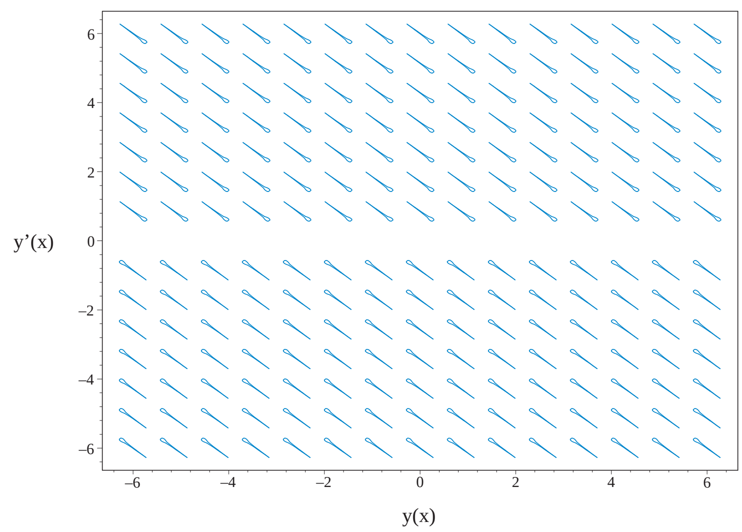


Figure 2.133: Slope field plot
$y'' + y' = x^3 + x^2 + x + 1$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = x^3 + x^2 + x + 1$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + r = 0$
- Factor the characteristic polynomial
  $r(r+1) = 0$
- Roots of the characteristic polynomial
  $r = (-1, 0)$
- 1st solution of the homogeneous ODE
  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE
  $y_2(x) = 1$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1\,\mathrm{e}^{-x} + C2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^3 + x^2 + x + 1\right]$
  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix}$$

- Compute Wronskian
  $W(y_1(x), y_2(x)) = e^{-x}$
- Substitute functions into equation for $y_p(x)$
  $y_p(x) = -e^{-x}\left(\int e^x(x+1)(x^2+1)\,dx\right) + \int (x^3+x^2+x+1)\,dx$
- Compute integrals
  $y_p(x) = -\frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x + 4 + \frac{1}{4}x^4$

- Substitute particular solution into general solution to ODE
  $y(x) = C1\,e^{-x} + C2 - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x + 4 + \frac{x^4}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^3+_a^2-_b(_a)+_a+1, _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 31

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = x^3+x^2+x+1,y(x),singsol=all)
```

$$y = \frac{x^4}{4} - c_1 e^{-x} + \frac{5x^2}{2} - \frac{2x^3}{3} - 4x + c_2$$

Mathematica DSolve solution

Solving time : 0.138 (sec)
Leaf size : 41

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==1+x+x^2+x^3,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - 4x - c_1 e^{-x} + c_2$$

2.1.36 Problem 36

Internal problem ID [9107]
Book : Second order enumerated odes
Section : section 1
Problem number : 36
Date solved : `Monday, January 27, 2025 at 05:42:02 PM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$y'' + y' = \sin(x)$$

Solved as second order linear constant coeff ode

Time used: 0.138 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{1^2 - (4)(1)(0)} \\
&= -\frac{1}{2} \pm \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{1}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= -1
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(0)x} + c_2 e^{(-1)x}
\end{aligned}
$$

Or

$$
y = c_1 + c_2 \,\mathrm{e}^{-x}
$$

Therefore the homogeneous solution y_h is

$$
y_h = c_1 + c_2 \,\mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x) + A_2 \sin(x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-A_1 \cos(x) - A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = \sin(x)
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{2}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 \,\mathrm{e}^{-x}\right) + \left(-\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)}{2} - \frac{\sin(x)}{2} + c_1 + c_2 \,\mathrm{e}^{-x}$$

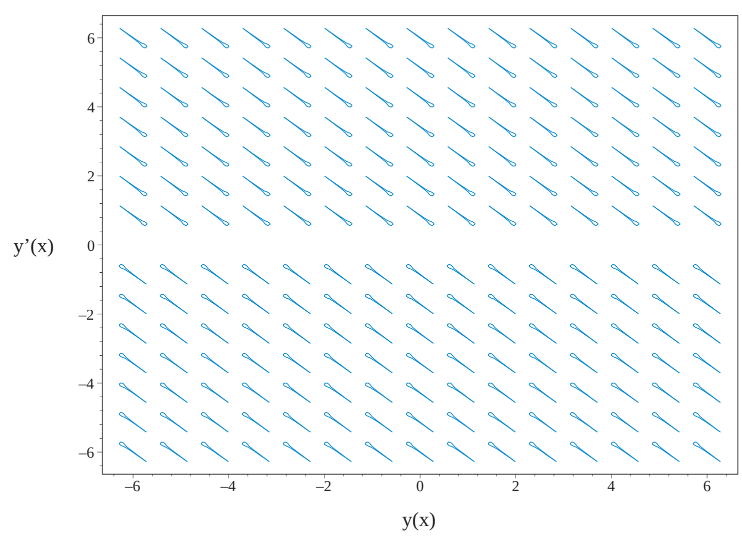


Figure 2.134: Slope field plot
$y'' + y' = \sin(x)$

Solved as second order linear exact ode

Time used: 0.163 (sec)

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= \sin(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\,dx$$

Substituting the above values for p,q,r,s gives

$$y'+y=\int \sin(x)\,dx$$

We now have a first order ode to solve which is

$$y'+y=-\cos(x)+c_1$$

In canonical form a linear first order is

$$y'+q(x)y=p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x)&=1\\ p(x)&=-\cos(x)+c_1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu&=e^{\int q\,dx}\\ &=e^{\int 1dx}\\ &=e^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y)&=\mu p\\ \frac{d}{dx}(\mu y)&=(\mu)(-\cos(x)+c_1)\\ \frac{d}{dx}(y\,e^{x})&=(e^{x})(-\cos(x)+c_1)\\ d(y\,e^{x})&=((-\cos(x)+c_1)\,e^{x})\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,e^{x}&=\int (-\cos(x)+c_1)\,e^{x}\,dx\\ &=-\frac{e^{x}\cos(x)}{2}-\frac{\sin(x)\,e^{x}}{2}+e^{x}c_1+c_2 \end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y=-\frac{\cos(x)}{2}-\frac{\sin(x)}{2}+c_1+c_2\,e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y=-\frac{\cos(x)}{2}-\frac{\sin(x)}{2}+c_1+c_2\,e^{-x}$$

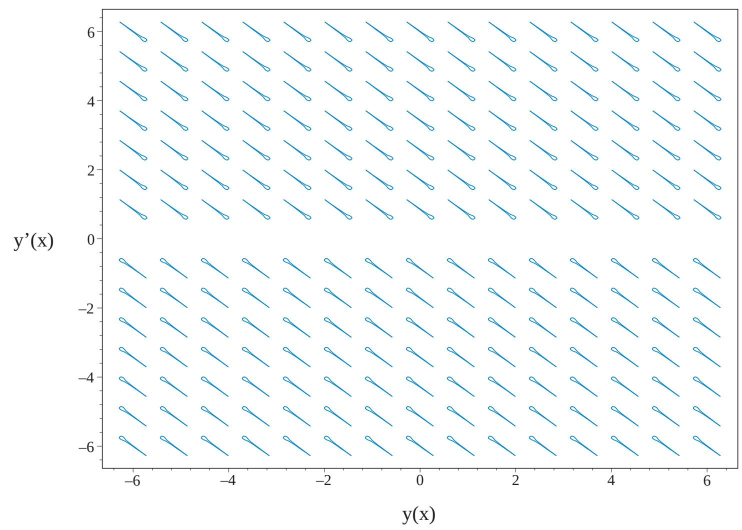


Figure 2.135: Slope field plot
$y'' + y' = \sin(x)$

Solved as second order missing y ode

Time used: 0.227 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) - \sin(x) = 0$$

Which is now solved for $u(x)$ as first order ode.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = \sin(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= e^{\int 1dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu u) &= \mu p \\ \frac{d}{dx}(\mu u) &= (\mu)(\sin(x)) \\ \frac{d}{dx}(u\,e^{x}) &= (e^{x})(\sin(x)) \\ d(u\,e^{x}) &= (\sin(x)\,e^{x})\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned} u\,\mathrm{e}^{x} &= \int \sin(x)\,\mathrm{e}^{x}\,dx \\ &= -\frac{\mathrm{e}^{x}\cos(x)}{2}+\frac{\sin(x)\,\mathrm{e}^{x}}{2}+c_1 \end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$u(x)=\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}$$

In summary, these are the solution found for $u(x)$

$$u(x)=\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}$$

For solution $u(x)=\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}$, since $u=y'$ then we now have a new first order ode to solve which is

$$y'=\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}$$

Since the ode has the form $y'=f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dy &= \int \frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}\,dx \\ y &= -c_1\,\mathrm{e}^{-x}-\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_2 \end{aligned}$$

In summary, these are the solution found for (y)

$$y=-c_1\,\mathrm{e}^{-x}-\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y=-c_1\,\mathrm{e}^{-x}-\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_2$$

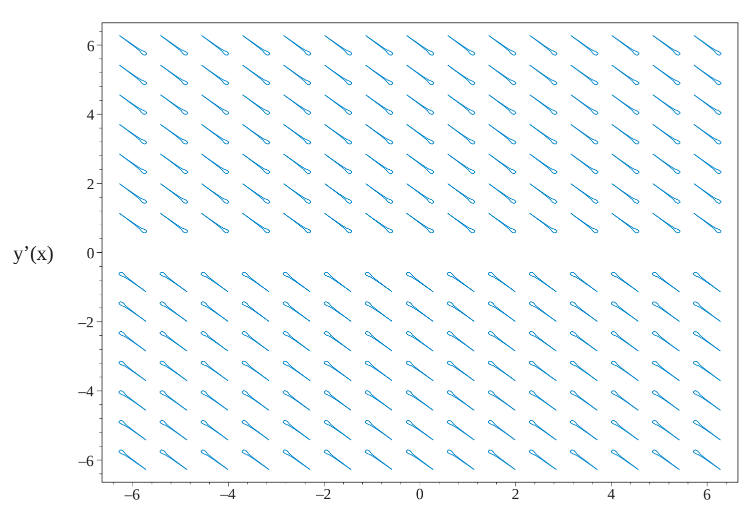


Figure 2.136: Slope field plot
$y''+y'=\sin(x)$

Solved as second order integrable as is ode

Time used: 0.060 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int (y''+y')\,dx &= \int \sin(x)\,dx\\ y'+y &= -\cos(x)+c_1\end{aligned}$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= 1\\ p(x) &= -\cos(x)+c_1\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= \mathrm{e}^{\int 1dx}\\ &= \mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(-\cos(x)+c_1)\\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})\,(-\cos(x)+c_1)\\ \mathrm{d}(y\,\mathrm{e}^{x}) &= ((-\cos(x)+c_1)\,\mathrm{e}^{x})\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{x} &= \int (-\cos(x)+c_1)\,\mathrm{e}^{x}\,dx\\ &= -\frac{\mathrm{e}^{x}\cos(x)}{2}-\frac{\sin(x)\,\mathrm{e}^{x}}{2}+\mathrm{e}^{x}c_1+c_2\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y = -\frac{\cos(x)}{2}-\frac{\sin(x)}{2}+c_1+c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)}{2}-\frac{\sin(x)}{2}+c_1+c_2\,\mathrm{e}^{-x}$$

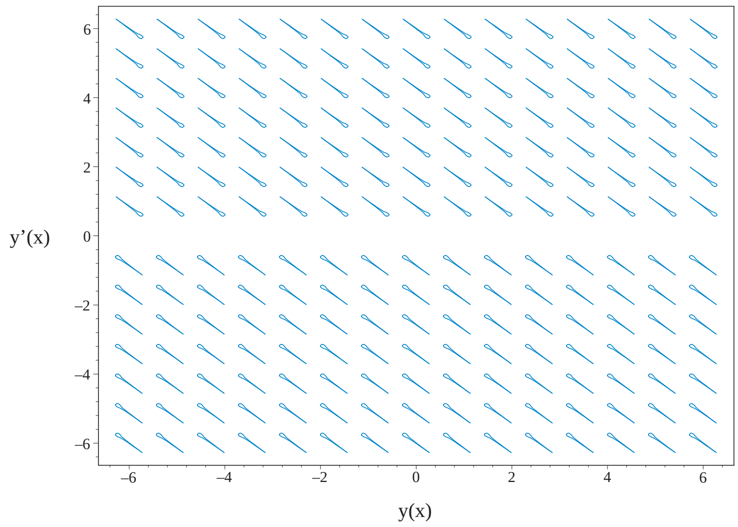


Figure 2.137: Slope field plot
$y'' + y' = \sin(x)$

Solved as second order integrable as is ode (ABC method)

Time used: 0.060 (sec)

Writing the ode as

$$y'' + y' = \sin(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int \sin(x)\, dx$$
$$y' + y = -\cos(x) + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= 1 \\ p(x) &= -\cos(x) + c_1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= \mathrm{e}^{\int 1 dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(-\cos(x) + c_1) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})(-\cos(x) + c_1) \\ \mathrm{d}(y\,\mathrm{e}^{x}) &= ((-\cos(x) + c_1)\,\mathrm{e}^{x})\, \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{x} &= \int (-\cos(x) + c_1)\,\mathrm{e}^{x}\, dx \\ &= -\frac{\mathrm{e}^{x}\cos(x)}{2} - \frac{\sin(x)\,\mathrm{e}^{x}}{2} + \mathrm{e}^{x} c_1 + c_2 \end{aligned}$$

Dividing throughout by the integrating factor e^x gives the final solution

$$y = -\frac{\cos(x)}{2} - \frac{\sin(x)}{2} + c_1 + c_2 e^{-x}$$

Will add steps showing solving for IC soon.

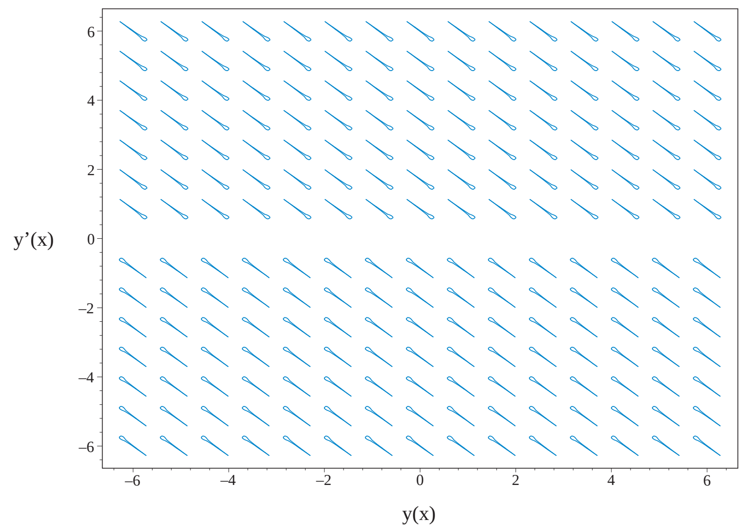


Figure 2.138: Slope field plot
$y'' + y' = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.142 (sec)

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(e^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1 (\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} (\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \,\mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin (x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos (x), \sin (x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos (x) + A_2 \sin (x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos (x) - A_2 \sin (x) - A_1 \sin (x) + A_2 \cos (x) = \sin (x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2) + \left(-\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{-x} + c_2 - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

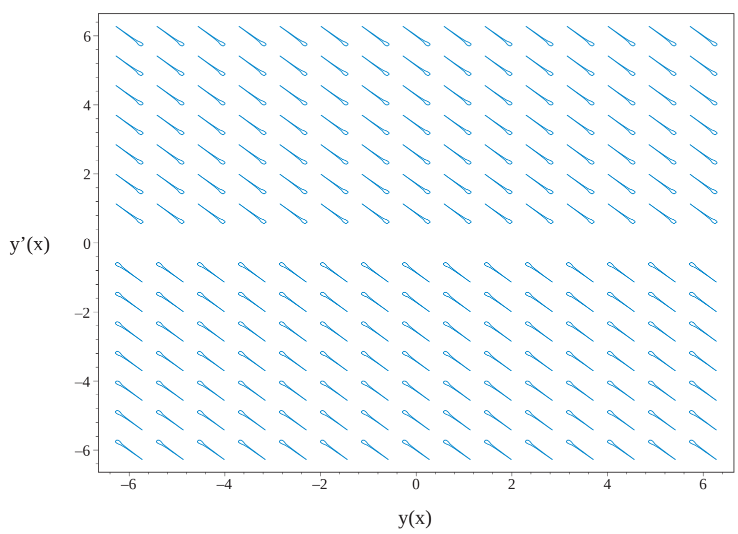


Figure 2.139: Slope field plot
$y'' + y' = \sin(x)$

Solved as second order ode adjoint method

Time used: 0.635 (sec)

In normal form the ode

$$y'' + y' = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (0) &= 0 \\ \xi''(x) - \xi'(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)}\\ &= \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$\xi = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$\xi = c_1 e^{(1)x} + c_2 e^{(0)x}$$

Or

$$\xi = \mathrm{e}^x c_1 + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}\end{aligned}$$

Or

$$y' + y\left(1 - \frac{\mathrm{e}^x c_1}{\mathrm{e}^x c_1 + c_2}\right) = \frac{c_1\left(-\frac{\mathrm{e}^x \cos(x)}{2} + \frac{\sin(x)\mathrm{e}^x}{2}\right) - \cos\,(x)\, c_2}{\mathrm{e}^x c_1 + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{c_2}{e^x c_1 + c_2} \\
p(x) &= \frac{-c_1(-\sin(x) + \cos(x))\, e^x - 2\cos(x)\, c_2}{2\, e^x c_1 + 2c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int \frac{c_2}{e^x c_1 + c_2} dx} \\
&= \frac{e^x}{e^x c_1 + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= \mu p \\
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{-c_1(-\sin(x) + \cos(x))\, e^x - 2\cos(x)\, c_2}{2\, e^x c_1 + 2c_2}\right) \\
\frac{d}{dx}\left(\frac{y\, e^x}{e^x c_1 + c_2}\right) &= \left(\frac{e^x}{e^x c_1 + c_2}\right)\left(\frac{-c_1(-\sin(x) + \cos(x))\, e^x - 2\cos(x)\, c_2}{2\, e^x c_1 + 2c_2}\right) \\
d\left(\frac{y\, e^x}{e^x c_1 + c_2}\right) &= \left(\frac{(-c_1(-\sin(x) + \cos(x))\, e^x - 2\cos(x)\, c_2)\, e^x}{(2\, e^x c_1 + 2c_2)(e^x c_1 + c_2)}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\, e^x}{e^x c_1 + c_2} &= \int \frac{(-c_1(-\sin(x) + \cos(x))\, e^x - 2\cos(x)\, c_2)\, e^x}{(2\, e^x c_1 + 2c_2)(e^x c_1 + c_2)}\, dx \\
&= \frac{-e^x - e^x \tan\left(\frac{x}{2}\right) - \frac{c_2}{2c_1} - \frac{c_2 \tan\left(\frac{x}{2}\right)^2}{2c_1}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)(e^x c_1 + c_2)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{e^x}{e^x c_1 + c_2}$ gives the final solution

$$
y = \frac{c_2(2c_3 c_1 - 1)\, e^{-x} + 2\left(c_3 c_1 - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} - \frac{1}{2}\right) c_1}{2c_1}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{c_2(2c_3 c_1 - 1)\, e^{-x} + 2\left(c_3 c_1 - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} - \frac{1}{2}\right) c_1}{2c_1}
$$

The constants can be merged to give

$$
y = \frac{c_2(2c_1 - 1)\, e^{-x} + 2\left(c_1 - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} - \frac{1}{2}\right) c_1}{2c_1}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{c_2(2c_1 - 1)\, e^{-x} + 2\left(c_1 - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} - \frac{1}{2}\right) c_1}{2c_1}
$$

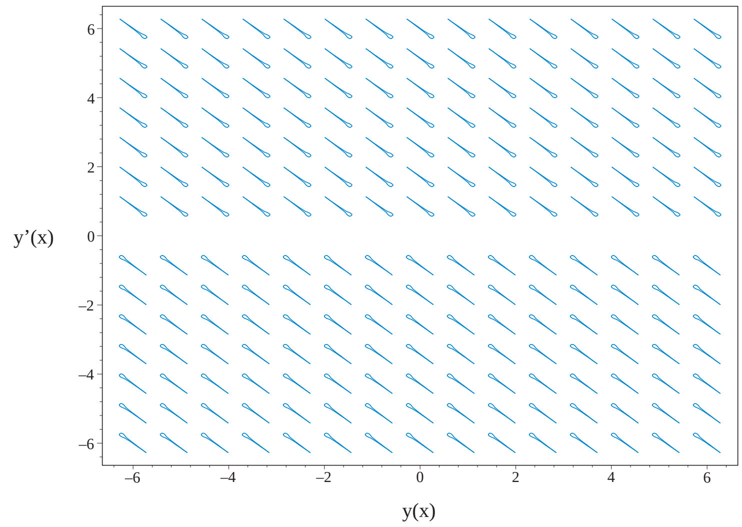


Figure 2.140: Slope field plot
$y'' + y' = \sin(x)$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = \sin(x)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + r = 0$
- Factor the characteristic polynomial
  $r(r+1) = 0$
- Roots of the characteristic polynomial
  $r = (-1, 0)$
- 1st solution of the homogeneous ODE
  $y_1(x) = e^{-x}$
- 2nd solution of the homogeneous ODE
  $y_2(x) = 1$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1\,e^{-x} + C2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$
- ○ Wronskian of solutions of the homogeneous equation
  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix}$
- ○ Compute Wronskian
  $W(y_1(x), y_2(x)) = e^{-x}$
- ○ Substitute functions into equation for $y_p(x)$
  $y_p(x) = -e^{-x}\left(\int e^x \sin(x)\,dx\right) + \int \sin(x)\,dx$
- ○ Compute integrals
  $y_p(x) = -\frac{\sin(x)}{2} - \frac{\cos(x)}{2}$

- Substitute particular solution into general solution to ODE

$$y(x) = \mathit{C1}\,\mathrm{e}^{-x} + \mathit{C2} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+sin(_a), _b(_a)`   *** Subl
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 21

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = sin(x),y(x),singsol=all)
```

$$y = -c_1 \mathrm{e}^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2$$

Mathematica DSolve solution

Solving time : 0.11 (sec)
Leaf size : 29

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==Sin[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1\left(-e^{-x}\right) + c_2$$

2.1.37 Problem 37

Internal problem ID [9108]
Book : Second order enumerated odes
Section : section 1
Problem number : 37
Date solved : `Monday, January 27, 2025 at 05:42:05 PM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$y'' + y' = \cos(x)$$

Solved as second order linear constant coeff ode

Time used: 0.131 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2\, \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2\, \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(x) - A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)}{2} + \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2\,\mathrm{e}^{-x}) + \left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)}{2} + \frac{\sin(x)}{2} + c_1 + c_2\,\mathrm{e}^{-x}$$

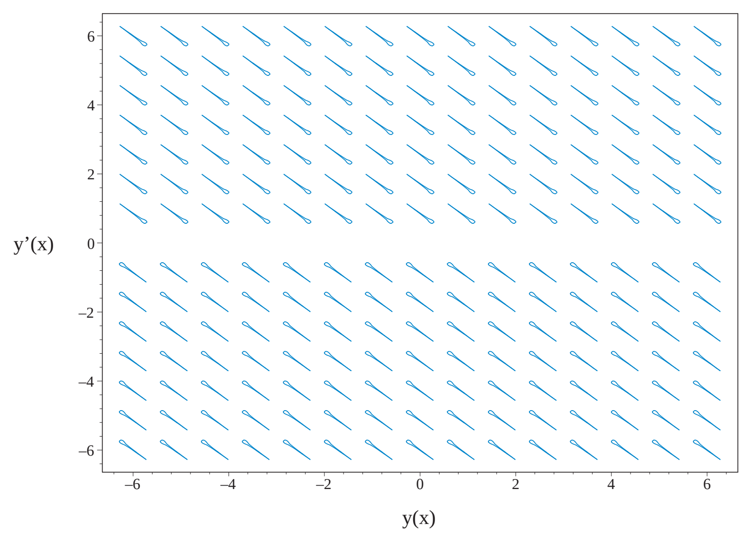


Figure 2.141: Slope field plot
$y'' + y' = \cos(x)$

Solved as second order linear exact ode

Time used: 0.164 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= \cos(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$y'+y=\int \cos{(x)}\;dx$$

We now have a first order ode to solve which is

$$y'+y=\sin{(x)}+c_1$$

In canonical form a linear first order is

$$y'+q(x)y=p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x)&=1\\ p(x)&=\sin{(x)}+c_1\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int q\,dx}\\ &=\mathrm{e}^{\int 1dx}\\ &=\mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=\mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\,(\sin{(x)}+c_1)\\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x})&=(\mathrm{e}^{x})\,(\sin{(x)}+c_1)\\ \mathrm{d}(y\,\mathrm{e}^{x})&=((\sin{(x)}+c_1)\,\mathrm{e}^{x})\;\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{x}&=\int (\sin{(x)}+c_1)\,\mathrm{e}^{x}\,dx\\ &=-\frac{\cos{(x)}\,\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{x}\sin{(x)}}{2}+\mathrm{e}^{x}c_1+c_2\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y=-\frac{\cos{(x)}}{2}+\frac{\sin{(x)}}{2}+c_1+c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y=-\frac{\cos{(x)}}{2}+\frac{\sin{(x)}}{2}+c_1+c_2\,\mathrm{e}^{-x}$$

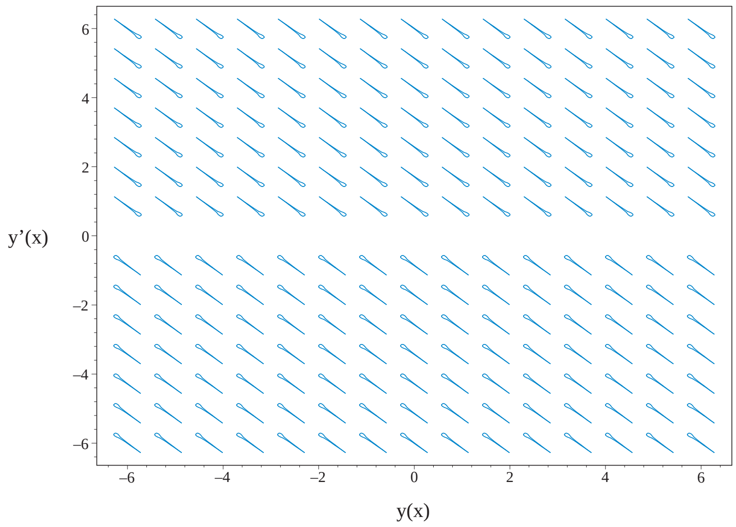


Figure 2.142: Slope field plot
$y'' + y' = \cos(x)$

Solved as second order missing y ode

Time used: 0.231 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + u(x) - \cos(x) = 0$$

Which is now solved for $u(x)$ as first order ode.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = \cos(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= e^{\int 1 dx}\\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu u) &= \mu p\\ \frac{d}{dx}(\mu u) &= (\mu)(\cos(x))\\ \frac{d}{dx}(u\,e^{x}) &= (e^{x})(\cos(x))\\ d(u\,e^{x}) &= (\cos(x)\,e^{x})\,dx\end{aligned}$$

Integrating gives

$$
\begin{aligned}
u\,\mathrm{e}^{x} &= \int \cos(x)\,\mathrm{e}^{x}\,dx \\
&= \frac{\cos(x)\,\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{x}\sin(x)}{2}+c_1
\end{aligned}
$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$
u(x)=\frac{\sin(x)}{2}+\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}
$$

In summary, these are the solution found for $u(x)$

$$
u(x)=\frac{\sin(x)}{2}+\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}
$$

For solution $u(x)=\frac{\sin(x)}{2}+\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}$, since $u=y'$ then we now have a new first order ode to solve which is

$$
y'=\frac{\sin(x)}{2}+\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}
$$

Since the ode has the form $y'=f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int \frac{\sin(x)}{2}+\frac{\cos(x)}{2}+c_1\,\mathrm{e}^{-x}\,dx \\
y &= -c_1\,\mathrm{e}^{-x}+\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_2
\end{aligned}
$$

In summary, these are the solution found for (y)

$$
y=-c_1\,\mathrm{e}^{-x}+\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_2
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y=-c_1\,\mathrm{e}^{-x}+\frac{\sin(x)}{2}-\frac{\cos(x)}{2}+c_2
$$

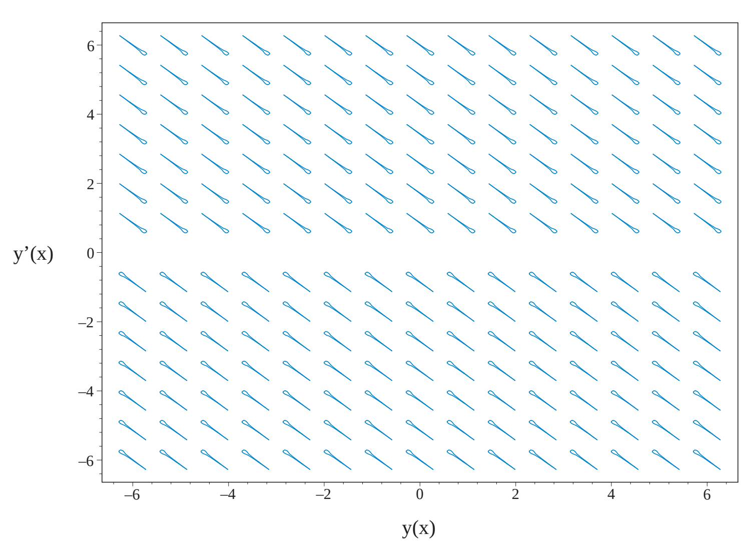


Figure 2.143: Slope field plot
$y''+y'=\cos(x)$

Solved as second order integrable as is ode

Time used: 0.056 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int (y''+y')\,dx &= \int \cos(x)\,dx\\ y'+y &= \sin(x)+c_1\end{aligned}$$

Which is now solved for y. In canonical form a linear first order is

$$y'+q(x)y=p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x)&=1\\ p(x)&=\sin(x)+c_1\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int q\,dx}\\ &=\mathrm{e}^{\int 1dx}\\ &=\mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=\mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)(\sin(x)+c_1)\\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x})&=(\mathrm{e}^{x})(\sin(x)+c_1)\\ \mathrm{d}(y\,\mathrm{e}^{x})&=((\sin(x)+c_1)\,\mathrm{e}^{x})\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{x}&=\int(\sin(x)+c_1)\,\mathrm{e}^{x}\,dx\\ &=-\frac{\cos(x)\,\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{x}\sin(x)}{2}+\mathrm{e}^{x}c_1+c_2\end{aligned}$$

Dividing throughout by the integrating factor e^{x} gives the final solution

$$y=-\frac{\cos(x)}{2}+\frac{\sin(x)}{2}+c_1+c_2\,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y=-\frac{\cos(x)}{2}+\frac{\sin(x)}{2}+c_1+c_2\,\mathrm{e}^{-x}$$

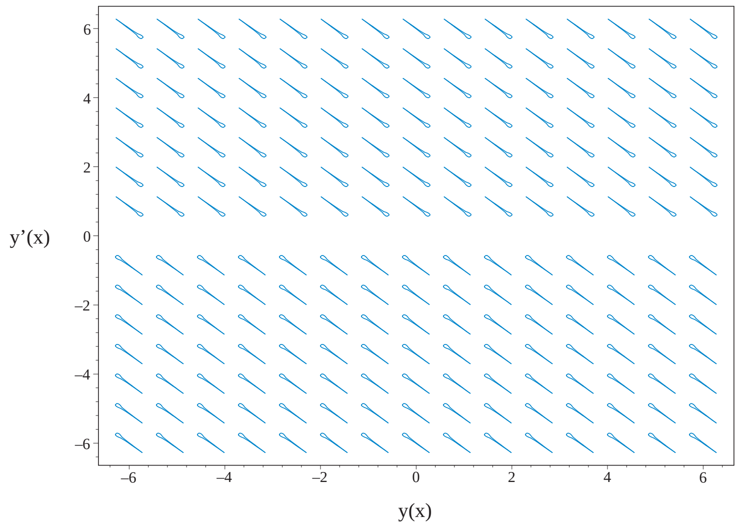


Figure 2.144: Slope field plot
$y'' + y' = \cos(x)$

Solved as second order integrable as is ode (ABC method)

Time used: 0.062 (sec)

Writing the ode as

$$y'' + y' = \cos(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\,dx = \int \cos(x)\,dx$$
$$y' + y = \sin(x) + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = 1$$
$$p(x) = \sin(x) + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int 1 dx} \\ &= \mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(\sin(x) + c_1) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})(\sin(x) + c_1) \\ \mathrm{d}(y\,\mathrm{e}^{x}) &= ((\sin(x) + c_1)\,\mathrm{e}^{x})\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{x} &= \int (\sin(x) + c_1)\,\mathrm{e}^{x}\,dx \\ &= -\frac{\cos(x)\,\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{x}\sin(x)}{2} + \mathrm{e}^{x} c_1 + c_2\end{aligned}$$

Dividing throughout by the integrating factor e^x gives the final solution

$$y = -\frac{\cos(x)}{2} + \frac{\sin(x)}{2} + c_1 + c_2 e^{-x}$$

Will add steps showing solving for IC soon.

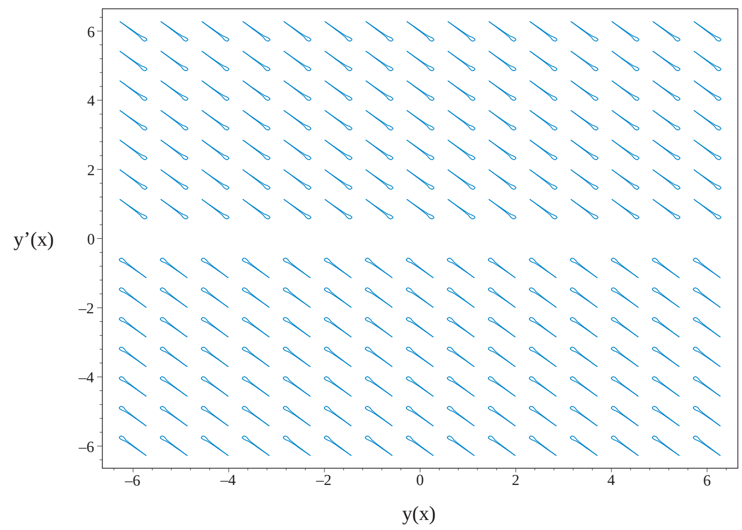


Figure 2.145: Slope field plot
$y'' + y' = \cos(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.079 (sec)

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{1}{1}\, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1 (\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} (\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\, \mathrm{e}^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(x) - A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)}{2} + \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \,\mathrm{e}^{-x} + c_2) + \left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{-x} + c_2 - \frac{\cos(x)}{2} + \frac{\sin(x)}{2}$$

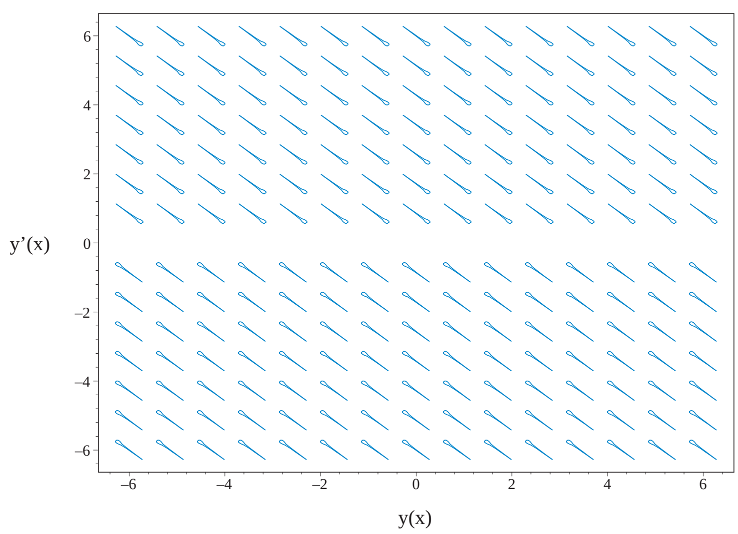


Figure 2.146: Slope field plot
$y'' + y' = \cos(x)$

Solved as second order ode adjoint method

Time used: 0.711 (sec)

In normal form the ode

$$y'' + y' = \cos(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= \cos(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (0) &= 0 \\ \xi''(x) - \xi'(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)}\\ &= \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ \xi &= c_1 e^{(1)x} + c_2 e^{(0)x}\end{aligned}$$

Or

$$\xi = \mathrm{e}^{x} c_1 + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}\end{aligned}$$

Or

$$y' + y\left(1 - \frac{\mathrm{e}^{x} c_1}{\mathrm{e}^{x} c_1 + c_2}\right) = \frac{c_1\left(\frac{\cos(x)\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{x}\sin(x)}{2}\right) + \sin\,(x)\, c_2}{\mathrm{e}^{x} c_1 + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{c_2}{e^x c_1 + c_2} \\
p(x) &= \frac{(\cos(x) + \sin(x)) e^x c_1 + 2\sin(x) c_2}{2 e^x c_1 + 2c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= e^{\int \frac{c_2}{e^x c_1 + c_2} dx} \\
&= \frac{e^x}{e^x c_1 + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= \mu p \\
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{(\cos(x) + \sin(x)) e^x c_1 + 2\sin(x) c_2}{2 e^x c_1 + 2c_2}\right) \\
\frac{d}{dx}\left(\frac{y e^x}{e^x c_1 + c_2}\right) &= \left(\frac{e^x}{e^x c_1 + c_2}\right)\left(\frac{(\cos(x) + \sin(x)) e^x c_1 + 2\sin(x) c_2}{2 e^x c_1 + 2c_2}\right) \\
d\left(\frac{y e^x}{e^x c_1 + c_2}\right) &= \left(\frac{((\cos(x) + \sin(x)) e^x c_1 + 2\sin(x) c_2) e^x}{(2 e^x c_1 + 2c_2)(e^x c_1 + c_2)}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y e^x}{e^x c_1 + c_2} &= \int \frac{((\cos(x) + \sin(x)) e^x c_1 + 2\sin(x) c_2) e^x}{(2 e^x c_1 + 2c_2)(e^x c_1 + c_2)} dx \\
&= \frac{-e^x + e^x \tan\left(\frac{x}{2}\right) - \frac{c_2}{2c_1} - \frac{c_2 \tan\left(\frac{x}{2}\right)^2}{2c_1}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)(e^x c_1 + c_2)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{e^x}{e^x c_1 + c_2}$ gives the final solution

$$
y = \frac{c_2(2c_3c_1 - 1) e^{-x} + 2c_1\left(c_3c_1 - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} - \frac{1}{2}\right)}{2c_1}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{c_2(2c_3c_1 - 1) e^{-x} + 2c_1\left(c_3c_1 - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} - \frac{1}{2}\right)}{2c_1}
$$

The constants can be merged to give

$$
y = \frac{c_2(2c_1 - 1) e^{-x} + 2c_1\left(c_1 - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} - \frac{1}{2}\right)}{2c_1}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{c_2(2c_1 - 1) e^{-x} + 2c_1\left(c_1 - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} - \frac{1}{2}\right)}{2c_1}
$$

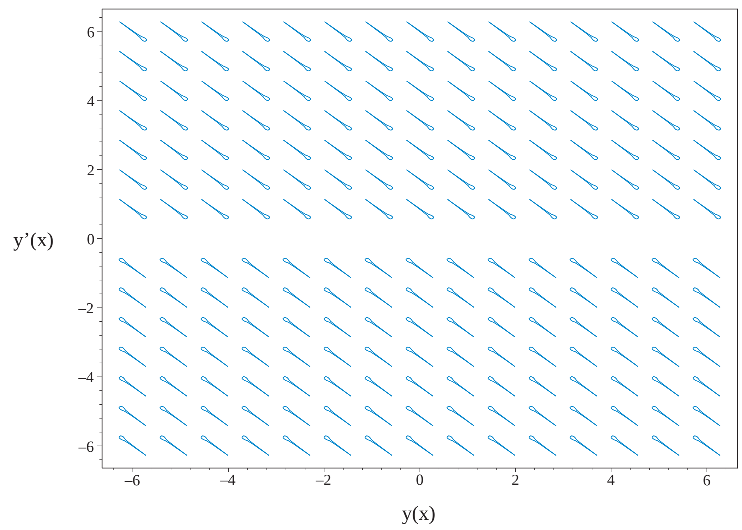


Figure 2.147: Slope field plot
$y'' + y' = \cos(x)$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = \cos(x)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + r = 0$
- Factor the characteristic polynomial
  $r(r+1) = 0$
- Roots of the characteristic polynomial
  $r = (-1, 0)$
- 1st solution of the homogeneous ODE
  $y_1(x) = e^{-x}$
- 2nd solution of the homogeneous ODE
  $y_2(x) = 1$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1\, e^{-x} + C2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos(x)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = e^{-x}$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -e^{-x}\left(\int e^{x}\cos(x)\, dx\right) + \int \cos(x)\, dx$
  - Compute integrals
    $y_p(x) = -\frac{\cos(x)}{2} + \frac{\sin(x)}{2}$

- Substitute particular solution into general solution to ODE

$$y(x) = \mathit{C1}\, \mathrm{e}^{-x} + \mathit{C2} - \frac{\cos(x)}{2} + \frac{\sin(x)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+cos(_a), _b(_a)`   *** Subl
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 21

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = cos(x),y(x),singsol=all)
```

$$y = -c_1 \mathrm{e}^{-x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2$$

Mathematica DSolve solution

Solving time : 0.079 (sec)
Leaf size : 28

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]==Cos[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}\left(\sin(x) - \cos(x) - 2c_1 e^{-x}\right) + c_2$$

2.1.38 Problem 38

Internal problem ID [9109]
Book : Second order enumerated odes
Section : section 1
Problem number : 38
Date solved : `Monday, January 27, 2025 at 05:42:07 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y = 1$$

Solved as second order linear constant coeff ode

Time used: 0.108 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 1 + c_1 \cos(x) + c_2 \sin(x)$$

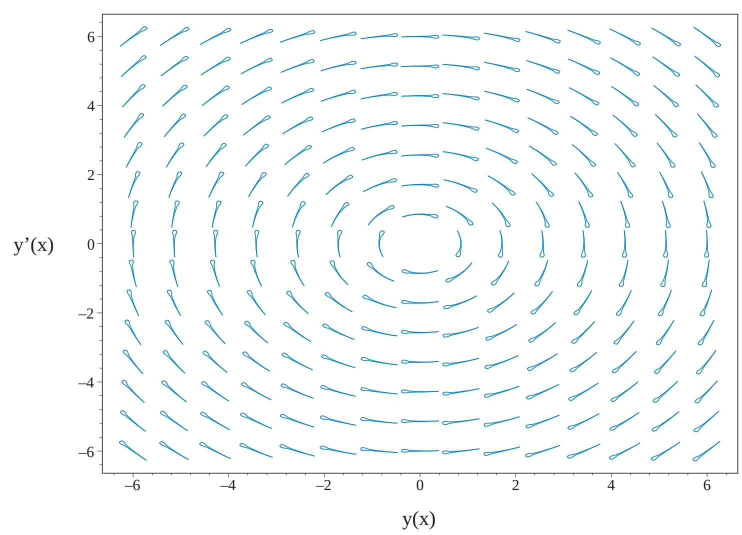


Figure 2.148: Slope field plot
$y'' + y = 1$

Solved as second order can be made integrable

Time used: 0.903 (sec)

Multiplying the ode by y' gives

$$y'y'' + y'y - y' = 0$$

Integrating the above w.r.t x gives

$$\begin{aligned} \int (y'y'' + y'y - y')\, dx &= 0 \\ \frac{y'^2}{2} + \frac{y^2}{2} - y &= c_1 \end{aligned}$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{-y^2 + 2y + 2c_1} \tag{1}$$

$$y' = -\sqrt{-y^2 + 2y + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\begin{aligned} \int \frac{1}{\sqrt{-y^2 + 2c_1 + 2y}} dy &= dx \\ \arctan\left(\frac{y-1}{\sqrt{-y^2 + 2c_1 + 2y}}\right) &= x + c_2 \end{aligned}$$

Singular solutions are found by solving

$$\sqrt{-y^2+2c_1+2y}=0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y=1-\sqrt{1+2c_1}$$
$$y=1+\sqrt{1+2c_1}$$

Solving for y gives

$$y=1-\sqrt{1+2c_1}$$
$$y=1+\sqrt{1+2c_1}$$
$$y=\tan(x+c_2)\sqrt{\frac{1+2c_1}{\tan(x+c_2)^2+1}}+1$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{-y^2+2c_1+2y}}dy=dx$$
$$-\arctan\left(\frac{y-1}{\sqrt{-y^2+2c_1+2y}}\right)=x+c_3$$

Singular solutions are found by solving

$$-\sqrt{-y^2+2c_1+2y}=0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y=1-\sqrt{1+2c_1}$$
$$y=1+\sqrt{1+2c_1}$$

Solving for y gives

$$y=1-\sqrt{1+2c_1}$$
$$y=1+\sqrt{1+2c_1}$$
$$y=-\tan(x+c_3)\sqrt{\frac{1+2c_1}{\tan(x+c_3)^2+1}}+1$$

Will add steps showing solving for IC soon.

The solution

$$y=1-\sqrt{1+2c_1}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y=1+\sqrt{1+2c_1}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y=\tan(x+c_2)\sqrt{\frac{1+2c_1}{\tan(x+c_2)^2+1}}+1$$
$$y=-\tan(x+c_3)\sqrt{\frac{1+2c_1}{\tan(x+c_3)^2+1}}+1$$

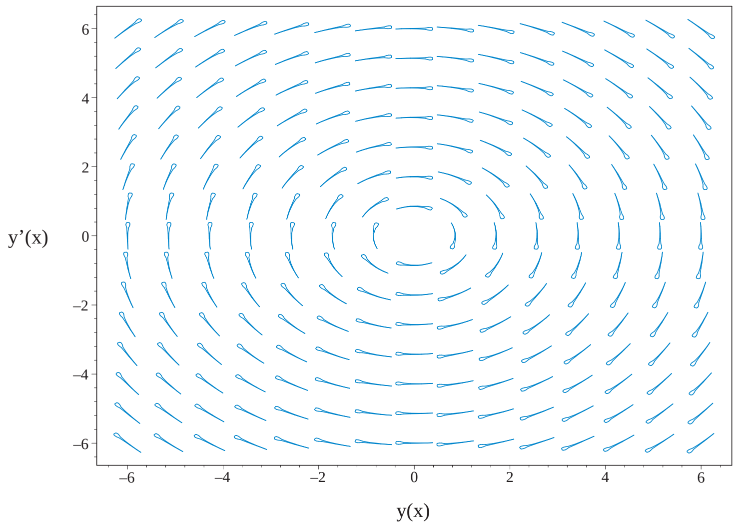


Figure 2.149: Slope field plot
$y'' + y = 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.109 (sec)

Writing the ode as

$$
\begin{aligned}
y'' + y &= 0 & (1)\\
Ay'' + By' + Cy &= 0 & (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 0 \qquad (3)\\
C &= 1
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1\\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} \, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 1 + c_1 \cos(x) + c_2 \sin(x)$$

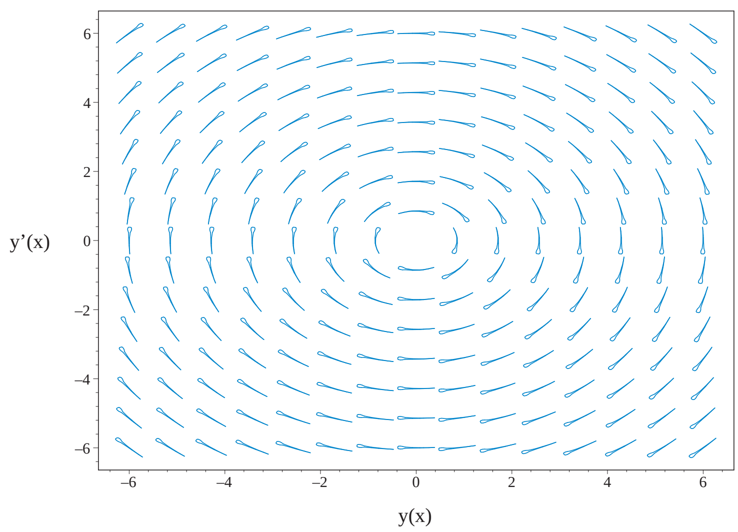


Figure 2.150: Slope field plot
$y'' + y = 1$

Solved as second order ode adjoint method

Time used: 0.826 (sec)

In normal form the ode

$$y'' + y = 1 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{c_1 \sin(x) - c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{c_1 \sin(x) - c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= e^{\int -\frac{-c_1 \sin(x)+c_2 \cos(x)}{c_1 \cos(x)+c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(\frac{c_1 \sin(x) - c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) &= \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right) \left(\frac{c_1 \sin(x) - c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)}\right) \\
\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) &= \left(\frac{c_1 \sin(x) - c_2 \cos(x)}{(c_1 \cos(x) + c_2 \sin(x))^2}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{c_1 \sin(x) - c_2 \cos(x)}{(c_1 \cos(x) + c_2 \sin(x))^2} dx \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x)+c_2 \sin(x)}$ gives the final solution

$$
y = \sin(x) c_2 c_3 + \cos(x) c_1 c_3 + 1
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \sin(x) c_2 c_3 + \cos(x) c_1 c_3 + 1
$$

The constants can be merged to give

$$
y = 1 + c_1 \cos(x) + c_2 \sin(x)
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = 1 + c_1 \cos(x) + c_2 \sin(x)
$$

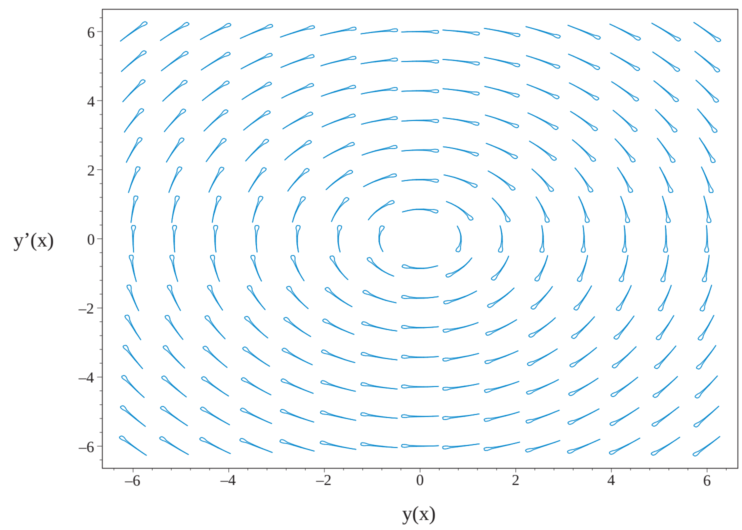


Figure 2.151: Slope field plot
$y'' + y = 1$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + y(x) = 1$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
$r^2 + 1 = 0$
- Use quadratic formula to solve for r
$r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
$r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE
$y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE
$y_2(x) = \sin(x)$
- General solution of the ODE
$y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
$y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1\right]$
  ◦ Wronskian of solutions of the homogeneous equation
  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  ◦ Compute Wronskian
  $W(y_1(x), y_2(x)) = 1$
  ◦ Substitute functions into equation for $y_p(x)$
  $y_p(x) = -\cos(x)\left(\int \sin(x)\, dx\right) + \sin(x)\left(\int \cos(x)\, dx\right)$
  ◦ Compute integrals
  $y_p(x) = 1$

- Substitute particular solution into general solution to ODE
$y(x) = C1 \cos(x) + C2 \sin(x) + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)+y(x) = 1,y(x),singsol=all)
```

$$y = \cos(x) c_1 + \sin(x) c_2 + 1$$

Mathematica DSolve solution

Solving time : 0.012 (sec)
Leaf size : 17

```
DSolve[{D[y[x],{x,2}]+y[x]==1,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \cos(x) + c_2 \sin(x) + 1$$

2.1.39 Problem 39

Internal problem ID [9110]
Book : Second order enumerated odes
Section : section 1
Problem number : 39
Date solved : `Monday, January 27, 2025 at 05:42:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + y = x$$

Solved as second order linear constant coeff ode

Time used: 0.119 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x + c_1 \cos(x) + c_2 \sin(x)$$

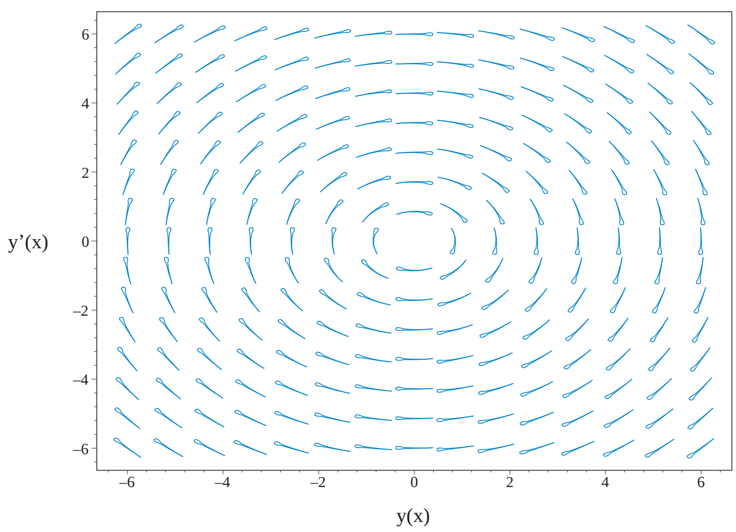


Figure 2.152: Slope field plot
$y'' + y = x$

Solved as second order ode using Kovacic algorithm

Time used: 0.144 (sec)

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x + c_1 \cos(x) + c_2 \sin(x)$$

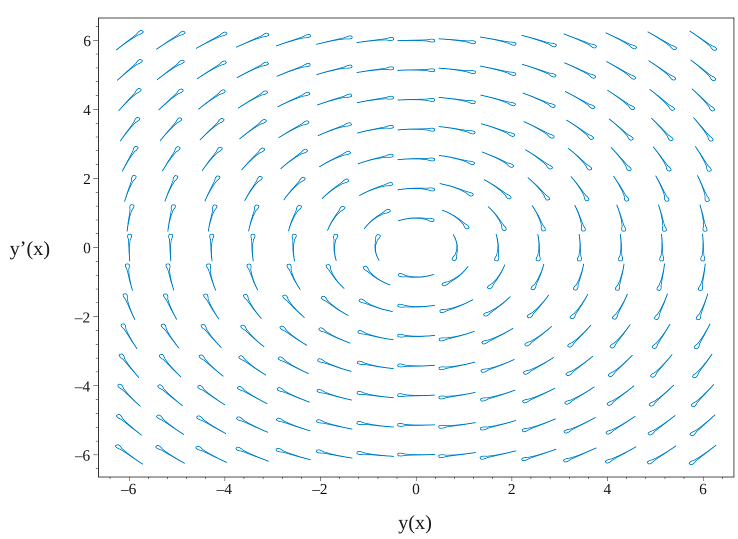


Figure 2.153: Slope field plot
$y'' + y = x$

Solved as second order ode adjoint method

Time used: 1.311 (sec)

In normal form the ode

$$y'' + y = x \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= x \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{c_1(\cos(x) + x\sin(x)) + c_2(\sin(x) - x\cos(x))}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\ p(x) &= \frac{(-c_2 x + c_1)\cos(x) + \sin(x)(c_1 x + c_2)}{c_1 \cos(x) + c_2 \sin(x)} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int -\frac{-c_1 \sin(x)+c_2 \cos(x)}{c_1 \cos(x)+c_2 \sin(x)} dx} \\ &= \frac{1}{c_1 \cos(x) + c_2 \sin(x)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(-c_2 x + c_1)\cos(x) + \sin(x)(c_1 x + c_2)}{c_1 \cos(x) + c_2 \sin(x)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) \\ &= \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right)\left(\frac{(-c_2 x + c_1)\cos(x) + \sin(x)(c_1 x + c_2)}{c_1 \cos(x) + c_2 \sin(x)}\right) \\ \mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) &= \left(\frac{(-c_2 x + c_1)\cos(x) + \sin(x)(c_1 x + c_2)}{(c_1 \cos(x) + c_2 \sin(x))^2}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{(-c_2 x + c_1)\cos(x) + \sin(x)(c_1 x + c_2)}{(c_1 \cos(x) + c_2 \sin(x))^2}\,dx \\ &= \frac{-x - 2x\tan\left(\frac{x}{2}\right)^2 - x\tan\left(\frac{x}{2}\right)^4}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)\left(\tan\left(\frac{x}{2}\right)^2 c_1 - 2\tan\left(\frac{x}{2}\right) c_2 - c_1\right)} + c_3 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x)+c_2 \sin(x)}$ gives the final solution

$$y = c_2 c_3 \sin(x) + c_1 c_3 \cos(x) + x$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_2 c_3 \sin(x) + c_1 c_3 \cos(x) + x$$

The constants can be merged to give

$$y = x + c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x + c_1 \cos(x) + c_2 \sin(x)$$

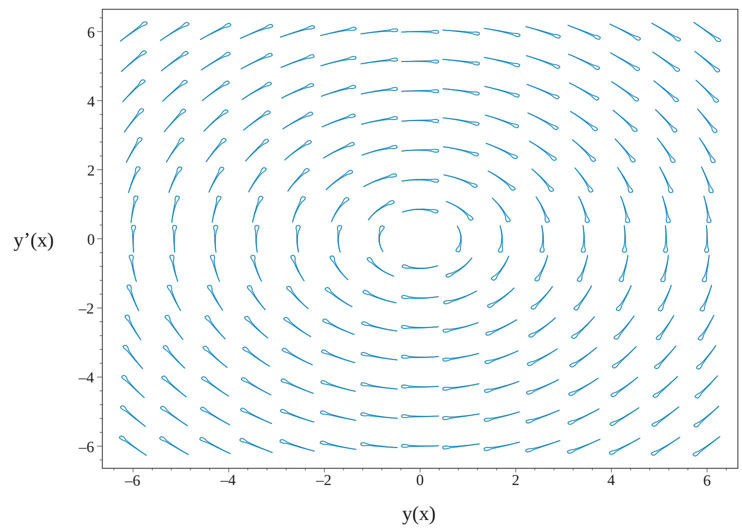


Figure 2.154: Slope field plot
$y'' + y = x$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + y(x) = x$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
$r^2 + 1 = 0$
- Use quadratic formula to solve for r
$r = \frac{0\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
$r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE
$y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE
$y_2(x) = \sin(x)$
- General solution of the ODE
$y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
$y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\right]$
  - Wronskian of solutions of the homogeneous equation
  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian
  $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$
  $y_p(x) = -\cos(x)\left(\int \sin(x)\, x dx\right) + \sin(x)\left(\int x\cos(x)\, dx\right)$
  - Compute integrals
  $y_p(x) = x$

- Substitute particular solution into general solution to ODE
$y(x) = C1 \cos(x) + C2 \sin(x) + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)+y(x) = x,y(x),singsol=all)
```

$$y = \sin\left(x\right) c_2 + \cos\left(x\right) c_1 + x$$

Mathematica DSolve solution

Solving time : 0.012 (sec)
Leaf size : 17

```
DSolve[{D[y[x],{x,2}]+y[x]==x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x + c_1 \cos(x) + c_2 \sin(x)$$

2.1.40 Problem 40

Internal problem ID [9111]
Book : Second order enumerated odes
Section : section 1
Problem number : 40
Date solved : `Monday, January 27, 2025 at 05:42:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + y = 1 + x$$

Solved as second order linear constant coeff ode

Time used: 0.120 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 1 + x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1 + x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 1 + x + c_1 \cos(x) + c_2 \sin(x)$$

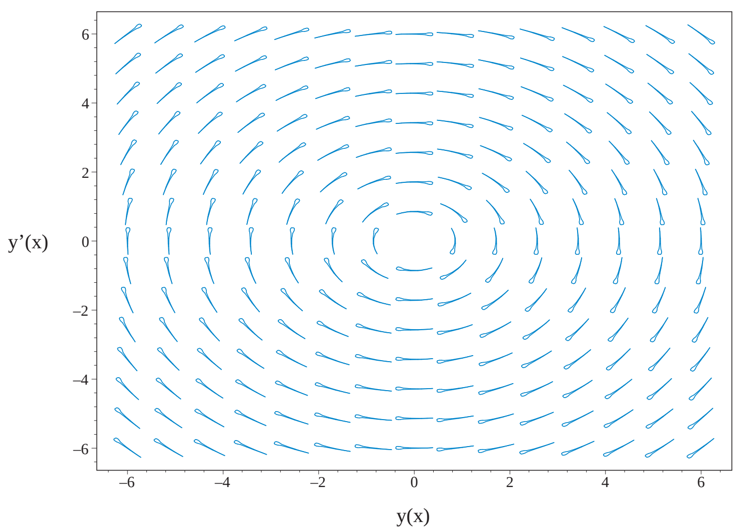


Figure 2.155: Slope field plot
$y'' + y = 1 + x$

Solved as second order ode using Kovacic algorithm

Time used: 0.147 (sec)

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = 1 + x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1 + x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 1 + x + c_1 \cos(x) + c_2 \sin(x)$$

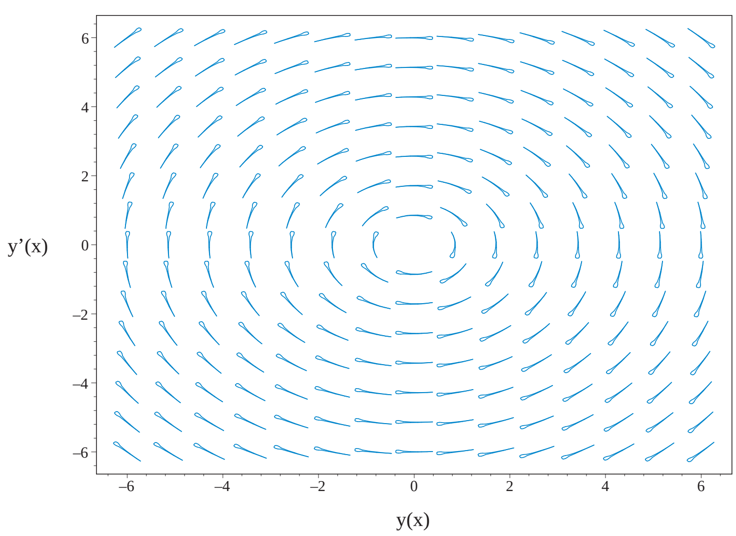


Figure 2.156: Slope field plot
$y'' + y = 1 + x$

Solved as second order ode adjoint method

Time used: 1.506 (sec)

In normal form the ode

$$y'' + y = 1 + x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= 1 + x \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{c_1 \sin(x) + c_1(\cos(x) + x\sin(x)) - c_2 \cos(x) + c_2(\sin(x) - x\cos(x))}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{(c_1 + (-x-1)c_2)\cos(x) + \sin(x)((1+x)c_1 + c_2)}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= e^{\int -\frac{-c_1 \sin(x)+c_2 \cos(x)}{c_1 \cos(x)+c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
&\frac{d}{dx}(\mu y) = \mu p \\
&\frac{d}{dx}(\mu y) = (\mu)\left(\frac{(c_1 + (-x-1)c_2)\cos(x) + \sin(x)((1+x)c_1 + c_2)}{c_1 \cos(x) + c_2 \sin(x)}\right) \\
&\frac{d}{dx}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) \\
&\quad = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right)\left(\frac{(c_1 + (-x-1)c_2)\cos(x) + \sin(x)((1+x)c_1 + c_2)}{c_1 \cos(x) + c_2 \sin(x)}\right) \\
&d\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{(c_1 + (-x-1)c_2)\cos(x) + \sin(x)((1+x)c_1 + c_2)}{(c_1 \cos(x) + c_2 \sin(x))^2}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{(c_1 + (-x-1)c_2)\cos(x) + \sin(x)((1+x)c_1 + c_2)}{(c_1 \cos(x) + c_2 \sin(x))^2} dx \\
&= \frac{-x - 2\tan\left(\frac{x}{2}\right)^2 - 2x\tan\left(\frac{x}{2}\right)^2 - x\tan\left(\frac{x}{2}\right)^4 - 1 - \tan\left(\frac{x}{2}\right)^4}{\left(1+\tan\left(\frac{x}{2}\right)^2\right)\left(\tan\left(\frac{x}{2}\right)^2 c_1 - 2\tan\left(\frac{x}{2}\right)c_2 - c_1\right)} + c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x)+c_2 \sin(x)}$ gives the final solution

$$y = c_2 c_3 \sin(x) + c_1 c_3 \cos(x) + x + 1$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_2 c_3 \sin(x) + c_1 c_3 \cos(x) + x + 1$$

The constants can be merged to give

$$y = 1 + x + c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 1 + x + c_1 \cos(x) + c_2 \sin(x)$$

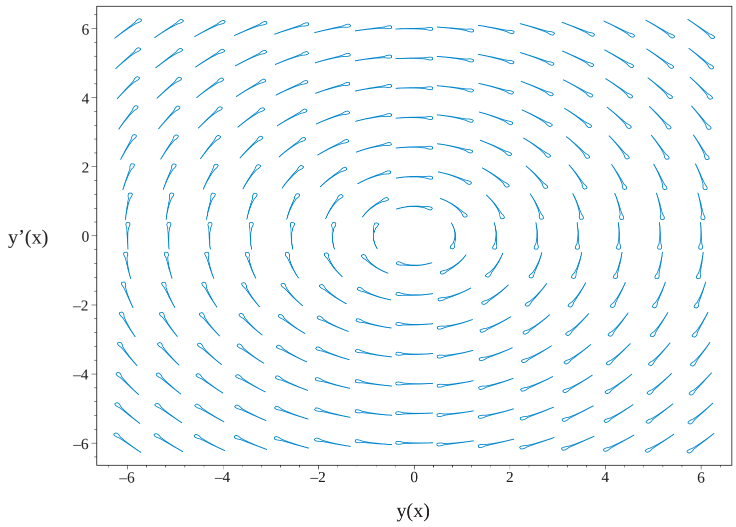


Figure 2.157: Slope field plot
$y'' + y = 1 + x$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + y(x) = x + 1$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = \sin(x)$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x + 1\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -\cos(x)\left(\int (x+1)\sin(x)\,dx\right) + \sin(x)\left(\int (x+1)\cos(x)\,dx\right)$
  - Compute integrals
    $y_p(x) = x + 1$

- Substitute particular solution into general solution to ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + x + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x)+y(x) = x+1,y(x),singsol=all)
```

$$y = \sin(x) c_2 + \cos(x) c_1 + x + 1$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]+y[x]==1+x,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x + c_1 \cos(x) + c_2 \sin(x) + 1$$

2.1.41 Problem 41

Internal problem ID [9112]
Book : Second order enumerated odes
Section : section 1
Problem number : 41
Date solved : `Monday, January 27, 2025 at 05:42:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + y = x^2 + x + 1$$

Solved as second order linear constant coeff ode

Time used: 0.130 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3x^2 + A_2x + A_1 + 2A_3 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 + x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x^2 + x - 1) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^2 + x - 1 + c_1 \cos(x) + c_2 \sin(x)$$

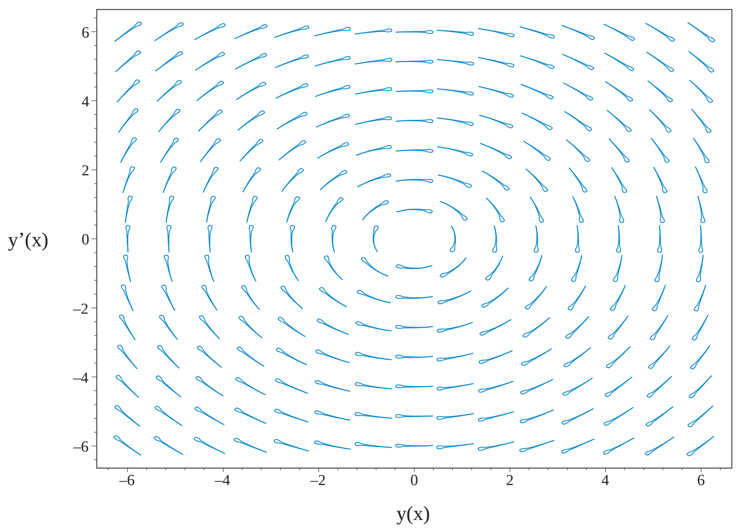


Figure 2.158: Slope field plot
$y'' + y = x^2 + x + 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.148 (sec)

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.60: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3x^2 + A_2x + A_1 + 2A_3 = x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 + x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x^2 + x - 1) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^2 + x - 1 + c_1 \cos(x) + c_2 \sin(x)$$

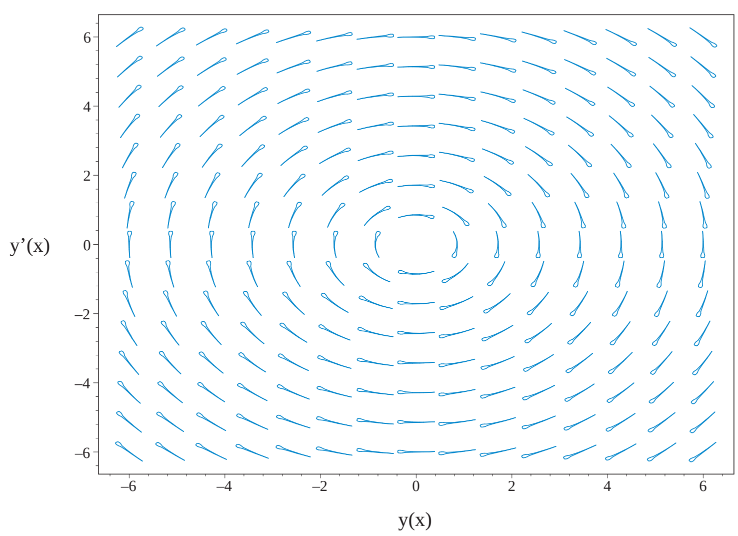


Figure 2.159: Slope field plot
$y'' + y = x^2 + x + 1$

Solved as second order ode adjoint method

Time used: 1.524 (sec)

In normal form the ode

$$y'' + y = x^2 + x + 1 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= x^2 + x + 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i \\ \lambda_2 &= -i \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y = \int \xi(x)\, r(x)\, dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{c_1(x^2 \sin(x) - 2\sin(x) + 2x\cos(x)) + c_1(\cos(x) + x\sin(x)) + c_1 \sin(x) +}{c_1 \cos(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)}$$
$$p(x) = \frac{((2x+1)\, c_1 - c_2(x^2 + x - 1)) \cos(x) + ((x^2 + x - 1)\, c_1 + c_2(2x+1)) \sin(x)}{c_1 \cos(x) + c_2 \sin(x)}$$

The integrating factor μ is

$$\mu = e^{\int q\, dx}$$
$$= e^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx}$$
$$= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{((2x+1)\, c_1 - c_2(x^2 + x - 1)) \cos(x) + ((x^2 + x - 1)\, c_1 + c_2(2x+1)) \sin(x)}{c_1 \cos(x) + c_2 \sin(x)}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right)\left(\frac{((2x+1)\, c_1 - c_2(x^2 + x - 1)) \cos(x) + ((x^2 + x - 1)\, c_1 + c_2(2x+1)) \sin(x)}{c_1 \cos(x) + c_2 \sin(x)}\right)$$

$$\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{((2x+1)\, c_1 - c_2(x^2 + x - 1)) \cos(x) + ((x^2 + x - 1)\, c_1 + c_2(2x+1)) \sin(x)}{(c_1 \cos(x) + c_2 \sin(x))^2}\right) \mathrm{d}x$$

Integrating gives

$$\frac{y}{c_1 \cos(x) + c_2 \sin(x)} = \int \frac{((2x+1)\, c_1 - c_2(x^2 + x - 1)) \cos(x) + ((x^2 + x - 1)\, c_1 + c_2(2x+1)) \sin(x)}{(c_1 \cos(x) + c_2 \sin(x))^2} dx$$
$$= \frac{-x - x^2 + 2\tan\left(\frac{x}{2}\right)^2 - 2x\tan\left(\frac{x}{2}\right)^2 - x\tan\left(\frac{x}{2}\right)^4 - 2x^2\tan\left(\frac{x}{2}\right)^2 - x^2\tan\left(\frac{x}{2}\right)^4 + 1 +}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)\left(\tan\left(\frac{x}{2}\right)^2 c_1 - 2\tan\left(\frac{x}{2}\right) c_2 - c_1\right)}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x) + c_2 \sin(x)}$ gives the final solution

$$y = c_2 c_3 \sin(x) + c_1 c_3 \cos(x) + x^2 + x - 1$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_2c_3 \sin(x) + c_1c_3 \cos(x) + x^2 + x - 1$$

The constants can be merged to give

$$y = x^2 + x - 1 + c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^2 + x - 1 + c_1 \cos(x) + c_2 \sin(x)$$

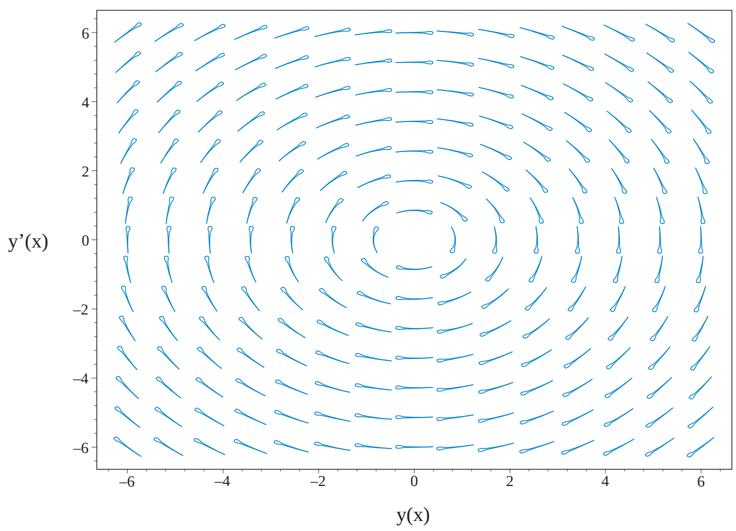


Figure 2.160: Slope field plot
$y'' + y = x^2 + x + 1$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + y(x) = x^2 + x + 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 + x + 1\right]$$
- Wronskian of solutions of the homogeneous equation
  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$
- Compute Wronskian
  $$W(y_1(x), y_2(x)) = 1$$
- Substitute functions into equation for $y_p(x)$
  $$y_p(x) = -\cos(x)\left(\int \sin(x)(x^2 + x + 1)\, dx\right) + \sin(x)\left(\int \cos(x)(x^2 + x + 1)\, dx\right)$$
- Compute integrals
  $$y_p(x) = x^2 + x - 1$$

- Substitute particular solution into general solution to ODE
  $$y(x) = C1 \cos(x) + C2 \sin(x) + x^2 + x - 1$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 18

```
dsolve(diff(diff(y(x),x),x)+y(x) = x^2+x+1,y(x),singsol=all)
```

$$y = \sin(x) c_2 + \cos(x) c_1 + x^2 + x - 1$$

Mathematica DSolve solution

Solving time : 0.015 (sec)
Leaf size : 21

```
DSolve[{D[y[x],{x,2}]+y[x]==1+x+x^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^2 + x + c_1 \cos(x) + c_2 \sin(x) - 1$$

2.1.42 Problem 42

Internal problem ID [9113]
Book : Second order enumerated odes
Section : section 1
Problem number : 42
Date solved : `Monday, January 27, 2025 at 05:42:18 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = x^3 + x^2 + x + 1$$

Solved as second order linear constant coeff ode

Time used: 0.166 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x^3 + x^2 + x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4x^3 + A_3x^2 + A_2x + 6xA_4 + A_1 + 2A_3 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = -5, A_3 = 1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 + x^2 - 5x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x^3 + x^2 - 5x - 1) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^3 + x^2 - 5x - 1 + c_1 \cos(x) + c_2 \sin(x)$$

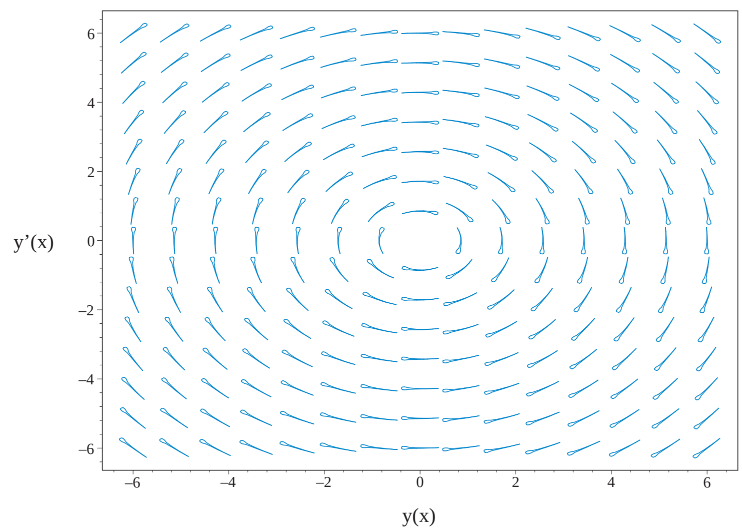


Figure 2.161: Slope field plot
$y'' + y = x^3 + x^2 + x + 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.151 (sec)

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.62: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4x^3 + A_3x^2 + A_2x + 6xA_4 + A_1 + 2A_3 = x^3 + x^2 + x + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = -5, A_3 = 1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 + x^2 - 5x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(x^3 + x^2 - 5x - 1\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^3 + x^2 - 5x - 1 + c_1 \cos(x) + c_2 \sin(x)$$

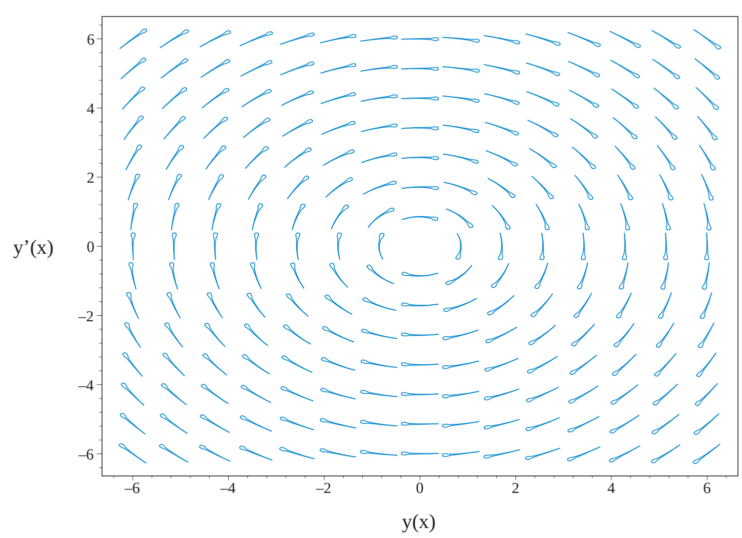


Figure 2.162: Slope field plot
$y'' + y = x^3 + x^2 + x + 1$

Solved as second order ode adjoint method

Time used: 1.753 (sec)

In normal form the ode

$$y'' + y = x^3 + x^2 + x + 1 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= x^3 + x^2 + x + 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0\\
\xi'' - (0)' + (\xi(x)) &= 0\\
\xi''(x) + \xi(x) &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\
&= \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +i\\
\lambda_2 &= -i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= i\\
\lambda_2 &= -i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y = \int \xi(x)\, r(x)\, dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{c_1(x^3 \sin(x) + 3x^2 \cos(x) - 6\cos(x) - 6x \sin(x)) + c_1(x^2 \sin(x) - 2\sin(}{}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)}$$
$$p(x) = \frac{(-c_2\, x^3 + (3c_1 - c_2)\, x^2 + (2c_1 + 5c_2)\, x - 5c_1 + c_2) \cos(x) + \sin(x) (c_1\, x^3 + (c_1 + 3c_2)\, x^2 + (-5c_1 + 2}{c_1 \cos(x) + c_2 \sin(x)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= e^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\ &= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$
$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{(-c_2\, x^3 + (3c_1 - c_2)\, x^2 + (2c_1 + 5c_2)\, x - 5c_1 + c_2) \cos(x) + \sin(x) (c_1\, x^3 + (c_1 + 3c_2)\, x^2 + (-5c_1 + 2}{c_1 \cos(x) + c_2 \sin(x)}\right.$$
$$\frac{d}{dx}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right)\left(\frac{(-c_2\, x^3 + (3c_1 - c_2)\, x^2 + (2c_1 + 5c_2)\, x - 5c_1 + c_2) \cos(x) + \sin(x) (c_1\, x^3 + (c_1}{c_1 \cos(x) + c_2 \sin(x)}\right.$$
$$d\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{(-c_2\, x^3 + (3c_1 - c_2)\, x^2 + (2c_1 + 5c_2)\, x - 5c_1 + c_2) \cos(x) + \sin(x) (c_1\, x^3 + (c_1 + 3c_2)\, x^2 + (-5c_1 + 2c_2)}{(c_1 \cos(x) + c_2 \sin(x))^2}\right.$$

Integrating gives

$$\frac{y}{c_1 \cos(x) + c_2 \sin(x)} = \int \frac{(-c_2\, x^3 + (3c_1 - c_2)\, x^2 + (2c_1 + 5c_2)\, x - 5c_1 + c_2) \cos(x) + \sin(x) (c_1\, x^3 + (c_1 +}{(c_1 \cos(x) + c_2 \sin(x))^2}$$
$$= \frac{5x - x^2 - x^3 + 2\tan\left(\frac{x}{2}\right)^2 + 10x \tan\left(\frac{x}{2}\right)^2 + 5x \tan\left(\frac{x}{2}\right)^4 - 2x^2 \tan\left(\frac{x}{2}\right)^2 - x^2 \tan\left(\frac{x}{2}\right)}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)\left(\tan\left(\frac{x}{2}\right)^2 c_1 - 2\tan\left(\frac{x}{2}\right) c_2 -\right.}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x) + c_2 \sin(x)}$ gives the final solution

$$y = c_2 c_3 \sin(x) + c_1 c_3 \cos(x) + x^3 + x^2 - 5x - 1$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_2c_3 \sin(x) + c_1c_3 \cos(x) + x^3 + x^2 - 5x - 1$$

The constants can be merged to give

$$y = x^3 + x^2 - 5x - 1 + c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^3 + x^2 - 5x - 1 + c_1 \cos(x) + c_2 \sin(x)$$

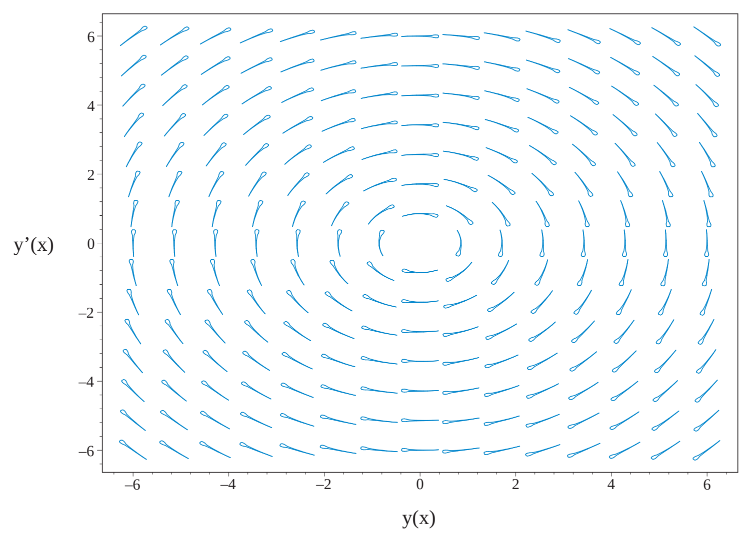


Figure 2.163: Slope field plot
$y'' + y = x^3 + x^2 + x + 1$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + y(x) = x^3 + x^2 + x + 1$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = \sin(x)$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^3 + x^2 + x + 1\right]$
- Wronskian of solutions of the homogeneous equation
  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- Compute Wronskian
  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$
  $y_p(x) = -\cos(x)\left(\int \sin(x)(x+1)(x^2+1)\,dx\right) + \sin(x)\left(\int \cos(x)(x+1)(x^2+1)\,dx\right)$
- Compute integrals
  $y_p(x) = x^3 + x^2 - 5x - 1$

- Substitute particular solution into general solution to ODE
  $y(x) = C1\cos(x) + C2\sin(x) + x^3 + x^2 - 5x - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 23

```
dsolve(diff(diff(y(x),x),x)+y(x) = x^3+x^2+x+1,y(x),singsol=all)
```

$$y = \sin(x)\,c_2 + \cos(x)\,c_1 + x^3 + x^2 - 5x - 1$$

Mathematica DSolve solution

Solving time : 0.014 (sec)
Leaf size : 26

```
DSolve[{D[y[x],{x,2}]+y[x]==1+x+x^2+x^3,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^3 + x^2 - 5x + c_1\cos(x) + c_2\sin(x) - 1$$

2.1.43 Problem 43

Internal problem ID [9114]
Book : Second order enumerated odes
Section : section 1
Problem number : 43
Date solved : `Monday, January 27, 2025 at 05:42:21 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \sin(x)$$

Solved as second order linear constant coeff ode

Time used: 0.132 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\,x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\,x}{2} + c_1 \cos(x) + c_2 \sin(x)$$

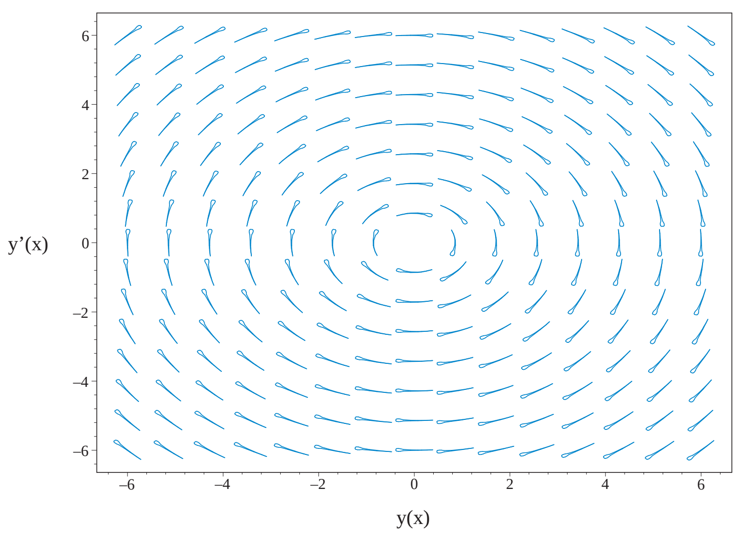


Figure 2.164: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.138 (sec)

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.64: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\,x, \sin(x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\,x + A_2 \sin(x)\,x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\,x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\,x}{2} + c_1 \cos(x) + c_2 \sin(x)$$

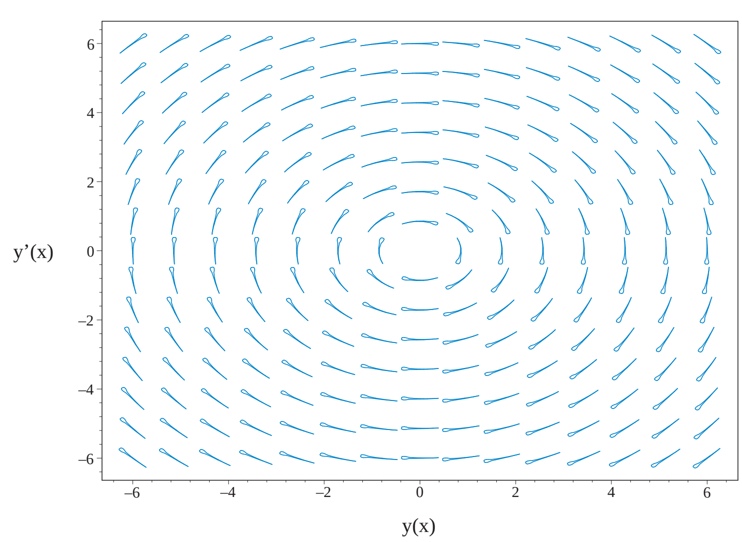


Figure 2.165: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode adjoint method

Time used: 2.438 (sec)

In normal form the ode

$$y'' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i \\ \lambda_2 &= -i \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{-\frac{\cos(x)^2 c_1}{2} + c_2\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{-c_2 \cos(x)\sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{-c_2 \cos(x)\sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right)$$

$$\frac{d}{dx}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right)\left(\frac{-c_2 \cos(x)\sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right)$$

$$d\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{-c_2 \cos(x)\sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))}\right) dx$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))} dx \\
&= \frac{\frac{x}{2} + \frac{x \tan(\frac{x}{2})^2}{2} - \frac{x \tan(\frac{x}{2})^4}{2} - \frac{x \tan(\frac{x}{2})^6}{2}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)^2 \left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1\right)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x)+c_2 \sin(x)}$ gives the final solution

$$
y = \frac{(2c_1c_3 - x) \cos(x)}{2} + c_2c_3 \sin(x)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{(2c_1c_3 - x) \cos(x)}{2} + c_2c_3 \sin(x)
$$

The constants can be merged to give

$$
y = \frac{(2c_1 - x) \cos(x)}{2} + c_2 \sin(x)
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{(2c_1 - x) \cos(x)}{2} + c_2 \sin(x)
$$

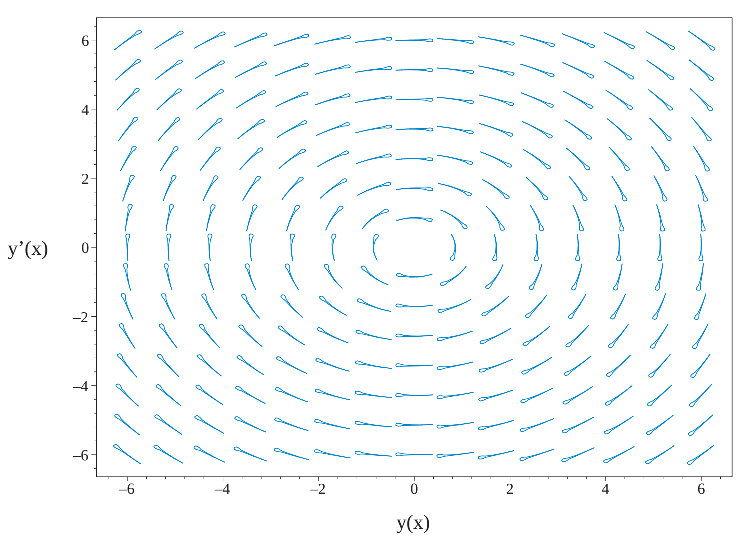


Figure 2.166: Slope field plot
$y'' + y = \sin(x)$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + y(x) = \sin(x)$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

$r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \sin(x)^2 dx\right) + \frac{\sin(x)\left(\int \sin(2x) dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) + \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x)+y(x) = sin(x),y(x),singsol=all)
```

$$y = \frac{(-x + 2c_1)\cos(x)}{2} + \frac{\sin(x)(2c_2 + 1)}{2}$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 22

```
DSolve[{D[y[x],{x,2}]+y[x]==Sin[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \left(-\frac{x}{2} + c_1\right)\cos(x) + c_2\sin(x)$$

2.1.44 Problem 44

Internal problem ID [9115]
Book : Second order enumerated odes
Section : section 1
Problem number : 44
Date solved : `Monday, January 27, 2025 at 05:42:25 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \cos(x)$$

Solved as second order linear constant coeff ode

Time used: 0.138 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{\sin(x)\, x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\sin(x)\, x}{2} + c_1 \cos(x) + c_2 \sin(x)$$

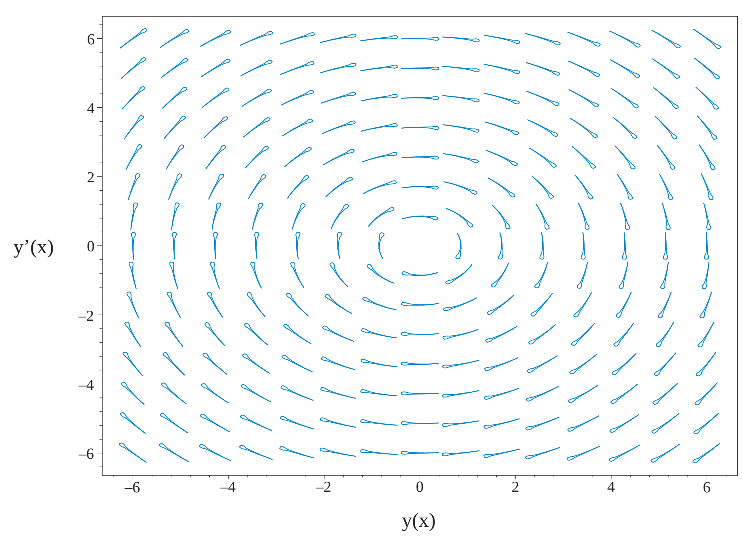


Figure 2.167: Slope field plot
$y'' + y = \cos(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.131 (sec)

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.66: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)x, \sin(x)x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)x + A_2 \sin(x)x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{\sin(x)x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\sin(x)x}{2} + c_1 \cos(x) + c_2 \sin(x)$$

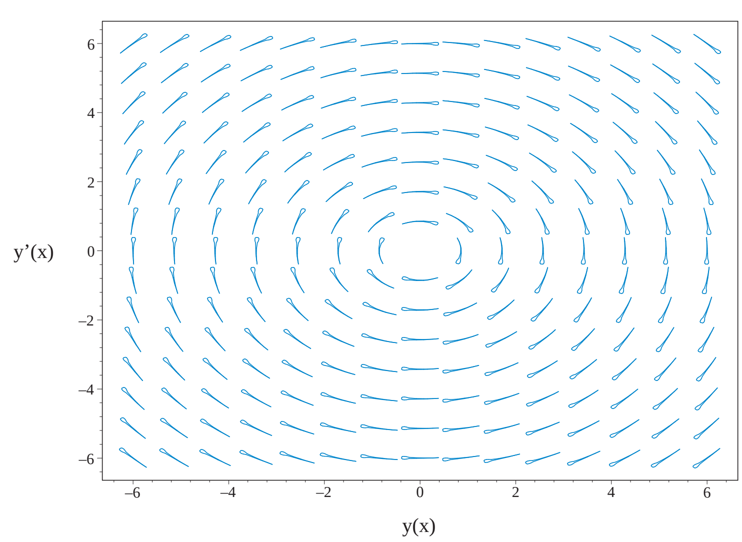


Figure 2.168: Slope field plot
$y'' + y = \cos(x)$

Solved as second order ode adjoint method

Time used: 2.557 (sec)

In normal form the ode

$$y'' + y = \cos(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= \cos(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i \\ \lambda_2 &= -i \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{c_1\left(\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right) - \frac{\cos(x)^2 c_2}{2}}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{c_1 \cos(x)\sin(x) - \cos(x)^2 c_2 + c_1 x}{2c_2 \sin(x) + 2c_1 \cos(x)}
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1 \cos(x)\sin(x) - \cos(x)^2 c_2 + c_1 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) &= \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right)\left(\frac{c_1 \cos(x)\sin(x) - \cos(x)^2 c_2 + c_1 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right) \\
\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) &= \left(\frac{c_1 \cos(x)\sin(x) - \cos(x)^2 c_2 + c_1 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))}\right) \mathrm{d}x
\end{aligned}$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{c_1 \cos(x) \sin(x) - \cos(x)^2 c_2 + c_1 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))} dx \\
&= \frac{-x \tan\left(\frac{x}{2}\right) - 2x \tan\left(\frac{x}{2}\right)^3 - x \tan\left(\frac{x}{2}\right)^5 - \frac{1}{2} - \frac{\tan\left(\frac{x}{2}\right)^2}{2} + \frac{\tan\left(\frac{x}{2}\right)^4}{2} + \frac{\tan\left(\frac{x}{2}\right)^6}{2}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)^2 \left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2\tan\left(\frac{x}{2}\right) c_2 - c_1\right)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x) + c_2 \sin(x)}$ gives the final solution

$$
y = -\frac{1}{2} + (2c_1c_3 + 1)\cos\left(\frac{x}{2}\right)^2 + \sin\left(\frac{x}{2}\right)(2c_2c_3 + x)\cos\left(\frac{x}{2}\right) - c_1c_3
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = -\frac{1}{2} + (2c_1c_3 + 1)\cos\left(\frac{x}{2}\right)^2 + \sin\left(\frac{x}{2}\right)(2c_2c_3 + x)\cos\left(\frac{x}{2}\right) - c_1c_3
$$

The constants can be merged to give

$$
y = -\frac{1}{2} + (2c_1 + 1)\cos\left(\frac{x}{2}\right)^2 + \sin\left(\frac{x}{2}\right)(2c_2 + x)\cos\left(\frac{x}{2}\right) - c_1
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = -\frac{1}{2} + (2c_1 + 1)\cos\left(\frac{x}{2}\right)^2 + \sin\left(\frac{x}{2}\right)(2c_2 + x)\cos\left(\frac{x}{2}\right) - c_1
$$

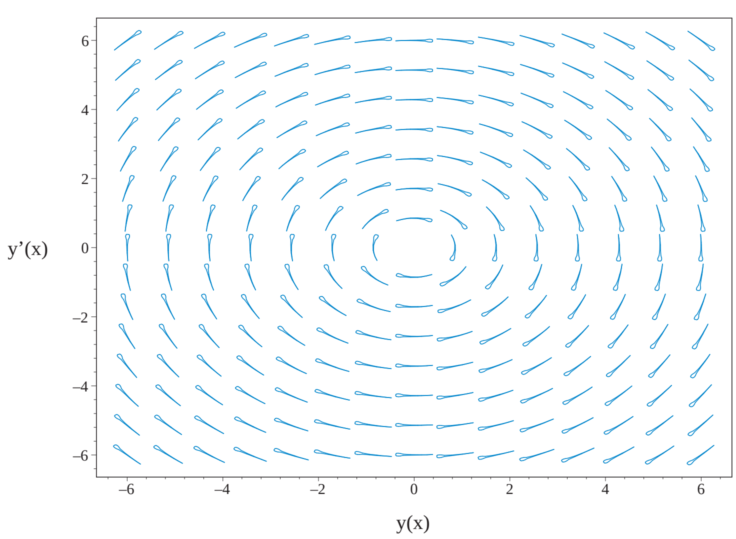


Figure 2.169: Slope field plot
$y'' + y = \cos(x)$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + y(x) = \cos(x)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r

$r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(x)\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- ∘ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(x)\left(\int \sin(2x) dx\right)}{2} + \sin(x) \left(\int \cos(x)^2 dx\right)$

- ∘ Compute integrals

  $y_p(x) = \frac{\cos(x)}{4} + \frac{\sin(x) x}{2}$

- Substitute particular solution into general solution to ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) + \frac{\cos(x)}{4} + \frac{\sin(x) x}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 18

```
dsolve(diff(diff(y(x),x),x)+y(x) = cos(x),y(x),singsol=all)
```

$$y = \frac{(2c_2 + x) \sin(x)}{2} + \cos(x) c_1$$

Mathematica DSolve solution

Solving time : 0.019 (sec)
Leaf size : 28

```
DSolve[{D[y[x],{x,2}]+y[x]==Cos[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}(x\sin(x) + \cos(x) + 2c_1\cos(x) + 2c_2\sin(x))$$

2.1.45 Problem 45

Internal problem ID [9116]
Book : Second order enumerated odes
Section : section 1
Problem number : 45
Date solved : `Monday, January 27, 2025 at 05:42:28 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$yy''^2 + y' = 0$$

Solved as second order missing x ode

Time used: 11.970 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y) = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p = 0 \tag{1}$$

$$p'^2 py + 1 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

<u>Solving equation (1)</u>

Solving for p from

$$p = 0$$

Solving gives $p = 0$

<u>Solving equation (2)</u>

Solving for the derivative gives these ODE's to solve

$$p' = -\frac{1}{\sqrt{-py}} \tag{1}$$

$$p' = \frac{1}{\sqrt{-py}} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y,p)\,\mathrm{d}y + N(y,p)\,\mathrm{d}p = 0 \tag{1A}$$

Therefore

$$\mathrm{d}p = \left(-\frac{1}{\sqrt{-py}}\right)\mathrm{d}y$$

$$\left(\frac{1}{\sqrt{-py}}\right)\mathrm{d}y + \mathrm{d}p = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(y,p) = \frac{1}{\sqrt{-py}}$$
$$N(y,p) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial p} &= \frac{\partial}{\partial p}\left(\frac{1}{\sqrt{-py}}\right)\\ &= -\frac{1}{2p\sqrt{-py}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial y} &= \frac{\partial}{\partial y}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial y}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial p} - \frac{\partial N}{\partial y}\right)\\ &= 1\left(\left(\frac{y}{2\left(-py\right)^{3/2}}\right) - (0)\right)\\ &= -\frac{1}{2p\sqrt{-py}}\end{aligned}$$

Since A depends on p, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial p}\right)\\ &= \sqrt{-py}\left((0) - \left(\frac{y}{2\left(-py\right)^{3/2}}\right)\right)\\ &= \frac{1}{2p}\end{aligned}$$

Since B does not depend on y, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}p}\\ &= e^{\int \frac{1}{2p} \,\mathrm{d}p}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{\ln(p)}{2}}\\ &= \sqrt{p}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \sqrt{p}\left(\frac{1}{\sqrt{-py}}\right)\\ &= \frac{\sqrt{p}}{\sqrt{-py}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \sqrt{p}(1)\\ &= \sqrt{p}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M}+\overline{N}\frac{\mathrm{d}p}{\mathrm{d}y}=0$$
$$\left(\frac{\sqrt{p}}{\sqrt{-py}}\right)+(\sqrt{p})\frac{\mathrm{d}p}{\mathrm{d}y}=0$$

The following equations are now set up to solve for the function $\phi(y,p)$

$$\frac{\partial\phi}{\partial y}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial p}=\overline{N} \tag{2}$$

Integrating (2) w.r.t. p gives

$$\int\frac{\partial\phi}{\partial p}\,\mathrm{d}p=\int\overline{N}\,\mathrm{d}p$$
$$\int\frac{\partial\phi}{\partial p}\,\mathrm{d}p=\int\sqrt{p}\,\mathrm{d}p$$
$$\phi=\frac{2p^{3/2}}{3}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both y and p. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial y}=\frac{\sqrt{p}}{\sqrt{-py}}$. Therefore equation (4) becomes

$$\frac{\sqrt{p}}{\sqrt{-py}}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\frac{\sqrt{p}}{\sqrt{-py}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int\left(\frac{\sqrt{p}}{\sqrt{-py}}\right)\mathrm{d}y$$
$$f(y)=-\frac{2\sqrt{-py}}{\sqrt{p}}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=\frac{2p^{3/2}}{3}-\frac{2\sqrt{-py}}{\sqrt{p}}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1=\frac{2p^{3/2}}{3}-\frac{2\sqrt{-py}}{\sqrt{p}}$$

Solving Eq. (2)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y,p)\,\mathrm{d}y+N(y,p)\,\mathrm{d}p=0 \tag{1A}$$

Therefore

$$\mathrm{d}p=\left(\frac{1}{\sqrt{-py}}\right)\mathrm{d}y$$

$$\left(-\frac{1}{\sqrt{-py}}\right)\mathrm{d}y+\mathrm{d}p=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(y,p)&=-\frac{1}{\sqrt{-py}}\\ N(y,p)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial p}=\frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial p}&=\frac{\partial}{\partial p}\left(-\frac{1}{\sqrt{-py}}\right)\\ &=\frac{1}{2p\sqrt{-py}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial y}&=\frac{\partial}{\partial y}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial y}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial p} - \frac{\partial N}{\partial y}\right) \\ &= 1\left(\left(-\frac{y}{2(-py)^{3/2}}\right) - (0)\right) \\ &= \frac{1}{2p\sqrt{-py}} \end{aligned}$$

Since A depends on p, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial p}\right) \\ &= -\sqrt{-py}\left((0) - \left(-\frac{y}{2(-py)^{3/2}}\right)\right) \\ &= \frac{1}{2p} \end{aligned}$$

Since B does not depend on y, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}p} \\ &= e^{\int \frac{1}{2p} \, \mathrm{d}p} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\frac{\ln(p)}{2}} \\ &= \sqrt{p} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sqrt{p}\left(-\frac{1}{\sqrt{-py}}\right) \\ &= -\frac{\sqrt{p}}{\sqrt{-py}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sqrt{p}(1) \\ &= \sqrt{p} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}p}{\mathrm{d}y} &= 0 \\ \left(-\frac{\sqrt{p}}{\sqrt{-py}}\right) + (\sqrt{p})\frac{\mathrm{d}p}{\mathrm{d}y} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(y, p)$

$$\frac{\partial \phi}{\partial y} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial p} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. p gives

$$\int \frac{\partial \phi}{\partial p} \, \mathrm{d}p = \int \overline{N} \, \mathrm{d}p$$
$$\int \frac{\partial \phi}{\partial p} \, \mathrm{d}p = \int \sqrt{p} \, \mathrm{d}p$$
$$\phi = \frac{2p^{3/2}}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both y and p. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial y} = -\frac{\sqrt{p}}{\sqrt{-py}}$. Therefore equation (4) becomes

$$-\frac{\sqrt{p}}{\sqrt{-py}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{\sqrt{p}}{\sqrt{-py}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{\sqrt{p}}{\sqrt{-py}}\right) \mathrm{d}y$$
$$f(y) = \frac{2\sqrt{-py}}{\sqrt{p}} + c_2$$

Where c_2 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2p^{3/2}}{3} + \frac{2\sqrt{-py}}{\sqrt{p}} + c_2$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_2 and c_2 constants into the constant c_2 gives the solution as

$$c_2 = \frac{2p^{3/2}}{3} + \frac{2\sqrt{-py}}{\sqrt{p}}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{2y'^{3/2}}{3} - \frac{2\sqrt{-yy'}}{\sqrt{y'}} = c_1$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{\left(12c_1 + 24\sqrt{-y}\right)^{2/3}}{4} \tag{1}$$

$$y' = \left(-\frac{\left(12c_1 + 24\sqrt{-y}\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(12c_1 + 24\sqrt{-y}\right)^{1/3}}{4}\right)^2 \tag{2}$$

$$y' = \left(-\frac{\left(12c_1 + 24\sqrt{-y}\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(12c_1 + 24\sqrt{-y}\right)^{1/3}}{4}\right)^2 \tag{3}$$

$$y' = \frac{\left(12c_1 - 24\sqrt{-y}\right)^{2/3}}{4} \tag{4}$$

$$y' = \left(-\frac{\left(12c_1 - 24\sqrt{-y}\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(12c_1 - 24\sqrt{-y}\right)^{1/3}}{4}\right)^2 \tag{5}$$

$$y' = \left(-\frac{\left(12c_1 - 24\sqrt{-y}\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(12c_1 - 24\sqrt{-y}\right)^{1/3}}{4}\right)^2 \tag{6}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{4}{\left(12c_1 + 24\sqrt{-y}\right)^{2/3}} dy = dx$$

$$-\frac{\left(-3c_1 + 2\sqrt{-y}\right)\left(12c_1 + 24\sqrt{-y}\right)^{1/3}}{8} = x + c_3$$

Singular solutions are found by solving

$$\frac{\left(12c_1 + 24\sqrt{-y}\right)^{2/3}}{4} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1^2}{4}$$

Solving Eq. (2)

Integrating gives

$$\int \frac{16}{\left(12c_1 + 24\sqrt{-y}\right)^{2/3}\left(i\sqrt{3} - 1\right)^2} dy = dx$$

$$-\frac{\left(i\sqrt{3} - 1\right)\left(12c_1 + 24\sqrt{-y}\right)^{1/3}\left(-3c_1 + 2\sqrt{-y}\right)}{16} = x + c_4$$

Singular solutions are found by solving

$$\frac{\left(12c_1 + 24\sqrt{-y}\right)^{2/3}\left(i\sqrt{3} - 1\right)^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1^2}{4}$$

Solving Eq. (3)

Integrating gives

$$\int \frac{16}{(12c_1 + 24\sqrt{-y})^{2/3}\left(1+i\sqrt{3}\right)^2} dy = dx$$

$$-\frac{3(12c_1 + 24\sqrt{-y})^{1/3}\left(c_1 - \frac{2\sqrt{-y}}{3}\right)\left(1+i\sqrt{3}\right)}{16} = x + c_5$$

Singular solutions are found by solving

$$\frac{(12c_1 + 24\sqrt{-y})^{2/3}\left(1+i\sqrt{3}\right)^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1^2}{4}$$

Solving Eq. (4)

Integrating gives

$$\int \frac{4}{(12c_1 - 24\sqrt{-y})^{2/3}} dy = dx$$

$$\frac{(3c_1 + 2\sqrt{-y})\,(12c_1 - 24\sqrt{-y})^{1/3}}{8} = x + c_6$$

Singular solutions are found by solving

$$\frac{(12c_1 - 24\sqrt{-y})^{2/3}}{4} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1^2}{4}$$

Solving Eq. (5)

Integrating gives

$$\int \frac{16}{(12c_1 - 24\sqrt{-y})^{2/3}\left(1+i\sqrt{3}\right)^2} dy = dx$$

$$-\frac{3\left(c_1 + \frac{2\sqrt{-y}}{3}\right)(12c_1 - 24\sqrt{-y})^{1/3}\left(1+i\sqrt{3}\right)}{16} = x + c_7$$

Singular solutions are found by solving

$$\frac{(12c_1 - 24\sqrt{-y})^{2/3}\left(1+i\sqrt{3}\right)^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1^2}{4}$$

Solving Eq. (6)

Integrating gives

$$\int \frac{16}{(12c_1 - 24\sqrt{-y})^{2/3}\left(i\sqrt{3}-1\right)^2} dy = dx$$

$$\frac{\left(i\sqrt{3}-1\right)(12c_1 - 24\sqrt{-y})^{1/3}\,(3c_1 + 2\sqrt{-y})}{16} = x + c_8$$

Singular solutions are found by solving

$$\frac{\left(12c_1 - 24\sqrt{-y}\right)^{2/3} \left(i\sqrt{3}-1\right)^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1^2}{4}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{2y'^{3/2}}{3} + \frac{2\sqrt{-y y'}}{\sqrt{y'}} = c_2$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{\left(12c_2 + 24\sqrt{-y}\right)^{2/3}}{4} \tag{1}$$

$$y' = \left(-\frac{\left(12c_2 + 24\sqrt{-y}\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(12c_2 + 24\sqrt{-y}\right)^{1/3}}{4}\right)^2 \tag{2}$$

$$y' = \left(-\frac{\left(12c_2 + 24\sqrt{-y}\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(12c_2 + 24\sqrt{-y}\right)^{1/3}}{4}\right)^2 \tag{3}$$

$$y' = \frac{\left(12c_2 - 24\sqrt{-y}\right)^{2/3}}{4} \tag{4}$$

$$y' = \left(-\frac{\left(12c_2 - 24\sqrt{-y}\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(12c_2 - 24\sqrt{-y}\right)^{1/3}}{4}\right)^2 \tag{5}$$

$$y' = \left(-\frac{\left(12c_2 - 24\sqrt{-y}\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(12c_2 - 24\sqrt{-y}\right)^{1/3}}{4}\right)^2 \tag{6}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{4}{\left(12c_2 + 24\sqrt{-y}\right)^{2/3}} dy = dx$$

$$-\frac{\left(-3c_2 + 2\sqrt{-y}\right)\left(12c_2 + 24\sqrt{-y}\right)^{1/3}}{8} = x + c_9$$

Singular solutions are found by solving

$$\frac{\left(12c_2 + 24\sqrt{-y}\right)^{2/3}}{4} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_2^2}{4}$$

Solving Eq. (2)

Integrating gives

$$\int \frac{16}{\left(12c_2+24\sqrt{-y}\right)^{2/3}\left(i\sqrt{3}-1\right)^2}dy = dx$$

$$-\frac{\left(i\sqrt{3}-1\right)\left(12c_2+24\sqrt{-y}\right)^{1/3}\left(-3c_2+2\sqrt{-y}\right)}{16} = x + \_C10$$

Singular solutions are found by solving

$$\frac{\left(12c_2+24\sqrt{-y}\right)^{2/3}\left(i\sqrt{3}-1\right)^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_2^2}{4}$$

Solving Eq. (3)

Integrating gives

$$\int \frac{16}{\left(12c_2+24\sqrt{-y}\right)^{2/3}\left(1+i\sqrt{3}\right)^2}dy = dx$$

$$-\frac{3\left(12c_2+24\sqrt{-y}\right)^{1/3}\left(c_2-\frac{2\sqrt{-y}}{3}\right)\left(1+i\sqrt{3}\right)}{16} = x + \_C11$$

Singular solutions are found by solving

$$\frac{\left(12c_2+24\sqrt{-y}\right)^{2/3}\left(1+i\sqrt{3}\right)^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_2^2}{4}$$

Solving Eq. (4)

Integrating gives

$$\int \frac{4}{\left(12c_2-24\sqrt{-y}\right)^{2/3}}dy = dx$$

$$\frac{\left(3c_2+2\sqrt{-y}\right)\left(12c_2-24\sqrt{-y}\right)^{1/3}}{8} = x + \_C12$$

Singular solutions are found by solving

$$\frac{\left(12c_2-24\sqrt{-y}\right)^{2/3}}{4} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_2^2}{4}$$

Solving Eq. (5)

Integrating gives

$$\int \frac{16}{\left(12c_2-24\sqrt{-y}\right)^{2/3}\left(1+i\sqrt{3}\right)^2}dy = dx$$

$$-\frac{3\left(c_2+\frac{2\sqrt{-y}}{3}\right)\left(12c_2-24\sqrt{-y}\right)^{1/3}\left(1+i\sqrt{3}\right)}{16} = x + \_C13$$

Singular solutions are found by solving

$$\frac{(12c_2 - 24\sqrt{-y})^{2/3} (1 + i\sqrt{3})^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_2^2}{4}$$

Solving Eq. (6)

Integrating gives

$$\int \frac{16}{(12c_2 - 24\sqrt{-y})^{2/3} (i\sqrt{3} - 1)^2} dy = dx$$
$$\frac{(i\sqrt{3} - 1) (12c_2 - 24\sqrt{-y})^{1/3} (3c_2 + 2\sqrt{-y})}{16} = x + \_C14$$

Singular solutions are found by solving

$$\frac{(12c_2 - 24\sqrt{-y})^{2/3} (i\sqrt{3} - 1)^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_2^2}{4}$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0 \, dx + \_C15$$
$$y = \_C15$$

Will add steps showing solving for IC soon.

Solving for y from the above solution(s) gives (after possible removing of solutions that do not verify)

$$y = \_C15$$

$$y = -\frac{c_1^2}{4}$$

$$y = -\frac{c_2^2}{4}$$

$$y = -\frac{c_1 \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_3 + 6x\right)^3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_3 + 6x\right)^2 c_3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z\right.}{}$$

$$y = -\frac{c_1 \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_6 + 6x\right)^3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_6 + 6x\right)^2 c_6}{3} + \frac{2x \operatorname{RootOf}\left(\right.}{}$$

$$y = -\frac{c_2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6\_C12 + 6x\right)^3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6\_C12 + 6x\right)^2 \_C12}{3} + \frac{2\,\mathrm{I}}{}$$

$$y = -\frac{c_2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6c_9 + 6x\right)^3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6c_9 + 6x\right)^2 c_9}{3} + \frac{2 \operatorname{RootOf}\left(\_Z\right.}{}$$

$$y = -\frac{i\_C10\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C10 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C10 - 3x\right)^2}{3} - \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 -\right.}{}$$

$$y = -\frac{i\_C14\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C14 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C14 - 3x\right)^2}{3} - \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 -\right.}{}$$

$$y = -\frac{ic_4\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_4 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_4 - 3x\right)^2}{3} - \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_4 -\right.}{3}$$

$$y = -\frac{ic_8\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_8 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_8 - 3x\right)^2}{3} - \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_8 -\right.}{3}$$

$$y = \frac{i\_C11\sqrt{3} \operatorname{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C11 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C11 - 3x\right)^2}{3} + \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 + \right.}{}$$

$$y = \frac{i\_C13\sqrt{3} \operatorname{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C13 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C13 - 3x\right)^2}{3} + \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 + \right.}{}$$

$$y = \frac{ic_5\sqrt{3} \operatorname{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_5 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_5 - 3x\right)^2}{3} + \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_5 + 3\right.}{3}$$

$$y = \frac{ic_7\sqrt{3} \operatorname{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_7 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_7 - 3x\right)^2}{3} + \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_7 + 3\right.}{3}$$

Summary of solutions found

$$y = \_C15$$

$$y = -\frac{c_1^2}{4}$$

$$y = -\frac{c_2^2}{4}$$

$$y = -\frac{c_1 \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_3 + 6x\right)^3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_3 + 6x\right)^2 c_3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_3 + 6x\right)^2 x}{3} - \frac{c_1^2}{4}$$

$$y = -\frac{c_1 \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_6 + 6x\right)^3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_6 + 6x\right)^2 c_6}{3} + \frac{2x \operatorname{RootOf}\left(\_Z^4 - 6c_1\_Z + 6c_6 + 6x\right)^2}{3} - \frac{c_1^2}{4}$$

$$y = -\frac{c_2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6\_C12 + 6x\right)^3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6\_C12 + 6x\right)^2 \_C12}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6\_C12 + 6x\right)^2 x}{3} - \frac{c_2^2}{4}$$

$$y = -\frac{c_2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6c_9 + 6x\right)^3}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6c_9 + 6x\right)^2 c_9}{3} + \frac{2 \operatorname{RootOf}\left(\_Z^4 - 6c_2\_Z + 6c_9 + 6x\right)^2 x}{3} - \frac{c_2^2}{4}$$

$$y = -\frac{i\_C10\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C10 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C10 - 3x\right)^2}{3} - \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C10 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C10 - 3x\right)^2}{3} - \frac{c_2 \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C10 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C10 - 3x\right)^3}{3} - \frac{\_C10 \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C10 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C10 - 3x\right)^2}{3} - \frac{x \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C10 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C10 - 3x\right)^2}{3} - \frac{c_2^2}{4}$$

$$y = -\frac{i\_C14\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C14 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C14 - 3x\right)^2}{3} - \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C14 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C14 - 3x\right)^2}{3} - \frac{c_2 \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C14 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C14 - 3x\right)^3}{3} - \frac{\_C14 \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C14 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C14 - 3x\right)^2}{3} - \frac{x \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,\_C14 - 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C14 - 3x\right)^2}{3} - \frac{c_2^2}{4}$$

$$y = -\frac{ic_4\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_4 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_4 - 3x\right)^2}{3} - \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_4 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_4 - 3x\right)^2}{3} - \frac{c_1 \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_4 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_4 - 3x\right)^3}{3} - \frac{c_4 \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_4 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_4 - 3x\right)^2}{3} - \frac{x \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_4 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_4 - 3x\right)^2}{3} - \frac{c_1^2}{4}$$

$$y = -\frac{ic_8\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_8 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_8 - 3x\right)^2}{3} - \frac{ix\sqrt{3} \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_8 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_8 - 3x\right)^2}{3} - \frac{c_1 \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_8 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_8 - 3x\right)^3}{3} - \frac{c_8 \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_8 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_8 - 3x\right)^2}{3} - \frac{x \operatorname{RootOf}\left(\_Z^4 - 3i\sqrt{3}\,c_8 - 3i\sqrt{3}\,x - 6c_1\_Z - 3c_8 - 3x\right)^2}{3} - \frac{c_1^2}{4}$$

$$y = \frac{i\_C11\sqrt{3}\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C11 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C11 - 3x\right)^2}{3} + \frac{ix\sqrt{3}\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C11 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C11 - 3x\right)^2}{3} - \frac{c_2\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C11 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C11 - 3x\right)^3}{3} - \frac{\_C11\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C11 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C11 - 3x\right)^2}{3} - \frac{x\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C11 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C11 - 3x\right)^2}{3} - \frac{c_2^2}{4}$$

$$y = \frac{i\_C13\sqrt{3}\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C13 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C13 - 3x\right)^2}{3} + \frac{ix\sqrt{3}\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C13 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C13 - 3x\right)^2}{3} - \frac{c_2\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C13 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C13 - 3x\right)^3}{3} - \frac{\_C13\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C13 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C13 - 3x\right)^2}{3} - \frac{x\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,\_C13 + 3i\sqrt{3}\,x - 6c_2\_Z - 3\_C13 - 3x\right)^2}{3} - \frac{c_2^2}{4}$$

$$y = \frac{ic_5\sqrt{3}\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_5 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_5 - 3x\right)^2}{3} + \frac{ix\sqrt{3}\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_5 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_5 - 3x\right)^2}{3} - \frac{c_1\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_5 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_5 - 3x\right)^3}{3} - \frac{c_5\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_5 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_5 - 3x\right)^2}{3} - \frac{x\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_5 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_5 - 3x\right)^2}{3} - \frac{c_1^2}{4}$$

$$y = \frac{ic_7\sqrt{3}\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_7 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_7 - 3x\right)^2}{3} + \frac{ix\sqrt{3}\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_7 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_7 - 3x\right)^2}{3} - \frac{c_1\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_7 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_7 - 3x\right)^3}{3} - \frac{c_7\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_7 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_7 - 3x\right)^2}{3} - \frac{x\,\mathrm{RootOf}\left(\_Z^4 + 3i\sqrt{3}\,c_7 + 3i\sqrt{3}\,x - 6c_1\_Z - 3c_7 - 3x\right)^2}{3} - \frac{c_1^2}{4}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resul
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_b(_a)*_a)^(1/2)/_a
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[_a, 1/3*_b]
```

Maple dsolve solution

Solving time : `0.100` (sec)
Leaf size : 263

```
dsolve(y(x)*diff(diff(y(x),x),x)^2+diff(y(x),x) = 0,y(x),singsol=all)
```

$$y = c_1$$
$$y = 0$$
$$-\int^{y} \frac{\_a}{\left(\_a^{3/2}\left(c_1 - 3\sqrt{\_a}\right)\right)^{2/3}} d\_a - x - c_2 = 0$$
$$-\int^{y} \frac{\_a}{\left(\_a^{3/2}\left(c_1 + 3\sqrt{\_a}\right)\right)^{2/3}} d\_a - x - c_2 = 0$$
$$\frac{-4\left(\int^{y} \frac{\_a}{\left(\_a^{3/2}(c_1-3\sqrt{\_a})\right)^{2/3}} d\_a\right) + 2i(-x-c_2)\sqrt{3} + 2x + 2c_2}{\left(-i\sqrt{3}-1\right)^2} = 0$$
$$\frac{-4\left(\int^{y} \frac{\_a}{\left(\_a^{3/2}(c_1-3\sqrt{\_a})\right)^{2/3}} d\_a\right) + 2i(x+c_2)\sqrt{3} + 2x + 2c_2}{\left(1-i\sqrt{3}\right)^2} = 0$$
$$\frac{-4\left(\int^{y} \frac{\_a}{\left(\_a^{3/2}(c_1+3\sqrt{\_a})\right)^{2/3}} d\_a\right) + 2i(-x-c_2)\sqrt{3} + 2x + 2c_2}{\left(-i\sqrt{3}-1\right)^2} = 0$$
$$\frac{-4\left(\int^{y} \frac{\_a}{\left(\_a^{3/2}(c_1+3\sqrt{\_a})\right)^{2/3}} d\_a\right) + 2i(x+c_2)\sqrt{3} + 2x + 2c_2}{\left(1-i\sqrt{3}\right)^2} = 0$$

Mathematica DSolve solution

Solving time : 61.031 (sec)
Leaf size : 23861

```
DSolve[{y[x]*D[y[x],{x,2}]^2+D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

Too large to display

2.1.46 Problem 46

Internal problem ID [9117]
Book : Second order enumerated odes
Section : section 1
Problem number : 46
Date solved : `Monday, January 27, 2025 at 05:42:41 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$yy''^2 + y'^3 = 0$$

Solved as second order missing x ode

Time used: 2.266 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y)^3 = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p^2 = 0 \tag{1}$$

$$p'^2 y + p = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p^2 = 0$$

Solving gives $p = 0$

Solving equation (2)

Solving for p' gives

$$p' = \frac{\sqrt{-py}}{y} \tag{1}$$

$$p' = -\frac{\sqrt{-py}}{y} \tag{2}$$

In canonical form, the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= \frac{\sqrt{-py}}{y} \end{aligned} \tag{1}$$

An ode of the form $p' = \frac{M(y,p)}{N(y,p)}$ is called homogeneous if the functions $M(y,p)$ and $N(y,p)$ are both homogeneous functions and of the same order. Recall that a function $f(y,p)$ is homogeneous of order n if

$$f(t^n y, t^n p) = t^n f(y,p)$$

In this case, it can be seen that both $M = \sqrt{-py}$ and $N = y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{p}{y}$, or $p = uy$. Hence

$$\frac{dp}{dy} = \frac{du}{dy}y + u$$

Applying the transformation $p = uy$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dy}y + u &= \sqrt{-u} \\ \frac{du}{dy} &= \frac{\sqrt{-u(y)} - u(y)}{y} \end{aligned}$$

Or

$$u'(y) - \frac{\sqrt{-u(y)} - u(y)}{y} = 0$$

Or

$$u'(y)\,y - \sqrt{-u(y)} + u(y) = 0$$

Which is now solved as separable in $u(y)$.

The ode

$$u'(y) = \frac{\sqrt{-u(y)} - u(y)}{y} \tag{2.1}$$

is separable as it can be written as

$$\begin{aligned} u'(y) &= \frac{\sqrt{-u(y)} - u(y)}{y} \\ &= f(y)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= \frac{1}{y} \\ g(u) &= \sqrt{-u} - u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(y)\,dy \\ \int \frac{1}{\sqrt{-u} - u}\,du &= \int \frac{1}{y}\,dy \end{aligned}$$

$$\ln\left(\frac{1}{\left(\sqrt{-u(y)}+1\right)^{2}}\right)=\ln(y)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$\sqrt{-u}-u=0$$

for $u(y)$ gives

$$u(y)=0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(\frac{1}{\left(\sqrt{-u(y)}+1\right)^{2}}\right)=\ln(y)+c_1$$
$$u(y)=0$$

Converting $\ln\left(\frac{1}{\left(\sqrt{-u(y)}+1\right)^{2}}\right)=\ln(y)+c_1$ back to p gives

$$\ln\left(\frac{1}{\left(\sqrt{-\frac{p}{y}}+1\right)^{2}}\right)=\ln(y)+c_1$$

Converting $u(y)=0$ back to p gives

$$p=0$$

In canonical form, the ODE is

$$\begin{aligned} p' &= F(y,p)\\ &= -\frac{\sqrt{-py}}{y} \end{aligned} \tag{1}$$

An ode of the form $p'=\frac{M(y,p)}{N(y,p)}$ is called homogeneous if the functions $M(y,p)$ and $N(y,p)$ are both homogeneous functions and of the same order. Recall that a function $f(y,p)$ is homogeneous of order n if

$$f(t^n y, t^n p)=t^n f(y,p)$$

In this case, it can be seen that both $M=-\sqrt{-py}$ and $N=y$ are both homogeneous and of the same order $n=1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{p}{y}$, or $p=uy$. Hence

$$\frac{\mathrm{d}p}{\mathrm{d}y}=\frac{\mathrm{d}u}{\mathrm{d}y}y+u$$

Applying the transformation $p=uy$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}y}y+u=-\sqrt{-u}$$
$$\frac{\mathrm{d}u}{\mathrm{d}y}=\frac{-\sqrt{-u(y)}-u(y)}{y}$$

Or

$$u'(y) - \frac{-\sqrt{-u(y)} - u(y)}{y} = 0$$

Or

$$u'(y)\, y + \sqrt{-u(y)} + u(y) = 0$$

Which is now solved as separable in $u(y)$.

The ode

$$u'(y) = -\frac{\sqrt{-u(y)} + u(y)}{y} \tag{2.2}$$

is separable as it can be written as

$$\begin{aligned} u'(y) &= -\frac{\sqrt{-u(y)} + u(y)}{y} \\ &= f(y)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= \frac{1}{y} \\ g(u) &= -\sqrt{-u} - u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(y)\, dy \\ \int \frac{1}{-\sqrt{-u} - u}\, du &= \int \frac{1}{y}\, dy \end{aligned}$$

$$-2\ln\left(\sqrt{-u(y)} - 1\right) = \ln(y) + c_2$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-\sqrt{-u} - u = 0$$

for $u(y)$ gives

$$\begin{aligned} u(y) &= -1 \\ u(y) &= 0 \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} -2\ln\left(\sqrt{-u(y)} - 1\right) &= \ln(y) + c_2 \\ u(y) &= -1 \\ u(y) &= 0 \end{aligned}$$

Converting $-2\ln\left(\sqrt{-u(y)} - 1\right) = \ln(y) + c_2$ back to p gives

$$-2\ln\left(\sqrt{-\frac{p}{y}} - 1\right) = \ln(y) + c_2$$

Converting $u(y) = -1$ back to p gives

$$p = -y$$

Converting $u(y) = 0$ back to p gives

$$p = 0$$

Solving for p gives

$$\begin{aligned}
p &= 0 \\
p &= \left(-\frac{2y\,\mathrm{e}^{c_1}\left(\sqrt{y\,\mathrm{e}^{c_1}}-1\right)}{\sqrt{y\,\mathrm{e}^{c_1}}}+y\,\mathrm{e}^{c_1}-1\right)\mathrm{e}^{-c_1} \\
p &= \left(-\frac{2y\,\mathrm{e}^{c_1}\left(\sqrt{y\,\mathrm{e}^{c_1}}+1\right)}{\sqrt{y\,\mathrm{e}^{c_1}}}+y\,\mathrm{e}^{c_1}-1\right)\mathrm{e}^{-c_1} \\
p &= -y \\
p &= -\left(\frac{2y\,\mathrm{e}^{c_2}\left(\sqrt{y\,\mathrm{e}^{c_2}}-1\right)}{\sqrt{y\,\mathrm{e}^{c_2}}}-y\,\mathrm{e}^{c_2}+1\right)\mathrm{e}^{-c_2} \\
p &= -\left(\frac{2y\,\mathrm{e}^{c_2}\left(\sqrt{y\,\mathrm{e}^{c_2}}+1\right)}{\sqrt{y\,\mathrm{e}^{c_2}}}-y\,\mathrm{e}^{c_2}+1\right)\mathrm{e}^{-c_2}
\end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned}
\int dy &= \int 0\,dx + c_3 \\
y &= c_3
\end{aligned}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \left(-\frac{2y\,\mathrm{e}^{c_1}\left(\sqrt{y\,\mathrm{e}^{c_1}}-1\right)}{\sqrt{y\,\mathrm{e}^{c_1}}}+y\,\mathrm{e}^{c_1}-1\right)\mathrm{e}^{-c_1}$$

Integrating gives

$$\begin{aligned}
\int -\frac{\sqrt{y\,\mathrm{e}^{c_1}}\,\mathrm{e}^{c_1}}{y\,\mathrm{e}^{c_1}\sqrt{y\,\mathrm{e}^{c_1}}-2y\,\mathrm{e}^{c_1}+\sqrt{y\,\mathrm{e}^{c_1}}}dy &= dx \\
\frac{2+\left(-2\sqrt{y\,\mathrm{e}^{c_1}}+2\right)\ln\left(\sqrt{y\,\mathrm{e}^{c_1}}-1\right)}{\sqrt{y\,\mathrm{e}^{c_1}}-1} &= x + c_4
\end{aligned}$$

Singular solutions are found by solving

$$-\frac{\left(y\,\mathrm{e}^{c_1}\sqrt{y\,\mathrm{e}^{c_1}}-2y\,\mathrm{e}^{c_1}+\sqrt{y\,\mathrm{e}^{c_1}}\right)\mathrm{e}^{-c_1}}{\sqrt{y\,\mathrm{e}^{c_1}}} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \mathrm{e}^{-c_1}$$

Solving for y gives

$$\begin{aligned}
y &= \mathrm{e}^{-c_1} \\
y &= \frac{\left(\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_4}{2}+\frac{x}{2}}\right)^2+2\,\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_4}{2}+\frac{x}{2}}\right)+1\right)\mathrm{e}^{-c_1}}{\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_4}{2}+\frac{x}{2}}\right)^2}
\end{aligned}$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \left(-\frac{2y\,\mathrm{e}^{c_1}\left(1+\sqrt{y\,\mathrm{e}^{c_1}}\right)}{\sqrt{y\,\mathrm{e}^{c_1}}} + y\,\mathrm{e}^{c_1} - 1\right)\mathrm{e}^{-c_1}$$

Integrating gives

$$\int -\frac{\sqrt{y\,\mathrm{e}^{c_1}}\,\mathrm{e}^{c_1}}{y\,\mathrm{e}^{c_1}\sqrt{y\,\mathrm{e}^{c_1}} + 2y\,\mathrm{e}^{c_1} + \sqrt{y\,\mathrm{e}^{c_1}}}dy = dx$$
$$\frac{-2+\left(-2\sqrt{y\,\mathrm{e}^{c_1}}-2\right)\ln\left(\sqrt{y\,\mathrm{e}^{c_1}}+1\right)}{\sqrt{y\,\mathrm{e}^{c_1}}+1} = x + c_5$$

Solving for y gives

$$y = \frac{\left(\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_5}{2}+\frac{x}{2}}\right)^2 + 2\,\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_5}{2}+\frac{x}{2}}\right) + 1\right)\mathrm{e}^{-c_1}}{\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_5}{2}+\frac{x}{2}}\right)^2}$$

For solution (4) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -y$$

Integrating gives

$$\int -\frac{1}{y}dy = dx$$
$$-\ln(y) = x + c_6$$

Singular solutions are found by solving

$$-y = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$

Solving for y gives

$$y = 0$$
$$y = \mathrm{e}^{-x-c_6}$$

For solution (5) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\left(\frac{2y\,\mathrm{e}^{c_2}\left(\sqrt{y\,\mathrm{e}^{c_2}}-1\right)}{\sqrt{y\,\mathrm{e}^{c_2}}} - y\,\mathrm{e}^{c_2} + 1\right)\mathrm{e}^{-c_2}$$

Integrating gives

$$\int -\frac{\sqrt{y\,\mathrm{e}^{c_2}}\,\mathrm{e}^{c_2}}{y\,\mathrm{e}^{c_2}\sqrt{y\,\mathrm{e}^{c_2}} - 2y\,\mathrm{e}^{c_2} + \sqrt{y\,\mathrm{e}^{c_2}}}dy = dx$$
$$\frac{2+\left(-2\sqrt{y\,\mathrm{e}^{c_2}}+2\right)\ln\left(\sqrt{y\,\mathrm{e}^{c_2}}-1\right)}{\sqrt{y\,\mathrm{e}^{c_2}}-1} = x + c_7$$

Singular solutions are found by solving

$$-\frac{\left(y\,\mathrm{e}^{c_2}\sqrt{y\,\mathrm{e}^{c_2}} - 2y\,\mathrm{e}^{c_2} + \sqrt{y\,\mathrm{e}^{c_2}}\right)\mathrm{e}^{-c_2}}{\sqrt{y\,\mathrm{e}^{c_2}}} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \mathrm{e}^{-c_2}$$

Solving for y gives

$$y = \mathrm{e}^{-c_2}$$

$$y = \frac{\left(\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_7}{2}+\frac{x}{2}}\right)^2 + 2\,\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_7}{2}+\frac{x}{2}}\right) + 1\right)\mathrm{e}^{-c_2}}{\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_7}{2}+\frac{x}{2}}\right)^2}$$

For solution (6) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\left(\frac{2y\,\mathrm{e}^{c_2}(1+\sqrt{y\,\mathrm{e}^{c_2}})}{\sqrt{y\,\mathrm{e}^{c_2}}} - y\,\mathrm{e}^{c_2} + 1\right)\mathrm{e}^{-c_2}$$

Integrating gives

$$\int -\frac{\sqrt{y\,\mathrm{e}^{c_2}}\,\mathrm{e}^{c_2}}{y\,\mathrm{e}^{c_2}\sqrt{y\,\mathrm{e}^{c_2}} + 2y\,\mathrm{e}^{c_2} + \sqrt{y\,\mathrm{e}^{c_2}}}dy = dx$$

$$\frac{-2 + (-2\sqrt{y\,\mathrm{e}^{c_2}} - 2)\ln\left(\sqrt{y\,\mathrm{e}^{c_2}} + 1\right)}{\sqrt{y\,\mathrm{e}^{c_2}} + 1} = x + c_8$$

Solving for y gives

$$y = \frac{\left(\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_8}{2}+\frac{x}{2}}\right)^2 + 2\,\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_8}{2}+\frac{x}{2}}\right) + 1\right)\mathrm{e}^{-c_2}}{\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_8}{2}+\frac{x}{2}}\right)^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 0$$

$$y = c_3$$

$$y = \mathrm{e}^{-c_1}$$

$$y = \mathrm{e}^{-c_2}$$

$$y = \frac{\left(\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_5}{2}+\frac{x}{2}}\right)^2 + 2\,\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_5}{2}+\frac{x}{2}}\right) + 1\right)\mathrm{e}^{-c_1}}{\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_5}{2}+\frac{x}{2}}\right)^2}$$

$$y = \frac{\left(\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_8}{2}+\frac{x}{2}}\right)^2 + 2\,\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_8}{2}+\frac{x}{2}}\right) + 1\right)\mathrm{e}^{-c_2}}{\mathrm{LambertW}\left(-\mathrm{e}^{\frac{c_8}{2}+\frac{x}{2}}\right)^2}$$

$$y = \frac{\left(\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_4}{2}+\frac{x}{2}}\right)^2 + 2\,\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_4}{2}+\frac{x}{2}}\right) + 1\right)\mathrm{e}^{-c_1}}{\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_4}{2}+\frac{x}{2}}\right)^2}$$

$$y = \frac{\left(\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_7}{2}+\frac{x}{2}}\right)^2 + 2\,\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_7}{2}+\frac{x}{2}}\right) + 1\right)\mathrm{e}^{-c_2}}{\mathrm{LambertW}\left(\mathrm{e}^{\frac{c_7}{2}+\frac{x}{2}}\right)^2}$$

$$y = \mathrm{e}^{-x-c_6}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each res
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      Try integration with the canonical coordinates of the symmetry [0, y]
      -> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(-_b(_a))^(3/2)-_b(_a)^2,
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

Maple dsolve solution

Solving time : `0.161` (sec)
Leaf size : 166

```
dsolve(y(x)*diff(diff(y(x),x),x)^2+diff(y(x),x)^3 = 0,y(x),singsol=all)
```

$$y = c_1$$
$$y = 0$$
$$y = \frac{c_2 \left(\operatorname{LambertW}\left(c_1 e^{\frac{x}{2}-1}\right)+1\right)^2}{\operatorname{LambertW}\left(c_1 e^{\frac{x}{2}-1}\right)^2}$$
$$y = \frac{c_2 \left(\operatorname{LambertW}\left(-c_1 e^{\frac{x}{2}-1}\right)+1\right)^2}{\operatorname{LambertW}\left(-c_1 e^{\frac{x}{2}-1}\right)^2}$$
$$y = e^{-\int e^{2\operatorname{RootOf}\left(e^{\_Z}\ln\left(\left(e^{\_Z}+1\right)^2\right)+c_1 e^{\_Z}-2\_Z e^{\_Z}+x e^{\_Z}+\ln\left(\left(e^{\_Z}+1\right)^2\right)+c_1-2\_Z+x-2\right)}dx-2\left(\int e^{\operatorname{RootOf}\left(e^{\_Z}\ln\left(\left(e^{\_Z}+1\right)^2\right)+c_1 e^{\_Z}-2\_Z e^{\_Z}\cdots\right.}\right.}$$

Mathematica DSolve solution

Solving time : `1.974` (sec)
Leaf size : 361

```
DSolve[{y[x]*D[y[x],{x,2}]^2+D[y[x],x]^3==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \text{InverseFunction}\left[-4\left(\frac{1}{2}\log\left(2\sqrt{\#1}-ic_1\right)-\frac{ic_1}{2\left(2\sqrt{\#1}-ic_1\right)}\right)\&\right][x+c_2]$$
$$y(x) \rightarrow \text{InverseFunction}\left[-4\left(\frac{ic_1}{2\left(2\sqrt{\#1}+ic_1\right)}+\frac{1}{2}\log\left(2\sqrt{\#1}+ic_1\right)\right)\&\right][x+c_2]$$
$$y(x) \rightarrow \text{InverseFunction}\left[-4\left(\frac{1}{2}\log\left(2\sqrt{\#1}-i(-c_1)\right)-\frac{i(-c_1)}{2\left(2\sqrt{\#1}-i(-c_1)\right)}\right)\&\right][x+c_2]$$
$$y(x) \rightarrow \text{InverseFunction}\left[-4\left(\frac{i(-c_1)}{2\left(2\sqrt{\#1}+i(-1)c_1\right)}+\frac{1}{2}\log\left(2\sqrt{\#1}+i(-1)c_1\right)\right)\&\right][x+c_2]$$
$$y(x) \rightarrow \text{InverseFunction}\left[-4\left(\frac{1}{2}\log\left(2\sqrt{\#1}-ic_1\right)-\frac{ic_1}{2\left(2\sqrt{\#1}-ic_1\right)}\right)\&\right][x+c_2]$$
$$y(x) \rightarrow \text{InverseFunction}\left[-4\left(\frac{ic_1}{2\left(2\sqrt{\#1}+ic_1\right)}+\frac{1}{2}\log\left(2\sqrt{\#1}+ic_1\right)\right)\&\right][x+c_2]$$

2.1.47 Problem 47

Internal problem ID [9118]
Book : Second order enumerated odes
Section : section 1
Problem number : 47
Date solved : `Monday, January 27, 2025 at 05:42:44 PM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$y^2{y''}^2 + y' = 0$$

Solved as second order missing x ode

Time used: 5.069 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y^2 p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + p(y) = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p = 0 \tag{1}$$

$$p'^2 p y^2 + 1 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p = 0$$

Solving gives $p = 0$

Solving equation (2)

Solving for the derivative gives these ODE's to solve

$$p' = -\frac{1}{\sqrt{-p}\, y} \tag{1}$$

$$p' = \frac{1}{\sqrt{-p}\, y} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

The ode

$$p' = -\frac{1}{\sqrt{-p}\, y} \tag{2.3}$$

is separable as it can be written as

$$\begin{aligned} p' &= -\frac{1}{\sqrt{-p}\, y} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= -\frac{1}{y} \\ g(p) &= \frac{1}{\sqrt{-p}} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\, dp &= \int f(y)\, dy \\ \int \sqrt{-p}\, dp &= \int -\frac{1}{y}\, dy \end{aligned}$$

$$-\frac{2(-p)^{3/2}}{3} = \ln\left(\frac{1}{y}\right) + c_1$$

Solving for p gives

$$p = -\frac{\left(-12\ln\left(\frac{1}{y}\right) - 12c_1\right)^{2/3}}{4}$$

Solving Eq. (2)

The ode

$$p' = \frac{1}{\sqrt{-p}\, y} \tag{2.4}$$

is separable as it can be written as

$$\begin{aligned} p' &= \frac{1}{\sqrt{-p}\, y} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= \frac{1}{y} \\ g(p) &= \frac{1}{\sqrt{-p}} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(y)\,dy$$

$$\int \sqrt{-p}\,dp = \int \frac{1}{y}\,dy$$

$$-\frac{2(-p)^{3/2}}{3} = \ln(y) + c_2$$

Solving for p gives

$$p = -\frac{(-12\ln(y) - 12c_2)^{2/3}}{4}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_3$$

$$y = c_3$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{(-12\ln(y) - 12c_2)^{2/3}}{4}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} -\frac{4}{(-12\ln(\tau) - 12c_2)^{2/3}} d\tau = x + c_4$$

Singular solutions are found by solving

$$-\frac{(-12\ln(y) - 12c_2)^{2/3}}{4} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \mathrm{e}^{-c_2}$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\left(-12\ln\left(\frac{1}{y}\right) - 12c_1\right)^{2/3}}{4}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} -\frac{4}{\left(-12\ln\left(\frac{1}{\tau}\right) - 12c_1\right)^{2/3}} d\tau = x + c_5$$

Singular solutions are found by solving

$$-\frac{\left(-12\ln\left(\frac{1}{y}\right)-12c_1\right)^{2/3}}{4}=0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = e^{c_1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\int^{y} -\frac{4}{\left(-12\ln\left(\tau\right)-12c_2\right)^{2/3}}d\tau = x + c_4$$

$$\int^{y} -\frac{4}{\left(-12\ln\left(\frac{1}{\tau}\right)-12c_1\right)^{2/3}}d\tau = x + c_5$$

$$y = c_3$$

$$y = e^{-c_2}$$

$$y = e^{c_1}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resul
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_b(_a))^(1/2)/_a = 0
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

Maple dsolve solution

Solving time : 0.146 (sec)
Leaf size : 233

```
dsolve(y(x)^2*diff(diff(y(x),x),x)^2+diff(y(x),x) = 0,y(x),singsol=all)
```

$$y = c_1$$
$$y = 0$$
$$-4\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)+8c_1\right)^{2/3}}d\_a\right)-x-c_2=0$$
$$-4\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-8c_1\right)^{2/3}}d\_a\right)-x-c_2=0$$
$$\frac{-16\left(\int^{y}\frac{1}{(-12\ln(\_a)+8c_1)^{2/3}}d\_a\right)+2i(-x-c_2)\sqrt{3}+2x+2c_2}{\left(-i\sqrt{3}-1\right)^2}=0$$
$$\frac{-16\left(\int^{y}\frac{1}{(-12\ln(\_a)+8c_1)^{2/3}}d\_a\right)+2i(x+c_2)\sqrt{3}+2x+2c_2}{\left(1-i\sqrt{3}\right)^2}=0$$
$$\frac{-16\left(\int^{y}\frac{1}{(12\ln(\_a)-8c_1)^{2/3}}d\_a\right)+2i(-x-c_2)\sqrt{3}+2x+2c_2}{\left(-i\sqrt{3}-1\right)^2}=0$$
$$\frac{-16\left(\int^{y}\frac{1}{(12\ln(\_a)-8c_1)^{2/3}}d\_a\right)+2i(x+c_2)\sqrt{3}+2x+2c_2}{\left(1-i\sqrt{3}\right)^2}=0$$

Mathematica DSolve solution

Solving time : 2.388 (sec)
Leaf size : 449

```
DSolve[{y[x]^2*D[y[x],{x,2}]^2+D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3}e^{-ic_1}(-\log(\#1)-ic_1)^{2/3}\Gamma\left(\frac{1}{3},-ic_1-\log(\#1)\right)}{(c_1-i\log(\#1))^{2/3}}\&\right][x+c_2]$$
$$y(x)\to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3}e^{ic_1}(-\log(\#1)+ic_1)^{2/3}\Gamma\left(\frac{1}{3},ic_1-\log(\#1)\right)}{(i\log(\#1)+c_1)^{2/3}}\&\right][x+c_2]$$
$$y(x)\to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3}e^{-i(-c_1)}(-\log(\#1)-i(-1)c_1)^{2/3}\Gamma\left(\frac{1}{3},-i(-1)c_1-\log(\#1)\right)}{(-i\log(\#1)-c_1)^{2/3}}\&\right][x+c_2]$$
$$y(x)\to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3}e^{-ic_1}(-\log(\#1)-ic_1)^{2/3}\Gamma\left(\frac{1}{3},-ic_1-\log(\#1)\right)}{(c_1-i\log(\#1))^{2/3}}\&\right][x+c_2]$$
$$y(x)\to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3}e^{i(-c_1)}(-\log(\#1)+i(-c_1))^{2/3}\Gamma\left(\frac{1}{3},i(-c_1)-\log(\#1)\right)}{(i\log(\#1)-c_1)^{2/3}}\&\right][x+c_2]$$
$$y(x)\to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3}e^{ic_1}(-\log(\#1)+ic_1)^{2/3}\Gamma\left(\frac{1}{3},ic_1-\log(\#1)\right)}{(i\log(\#1)+c_1)^{2/3}}\&\right][x+c_2]$$

2.1.48 Problem 48

Internal problem ID [9119]
Book : Second order enumerated odes
Section : section 1
Problem number : 48
Date solved : `Monday, January 27, 2025 at 05:42:49 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$yy''^4 + y'^2 = 0$$

Solved as second order missing x ode

Time used: 61.114 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)^4 \left(\frac{d}{dy}p(y)\right)^4 + p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p^2 = 0 \tag{1}$$

$$p'^4 p^2 y + 1 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p^2 = 0$$

Solving gives $p = 0$

Solving equation (2)

Solving for the derivative gives these ODE's to solve

$$p' = \frac{(-p^2y^3)^{1/4}}{py} \tag{1}$$

$$p' = \frac{i(-p^2y^3)^{1/4}}{py} \tag{2}$$

$$p' = -\frac{(-p^2y^3)^{1/4}}{py} \tag{3}$$

$$p' = -\frac{i(-p^2y^3)^{1/4}}{py} \tag{4}$$

Now each of the above is solved separately.

Solving Eq. (1)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y,p)\,\mathrm{d}y + N(y,p)\,\mathrm{d}p = 0 \tag{1A}$$

Therefore

$$\mathrm{d}p = \left(\frac{(-p^2y^3)^{1/4}}{py}\right)\mathrm{d}y$$

$$\left(-\frac{(-p^2y^3)^{1/4}}{py}\right)\mathrm{d}y + \mathrm{d}p = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(y,p) = -\frac{(-p^2y^3)^{1/4}}{py}$$
$$N(y,p) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial p} &= \frac{\partial}{\partial p}\left(-\frac{(-p^2y^3)^{1/4}}{py}\right)\\ &= -\frac{y^2}{2\,(-p^2y^3)^{3/4}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial y} &= \frac{\partial}{\partial y}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial y}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial p} - \frac{\partial N}{\partial y}\right)\\ &= 1\left(\left(\frac{(-p^2y^3)^{1/4}}{p^2y} + \frac{y^2}{2\,(-p^2y^3)^{3/4}}\right) - (0)\right)\\ &= -\frac{y^2}{2\,(-p^2y^3)^{3/4}}\end{aligned}$$

Since A depends on p, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial p}\right)\\ &= -\frac{py}{(-p^2y^3)^{1/4}}\left((0) - \left(\frac{(-p^2y^3)^{1/4}}{p^2y} + \frac{y^2}{2\,(-p^2y^3)^{3/4}}\right)\right)\\ &= \frac{1}{2p}\end{aligned}$$

Since B does not depend on y, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}p}\\ &= e^{\int \frac{1}{2p} \,\mathrm{d}p}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{\ln(p)}{2}}\\ &= \sqrt{p}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \sqrt{p}\left(-\frac{(-p^2y^3)^{1/4}}{py}\right)\\ &= -\frac{(-p^2y^3)^{1/4}}{\sqrt{p}\,y}\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \sqrt{p}(1) \\
&= \sqrt{p}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}p}{\mathrm{d}y} &= 0 \\
\left(-\frac{\left(-p^{2} y^{3}\right)^{1/4}}{\sqrt{p}\, y}\right) + \left(\sqrt{p}\right)\frac{\mathrm{d}p}{\mathrm{d}y} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(y,p)$

$$\frac{\partial \phi}{\partial y} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial p} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. p gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial p}\,\mathrm{d}p &= \int \overline{N}\,\mathrm{d}p \\
\int \frac{\partial \phi}{\partial p}\,\mathrm{d}p &= \int \sqrt{p}\,\mathrm{d}p \\
\phi &= \frac{2p^{3/2}}{3} + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both y and p. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial y} = -\frac{\left(-p^{2} y^{3}\right)^{1/4}}{\sqrt{p}\, y}$. Therefore equation (4) becomes

$$-\frac{\left(-p^{2} y^{3}\right)^{1/4}}{\sqrt{p}\, y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{\left(-p^{2} y^{3}\right)^{1/4}}{\sqrt{p}\, y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{\left(-p^{2} y^{3}\right)^{1/4}}{\sqrt{p}\, y}\right)\mathrm{d}y \\
f(y) &= -\frac{4\left(-p^{2} y^{3}\right)^{1/4}}{3\sqrt{p}} + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2p^{3/2}}{3} - \frac{4(-p^2y^3)^{1/4}}{3\sqrt{p}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{2p^{3/2}}{3} - \frac{4(-p^2y^3)^{1/4}}{3\sqrt{p}}$$

Solving Eq. (2)

Writing the ode as

$$p' = \frac{i(-p^2y^3)^{1/4}}{py}$$
$$p' = \omega(y, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$
$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{i(-p^2y^3)^{1/4}(b_3 - a_2)}{py} + \frac{\sqrt{-p^2y^3}\, a_3}{p^2y^2} - \left(-\frac{i(-p^2y^3)^{1/4}}{p\, y^2} - \frac{3ipy}{4(-p^2y^3)^{3/4}}\right)(pa_3 + ya_2 + a_1) - \left(-\frac{i(-p^2y^3)^{1/4}}{p^2y} - \frac{iy^2}{2(-p^2y^3)^{3/4}}\right)(pb_3 + yb_2 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{ip^4y^3a_3 - 3ip^3y^4a_2 + 6ip^3y^4b_3 + 2ip^2y^5b_2 + ip^3y^3a_1 + 2ip^2y^4b_1 - 4b_2p^2y^2(-p^2y^3)^{3/4} - 4(-p^2y^3)^{5/4}\, a_3}{4p^2y^2(-p^2y^3)^{3/4}} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-ip^4y^3a_3 + 3ip^3y^4a_2 - 6ip^3y^4b_3 - 2ip^2y^5b_2 - ip^3y^3a_1 \\ &- 2ip^2y^4b_1 + 4b_2p^2y^2(-p^2y^3)^{3/4} + 4(-p^2y^3)^{5/4}\, a_3 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &-p^2y^2\Big(ip^2ya_3 - 3ip\,y^2a_2 + 6ip\,y^2b_3 + 2iy^3b_2 + ipya_1 \\ &+ 2iy^2b_1 - 4(-p^2y^3)^{3/4}\, b_2 + 4(-p^2y^3)^{1/4}\, ya_3\Big) = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$\left\{p, y, \left(-p^2 y^3\right)^{1/4}, \left(-p^2 y^3\right)^{3/4}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$\left\{p = v_1, y = v_2, \left(-p^2 y^3\right)^{1/4} = v_3, \left(-p^2 y^3\right)^{3/4} = v_4\right\}$$

The above PDE (6E) now becomes

$$-v_1^2 v_2^2 \left(-3 i v_1 v_2^2 a_2 + i v_1^2 v_2 a_3 + 2 i v_2^3 b_2 + 6 i v_1 v_2^2 b_3 + i v_1 v_2 a_1 + 2 i v_2^2 b_1 + 4 v_3 v_2 a_3 - 4 v_4 b_2\right) = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-i v_2^3 a_3 v_1^4 + (3 i a_2 - 6 i b_3) v_1^3 v_2^4 - i a_1 v_1^3 v_2^3 - 2 i b_2 v_1^2 v_2^5 - 2 i b_1 v_1^2 v_2^4 - 4 a_3 v_3 v_1^2 v_2^3 + 4 v_4 b_2 v_1^2 v_2^2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2 i b_1 &= 0 \\
-2 i b_2 &= 0 \\
-i a_1 &= 0 \\
-i a_3 &= 0 \\
-4 a_3 &= 0 \\
4 b_2 &= 0 \\
3 i a_2 - 6 i b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= \frac{a_2}{2}
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= y \\
\eta &= \frac{p}{2}
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dp}{dy} &= \frac{\eta}{\xi} \\ &= \frac{\frac{p}{2}}{y} \\ &= \frac{p}{2y}\end{aligned}$$

This is easily solved to give

$$p = c_1\sqrt{y}$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{p}{\sqrt{y}}$$

And S is found from

$$\begin{aligned}dS &= \frac{dy}{\xi} \\ &= \frac{dy}{y}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dy}{T} \\ &= \ln(y)\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y,p)S_p}{R_y + \omega(y,p)R_p} \tag{2}$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$\omega(y,p) = \frac{i(-p^2y^3)^{1/4}}{py}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_y &= -\frac{p}{2y^{3/2}} \\ R_p &= \frac{1}{\sqrt{y}} \\ S_y &= \frac{1}{y} \\ S_p &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2\sqrt{y}\,p}{2i\left(-p^2y^3\right)^{1/4} - p^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R}{(-1+i)\sqrt{2}\sqrt{R} - R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{2R}{i\sqrt{R}\sqrt{2} - \sqrt{R}\sqrt{2} - R^2}\, dR$$

$$S(R) = -\frac{(4i\sqrt{2}+4\sqrt{2})\sqrt{2}\left(\ln\left(\frac{R^3+2R^{3/2}\sqrt{2}+4}{R^3-2R^{3/2}\sqrt{2}+4}\right) + 2\arctan\left(\frac{R^{3/2}\sqrt{2}}{2}+1\right) + 2\arctan\left(\frac{R^{3/2}\sqrt{2}}{2}-1\right)\right)}{48} - \frac{(i\cdot$$

$$S(R) = \left(-\frac{1}{6}-\frac{i}{6}\right)\ln\left(\frac{R^3+2R^{3/2}\sqrt{2}+4}{R^3-2R^{3/2}\sqrt{2}+4}\right) + \left(\frac{1}{6}-\frac{i}{6}\right)\ln\left(\frac{R^3-2R^{3/2}\sqrt{2}+4}{R^3+2R^{3/2}\sqrt{2}+4}\right) + \frac{2i\arctan\left(\frac{R^3}{4}\right)}{3} - \cdot$$

To complete the solution, we just need to transform the above back to y, p coordinates. This results in

$$\ln(y) = \left(-\frac{1}{6}-\frac{i}{6}\right)\ln\left(\frac{\frac{p^3}{y^{3/2}}+2\left(\frac{p}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}{\frac{p^3}{y^{3/2}}-2\left(\frac{p}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}\right) + \left(\frac{1}{6}-\frac{i}{6}\right)\ln\left(\frac{\frac{p^3}{y^{3/2}}-2\left(\frac{p}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}{\frac{p^3}{y^{3/2}}+2\left(\frac{p}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}\right) + \frac{2i\operatorname{arct}}{}$$

Solving Eq. (3)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y,p)\,\mathrm{d}y + N(y,p)\,\mathrm{d}p = 0 \tag{1A}$$

Therefore

$$
\mathrm{d}p = \left(-\frac{(-p^2y^3)^{1/4}}{py}\right)\mathrm{d}y
$$

$$
\left(\frac{(-p^2y^3)^{1/4}}{py}\right)\mathrm{d}y + \mathrm{d}p = 0 \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(y,p) &= \frac{(-p^2y^3)^{1/4}}{py} \\
N(y,p) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial p} = \frac{\partial N}{\partial y}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial p} &= \frac{\partial}{\partial p}\left(\frac{(-p^2y^3)^{1/4}}{py}\right) \\
&= \frac{y^2}{2\,(-p^2y^3)^{3/4}}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial y} &= \frac{\partial}{\partial y}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial y}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial p} - \frac{\partial N}{\partial y}\right) \\
&= 1\left(\left(-\frac{(-p^2y^3)^{1/4}}{p^2y} - \frac{y^2}{2\,(-p^2y^3)^{3/4}}\right) - (0)\right) \\
&= \frac{y^2}{2\,(-p^2y^3)^{3/4}}
\end{aligned}
$$

Since A depends on p, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial p}\right) \\
&= \frac{py}{(-p^2y^3)^{1/4}}\left((0) - \left(-\frac{(-p^2y^3)^{1/4}}{p^2y} - \frac{y^2}{2\,(-p^2y^3)^{3/4}}\right)\right) \\
&= \frac{1}{2p}
\end{aligned}
$$

Since B does not depend on y, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}p} \\
&= e^{\int \frac{1}{2p} \,\mathrm{d}p}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{\ln(p)}{2}} \\
&= \sqrt{p}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \sqrt{p}\left(\frac{\left(-p^2y^3\right)^{1/4}}{py}\right)\\ &= \frac{\left(-p^2y^3\right)^{1/4}}{\sqrt{p}\,y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \sqrt{p}(1)\\ &= \sqrt{p}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}p}{\mathrm{d}y} &= 0\\ \left(\frac{\left(-p^2y^3\right)^{1/4}}{\sqrt{p}\,y}\right) + \left(\sqrt{p}\right)\frac{\mathrm{d}p}{\mathrm{d}y} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(y,p)$

$$\frac{\partial\phi}{\partial y} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial p} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. p gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial p}\,\mathrm{d}p &= \int\overline{N}\,\mathrm{d}p\\ \int\frac{\partial\phi}{\partial p}\,\mathrm{d}p &= \int\sqrt{p}\,\mathrm{d}p\end{aligned}$$

$$\phi = \frac{2p^{3/2}}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both y and p. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial y} = \frac{\left(-p^2y^3\right)^{1/4}}{\sqrt{p}\,y}$. Therefore equation (4) becomes

$$\frac{\left(-p^2y^3\right)^{1/4}}{\sqrt{p}\,y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{\left(-p^2y^3\right)^{1/4}}{\sqrt{p}\,y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{(-p^2y^3)^{1/4}}{\sqrt{p}\,y}\right) \mathrm{d}y$$

$$f(y) = \frac{4(-p^2y^3)^{1/4}}{3\sqrt{p}} + c_4$$

Where c_4 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2p^{3/2}}{3} + \frac{4(-p^2y^3)^{1/4}}{3\sqrt{p}} + c_4$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_4 and c_2 constants into the constant c_4 gives the solution as

$$c_4 = \frac{2p^{3/2}}{3} + \frac{4(-p^2y^3)^{1/4}}{3\sqrt{p}}$$

Solving Eq. (4)

Writing the ode as

$$p' = -\frac{i(-p^2y^3)^{1/4}}{py}$$
$$p' = \omega(y,p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ya_2 + a_1 \tag{1E}$$
$$\eta = pb_3 + yb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{i(-p^2y^3)^{1/4}(b_3 - a_2)}{py} + \frac{\sqrt{-p^2y^3}\,a_3}{p^2y^2} - \left(\frac{i(-p^2y^3)^{1/4}}{p\,y^2} + \frac{3ipy}{4\,(-p^2y^3)^{3/4}}\right)(pa_3 + ya_2 + a_1) - \left(\frac{i(-p^2y^3)^{1/4}}{p^2y} + \frac{iy^2}{2\,(-p^2y^3)^{3/4}}\right)(pb_3 + yb_2 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{ip^4y^3a_3 - 3ip^3y^4a_2 + 6ip^3y^4b_3 + 2ip^2y^5b_2 + 4b_2p^2y^2(-p^2y^3)^{3/4} + ip^3y^3a_1 + 2ip^2y^4b_1 + 4(-p^2y^3)^{5/4}\,a_3}{4p^2y^2\,(-p^2y^3)^{3/4}} = 0$$

Setting the numerator to zero gives

$$ip^4y^3a_3 - 3ip^3y^4a_2 + 6ip^3y^4b_3 + 2ip^2y^5b_2 + 4b_2p^2y^2(-p^2y^3)^{3/4} + ip^3y^3a_1 + 2ip^2y^4b_1 + 4(-p^2y^3)^{5/4}a_3 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$p^2y^2\left(ip^2ya_3 - 3ip\,y^2a_2 + 6ip\,y^2b_3 + 2iy^3b_2 + 4(-p^2y^3)^{3/4}b_2 + ipya_1 + 2iy^2b_1 - 4(-p^2y^3)^{1/4}ya_3\right) = 0$$

Looking at the above PDE shows the following are all the terms with $\{p, y\}$ in them.

$$\left\{p, y, (-p^2y^3)^{1/4}, (-p^2y^3)^{3/4}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{p, y\}$ in them

$$\left\{p = v_1, y = v_2, (-p^2y^3)^{1/4} = v_3, (-p^2y^3)^{3/4} = v_4\right\}$$

The above PDE (6E) now becomes

$$v_1^2v_2^2(-3iv_1v_2^2a_2 + iv_1^2v_2a_3 + 2iv_2^3b_2 + 6iv_1v_2^2b_3 + iv_1v_2a_1 + 2iv_2^2b_1 - 4v_3v_2a_3 + 4v_4b_2) = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$iv_2^3a_3v_1^4 + (-3ia_2 + 6ib_3)v_1^3v_2^4 + ia_1v_1^3v_2^3 + 2ib_2v_1^2v_2^5 + 2ib_1v_1^2v_2^4 - 4a_3v_3v_1^2v_2^3 + 4v_4b_2v_1^2v_2^2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} ia_1 &= 0 \\ ia_3 &= 0 \\ 2ib_1 &= 0 \\ 2ib_2 &= 0 \\ -4a_3 &= 0 \\ 4b_2 &= 0 \\ -3ia_2 + 6ib_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= \frac{a_2}{2} \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = y$$
$$\eta = \frac{p}{2}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dp}{dy} &= \frac{\eta}{\xi}\\ &= \frac{\frac{p}{2}}{y}\\ &= \frac{p}{2y}\end{aligned}$$

This is easily solved to give

$$p = c_1\sqrt{y}$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{p}{\sqrt{y}}$$

And S is found from

$$\begin{aligned}dS &= \frac{dy}{\xi}\\ &= \frac{dy}{y}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dy}{T}\\ &= \ln(y)\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y,p)S_p}{R_y + \omega(y,p)R_p} \tag{2}$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$\omega(y,p) = -\frac{i(-p^2y^3)^{1/4}}{py}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_y &= -\frac{p}{2y^{3/2}}\\
R_p &= \frac{1}{\sqrt{y}}\\
S_y &= \frac{1}{y}\\
S_p &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{2\sqrt{y}\,p}{2i\left(-p^2y^3\right)^{1/4} + p^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{2R}{(-1+i)\sqrt{2}\sqrt{R} + R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\int dS = \int -\frac{2R}{i\sqrt{R}\sqrt{2} - \sqrt{R}\sqrt{2} + R^2}\,dR
$$

$$
S(R) = -\frac{\left(-4i\sqrt{2} - 4\sqrt{2}\right)\sqrt{2}\left(\ln\left(\frac{R^3+2R^{3/2}\sqrt{2}+4}{R^3-2R^{3/2}\sqrt{2}+4}\right) + 2\arctan\left(\frac{R^{3/2}\sqrt{2}}{2}+1\right) + 2\arctan\left(\frac{R^{3/2}\sqrt{2}}{2}-1\right)\right)}{48} -
$$

$$
S(R) = \left(\frac{1}{6}+\frac{i}{6}\right)\ln\left(\frac{R^3+2R^{3/2}\sqrt{2}+4}{R^3-2R^{3/2}\sqrt{2}+4}\right) + \left(-\frac{1}{6}+\frac{i}{6}\right)\ln\left(\frac{R^3-2R^{3/2}\sqrt{2}+4}{R^3+2R^{3/2}\sqrt{2}+4}\right) + \frac{2i\arctan\left(\frac{R^3}{4}\right)}{3} +
$$

To complete the solution, we just need to transform the above back to y, p coordinates. This results in

$$
\ln(y) = \left(\frac{1}{6}+\frac{i}{6}\right)\ln\left(\frac{\frac{p^3}{y^{3/2}} + 2\left(\frac{p}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}{\frac{p^3}{y^{3/2}} - 2\left(\frac{p}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}\right) + \left(-\frac{1}{6}+\frac{i}{6}\right)\ln\left(\frac{\frac{p^3}{y^{3/2}} - 2\left(\frac{p}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}{\frac{p^3}{y^{3/2}} + 2\left(\frac{p}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}\right) + \frac{2i\,\mathrm{arct}}{}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
\frac{2y'^{3/2}}{3} - \frac{4\left(-y'^2y^3\right)^{1/4}}{3\sqrt{y'}} = c_1
$$

Solving for the derivative gives these ODE's to solve

$$
y' = \frac{\left(16(-y^3)^{1/4} + 12c_1\right)^{2/3}}{4} \tag{1}
$$

$$
y' = \left(-\frac{\left(16(-y^3)^{1/4} + 12c_1\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(16(-y^3)^{1/4} + 12c_1\right)^{1/3}}{4}\right)^2 \tag{2}
$$

$$y' = \left(-\frac{\left(16(-y^3)^{1/4}+12c_1\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(16(-y^3)^{1/4}+12c_1\right)^{1/3}}{4}\right)^2 \tag{3}$$

$$y' = \frac{\left(16i(-y^3)^{1/4}+12c_1\right)^{2/3}}{4} \tag{4}$$

$$y' = \left(-\frac{\left(16i(-y^3)^{1/4}+12c_1\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(16i(-y^3)^{1/4}+12c_1\right)^{1/3}}{4}\right)^2 \tag{5}$$

$$y' = \left(-\frac{\left(16i(-y^3)^{1/4}+12c_1\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(16i(-y^3)^{1/4}+12c_1\right)^{1/3}}{4}\right)^2 \tag{6}$$

$$y' = \frac{\left(-16(-y^3)^{1/4}+12c_1\right)^{2/3}}{4} \tag{7}$$

$$y' = \left(-\frac{\left(-16(-y^3)^{1/4}+12c_1\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(-16(-y^3)^{1/4}+12c_1\right)^{1/3}}{4}\right)^2 \tag{8}$$

$$y' = \left(-\frac{\left(-16(-y^3)^{1/4}+12c_1\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(-16(-y^3)^{1/4}+12c_1\right)^{1/3}}{4}\right)^2 \tag{9}$$

$$y' = \frac{\left(-16i(-y^3)^{1/4}+12c_1\right)^{2/3}}{4} \tag{10}$$

$$y' = \left(-\frac{\left(-16i(-y^3)^{1/4}+12c_1\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(-16i(-y^3)^{1/4}+12c_1\right)^{1/3}}{4}\right)^2 \tag{11}$$

$$y' = \left(-\frac{\left(-16i(-y^3)^{1/4}+12c_1\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(-16i(-y^3)^{1/4}+12c_1\right)^{1/3}}{4}\right)^2 \tag{12}$$

Now each of the above is solved separately.

Solving Eq. (1)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{4}{\left(16\left(-\tau^3\right)^{1/4}+12c_1\right)^{2/3}}d\tau = x + c_7$$

Solving Eq. (2)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{16}{\left(16\left(-\tau^3\right)^{1/4}+12c_1\right)^{2/3}\left(i\sqrt{3}-1\right)^2}d\tau = x + c_8$$

Solving Eq. (3)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(16 \left(-\tau^{3}\right)^{1/4}+12 c_{1}\right)^{2/3} \left(1+i \sqrt{3}\right)^{2}}d\tau = x +c_{9}$$

Solving Eq. (4)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{4}{\left(16 i \left(-\tau^{3}\right)^{1/4}+12 c_{1}\right)^{2/3}}d\tau = x +\_C10$$

Solving Eq. (5)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(16 i \left(-\tau^{3}\right)^{1/4}+12 c_{1}\right)^{2/3} \left(i \sqrt{3}-1\right)^{2}}d\tau = x +\_C11$$

Solving Eq. (6)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(16 i \left(-\tau^{3}\right)^{1/4}+12 c_{1}\right)^{2/3} \left(1+i \sqrt{3}\right)^{2}}d\tau = x +\_C12$$

Solving Eq. (7)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{4}{\left(-16 \left(-\tau^{3}\right)^{1/4}+12 c_{1}\right)^{2/3}}d\tau = x +\_C13$$

Singular solutions are found by solving

$$\frac{\left(-16(-y^{3})^{1/4}+12 c_{1}\right)^{2/3}}{4} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{3\left(-6 c_{1}\right)^{1/3} c_{1}}{8}$$

$$y = \frac{3\left(-\frac{\left(-6 c_{1}\right)^{1/3}}{4}-\frac{i \sqrt{3}\, \left(-6 c_{1}\right)^{1/3}}{4}\right) c_{1}}{4}$$

$$y = \frac{3\left(-\frac{\left(-6 c_{1}\right)^{1/3}}{4}+\frac{i \sqrt{3}\, \left(-6 c_{1}\right)^{1/3}}{4}\right) c_{1}}{4}$$

Solving Eq. (8)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(-16 \left(-\tau^{3}\right)^{1/4}+12 c_{1}\right)^{2/3} \left(1+i \sqrt{3}\right)^{2}}d\tau = x +\_C14$$

Singular solutions are found by solving

$$\frac{\left(-16(-y^3)^{1/4}+12c_1\right)^{2/3}\left(1+i\sqrt{3}\right)^2}{16}=0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y=\frac{3(-6c_1)^{1/3}\,c_1}{8}$$
$$y=\frac{3\left(-\frac{(-6c_1)^{1/3}}{4}-\frac{i\sqrt{3}\,(-6c_1)^{1/3}}{4}\right)c_1}{4}$$
$$y=\frac{3\left(-\frac{(-6c_1)^{1/3}}{4}+\frac{i\sqrt{3}\,(-6c_1)^{1/3}}{4}\right)c_1}{4}$$

Solving Eq. (9)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{16}{\left(-16\,(-\tau^3)^{1/4}+12c_1\right)^{2/3}\left(i\sqrt{3}-1\right)^2}d\tau = x + \_C15$$

Singular solutions are found by solving

$$\frac{\left(-16(-y^3)^{1/4}+12c_1\right)^{2/3}\left(i\sqrt{3}-1\right)^2}{16}=0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y=\frac{3(-6c_1)^{1/3}\,c_1}{8}$$
$$y=\frac{3\left(-\frac{(-6c_1)^{1/3}}{4}-\frac{i\sqrt{3}\,(-6c_1)^{1/3}}{4}\right)c_1}{4}$$
$$y=\frac{3\left(-\frac{(-6c_1)^{1/3}}{4}+\frac{i\sqrt{3}\,(-6c_1)^{1/3}}{4}\right)c_1}{4}$$

Solving Eq. (10)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{4}{\left(-16i\,(-\tau^3)^{1/4}+12c_1\right)^{2/3}}d\tau = x + \_C16$$

Solving Eq. (11)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{16}{\left(-16i\,(-\tau^3)^{1/4}+12c_1\right)^{2/3}\left(1+i\sqrt{3}\right)^2}d\tau = x + \_C17$$

Solving Eq. (12)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{16}{\left(-16i\,(-\tau^3)^{1/4}+12c_1\right)^{2/3}\left(i\sqrt{3}-1\right)^2}d\tau = x + \_C18$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{2y'^{3/2}}{3} + \frac{4\left(-y'^2 y^3\right)^{1/4}}{3\sqrt{y'}} = c_4$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{\left(16(-y^3)^{1/4} + 12c_4\right)^{2/3}}{4} \tag{1}$$

$$y' = \left(-\frac{\left(16(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(16(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4}\right)^2 \tag{2}$$

$$y' = \left(-\frac{\left(16(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(16(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4}\right)^2 \tag{3}$$

$$y' = \frac{\left(16i(-y^3)^{1/4} + 12c_4\right)^{2/3}}{4} \tag{4}$$

$$y' = \left(-\frac{\left(16i(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(16i(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4}\right)^2 \tag{5}$$

$$y' = \left(-\frac{\left(16i(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(16i(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4}\right)^2 \tag{6}$$

$$y' = \frac{\left(-16(-y^3)^{1/4} + 12c_4\right)^{2/3}}{4} \tag{7}$$

$$y' = \left(-\frac{\left(-16(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(-16(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4}\right)^2 \tag{8}$$

$$y' = \left(-\frac{\left(-16(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(-16(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4}\right)^2 \tag{9}$$

$$y' = \frac{\left(-16i(-y^3)^{1/4} + 12c_4\right)^{2/3}}{4} \tag{10}$$

$$y' = \left(-\frac{\left(-16i(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4} - \frac{i\sqrt{3}\left(-16i(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4}\right)^2 \tag{11}$$

$$y' = \left(-\frac{\left(-16i(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4} + \frac{i\sqrt{3}\left(-16i(-y^3)^{1/4} + 12c_4\right)^{1/3}}{4}\right)^2 \tag{12}$$

Now each of the above is solved separately.

Solving Eq. (1)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{4}{\left(16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}}d\tau = x + \_C19$$

Solving Eq. (2)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x + \_C20$$

Solving Eq. (3)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x + \_C21$$

Solving Eq. (4)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{4}{\left(16i\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}}d\tau = x + \_C22$$

Solving Eq. (5)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(16i\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x + \_C23$$

Solving Eq. (6)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(16i\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x + \_C24$$

Solving Eq. (7)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{4}{\left(-16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}}d\tau = x + \_C25$$

Singular solutions are found by solving

$$\frac{\left(-16(-y^{3})^{1/4}+12c_{4}\right)^{2/3}}{4} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{3(-6c_4)^{1/3} c_4}{8}$$
$$y = \frac{3\left(-\frac{(-6c_4)^{1/3}}{4} - \frac{i\sqrt{3}\,(-6c_4)^{1/3}}{4}\right) c_4}{4}$$
$$y = \frac{3\left(-\frac{(-6c_4)^{1/3}}{4} + \frac{i\sqrt{3}\,(-6c_4)^{1/3}}{4}\right) c_4}{4}$$

Solving Eq. (8)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(-16\,(-\tau^3)^{1/4} + 12c_4\right)^{2/3} \left(1 + i\sqrt{3}\right)^2} d\tau = x + \_C26$$

Singular solutions are found by solving

$$\frac{\left(-16(-y^3)^{1/4} + 12c_4\right)^{2/3} \left(1 + i\sqrt{3}\right)^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{3(-6c_4)^{1/3} c_4}{8}$$
$$y = \frac{3\left(-\frac{(-6c_4)^{1/3}}{4} - \frac{i\sqrt{3}\,(-6c_4)^{1/3}}{4}\right) c_4}{4}$$
$$y = \frac{3\left(-\frac{(-6c_4)^{1/3}}{4} + \frac{i\sqrt{3}\,(-6c_4)^{1/3}}{4}\right) c_4}{4}$$

Solving Eq. (9)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{16}{\left(-16\,(-\tau^3)^{1/4} + 12c_4\right)^{2/3} \left(i\sqrt{3} - 1\right)^2} d\tau = x + \_C27$$

Singular solutions are found by solving

$$\frac{\left(-16(-y^3)^{1/4} + 12c_4\right)^{2/3} \left(i\sqrt{3} - 1\right)^2}{16} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{3(-6c_4)^{1/3} c_4}{8}$$
$$y = \frac{3\left(-\frac{(-6c_4)^{1/3}}{4} - \frac{i\sqrt{3}\,(-6c_4)^{1/3}}{4}\right) c_4}{4}$$
$$y = \frac{3\left(-\frac{(-6c_4)^{1/3}}{4} + \frac{i\sqrt{3}\,(-6c_4)^{1/3}}{4}\right) c_4}{4}$$

Solving Eq. (10)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{4}{\left(-16i\left(-\tau^3\right)^{1/4}+12c_4\right)^{2/3}}d\tau = x + \_C28$$

Solving Eq. (11)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{16}{\left(-16i\left(-\tau^3\right)^{1/4}+12c_4\right)^{2/3}\left(1+i\sqrt{3}\right)^2}d\tau = x + \_C29$$

Solving Eq. (12)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{16}{\left(-16i\left(-\tau^3\right)^{1/4}+12c_4\right)^{2/3}\left(i\sqrt{3}-1\right)^2}d\tau = x + \_C30$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\ln(y) = \left(-\frac{1}{6}-\frac{i}{6}\right)\ln\left(\frac{\frac{y'^3}{y^{3/2}}+2\left(\frac{y'}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}{\frac{y'^3}{y^{3/2}}-2\left(\frac{y'}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}\right)+\left(\frac{1}{6}-\frac{i}{6}\right)\ln\left(\frac{\frac{y'^3}{y^{3/2}}-2\left(\frac{y'}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}{\frac{y'^3}{y^{3/2}}+2\left(\frac{y'}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}\right)+\frac{2i\arctan}{3}$$

Unable to solve. Terminating.

For solution (4) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\ln(y) = \left(\frac{1}{6}+\frac{i}{6}\right)\ln\left(\frac{\frac{y'^3}{y^{3/2}}+2\left(\frac{y'}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}{\frac{y'^3}{y^{3/2}}-2\left(\frac{y'}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}\right)+\left(-\frac{1}{6}+\frac{i}{6}\right)\ln\left(\frac{\frac{y'^3}{y^{3/2}}-2\left(\frac{y'}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}{\frac{y'^3}{y^{3/2}}+2\left(\frac{y'}{\sqrt{y}}\right)^{3/2}\sqrt{2}+4}\right)+\frac{2i\arctan}{3}$$

Unable to solve. Terminating.

For solution (5) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + \_C31$$

$$y = \_C31$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\int^{y}\frac{4}{\left(-16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}}d\tau = x+\_C25$$

$$\int^{y}\frac{4}{\left(16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}}d\tau = x+\_C19$$

$$\int^{y}\frac{4}{\left(-16\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}}d\tau = x+\_C13$$

$$\int^{y}\frac{4}{\left(16\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}}d\tau = x+c_{7}$$

$$\int^{y}\frac{4}{\left(-16i\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}}d\tau = x+\_C28$$

$$\int^{y}\frac{4}{\left(-16i\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}}d\tau = x+\_C16$$

$$\int^{y}\frac{4}{\left(16i\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}}d\tau = x+\_C22$$

$$\int^{y}\frac{4}{\left(16i\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}}d\tau = x+\_C10$$

$$\int^{y}\frac{16}{\left(-16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x+\_C27$$

$$\int^{y}\frac{16}{\left(16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x+\_C20$$

$$\int^{y}\frac{16}{\left(-16\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x+\_C15$$

$$\int^{y}\frac{16}{\left(-16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x+\_C26$$

$$\int^{y}\frac{16}{\left(-16\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x+\_C14$$

$$\int^{y}\frac{16}{\left(16\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x+c_{9}$$

$$\int^{y}\frac{16}{\left(16\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x+c_{8}$$

$$\int^{y}\frac{16}{\left(16\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x+\_C21$$

$$\int^{y}\frac{16}{\left(-16i\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x+\_C18$$

$$\int^{y}\frac{16}{\left(-16i\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x + \_C17$$

$$\int^{y}\frac{16}{\left(-16i\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x + \_C29$$

$$\int^{y}\frac{16}{\left(-16i\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x + \_C30$$

$$\int^{y}\frac{16}{\left(16i\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x + \_C24$$

$$\int^{y}\frac{16}{\left(16i\left(-\tau^{3}\right)^{1/4}+12c_{4}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x + \_C23$$

$$\int^{y}\frac{16}{\left(16i\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}\left(i\sqrt{3}-1\right)^{2}}d\tau = x + \_C11$$

$$\int^{y}\frac{16}{\left(16i\left(-\tau^{3}\right)^{1/4}+12c_{1}\right)^{2/3}\left(1+i\sqrt{3}\right)^{2}}d\tau = x + \_C12$$

$$y = \_C31$$

$$y = \frac{3(-6c_{1})^{1/3}c_{1}}{8}$$

$$y = \frac{3(-6c_{4})^{1/3}c_{4}}{8}$$

$$y = \frac{3\left(-\frac{(-6c_{1})^{1/3}}{4}-\frac{i\sqrt{3}\,(-6c_{1})^{1/3}}{4}\right)c_{1}}{4}$$

$$y = \frac{3\left(-\frac{(-6c_{1})^{1/3}}{4}+\frac{i\sqrt{3}\,(-6c_{1})^{1/3}}{4}\right)c_{1}}{4}$$

$$y = \frac{3\left(-\frac{(-6c_{4})^{1/3}}{4}-\frac{i\sqrt{3}\,(-6c_{4})^{1/3}}{4}\right)c_{4}}{4}$$

$$y = \frac{3\left(-\frac{(-6c_{4})^{1/3}}{4}+\frac{i\sqrt{3}\,(-6c_{4})^{1/3}}{4}\right)c_{4}}{4}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 4 solutions were found. Trying to solve each resul
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
```

```
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_a^3*_b(_a)^2)^(1/
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 1/2*_b]
```

Maple dsolve solution

Solving time : `13.932` (sec)
Leaf size : 2829

```
dsolve(y(x)*diff(diff(y(x),x),x)^4+diff(y(x),x)^2 = 0,y(x),singsol=all)
```

$$y = c_1$$
$$y = 0$$
$$\int^{y} \frac{\_a^{2}}{\sqrt{\_a^{3} \left(2\_a-\left(c_1\_a\right)^{1/4}\right) \left(-2\_a^{3}+\_a^{2} \left(c_1\_a\right)^{1/4}\right)^{1/3}}}d\_a -x-c_2 = 0$$
$$\int^{y} \frac{\_a^{2}}{\sqrt{-\_a^{3} \left(i \left(c_1\_a\right)^{1/4}-2\_a\right) \left(\left(i \left(c_1\_a\right)^{1/4}-2\_a\right) \_a^{2}\right)^{1/3}}}d\_a -x-c_2 = 0$$
$$\int^{y} \frac{\_a^{2}}{\sqrt{\_a^{3} \left(2\_a+\left(c_1\_a\right)^{1/4}\right) \left(-2\_a^{3}-\_a^{2} \left(c_1\_a\right)^{1/4}\right)^{1/3}}}d\_a -x-c_2 = 0$$
$$\int^{y} \frac{\_a^{2}}{\sqrt{\_a^{3} \left(i \left(c_1\_a\right)^{1/4}+2\_a\right) \left(-\left(i \left(c_1\_a\right)^{1/4}+2\_a\right) \_a^{2}\right)^{1/3}}}d\_a -x-c_2 = 0$$
$$\sqrt{2}\, \left(\int^{y} \frac{\_a^{2}}{\sqrt{\left(-2\_a+\left(c_1\_a\right)^{1/4}\right) \left(1+i \sqrt{3}\right) \_a^{3} \left(-2\_a^{3}+\_a^{2} \left(c_1\_a\right)^{1/4}\right)^{1/3}}}d\_a \right)-x-c_2 = 0$$
$$\sqrt{2}\, \left(\int^{y} \frac{\_a^{2}}{\sqrt{\left(i-\sqrt{3}\right) \_a^{3} \left(\left(i \left(c_1\_a\right)^{1/4}-2\_a\right) \_a^{2}\right)^{1/3} \left(\left(c_1\_a\right)^{1/4}+2 i \_a\right)}}d\_a \right)-x-c_2 = 0$$
$$\sqrt{2}\, \left(\int^{y} \frac{\_a^{2}}{\sqrt{-2\_a^{3} \left(-2\_a^{3}-\_a^{2} \left(c_1\_a\right)^{1/4}\right)^{1/3} \left(1+i \sqrt{3}\right) \left(\_a+\frac{\left(c_1\_a\right)^{1/4}}{2}\right)}}d\_a \right)-x-c_2 = 0$$
$$\sqrt{2}\, \left(\int^{y} \frac{\_a^{2}}{\sqrt{-\_a^{3} \left(i \left(c_1\_a\right)^{1/4}+2\_a\right) \left(-\left(i \left(c_1\_a\right)^{1/4}+2\_a\right) \_a^{2}\right)^{1/3} \left(1+i \sqrt{3}\right)}}d\_a \right)-x-c_2 = 0$$
$$-\int^{y} \frac{\_a^{2}}{\sqrt{\_a^{3} \left(2\_a-\left(c_1\_a\right)^{1/4}\right) \left(-2\_a^{3}+\_a^{2} \left(c_1\_a\right)^{1/4}\right)^{1/3}}}d\_a -x-c_2 = 0$$
$$-\int^{y} \frac{\_a^{2}}{\sqrt{-\_a^{3} \left(i \left(c_1\_a\right)^{1/4}-2\_a\right) \left(\left(i \left(c_1\_a\right)^{1/4}-2\_a\right) \_a^{2}\right)^{1/3}}}d\_a -x-c_2 = 0$$
$$-\int^{y} \frac{\_a^{2}}{\sqrt{\_a^{3} \left(2\_a+\left(c_1\_a\right)^{1/4}\right) \left(-2\_a^{3}-\_a^{2} \left(c_1\_a\right)^{1/4}\right)^{1/3}}}d\_a -x-c_2 = 0$$
$$-\int^{y} \frac{\_a^{2}}{\sqrt{\_a^{3} \left(i \left(c_1\_a\right)^{1/4}+2\_a\right) \left(-\left(i \left(c_1\_a\right)^{1/4}+2\_a\right) \_a^{2}\right)^{1/3}}}d\_a -x-c_2 = 0$$
$$-\sqrt{2}\, \left(\int^{y} \frac{\_a^{2}}{\sqrt{\left(1-i \sqrt{3}\right) \_a^{3} \left(-2\_a+\left(c_1\_a\right)^{1/4}\right) \left(-2\_a^{3}+\_a^{2} \left(c_1\_a\right)^{1/4}\right)^{1/3}}}d\_a \right)-x-c_2 = 0$$
$$\sqrt{2}\, \left(\int^{y} \frac{\_a^{2}}{\sqrt{\left(1-i \sqrt{3}\right) \_a^{3} \left(-2\_a+\left(c_1\_a\right)^{1/4}\right) \left(-2\_a^{3}+\_a^{2} \left(c_1\_a\right)^{1/4}\right)^{1/3}}}d\_a \right)-x-c_2 = 0$$
$$-\sqrt{2}\, \left(\int^{y} \frac{\_a^{2}}{\sqrt{\_a^{3} \left(\left(i \left(c_1\_a\right)^{1/4}-2\_a\right) \_a^{2}\right)^{1/3} \left(\sqrt{3}-i\right) \left(\left(c_1\_a\right)^{1/4}+2 i \_a\right)}}d\_a \right)-x-c_2 = 0$$

Mathematica DSolve solution

Solving time : 3.846 (sec)
Leaf size : 1237

```
DSolve[{y[x]*D[y[x],{x,2}]^4+D[y[x],x]^2==0,{}},y[x],x,IncludeSingularSolutions->True]
```

Too large to display

2.1.49 Problem 49

Internal problem ID [9120]
Book : Second order enumerated odes
Section : section 1
Problem number : 49
Date solved : Monday, January 27, 2025 at 05:43:51 PM
CAS classification :
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]

Solve

$$y^3 y''^2 + yy' = 0$$

Factoring the ode gives these factors

$$y = 0 \tag{1}$$

$$y''^2 y^2 + y' = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y = 0$$

Solving gives $y = 0$

Solving equation (2)

Solved as second order missing x ode

Time used: 5.480 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 y^2 + p(y) = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p = 0 \tag{1}$$

$$p'^2 p y^2 + 1 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p = 0$$

Solving gives $p = 0$

Solving equation (2)

Solving for the derivative gives these ODE's to solve

$$p' = -\frac{1}{\sqrt{-p}\, y} \tag{1}$$

$$p' = \frac{1}{\sqrt{-p}\, y} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

The ode

$$p' = -\frac{1}{\sqrt{-p}\, y} \tag{2.5}$$

is separable as it can be written as

$$\begin{aligned} p' &= -\frac{1}{\sqrt{-p}\, y} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= -\frac{1}{y} \\ g(p) &= \frac{1}{\sqrt{-p}} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\, dp &= \int f(y)\, dy \\ \int \sqrt{-p}\, dp &= \int -\frac{1}{y}\, dy \end{aligned}$$

$$-\frac{2(-p)^{3/2}}{3} = \ln\left(\frac{1}{y}\right) + c_1$$

Solving for p gives

$$p = -\frac{\left(-12\ln\left(\frac{1}{y}\right) - 12c_1\right)^{2/3}}{4}$$

Solving Eq. (2)

The ode

$$p' = \frac{1}{\sqrt{-p}\,y} \tag{2.6}$$

is separable as it can be written as

$$\begin{aligned} p' &= \frac{1}{\sqrt{-p}\,y} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= \frac{1}{y} \\ g(p) &= \frac{1}{\sqrt{-p}} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\,dp &= \int f(y)\,dy \\ \int \sqrt{-p}\,dp &= \int \frac{1}{y}\,dy \end{aligned}$$

$$-\frac{2(-p)^{3/2}}{3} = \ln(y) + c_2$$

Solving for p gives

$$p = -\frac{(-12\ln(y) - 12c_2)^{2/3}}{4}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dy &= \int 0\,dx + c_3 \\ y &= c_3 \end{aligned}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{(-12\ln(y) - 12c_2)^{2/3}}{4}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} -\frac{4}{(-12\ln(\tau) - 12c_2)^{2/3}}d\tau = x + c_4$$

Singular solutions are found by solving

$$-\frac{(-12\ln(y) - 12c_2)^{2/3}}{4} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \mathrm{e}^{-c_2}$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\left(-12\ln\left(\frac{1}{y}\right) - 12c_1\right)^{2/3}}{4}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y -\frac{4}{\left(-12\ln\left(\frac{1}{\tau}\right) - 12c_1\right)^{2/3}} d\tau = x + c_5$$

Singular solutions are found by solving

$$-\frac{\left(-12\ln\left(\frac{1}{y}\right) - 12c_1\right)^{2/3}}{4} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \mathrm{e}^{c_1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\int^y -\frac{4}{\left(-12\ln\left(\tau\right) - 12c_2\right)^{2/3}} d\tau = x + c_4$$

$$\int^y -\frac{4}{\left(-12\ln\left(\frac{1}{\tau}\right) - 12c_1\right)^{2/3}} d\tau = x + c_5$$

$$y = c_3$$

$$y = \mathrm{e}^{-c_2}$$

$$y = \mathrm{e}^{c_1}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each res
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
```

```
    `, `-> Computing symmetries using: way = 3
    <- differential order: 2; canonical coordinates successful
    <- differential order 2; missing variables successful
-------------------
* Tackling next ODE.
    *** Sublevel 3 ***
    Methods for second order ODEs:
    --- Trying classification methods ---
    trying 2nd order Liouville
    trying 2nd order WeierstrassP
    trying 2nd order JacobiSN
    differential order: 2; trying a linearization to 3rd order
    trying 2nd order ODE linearizable_by_differentiation
    trying 2nd order, 2 integrating factors of the form mu(x,y)
    trying differential order: 2; missing variables
    `, `-> Computing symmetries using: way = 3
    <- differential order: 2; canonical coordinates successful
    <- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : 0.102 (sec)
Leaf size : 233

```
dsolve(y(x)^3*diff(diff(y(x),x),x)^2+y(x)*diff(y(x),x) = 0,y(x),singsol=all)
```

$$y = c_1$$
$$y = 0$$
$$-4\left(\int^{y}\frac{1}{\left(-12\ln\left(\_a\right)+8c_1\right)^{2/3}}d\_a\right)-x-c_2 = 0$$
$$-4\left(\int^{y}\frac{1}{\left(12\ln\left(\_a\right)-8c_1\right)^{2/3}}d\_a\right)-x-c_2 = 0$$
$$\frac{-16\left(\int^{y}\frac{1}{\left(-12\ln(\_a)+8c_1\right)^{2/3}}d\_a\right)+2i(-x-c_2)\sqrt{3}+2x+2c_2}{\left(-i\sqrt{3}-1\right)^{2}} = 0$$
$$\frac{-16\left(\int^{y}\frac{1}{\left(-12\ln(\_a)+8c_1\right)^{2/3}}d\_a\right)+2i(x+c_2)\sqrt{3}+2x+2c_2}{\left(1-i\sqrt{3}\right)^{2}} = 0$$
$$\frac{-16\left(\int^{y}\frac{1}{\left(12\ln(\_a)-8c_1\right)^{2/3}}d\_a\right)+2i(-x-c_2)\sqrt{3}+2x+2c_2}{\left(-i\sqrt{3}-1\right)^{2}} = 0$$
$$\frac{-16\left(\int^{y}\frac{1}{\left(12\ln(\_a)-8c_1\right)^{2/3}}d\_a\right)+2i(x+c_2)\sqrt{3}+2x+2c_2}{\left(1-i\sqrt{3}\right)^{2}} = 0$$

Mathematica DSolve solution

Solving time : `2.307` (sec)
Leaf size : 459

```
DSolve[{y[x]^3*D[y[x],{x,2}]^2+y[x]*D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{-ic_1}(-\log(\#1) - ic_1)^{2/3}\Gamma\left(\frac{1}{3}, -ic_1 - \log(\#1)\right)}{(c_1 - i\log(\#1))^{2/3}}\&\right][x + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{ic_1}(-\log(\#1) + ic_1)^{2/3}\Gamma\left(\frac{1}{3}, ic_1 - \log(\#1)\right)}{(i\log(\#1) + c_1)^{2/3}}\&\right][x + c_2]$$

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{-i(-c_1)}(-\log(\#1) - i(-1)c_1)^{2/3}\Gamma\left(\frac{1}{3}, -i(-1)c_1 - \log(\#1)\right)}{(-i\log(\#1) - c_1)^{2/3}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{-ic_1}(-\log(\#1) - ic_1)^{2/3}\Gamma\left(\frac{1}{3}, -ic_1 - \log(\#1)\right)}{(c_1 - i\log(\#1))^{2/3}}\&\right][x + c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{i(-c_1)}(-\log(\#1) + i(-c_1))^{2/3}\Gamma\left(\frac{1}{3}, i(-c_1) - \log(\#1)\right)}{(i\log(\#1) - c_1)^{2/3}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\left(\frac{2}{3}\right)^{2/3} e^{ic_1}(-\log(\#1) + ic_1)^{2/3}\Gamma\left(\frac{1}{3}, ic_1 - \log(\#1)\right)}{(i\log(\#1) + c_1)^{2/3}}\&\right][x + c_2]$$

2.1.50 Problem 50

Internal problem ID [9121]
Book : Second order enumerated odes
Section : section 1
Problem number : 50
Date solved : `Monday, January 27, 2025 at 05:43:57 PM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible,`

Solve

$$yy'' + {y'}^3 = 0$$

Solved as second order missing x ode

Time used: 0.451 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y) \left(\frac{d}{dy}p(y)\right) + p(y)^3 = 0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p = 0 \tag{1}$$

$$p'y + p^2 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p = 0$$

Solving gives $p = 0$

Solving equation (2)

The ode

$$p' = -\frac{p^2}{y} \tag{2.7}$$

is separable as it can be written as

$$
\begin{aligned}
p' &= -\frac{p^2}{y} \\
&= f(y)g(p)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(y) &= -\frac{1}{y} \\
g(p) &= p^2
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(p)}\, dp &= \int f(y)\, dy \\
\int \frac{1}{p^2}\, dp &= \int -\frac{1}{y}\, dy
\end{aligned}
$$

$$
-\frac{1}{p} = \ln\left(\frac{1}{y}\right) + c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$
p^2 = 0
$$

for p gives

$$
p = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
-\frac{1}{p} &= \ln\left(\frac{1}{y}\right) + c_1 \\
p &= 0
\end{aligned}
$$

Solving for p gives

$$
\begin{aligned}
p &= 0 \\
p &= -\frac{1}{\ln\left(\frac{1}{y}\right) + c_1}
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = 0
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int 0\, dx + c_2 \\
y &= c_2
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{1}{\ln\left(\frac{1}{y}\right) + c_1}$$

Integrating gives

$$\int \left(-\ln\left(\frac{1}{y}\right) - c_1\right) dy = dx$$
$$-y\left(\ln\left(\frac{1}{y}\right) + c_1 + 1\right) = x + c_3$$

Solving for y gives

$$y = \frac{x + c_3}{\operatorname{LambertW}\left((x + c_3)\,\mathrm{e}^{-c_1 - 1}\right)}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2$$
$$y = \frac{x + c_3}{\operatorname{LambertW}\left((x + c_3)\,\mathrm{e}^{-c_1 - 1}\right)}$$

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)y(x) + \left(\frac{d}{dx}y(x)\right)^3 = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Define new dependent variable u
  $u(x) = \frac{d}{dx}y(x)$
- Compute $\frac{d^2}{dx^2}y(x)$
  $\frac{d}{dx}u(x) = \frac{d^2}{dx^2}y(x)$
- Use chain rule on the lhs
  $\left(\frac{d}{dx}y(x)\right)\left(\frac{d}{dy}u(y)\right) = \frac{d^2}{dx^2}y(x)$
- Substitute in the definition of u
  $u(y)\left(\frac{d}{dy}u(y)\right) = \frac{d^2}{dx^2}y(x)$
- Make substitutions $\frac{d}{dx}y(x) = u(y)$, $\frac{d^2}{dx^2}y(x) = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE
  $u(y)\left(\frac{d}{dy}u(y)\right)y + u(y)^3 = 0$
- Solve for the highest derivative
  $\frac{d}{dy}u(y) = -\frac{u(y)^2}{y}$
- Separate variables
  $\frac{\frac{d}{dy}u(y)}{u(y)^2} = -\frac{1}{y}$
- Integrate both sides with respect to y
  $\int \frac{\frac{d}{dy}u(y)}{u(y)^2}dy = \int -\frac{1}{y}dy + C1$
- Evaluate integral
  $-\frac{1}{u(y)} = -\ln(y) + C1$

- Solve for $u(y)$

  $u(y) = \frac{1}{\ln(y) - C1}$
- Solve 1st ODE for $u(y)$

  $u(y) = \frac{1}{\ln(y) - C1}$
- Revert to original variables with substitution $u(y) = \frac{d}{dx}y(x)\,, y = y(x)$

  $\frac{d}{dx}y(x) = \frac{1}{\ln(y(x)) - C1}$
- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{1}{\ln(y(x)) - C1}$
- Separate variables

  $\left(\frac{d}{dx}y(x)\right)\left(\ln\left(y(x)\right) - C1\right) = 1$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right)\left(\ln\left(y(x)\right) - C1\right) dx = \int 1 dx + C2$
- Evaluate integral

  $-C1\, y(x) + y(x)\ln\left(y(x)\right) - y(x) = x + C2$
- Solve for $y(x)$

  $y(x) = \mathrm{e}^{\mathit{LambertW}\left((x+C2)\mathrm{e}^{-C1-1}\right)+C1+1}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 27

```
dsolve(y(x)*diff(diff(y(x),x),x)+diff(y(x),x)^3 = 0,y(x),singsol=all)
```

$$y = 0$$
$$y = c_1$$
$$y = \frac{x + c_2}{\mathrm{LambertW}\left((x + c_2)\,\mathrm{e}^{c_1 - 1}\right)}$$

Mathematica DSolve solution

Solving time : 60.104 (sec)
Leaf size : 26

```
DSolve[{y[x]*D[y[x],{x,2}]+D[y[x],x]^3==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x + c_2}{W\left(e^{-1-c_1}(x + c_2)\right)}$$

2.1.51 Problem 51

Internal problem ID [9122]
Book : Second order enumerated odes
Section : section 1
Problem number : 51
Date solved : `Monday, January 27, 2025 at 05:43:58 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y{y''}^{3}+y^{3}y'=0$$

Factoring the ode gives these factors

$$y=0 \tag{1}$$

$${y''}^{3}+y'y^{2}=0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y=0$$

Solving gives $y=0$

Solving equation (2)

Solved as second order missing x ode

Time used: 84.304 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y'=p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^{3}\left(\frac{d}{dy}p(y)\right)^{3}+p(y)\,y^{2}=0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p=0 \tag{1}$$

$${p'}^{3}p^{2}+y^{2}=0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p = 0$$

Solving gives $p = 0$

Solving equation (2)

Let $p = p'$ the ode becomes

$$p^3 p^2 + y^2 = 0$$

Solving for p from the above results in

$$p = -\frac{y}{\sqrt{-p}\, p} \tag{1}$$

$$p = \frac{y}{\sqrt{-p}\, p} \tag{2}$$

This has the form

$$p = yf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = p'(y)$. Each of the above ode's is dAlembert ode which is now solved.

Solving ode 1A

Taking derivative of (*) w.r.t. y gives

$$p = f + (yf' + g')\frac{dp}{dy}$$

$$p - f = (yf' + g')\frac{dp}{dy} \tag{2}$$

Comparing the form $p = yf + g$ to (1A) shows that

$$f = \frac{1}{(-p)^{3/2}}$$

$$g = 0$$

Hence (2) becomes

$$p - \frac{1}{(-p)^{3/2}} = \frac{3yp'(y)}{2(-p)^{5/2}} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dy} = 0$ in the above which gives

$$p - \frac{1}{(-p)^{3/2}} = 0$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}y} \neq 0$. From eq. (2A). This results in

$$p'(y) = \frac{2\left(p(y) - \frac{1}{(-p(y))^{3/2}}\right)(-p(y))^{5/2}}{3y} \tag{3}$$

This ODE is now solved for $p(y)$. No inversion is needed.

The ode

$$p'(y) = \frac{2\left(\left(-p(y)\right)^{5/2}+1\right)p(y)}{3y} \tag{2.8}$$

is separable as it can be written as

$$\begin{aligned} p'(y) &= \frac{2\left(\left(-p(y)\right)^{5/2}+1\right)p(y)}{3y} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= \frac{2}{3y} \\ g(p) &= \left(\left(-p\right)^{5/2}+1\right)p \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\,dp &= \int f(y)\,dy \\ \int \frac{1}{\left(\left(-p\right)^{5/2}+1\right)p}\,dp &= \int \frac{2}{3y}\,dy \end{aligned}$$

$$-\frac{2\ln\left(p(y)^2-\left(-p(y)\right)^{3/2}-p(y)-\sqrt{-p\left(y\right)}+1\right)}{5} + \ln\left(-p(y)\right) - \frac{2\ln\left(\sqrt{-p\left(y\right)}+1\right)}{5} = \ln\left(y^{2/3}\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or

$$\left(\left(-p\right)^{5/2}+1\right)p = 0$$

for $p(y)$ gives

$$\begin{aligned} p(y) &= 0 \\ p(y) &= -\operatorname{RootOf}\left(\_Z^4-\_Z^3+\_Z^2-\_Z+1, \text{index}=1\right)^2 \\ p(y) &= -\operatorname{RootOf}\left(\_Z^4-\_Z^3+\_Z^2-\_Z+1, \text{index}=4\right)^2 \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $-\operatorname{RootOf}\left(\_Z^4-\_Z^3+\_Z^2-\_Z+1, \text{index}=1\right)^2$ will not be used

The solution $-\operatorname{RootOf}\left(\_Z^4-\_Z^3+\_Z^2-\_Z+1, \text{index}=4\right)^2$ will not be used

Therefore the solutions found are

$$-\frac{2\ln\left(p(y)^2-\left(-p(y)\right)^{3/2}-p(y)-\sqrt{-p\left(y\right)}+1\right)}{5} + \ln\left(-p(y)\right) - \frac{2\ln\left(\sqrt{-p\left(y\right)}+1\right)}{5} = \ln\left(y^{2/3}\right)+c_1$$

$$p(y) = 0$$

Substituing the above solution for p in (2A) gives

$$p = \frac{}{\left(\left(\operatorname{RootOf}\left(-1+\left(y^5 e^{\frac{15c_1}{2}}+1\right)\_Z^{75}+\left(-15y^5 e^{\frac{15c_1}{2}}-15\right)\_Z^{70}+\left(105y^5 e^{\frac{15c_1}{2}}+105\right)\_Z^{65}+\left(-4\right.\right.\right.\right.}$$

Solving ode 2A

Taking derivative of (*) w.r.t. y gives

$$\begin{aligned} p &= f + (yf' + g')\frac{dp}{dy} \\ p - f &= (yf' + g')\frac{dp}{dy} \end{aligned} \tag{2}$$

Comparing the form $p = yf + g$ to (1A) shows that

$$\begin{aligned} f &= -\frac{1}{(-p)^{3/2}} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p + \frac{1}{(-p)^{3/2}} = -\frac{3yp'(y)}{2(-p)^{5/2}} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dy} = 0$ in the above which gives

$$p + \frac{1}{(-p)^{3/2}} = 0$$

Solving the above for p results in

$$p_1 = -1$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$p = -y$$

The general solution is found when $\frac{\mathrm{dp}}{\mathrm{dy}} \neq 0$. From eq. (2A). This results in

$$p'(y) = -\frac{2\left(p(y) + \frac{1}{(-p(y))^{3/2}}\right)(-p(y))^{5/2}}{3y} \tag{3}$$

This ODE is now solved for $p(y)$. No inversion is needed.

The ode

$$p'(y) = -\frac{2\left((-p(y))^{5/2} - 1\right)p(y)}{3y} \tag{2.9}$$

is separable as it can be written as

$$\begin{aligned} p'(y) &= -\frac{2\left((-p(y))^{5/2} - 1\right)p(y)}{3y} \\ &= f(y)g(p) \end{aligned}$$

Where

$$f(y) = -\frac{2}{3y}$$
$$g(p) = \left((-p)^{5/2} - 1\right) p$$

Integrating gives

$$\int \frac{1}{g(p)}\, dp = \int f(y)\, dy$$
$$\int \frac{1}{\left((-p)^{5/2} - 1\right) p}\, dp = \int -\frac{2}{3y}\, dy$$

$$\frac{2 \ln \left(p(y)^2 + (-p(y))^{3/2} - p(y) + \sqrt{-p\left(y\right)} + 1\right)}{5} - \ln\left(-p(y)\right) + \frac{2 \ln \left(\sqrt{-p\left(y\right)} - 1\right)}{5} = \ln\left(\frac{1}{y^{2/3}}\right) + c_2$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$\left((-p)^{5/2} - 1\right) p = 0$$

for $p(y)$ gives

$$p(y) = -1$$
$$p(y) = 0$$
$$p(y) = -\operatorname{RootOf}\left(\_Z^4 + \_Z^3 + \_Z^2 + \_Z + 1, \text{index} = 1\right)^2$$
$$p(y) = -\operatorname{RootOf}\left(\_Z^4 + \_Z^3 + \_Z^2 + \_Z + 1, \text{index} = 4\right)^2$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $-\operatorname{RootOf}\left(\_Z^4 + \_Z^3 + \_Z^2 + \_Z + 1, \text{index} = 1\right)^2$ will not be used

The solution $-\operatorname{RootOf}\left(\_Z^4 + \_Z^3 + \_Z^2 + \_Z + 1, \text{index} = 4\right)^2$ will not be used

Therefore the solutions found are

$$\frac{2 \ln \left(p(y)^2 + (-p(y))^{3/2} - p(y) + \sqrt{-p\left(y\right)} + 1\right)}{5} - \ln\left(-p(y)\right) + \frac{2 \ln \left(\sqrt{-p\left(y\right)} - 1\right)}{5} = \ln\left(\frac{1}{y^{2/3}}\right) + c_2$$
$$p(y) = -1$$
$$p(y) = 0$$

Substituing the above solution for p in (2A) gives

$$p = -\frac{}{\left(\left(\operatorname{RootOf}\left(\left(y^5 + e^{\frac{15c_2}{2}}\right)\_Z^{75} + \left(15y^5 + 15\, e^{\frac{15c_2}{2}}\right)\_Z^{70} + \left(105y^5 + 105\, e^{\frac{15c_2}{2}}\right)\_Z^{65} + \left(455y^5 + 455\right.\right.\right.\right.}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_3$$
$$y = c_3$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{}{\left(\left(\operatorname{RootOf}\left(-1+\left(y^5 e^{\frac{15c_1}{2}}+1\right)\_Z^{75}+\left(-15y^5 e^{\frac{15c_1}{2}}-15\right)\_Z^{70}+\left(105y^5 e^{\frac{15c_1}{2}}+105\right)\_Z^{65}+\left(-\right.\right.\right.\right.}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{\left(\left(\operatorname{RootOf}\left(-1+\left(\tau^5 e^{\frac{15c_1}{2}}+1\right)\_Z^{75}+\left(-15\tau^5 e^{\frac{15c_1}{2}}-15\right)\_Z^{70}+\left(105\tau^5 e^{\frac{15c_1}{2}}+105\right)\_Z^{65}+\left(-45\right.\right.\right.\right.}{}$$

Solving for y gives

$$y = \operatorname{RootOf}\left(-\int^{\_Z} \frac{\left(\left(\operatorname{RootOf}\left(-1+\left(\tau^5 e^{\frac{15c_1}{2}}+1\right)\_Z^{75}+\left(-15\tau^5 e^{\frac{15c_1}{2}}-15\right)\_Z^{70}+\left(105\tau^5 e^{\frac{15c_1}{2}}+10\right.\right.\right.\right.}{} + x + c_4\right)$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -y$$

Integrating gives

$$\int -\frac{1}{y}dy = dx$$
$$-\ln(y) = x + c_5$$

Singular solutions are found by solving

$$-y = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$

Solving for y gives

$$y = 0$$

$$y = \mathrm{e}^{-x-c_5}$$

For solution (4) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\ }{\left(\left(\operatorname{RootOf}\left(\left(y^5+\mathrm{e}^{\frac{15c_2}{2}}\right)\_Z^{75}+\left(15y^5+15\,\mathrm{e}^{\frac{15c_2}{2}}\right)\_Z^{70}+\left(105y^5+105\,\mathrm{e}^{\frac{15c_2}{2}}\right)\_Z^{65}+\left(455y^5+455\right.\right.\right.\right.}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} -\frac{\left(\left(\operatorname{RootOf}\left(\left(\tau^5+\mathrm{e}^{\frac{15c_2}{2}}\right)\_Z^{75}+\left(15\tau^5+15\,\mathrm{e}^{\frac{15c_2}{2}}\right)\_Z^{70}+\left(105\tau^5+105\,\mathrm{e}^{\frac{15c_2}{2}}\right)\_Z^{65}+\left(455\tau^5+455\right.\right.\right.\right.}{\ }$$

Solving for y gives

$$y = \operatorname{RootOf}\left(-\int^{\_Z} -\frac{\left(\left(\operatorname{RootOf}\left(\left(\tau^5+\mathrm{e}^{\frac{15c_2}{2}}\right)\_Z^{75}+\left(15\tau^5+15\,\mathrm{e}^{\frac{15c_2}{2}}\right)\_Z^{70}+\left(105\tau^5+105\,\mathrm{e}^{\frac{15c_2}{2}}\right)\_Z^{65}+\left(455\tau^5+455\,\mathrm{e}\right.\right.\right.\right.}{\ } + x + c_6\right)$$

Will add steps showing solving for IC soon.

The solution

$$y = \operatorname{RootOf}\left(-\int^{\_Z} \frac{\left(\left(\operatorname{RootOf}\left(-1+\left(\tau^5\mathrm{e}^{\frac{15c_1}{2}}+1\right)\_Z^{75}+\left(-15\tau^5\mathrm{e}^{\frac{15c_1}{2}}-15\right)\_Z^{70}+\left(105\tau^5\mathrm{e}^{\frac{15c_1}{2}}+105\right)\right.\right.\right.}{\ } + x + c_4\right)$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = \text{RootOf}\left(-\int^{\_Z} -\frac{\left(\left(\text{RootOf}\left(\left(\tau^5 + e^{\frac{15c_2}{2}}\right)\_Z^{75} + \left(15\tau^5 + 15\,e^{\frac{15c_2}{2}}\right)\_Z^{70} + \left(105\tau^5 + 105\,e^{\frac{15c_2}{2}}\right)\_Z^{65} + \left(455\tau^5 + 45\right.\right.\right.\right.}{} + x + c_6\right)$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = 0$$
$$y = c_3$$
$$y = e^{-x-c_5}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 3 solutions were found. Trying to solve each res
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      Try integration with the canonical coordinates of the symmetry [0, y]
      -> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)^2+(-_b(_a))^(1/3),
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

Maple dsolve solution

Solving time : 0.184 (sec)
Leaf size : 126

```
dsolve(y(x)*diff(diff(y(x),x),x)^3+diff(y(x),x)*y(x)^3 = 0,y(x),singsol=all)
```

$$y = 0$$
$$y = c_1$$
$$y = e^{\int \operatorname{RootOf}\left(x - \int^{\_Z} -\frac{1}{\_f^2 - (-\_f)^{1/3}} d\_f + c_1\right) dx + c_2}$$
$$y = e^{\int \operatorname{RootOf}\left(x + 2\left(\int^{\_Z} \frac{1}{i(-\_f)^{1/3}\sqrt{3} + 2\_f^2 + (-\_f)^{1/3}} d\_f\right) + c_1\right) dx + c_2}$$
$$y = e^{\int \operatorname{RootOf}\left(x - 2\left(\int^{\_Z} \frac{1}{i(-\_f)^{1/3}\sqrt{3} - 2\_f^2 - (-\_f)^{1/3}} d\_f\right) + c_1\right) dx + c_2}$$

Mathematica DSolve solution

Solving time : 2.742 (sec)
Leaf size : 800

```
DSolve[{y[x]*D[y[x],{x,2}]^3+y[x]^3*D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to 0$$

$$y(x)\to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3\#1^{5/3}}{5c_1}\right)}{\left(-\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x)\to \text{InverseFunction}\left[\frac{\#1\left(1+\frac{3\sqrt[3]{-1}\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},-\frac{3\sqrt[3]{-1}\#1^{5/3}}{5c_1}\right)}{\left(\sqrt[3]{-1}\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x)\to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3(-1)^{2/3}\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3(-1)^{2/3}\#1^{5/3}}{5c_1}\right)}{\left(-(-1)^{2/3}\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x)\to 0$$

$$y(x)\to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3\#1^{5/3}}{5(-c_1)}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3\#1^{5/3}}{5(-c_1)}\right)}{\left(-\#1^{5/3}+\frac{5(-c_1)}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x)\to \text{InverseFunction}\left[\frac{\#1\left(1+\frac{3\sqrt[3]{-1}\#1^{5/3}}{5(-c_1)}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},-\frac{3\sqrt[3]{-1}\#1^{5/3}}{5(-c_1)}\right)}{\left(\sqrt[3]{-1}\#1^{5/3}+\frac{5}{3}(-1)c_1\right)^{3/5}}\&\right][x+c_2]$$

$$y(x)\to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3(-1)^{2/3}\#1^{5/3}}{5(-c_1)}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3(-1)^{2/3}\#1^{5/3}}{5(-c_1)}\right)}{\left(-(-1)^{2/3}\#1^{5/3}+\frac{5(-c_1)}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x)\to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3\#1^{5/3}}{5c_1}\right)}{\left(-\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x)\to \text{InverseFunction}\left[\frac{\#1\left(1+\frac{3\sqrt[3]{-1}\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},-\frac{3\sqrt[3]{-1}\#1^{5/3}}{5c_1}\right)}{\left(\sqrt[3]{-1}\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

$$y(x)\to \text{InverseFunction}\left[\frac{\#1\left(1-\frac{3(-1)^{2/3}\#1^{5/3}}{5c_1}\right)^{3/5}\text{Hypergeometric2F1}\left(\frac{3}{5},\frac{3}{5},\frac{8}{5},\frac{3(-1)^{2/3}\#1^{5/3}}{5c_1}\right)}{\left(-(-1)^{2/3}\#1^{5/3}+\frac{5c_1}{3}\right)^{3/5}}\&\right][x+c_2]$$

2.1.52 Problem 52

Internal problem ID [9123]
Book : Second order enumerated odes
Section : section 1
Problem number : 52
Date solved : `Monday, January 27, 2025 at 05:45:23 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y{y''}^{3}+y^{3}{y'}^{5}=0$$

Factoring the ode gives these factors

$$y=0 \tag{1}$$

$${y'}^{5}y^{2}+{y''}^{3}=0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y=0$$

Solving gives $y=0$

Solving equation (2)

Solved as second order missing x ode

Time used: 23.041 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y'=p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^{5}\, y^{2}+p(y)^{3}\left(\frac{d}{dy}p(y)\right)^{3}=0$$

Which is now solved as first order ode for $p(y)$.

Factoring the ode gives these factors

$$p^{3}=0 \tag{1}$$

$$p^{2}y^{2}+{p'}^{3}=0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for p from

$$p^3 = 0$$

Solving gives $p = 0$

Solving equation (2)

The ode has the form

$$(p')^{\frac{n}{m}} = f(y)g(p) \tag{1}$$

Where $n = 3, m = 1, f = -y^2, g = p^2$. Hence the ode is

$$(p')^3 = -p^2y^2$$

Solving for p' from (1) gives

$$\begin{aligned}
p' &= (fg)^{1/3} \\
p' &= -\frac{(fg)^{1/3}}{2} + \frac{i\sqrt{3}\,(fg)^{1/3}}{2} \\
p' &= -\frac{(fg)^{1/3}}{2} - \frac{i\sqrt{3}\,(fg)^{1/3}}{2}
\end{aligned}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\begin{aligned}
-y^2 &> 0 \\
p^2 &> 0
\end{aligned}$$

Under the above assumption the differential equations become separable and can be written as

$$\begin{aligned}
p' &= f^{1/3}g^{1/3} \\
p' &= \frac{f^{1/3}g^{1/3}(-1 + i\sqrt{3})}{2} \\
p' &= -\frac{f^{1/3}g^{1/3}(1 + i\sqrt{3})}{2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{g^{1/3}}\, dp &= \left(f^{1/3}\right)\, dy \\
\frac{2}{g^{1/3}\left(-1 + i\sqrt{3}\right)}\, dp &= \left(f^{1/3}\right)\, dy \\
-\frac{2}{g^{1/3}\left(1 + i\sqrt{3}\right)}\, dp &= \left(f^{1/3}\right)\, dy
\end{aligned}$$

Replacing $f(y), g(p)$ by their values gives

$$\begin{aligned}
\frac{1}{(p^2)^{1/3}}\, dp &= \left(\left(-y^2\right)^{1/3}\right)\, dy \\
\frac{2}{(p^2)^{1/3}\left(-1 + i\sqrt{3}\right)}\, dp &= \left(\left(-y^2\right)^{1/3}\right)\, dy \\
-\frac{2}{(p^2)^{1/3}\left(1 + i\sqrt{3}\right)}\, dp &= \left(\left(-y^2\right)^{1/3}\right)\, dy
\end{aligned}$$

Integrating now gives the following solutions

$$\int \frac{1}{\left(p^2\right)^{1/3}} dp = \int \left(-y^2\right)^{1/3} dy + c_1$$
$$\frac{3\left(p^2\right)^{2/3}}{p} = \frac{3y\left(-y^2\right)^{1/3}}{5}$$
$$\int \frac{2}{\left(p^2\right)^{1/3}\left(-1+i\sqrt{3}\right)} dp = \int \left(-y^2\right)^{1/3} dy + c_1$$
$$-\frac{3\left(p^2\right)^{2/3}\left(1+i\sqrt{3}\right)}{2p} = \frac{3y\left(-y^2\right)^{1/3}}{5}$$
$$\int -\frac{2}{\left(p^2\right)^{1/3}\left(1+i\sqrt{3}\right)} dp = \int \left(-y^2\right)^{1/3} dy + c_1$$
$$\frac{3\left(p^2\right)^{2/3}\left(-1+i\sqrt{3}\right)}{2p} = \frac{3y\left(-y^2\right)^{1/3}}{5}$$

Therefore

$$\frac{3\left(p^2\right)^{2/3}}{p} = \frac{3y\left(-y^2\right)^{1/3}}{5} + c_1$$
$$p = -\frac{y^5}{125} + \frac{\left(-y^2\right)^{2/3} c_1 y^2}{25} + \frac{\left(-y^2\right)^{1/3} c_1^2 y}{15} + \frac{c_1^3}{27}$$
$$p = -\frac{y^5}{125} + \frac{\left(-y^2\right)^{2/3} c_1 y^2}{25} + \frac{\left(-y^2\right)^{1/3} c_1^2 y}{15} + \frac{c_1^3}{27}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{3\left(y'^2\right)^{2/3}}{y'} = \frac{3y\left(-y^2\right)^{1/3}}{5} + c_1$$

Solving for the derivative gives these ODE's to solve

$$y' = 0 \tag{1}$$
$$y' = \frac{\sqrt{15}\left(\frac{y^2\left(-y^2\right)^{2/3}\sqrt{15}}{25} + \frac{2c_1\sqrt{15}\, y\left(-y^2\right)^{1/3}}{15} + \frac{\sqrt{15}\, c_1^2}{9}\right)\left(3y\left(-y^2\right)^{1/3} + 5c_1\right)}{225} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_2$$
$$y = c_2$$

Solving Eq. (2)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{3375}{\left(9\tau^2\left(-\tau^2\right)^{2/3} + 30c_1\left(-\tau^2\right)^{1/3}\tau + 25c_1^2\right)\left(3\tau\left(-\tau^2\right)^{1/3} + 5c_1\right)} d\tau = x + c_3$$

Singular solutions are found by solving

$$\frac{\left(9y^2\left(-y^2\right)^{2/3} + 30c_1\left(-y^2\right)^{1/3} y + 25c_1^2\right)\left(3y\left(-y^2\right)^{1/3} + 5c_1\right)}{3375} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{\left(-\frac{\sqrt{5}}{4} - \frac{1}{4} - \frac{i\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right) 5^{3/5} 3^{2/5} c_1^{3/5}}{3}$$
$$y = \frac{\left(-\frac{\sqrt{5}}{4} - \frac{1}{4} + \frac{i\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right) 5^{3/5} 3^{2/5} c_1^{3/5}}{3}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_4$$
$$y = c_4$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{y^5}{125} + \frac{y^2 (-y^2)^{2/3} c_1}{25} + \frac{y (-y^2)^{1/3} c_1^2}{15} + \frac{c_1^3}{27}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{1}{-\frac{\tau^5}{125} + \frac{(-\tau^2)^{2/3} c_1 \tau^2}{25} + \frac{(-\tau^2)^{1/3} c_1^2 \tau}{15} + \frac{c_1^3}{27}} d\tau = x + c_5$$

Singular solutions are found by solving

$$-\frac{y^5}{125} + \frac{(-y^2)^{2/3} c_1 y^2}{25} + \frac{(-y^2)^{1/3} c_1^2 y}{15} + \frac{c_1^3}{27} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{\left(-\frac{\sqrt{5}}{4} - \frac{1}{4} - \frac{i\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right) 5^{3/5} 3^{2/5} c_1^{3/5}}{3}$$
$$y = \frac{\left(-\frac{\sqrt{5}}{4} - \frac{1}{4} + \frac{i\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right) 5^{3/5} 3^{2/5} c_1^{3/5}}{3}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\int^y \frac{1}{-\frac{\tau^5}{125} + \frac{(-\tau^2)^{2/3} c_1 \tau^2}{25} + \frac{(-\tau^2)^{1/3} c_1^2 \tau}{15} + \frac{c_1^3}{27}} d\tau = x + c_5$$
$$\int^y \frac{3375}{\left(9\tau^2 (-\tau^2)^{2/3} + 30 c_1 (-\tau^2)^{1/3} \tau + 25 c_1^2\right) \left(3\tau (-\tau^2)^{1/3} + 5 c_1\right)} d\tau = x + c_3$$
$$y = c_2$$
$$y = c_4$$
$$y = \frac{\left(-\frac{\sqrt{5}}{4} - \frac{1}{4} - \frac{i\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right) 5^{3/5} 3^{2/5} c_1^{3/5}}{3}$$
$$y = \frac{\left(-\frac{\sqrt{5}}{4} - \frac{1}{4} + \frac{i\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right) 5^{3/5} 3^{2/5} c_1^{3/5}}{3}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 3 solutions were found. Trying to solve each resul
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(-_a^2*_b(_a)^2)^(1/3)
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[_a, 5*_b]
```

Maple dsolve solution

Solving time : `0.214` (sec)
Leaf size : 208

```
dsolve(y(x)*diff(diff(y(x),x),x)^3+y(x)^3*diff(y(x),x)^5 = 0,y(x),singsol=all)
```

$$y = 0$$

$$y = c_1$$

$$\int^{y} \frac{1}{\operatorname{RootOf}\left(5\left(\int_{\_g}^{\_Z} \frac{1}{\_a\left(-\_a^2\_f^2\right)^{1/3}-5\_f} d\_f\right)-\ln\left(\_a^5+125\right)+5c_1\right)} d\_a-x-c_2=0$$

$$\int^{y} \frac{1}{\operatorname{RootOf}\left(\sqrt{3}\,\ln\left(\_a^5+125\right)-i\ln\left(\_a^5+125\right)+20\left(\int_{\_g}^{\_Z} \frac{1}{2i\_a\left(-\_a^2\_f^2\right)^{1/3}+5i\_f+5\sqrt{3}\,\_f} d\_f\right)-20c_1\right)}$$

$$-x-c_2=0$$

$$\int^{y} \frac{1}{\operatorname{RootOf}\left(\sqrt{3}\,\ln\left(\_a^5+125\right)+i\ln\left(\_a^5+125\right)+20\left(\int_{\_g}^{\_Z} \frac{1}{-2i\_a\left(-\_a^2\_f^2\right)^{1/3}+5\sqrt{3}\,\_f-5i\_f} d\_f\right)-20c_1\right.}$$

$$-x-c_2=0$$

Mathematica DSolve solution

Solving time : 24.151 (sec)
Leaf size : 449

```
DSolve[{y[x]*D[y[x],{x,2}]^3+y[x]^3*D[y[x],x]^5==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3\#1^{5/3}}{5c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},-\frac{3i\left(-i+\sqrt{3}\right)\#1^{5/3}}{10c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3i\left(i+\sqrt{3}\right)\#1^{5/3}}{10c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

$$y(x) \to 0$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3\#1^{5/3}}{5(-c_1)}\right)}{(-c_1)^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},-\frac{3i\left(-i+\sqrt{3}\right)\#1^{5/3}}{10(-c_1)}\right)}{(-c_1)^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3i\left(i+\sqrt{3}\right)\#1^{5/3}}{10(-c_1)}\right)}{(-c_1)^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3\#1^{5/3}}{5c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},-\frac{3i\left(-i+\sqrt{3}\right)\#1^{5/3}}{10c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{27\#1\,\text{Hypergeometric2F1}\left(\frac{3}{5},3,\frac{8}{5},\frac{3i\left(i+\sqrt{3}\right)\#1^{5/3}}{10c_1}\right)}{c_1{}^3}\&\right][x+c_2]$$

2.2 section 2

2.2.1 Problem 1

Internal problem ID [9124]
Book : Second order enumerated odes
Section : section 2
Problem number : 1
Date solved : `Monday, January 27, 2025 at 05:45:46 PM`
CAS classification : `[_Liouville, [_2nd_order, _reducible, _mu_xy]]`

Solve

$$y'' + xy' + yy'^2 = 0$$

Solved as second nonlinear ode solved by Mainardi Lioville method

Time used: 0.203 (sec)

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = x$$
$$g(y) = y$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx} \ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = y$ and $f = x$, then

$$\begin{aligned}\int -g dy &= \int -y dy \\ &= -\frac{y^2}{2} \\ \int -f dx &= \int -x dx \\ &= -\frac{x^2}{2}\end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = c_2 \,\mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{-\frac{x^2}{2}}$$

Which is now solved as first order separable ode. The ode

$$y' = c_2 \,\mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{-\frac{x^2}{2}} \tag{2.10}$$

is separable as it can be written as

$$\begin{aligned}y' &= c_2 \,\mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{-\frac{x^2}{2}} \\ &= f(x) g(y)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= \mathrm{e}^{-\frac{x^2}{2}} c_2 \\ g(y) &= \mathrm{e}^{-\frac{y^2}{2}}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int \mathrm{e}^{\frac{y^2}{2}}\, dy &= \int \mathrm{e}^{-\frac{x^2}{2}} c_2\, dx\end{aligned}$$

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} = \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} + c_3$$

Will add steps showing solving for IC soon.

Solving for y from the above solution(s) gives (after possible removing of solutions that do not verify)

$$y = -i\,\mathrm{RootOf}\left(i\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) c_2 + i\sqrt{2}\,c_3 - \mathrm{erf}\left(\_Z\right)\sqrt{\pi}\right)\sqrt{2}$$

Summary of solutions found

$$y = -i\,\mathrm{RootOf}\left(i\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) c_2 + i\sqrt{2}\,c_3 - \mathrm{erf}\left(\_Z\right)\sqrt{\pi}\right)\sqrt{2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+y(x)*diff(y(x),x)^2 = 0,y(x),singsol=all)
```

$$y = -i \,\mathrm{RootOf}\left(i\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) c_1 + i\sqrt{2}\,c_2 - \mathrm{erf}\left(\_Z\right)\sqrt{\pi}\right)\sqrt{2}$$

Mathematica DSolve solution

Solving time : 1.496 (sec)
Leaf size : 44

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+y[x]*(D[y[x],x])^2==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to -i\sqrt{2}\mathrm{erf}^{-1}\left(i\left(\sqrt{\frac{2}{\pi}}c_2 - c_1\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)\right)$$

2.2.2 Problem 2

Internal problem ID [9125]
Book : Second order enumerated odes
Section : section 2
Problem number : 2
Date solved : `Monday, January 27, 2025 at 05:45:47 PM`
CAS classification : `[_Liouville, [_2nd_order, _reducible, _mu_xy]]`

Solve

$$y'' + \sin(x)\, y' + y{y'}^2 = 0$$

Solved as second nonlinear ode solved by Mainardi Lioville method

Time used: 0.244 (sec)

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y){y'}^2 = 0 \tag{1A}$$

Where in this problem

$$\begin{aligned} f(x) &= \sin(x) \\ g(y) &= y \end{aligned}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy}\, e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = y$ and $f = \sin(x)$, then

$$
\begin{aligned}
\int -g dy &= \int -y dy \\
&= -\frac{y^2}{2} \\
\int -f dx &= \int -\sin(x)\, dx \\
&= \cos(x)
\end{aligned}
$$

Substituting the above into Eq(6A) gives

$$y' = c_2 \,\mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{\cos(x)}$$

Which is now solved as first order separable ode. The ode

$$y' = c_2 \,\mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{\cos(x)} \tag{2.11}$$

is separable as it can be written as

$$
\begin{aligned}
y' &= c_2 \,\mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{\cos(x)} \\
&= f(x) g(y)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \mathrm{e}^{\cos(x)} c_2 \\
g(y) &= \mathrm{e}^{-\frac{y^2}{2}}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\
\int \mathrm{e}^{\frac{y^2}{2}}\, dy &= \int \mathrm{e}^{\cos(x)} c_2\, dx
\end{aligned}
$$

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} = \int \mathrm{e}^{\cos(x)} c_2 dx + 2c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} = \int \mathrm{e}^{\cos(x)} c_2 dx + 2c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 35

```
dsolve(diff(diff(y(x),x),x)+sin(x)*diff(y(x),x)+y(x)*diff(y(x),x)^2 = 0,y(x),singsol=all
```

$$y = -i \operatorname{RootOf}\left(i\sqrt{2}\, c_1 \left(\int \mathrm{e}^{\cos(x)} dx\right) + i\sqrt{2}\, c_2 - \operatorname{erf}\left(\_Z\right)\sqrt{\pi}\right)\sqrt{2}$$

Mathematica DSolve solution

Solving time : 66.642 (sec)
Leaf size : 76

```
DSolve[{D[y[x],{x,2}]+Sin[x]*D[y[x],x]+y[x]*(D[y[x],x])^2==0,{}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to -i\sqrt{2}\text{erf}^{-1}\left(i\sqrt{\frac{2}{\pi}}\left(\int_1^x -e^{\cos(K[2])}c_1 dK[2] + c_2\right)\right)$$

$$y(x) \to -i\sqrt{2}\text{erf}^{-1}\left(i\sqrt{\frac{2}{\pi}}c_2\right)$$

2.2.3 Problem 3

Internal problem ID [9126]
Book : Second order enumerated odes
Section : section 2
Problem number : 3
Date solved : Tuesday, January 28, 2025 at 04:00:07 PM
CAS classification :
[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]

Solve

$$y'' + (1-x)\, y' + y^2 {y'}^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 61

```
dsolve(diff(diff(y(x),x),x)+(1-x)*diff(y(x),x)+y(x)^2*diff(y(x),x)^2 = 0,y(x),singsol=
```

$$c_1 \operatorname{erf}\left(\frac{i\sqrt{2}\,(x-1)}{2}\right) - c_2 + \frac{2\, 3^{5/6} y \pi}{9\Gamma\left(\frac{2}{3}\right)(-y^3)^{1/3}} - \frac{y\Gamma\left(\frac{1}{3}, -\frac{y^3}{3}\right) 3^{1/3}}{3\,(-y^3)^{1/3}} = 0$$

Mathematica DSolve solution

Solving time : 16.947 (sec)
Leaf size : 64

```
DSolve[{D[y[x],{x,2}]+(1-x)*D[y[x],x]+y[x]^2*(D[y[x],x])^2==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \text{InverseFunction}\left[-\frac{\#1\Gamma\left(\frac{1}{3}, -\frac{\#1^3}{3}\right)}{3^{2/3}\sqrt[3]{-\#1^3}}\&\right]\left[\int_1^x -e^{\frac{1}{2}(K[2]-2)K[2]} c_1 dK[2] + c_2\right]$$

2.2.4 Problem 4

Internal problem ID [9127]
Book : Second order enumerated odes
Section : section 2
Problem number : 4
Date solved : Tuesday, January 28, 2025 at 04:00:07 PM
CAS classification :
[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]

Solve

$$y'' + (\sin(x) + 2x)\, y' + \cos(y)\, y y'^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 34

```
dsolve(diff(diff(y(x),x),x)+(sin(x)+2*x)*diff(y(x),x)+cos(y(x))*y(x)*diff(y(x),x)^2 = 0,
```

$$\int^{y} e^{\cos(\_a)+\_a\sin(\_a)} d\_a - c_1\left(\int e^{-x^2+\cos(x)} dx\right) - c_2 = 0$$

Mathematica DSolve solution

Solving time : 1.353 (sec)
Leaf size : 68

```
DSolve[{D[y[x],{x,2}]+(Sin[x]+2*x)*D[y[x],x]+Cos[y[x]]*y[x]*(D[y[x],x])^2==0,{}},y[x],x,Includ
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \exp\left(-\int_1^{K[3]} -\cos(K[1])K[1]dK[1]\right) dK[3]\&\right]\left[\int_1^x -\exp\left(-\int_1^{K[4]} (2K[2]+\sin(K[2]))dK[2]\right) c_1 dK[4] + c_2\right]$$

2.2.5 Problem 5

Internal problem ID [9128]
Book : Second order enumerated odes
Section : section 2
Problem number : 5
Date solved : `Monday, January 27, 2025 at 05:45:49 PM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$y''y' + y^2 = 0$$

Solved as second order missing x ode

Time used: 1.108 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right) + y^2 = 0$$

Which is now solved as first order ode for $p(y)$.

The ode

$$p' = -\frac{y^2}{p^2} \tag{2.12}$$

is separable as it can be written as

$$\begin{aligned} p' &= -\frac{y^2}{p^2} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= -y^2 \\ g(p) &= \frac{1}{p^2} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)} dp = \int f(y) \, dy$$
$$\int p^2 \, dp = \int -y^2 \, dy$$

$$\frac{p^3}{3} = -\frac{y^3}{3} + c_1$$

Solving for p gives

$$p = \left(-y^3 + 3c_1\right)^{1/3}$$
$$p = -\frac{\left(-y^3 + 3c_1\right)^{1/3}}{2} - \frac{i\sqrt{3}\left(-y^3 + 3c_1\right)^{1/3}}{2}$$
$$p = -\frac{\left(-y^3 + 3c_1\right)^{1/3}}{2} + \frac{i\sqrt{3}\left(-y^3 + 3c_1\right)^{1/3}}{2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \left(-y^3 + 3c_1\right)^{1/3}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{1}{\left(-\tau^3 + 3c_1\right)^{1/3}} d\tau = x + c_2$$

Singular solutions are found by solving

$$\left(-y^3 + 3c_1\right)^{1/3} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 3^{1/3} c_1^{1/3}$$
$$y = -\frac{3^{1/3} c_1^{1/3}}{2} - \frac{i 3^{5/6} c_1^{1/3}}{2}$$
$$y = -\frac{3^{1/3} c_1^{1/3}}{2} + \frac{i 3^{5/6} c_1^{1/3}}{2}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\left(-y^3 + 3c_1\right)^{1/3}}{2} - \frac{i\sqrt{3}\left(-y^3 + 3c_1\right)^{1/3}}{2}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{1}{-\frac{\left(-\tau^3+3c_1\right)^{1/3}}{2} - \frac{i\sqrt{3}\left(-\tau^3+3c_1\right)^{1/3}}{2}} d\tau = x + c_3$$

Singular solutions are found by solving

$$-\frac{\left(-y^3 + 3c_1\right)^{1/3}}{2} - \frac{i\sqrt{3}\left(-y^3 + 3c_1\right)^{1/3}}{2} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
\begin{aligned}
y &= 3^{1/3}c_1^{1/3} \\
y &= -\frac{3^{1/3}c_1^{1/3}}{2} - \frac{i3^{5/6}c_1^{1/3}}{2} \\
y &= -\frac{3^{1/3}c_1^{1/3}}{2} + \frac{i3^{5/6}c_1^{1/3}}{2}
\end{aligned}
$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = -\frac{(-y^3+3c_1)^{1/3}}{2} + \frac{i\sqrt{3}\,(-y^3+3c_1)^{1/3}}{2}
$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$
\int^y \frac{1}{-\frac{(-\tau^3+3c_1)^{1/3}}{2} + \frac{i\sqrt{3}\,(-\tau^3+3c_1)^{1/3}}{2}} d\tau = x + c_4
$$

Singular solutions are found by solving

$$
-\frac{(-y^3+3c_1)^{1/3}}{2} + \frac{i\sqrt{3}\,(-y^3+3c_1)^{1/3}}{2} = 0
$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
\begin{aligned}
y &= 3^{1/3}c_1^{1/3} \\
y &= -\frac{3^{1/3}c_1^{1/3}}{2} - \frac{i3^{5/6}c_1^{1/3}}{2} \\
y &= -\frac{3^{1/3}c_1^{1/3}}{2} + \frac{i3^{5/6}c_1^{1/3}}{2}
\end{aligned}
$$

Will add steps showing solving for IC soon.

The solution

$$
y = 3^{1/3}c_1^{1/3}
$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$
y = -\frac{3^{1/3}c_1^{1/3}}{2} - \frac{i3^{5/6}c_1^{1/3}}{2}
$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$
y = -\frac{3^{1/3}c_1^{1/3}}{2} + \frac{i3^{5/6}c_1^{1/3}}{2}
$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$
\begin{aligned}
&\int^y \frac{1}{-\frac{(-\tau^3+3c_1)^{1/3}}{2} + \frac{i\sqrt{3}\,(-\tau^3+3c_1)^{1/3}}{2}} d\tau = x + c_4 \\
&\int^y \frac{1}{-\frac{(-\tau^3+3c_1)^{1/3}}{2} - \frac{i\sqrt{3}\,(-\tau^3+3c_1)^{1/3}}{2}} d\tau = x + c_3 \\
&\int^y \frac{1}{(-\tau^3+3c_1)^{1/3}} d\tau = x + c_2
\end{aligned}
$$

Solved as second order can be made integrable

Time used: 0.602 (sec)

Multiplying the ode by y' gives

$$y^2y' + y'^2y'' = 0$$

Integrating the above w.r.t x gives

$$\int \left(y^2y' + y'^2y''\right) dx = 0$$
$$\frac{y^3}{3} + \frac{y'^3}{3} = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \left(-y^3 + 3c_1\right)^{1/3} \tag{1}$$
$$y' = -\frac{\left(-y^3 + 3c_1\right)^{1/3}}{2} - \frac{i\sqrt{3}\left(-y^3 + 3c_1\right)^{1/3}}{2} \tag{2}$$
$$y' = -\frac{\left(-y^3 + 3c_1\right)^{1/3}}{2} + \frac{i\sqrt{3}\left(-y^3 + 3c_1\right)^{1/3}}{2} \tag{3}$$

Now each of the above is solved separately.

Solving Eq. (1)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{1}{\left(-\tau^3 + 3c_1\right)^{1/3}} d\tau = x + c_2$$

Singular solutions are found by solving

$$\left(-y^3 + 3c_1\right)^{1/3} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 3^{1/3}c_1^{1/3}$$
$$y = -\frac{3^{1/3}c_1^{1/3}}{2} - \frac{i3^{5/6}c_1^{1/3}}{2}$$
$$y = -\frac{3^{1/3}c_1^{1/3}}{2} + \frac{i3^{5/6}c_1^{1/3}}{2}$$

Solving Eq. (2)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{1}{-\frac{\left(-\tau^3+3c_1\right)^{1/3}}{2} - \frac{i\sqrt{3}\left(-\tau^3+3c_1\right)^{1/3}}{2}} d\tau = x + c_3$$

Singular solutions are found by solving

$$-\frac{\left(-y^3 + 3c_1\right)^{1/3}}{2} - \frac{i\sqrt{3}\left(-y^3 + 3c_1\right)^{1/3}}{2} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 3^{1/3}c_1^{1/3}$$
$$y = -\frac{3^{1/3}c_1^{1/3}}{2} - \frac{i3^{5/6}c_1^{1/3}}{2}$$
$$y = -\frac{3^{1/3}c_1^{1/3}}{2} + \frac{i3^{5/6}c_1^{1/3}}{2}$$

Solving Eq. (3)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{1}{-\frac{(-\tau^3+3c_1)^{1/3}}{2} + \frac{i\sqrt{3}\,(-\tau^3+3c_1)^{1/3}}{2}} d\tau = x + c_4$$

Singular solutions are found by solving

$$-\frac{(-y^3+3c_1)^{1/3}}{2} + \frac{i\sqrt{3}\,(-y^3+3c_1)^{1/3}}{2} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 3^{1/3} c_1^{1/3}$$
$$y = -\frac{3^{1/3} c_1^{1/3}}{2} - \frac{i 3^{5/6} c_1^{1/3}}{2}$$
$$y = -\frac{3^{1/3} c_1^{1/3}}{2} + \frac{i 3^{5/6} c_1^{1/3}}{2}$$

Will add steps showing solving for IC soon.

The solution

$$y = 3^{1/3} c_1^{1/3}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{3^{1/3} c_1^{1/3}}{2} - \frac{i 3^{5/6} c_1^{1/3}}{2}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{3^{1/3} c_1^{1/3}}{2} + \frac{i 3^{5/6} c_1^{1/3}}{2}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\int^{y} \frac{1}{-\frac{(-\tau^3+3c_1)^{1/3}}{2} + \frac{i\sqrt{3}\,(-\tau^3+3c_1)^{1/3}}{2}} d\tau = x + c_4$$

$$\int^{y} \frac{1}{-\frac{(-\tau^3+3c_1)^{1/3}}{2} - \frac{i\sqrt{3}\,(-\tau^3+3c_1)^{1/3}}{2}} d\tau = x + c_3$$

$$\int^{y} \frac{1}{(-\tau^3+3c_1)^{1/3}} d\tau = x + c_2$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
Try integration with the canonical coordinates of the symmetry [0, y]
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)^3+1)/_b(_a), _b(_a), expli
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

Maple dsolve solution

Solving time : 0.052 (sec)
Leaf size : 61

```
dsolve(diff(y(x),x)*diff(diff(y(x),x),x)+y(x)^2 = 0,y(x),singsol=all)
```

$$y = 0$$

$$y = e^{\frac{\sqrt{3}\left(\int \tan\left(\operatorname{RootOf}\left(-\sqrt{3}\,\ln\left(\cos\left(\_Z\right)^{2}\right)-2\sqrt{3}\,\ln\left(\tan\left(\_Z\right)+\sqrt{3}\right)+6\sqrt{3}\,c_1+6\sqrt{3}\,x+6\_Z\right)\right)dx\right)}{2}+c_2+\frac{x}{2}}$$

Mathematica DSolve solution

Solving time : 0.503 (sec)
Leaf size : 55

```
DSolve[{D[y[x],{x,2}]*D[y[x],x]+y[x]^2==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 \exp\left(\int_1^x \text{InverseFunction}\left[\int_1^{\#1} \frac{K[1]}{(K[1]+1)\left(K[1]^2-K[1]+1\right)}dK[1]\&\right][c_1 - K[2]]dK[2]\right)$$

2.2.6 Problem 6

Internal problem ID [9129]
Book : Second order enumerated odes
Section : section 2
Problem number : 6
Date solved : `Monday, January 27, 2025 at 05:45:51 PM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$y''y' + y^n = 0$$

Solved as second order missing x ode

Time used: 1.511 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right) + y^n = 0$$

Which is now solved as first order ode for $p(y)$.

The ode

$$p' = -\frac{y^n}{p^2} \tag{2.13}$$

is separable as it can be written as

$$\begin{aligned} p' &= -\frac{y^n}{p^2} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= -y^n \\ g(p) &= \frac{1}{p^2} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\, dp = \int f(y)\, dy$$

$$\int p^2\, dp = \int -y^n\, dy$$

$$\frac{p^3}{3} = -\frac{y^{n+1}}{n+1} + c_1$$

Solving for p gives

$$p = \frac{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{n+1}$$

$$p = -\frac{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{2(n+1)} - \frac{i\sqrt{3}\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{2(n+1)}$$

$$p = -\frac{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{2(n+1)} + \frac{i\sqrt{3}\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{2n+2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{n+1}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{n+1}{\left(\left(3c_1 n - 3\tau^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}} d\tau = x + c_2$$

Singular solutions are found by solving

$$\frac{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{n+1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \mathrm{e}^{\frac{\ln(c_1 n + c_1)}{n+1}}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{2(n+1)} - \frac{i\sqrt{3}\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{2(n+1)}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y -\frac{2(n+1)}{\left(\left(3c_1 n - 3\tau^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}\left(1 + i\sqrt{3}\right)} d\tau = x + c_3$$

Singular solutions are found by solving

$$-\frac{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}\left(1 + i\sqrt{3}\right)}{2(n+1)} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \mathrm{e}^{\frac{\ln(c_1 n + c_1)}{n+1}}$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{2\,(n+1)} + \frac{i\sqrt{3}\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}}{2n+2}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{2n+2}{\left(\left(3c_1 n - 3\tau^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}\left(i\sqrt{3} - 1\right)} d\tau = x + c_4$$

Singular solutions are found by solving

$$\frac{\left(\left(3c_1 n - 3y^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}\left(i\sqrt{3} - 1\right)}{2n+2} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \mathrm{e}^{\frac{\ln(c_1 n + c_1)}{n+1}}$$

Will add steps showing solving for IC soon.

The solution

$$y = \mathrm{e}^{\frac{\ln(c_1 n + c_1)}{n+1}}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\int^y \frac{n+1}{\left(\left(3c_1 n - 3\tau^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}} d\tau = x + c_2$$

$$\int^y \frac{2n+2}{\left(\left(3c_1 n - 3\tau^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}\left(i\sqrt{3} - 1\right)} d\tau = x + c_4$$

$$\int^y -\frac{2(n+1)}{\left(\left(3c_1 n - 3\tau^{n+1} + 3c_1\right)(n+1)^2\right)^{1/3}\left(1 + i\sqrt{3}\right)} d\tau = x + c_3$$

Solved as second order can be made integrable

Time used: 1.646 (sec)

Multiplying the ode by y' gives

$$y'^2 y'' + y^{n-1} y' y = 0$$

Integrating the above w.r.t x gives

$$\int \left(y'^2 y'' + y^{n-1} y' y\right) dx = 0$$

$$\frac{y'^3}{3} + \frac{y^2 \mathrm{e}^{(n-1)\ln(y)}}{n+1} = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \frac{\left(\left(-3y^2 e^{(n-1)\ln(y)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}}{n+1} \tag{1}$$

$$y' = -\frac{\left(\left(-3y^2 e^{(n-1)\ln(y)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}}{2(n+1)} - \frac{i\sqrt{3}\left(\left(-3y^2 e^{(n-1)\ln(y)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}}{2(n+1)} \tag{2}$$

$$y' = -\frac{\left(\left(-3y^2 e^{(n-1)\ln(y)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}}{2(n+1)} + \frac{i\sqrt{3}\left(\left(-3y^2 e^{(n-1)\ln(y)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}}{2n+2} \tag{3}$$

Now each of the above is solved separately.

Solving Eq. (1)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{n+1}{\left(\left(-3\tau^2 e^{(n-1)\ln(\tau)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}} d\tau = x + c_2$$

Singular solutions are found by solving

$$\frac{\left(\left(-3y^2 e^{(n-1)\ln(y)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}}{n+1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = e^{\frac{\ln(c_1(n+1))}{n+1}}$$

Solving Eq. (2)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y -\frac{2(n+1)}{\left(\left(-3\tau^2 e^{(n-1)\ln(\tau)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}\left(1+i\sqrt{3}\right)} d\tau = x + c_3$$

Singular solutions are found by solving

$$-\frac{\left(\left(-3y^2 e^{(n-1)\ln(y)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}\left(1+i\sqrt{3}\right)}{2(n+1)} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = e^{\frac{\ln(c_1(n+1))}{n+1}}$$

Solving Eq. (3)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{2n+2}{\left(\left(-3\tau^2 e^{(n-1)\ln(\tau)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}\left(i\sqrt{3}-1\right)} d\tau = x + c_4$$

Singular solutions are found by solving

$$\frac{\left(\left(-3y^2 \mathrm{e}^{(n-1)\ln(y)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3} \left(i\sqrt{3}-1\right)}{2n+2} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \mathrm{e}^{\frac{\ln(c_1(n+1))}{n+1}}$$

Will add steps showing solving for IC soon.

The solution

$$y = \mathrm{e}^{\frac{\ln(c_1(n+1))}{n+1}}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\int^{y} \frac{n+1}{\left(\left(-3\tau^2 \mathrm{e}^{(n-1)\ln(\tau)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3}} d\tau = x + c_2$$

$$\int^{y} -\frac{2(n+1)}{\left(\left(-3\tau^2 \mathrm{e}^{(n-1)\ln(\tau)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3} \left(1+i\sqrt{3}\right)} d\tau = x + c_3$$

$$\int^{y} \frac{2n+2}{\left(\left(-3\tau^2 \mathrm{e}^{(n-1)\ln(\tau)} + 3c_1 n + 3c_1\right)(n+1)^2\right)^{1/3} \left(i\sqrt{3}-1\right)} d\tau = x + c_4$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_a^n/_b(_a) = 0, _b(_a), H
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[-3/(n-2)*_a, -_b*(n+1
```

Maple dsolve solution

Solving time : 0.028 (sec)
Leaf size : 169

```
dsolve(diff(y(x),x)*diff(diff(y(x),x),x)+y(x)^n = 0,y(x),singsol=all)
```

$$\frac{(-2n-2)\left(\int^{y}\frac{1}{\left(-\left(3\_a^{n+1}-c_1\right)(n+1)^2\right)^{1/3}}d\_a\right)-\left(1+i\sqrt{3}\right)(x+c_2)}{1+i\sqrt{3}}=0$$

$$-\frac{2i(n+1)\left(\int^{y}\frac{1}{\left(-\left(3\_a^{n+1}-c_1\right)(n+1)^2\right)^{1/3}}d\_a\right)+(x+c_2)\left(\sqrt{3}+i\right)}{\sqrt{3}+i}=0$$

$$\left(\int^{y}\frac{1}{\left(-\left(3\_a^{n+1}-c_1\right)(n+1)^2\right)^{1/3}}d\_a\right)n+\int^{y}\frac{1}{\left(-\left(3\_a^{n+1}-c_1\right)(n+1)^2\right)^{1/3}}d\_a-c_2-x=0$$

Mathematica DSolve solution

Solving time : 2.216 (sec)
Leaf size : 910

```
DSolve[{D[y[x],{x,2}]*D[y[x],x]+y[x]^n==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{c_1(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)c_1}\right)}{\sqrt[3]{-3\#1^{n+1}+3c_1(n+1)}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{(-1)^{2/3}\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{c_1(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)c_1}\right)}{\sqrt[3]{-3\#1^{n+1}+3c_1(n+1)}}\&\right]+c_2]$$

$$y(x) \to \text{InverseFunction}\left[-\frac{\sqrt[3]{-\frac{1}{3}}\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{c_1(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)c_1}\right)}{\sqrt[3]{-\#1^{n+1}+c_1(n+1)}}\&\right]+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{(-c_1)(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)(-c_1)}\right)}{\sqrt[3]{-3\#1^{n+1}+3(-c_1)(n+1)}}\&\right]+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{(-1)^{2/3}\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{(-c_1)(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)(-c}\right.}{\sqrt[3]{-3\#1^{n+1}+3(-c_1)(n+1)}}\right.+c_2]$$

$$y(x) \to \text{InverseFunction}\left[-\frac{\sqrt[3]{-\frac{1}{3}}\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{(-c_1)(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)(-c}\right.}{\sqrt[3]{-\#1^{n+1}+(-c_1)(n+1)}}\right.+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{c_1(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)c_1}\right)}{\sqrt[3]{-3\#1^{n+1}+3c_1(n+1)}}\&\right][x+c_2]$$

$$y(x) \to \text{InverseFunction}\left[\frac{(-1)^{2/3}\#1\sqrt[3]{n+1}\sqrt[3]{1-\frac{\#1^{n+1}}{c_1(n+1)}}\,\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{n+1},1+\frac{1}{n+1},\frac{\#1^{n+1}}{(n+1)c_1}\right)}{\sqrt[3]{-3\#1^{n+1}+3c_1(n+1)}}\&\right.$$

2.2.7 Problem 8

Internal problem ID [9130]
Book : Second order enumerated odes
Section : section 2
Problem number : 8
Date solved : `Monday, January 27, 2025 at 05:45:55 PM`
CAS classification : `[[_homogeneous, ‘class C‘], _dAlembert]`

Solve

$$y' = (x + y)^4$$

Solved as first order homogeneous class C ode

Time used: 0.544 (sec)

Let

$$z = x + y \tag{1}$$

Then

$$z'(x) = 1 + y'$$

Therefore

$$y' = z'(x) - 1$$

Hence the given ode can now be written as

$$z'(x) - 1 = z^4$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^4 + 1} dz$$

$$x + c_1 = \frac{\sqrt{2}\left(\ln\left(\frac{z^2+z\sqrt{2}+1}{z^2-z\sqrt{2}+1}\right) + 2\arctan\left(z\sqrt{2}+1\right) + 2\arctan\left(z\sqrt{2}-1\right)\right)}{8}$$

Replacing z back by its value from (1) then the above gives the solution as

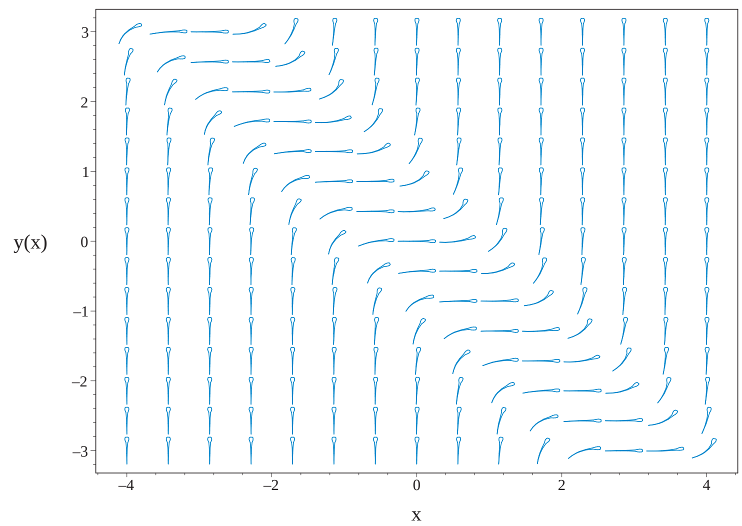


Figure 2.170: Slope field plot
$y' = (x+y)^4$

Summary of solutions found

$$\frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2+(x+y)\sqrt{2}+1}{(x+y)^2-(x+y)\sqrt{2}+1}\right)+2\arctan\left((x+y)\sqrt{2}+1\right)+2\arctan\left((x+y)\sqrt{2}-1\right)\right)}{8} = x+c_1$$

Solved using Lie symmetry for first order ode

Time used: 1.301 (sec)

Writing the ode as

$$y' = (x+y)^4$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (x+y)^4(b_3-a_2) - (x+y)^8 a_3 - 4(x+y)^3(xa_2+ya_3+a_1) - 4(x+y)^3(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-x^8a_3 - 8x^7ya_3 - 28x^6y^2a_3 - 56x^5y^3a_3 - 70x^4y^4a_3 - 56x^3y^5a_3 - 28x^2y^6a_3 - 8x\,y^7a_3 - y^8a_3 - 5x^4a_2 - 4x^4b_2 + x^4b_3 - 16x^3ya_2 - 4x^3ya_3 - 12x^3yb_2 - 18x^2y^2a_2 - 12x^2y^2a_3 - 12x^2y^2b_2 - 6x^2y^2b_3 - 8x\,y^3a_2 - 12x\,y^3a_3 - 4x\,y^3b_2 - 8x\,y^3b_3 - y^4a_2 - 4y^4a_3 - 3y^4b_3 - 4x^3a_1 - 4x^3b_1 - 12x^2ya_1 - 12x^2yb_1 - 12x\,y^2a_1 - 12x\,y^2b_1 - 4y^3a_1 - 4y^3b_1 + b_2 = 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^8a_3 - 8x^7ya_3 - 28x^6y^2a_3 - 56x^5y^3a_3 - 70x^4y^4a_3 - 56x^3y^5a_3 \\
&- 28x^2y^6a_3 - 8x\,y^7a_3 - y^8a_3 - 5x^4a_2 - 4x^4b_2 + x^4b_3 - 16x^3ya_2 - 4x^3ya_3 \\
&- 12x^3yb_2 - 18x^2y^2a_2 - 12x^2y^2a_3 - 12x^2y^2b_2 - 6x^2y^2b_3 - 8x\,y^3a_2 \\
&- 12x\,y^3a_3 - 4x\,y^3b_2 - 8x\,y^3b_3 - y^4a_2 - 4y^4a_3 - 3y^4b_3 - 4x^3a_1 - 4x^3b_1 \\
&- 12x^2ya_1 - 12x^2yb_1 - 12x\,y^2a_1 - 12x\,y^2b_1 - 4y^3a_1 - 4y^3b_1 + b_2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_3v_1^8 - 8a_3v_1^7v_2 - 28a_3v_1^6v_2^2 - 56a_3v_1^5v_2^3 - 70a_3v_1^4v_2^4 - 56a_3v_1^3v_2^5 - 28a_3v_1^2v_2^6 \\
&- 8a_3v_1v_2^7 - a_3v_2^8 - 5a_2v_1^4 - 16a_2v_1^3v_2 - 18a_2v_1^2v_2^2 - 8a_2v_1v_2^3 - a_2v_2^4 \\
&- 4a_3v_1^3v_2 - 12a_3v_1^2v_2^2 - 12a_3v_1v_2^3 - 4a_3v_2^4 - 4b_2v_1^4 - 12b_2v_1^3v_2 - 12b_2v_1^2v_2^2 \\
&- 4b_2v_1v_2^3 + b_3v_1^4 - 6b_3v_1^2v_2^2 - 8b_3v_1v_2^3 - 3b_3v_2^4 - 4a_1v_1^3 - 12a_1v_1^2v_2 \\
&- 12a_1v_1v_2^2 - 4a_1v_2^3 - 4b_1v_1^3 - 12b_1v_1^2v_2 - 12b_1v_1v_2^2 - 4b_1v_2^3 + b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-a_3v_1^8 - 8a_3v_1^7v_2 - 28a_3v_1^6v_2^2 - 56a_3v_1^5v_2^3 - 70a_3v_1^4v_2^4 + (-5a_2 - 4b_2 + b_3)\,v_1^4 \\
&- 56a_3v_1^3v_2^5 + (-16a_2 - 4a_3 - 12b_2)\,v_1^3v_2 + (-4a_1 - 4b_1)\,v_1^3 \\
&- 28a_3v_1^2v_2^6 + (-18a_2 - 12a_3 - 12b_2 - 6b_3)\,v_1^2v_2^2 + (-12a_1 - 12b_1)\,v_1^2v_2 \\
&- 8a_3v_1v_2^7 + (-8a_2 - 12a_3 - 4b_2 - 8b_3)\,v_1v_2^3 + (-12a_1 - 12b_1)\,v_1v_2^2 \\
&- a_3v_2^8 + (-a_2 - 4a_3 - 3b_3)\,v_2^4 + (-4a_1 - 4b_1)\,v_2^3 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-70a_3 &= 0 \\
-56a_3 &= 0 \\
-28a_3 &= 0 \\
-8a_3 &= 0 \\
-a_3 &= 0 \\
-12a_1 - 12b_1 &= 0 \\
-4a_1 - 4b_1 &= 0 \\
-16a_2 - 4a_3 - 12b_2 &= 0 \\
-5a_2 - 4b_2 + b_3 &= 0 \\
-a_2 - 4a_3 - 3b_3 &= 0 \\
-18a_2 - 12a_3 - 12b_2 - 6b_3 &= 0 \\
-8a_2 - 12a_3 - 4b_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{1}{-1} \\
&= -1
\end{aligned}
$$

This is easily solved to give

$$
y = -x + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = x + y
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{-1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -x
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = (x + y)^4$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 1 \\ S_x &= -1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{1 + (x + y)^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^4 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

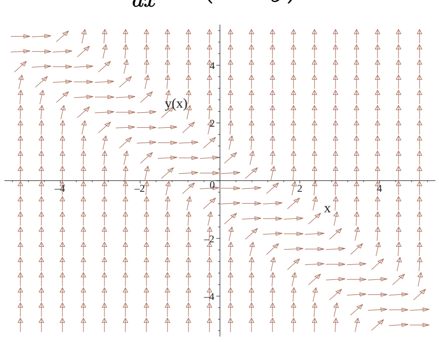
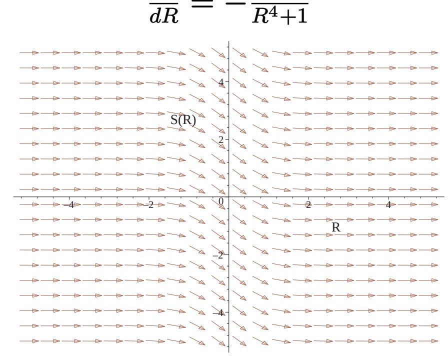
Since the ode has the form $\frac{d}{dR} S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int -\frac{1}{R^4 + 1}\, dR \\ S(R) &= -\frac{\sqrt{2}\left(\ln\left(\frac{R^2 + R\sqrt{2} + 1}{R^2 - R\sqrt{2} + 1}\right) + 2\arctan\left(R\sqrt{2} + 1\right) + 2\arctan\left(R\sqrt{2} - 1\right)\right)}{8} + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-x = -\frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2 + (x+y)\sqrt{2} + 1}{(x+y)^2 - (x+y)\sqrt{2} + 1}\right) + 2\arctan\left((x + y)\sqrt{2} + 1\right) + 2\arctan\left((x + y)\sqrt{2} - 1\right)\right)}{8} + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (x + y)^4$ 	$R = x + y$ $S = -x$	$\frac{dS}{dR} = -\frac{1}{R^4+1}$ 

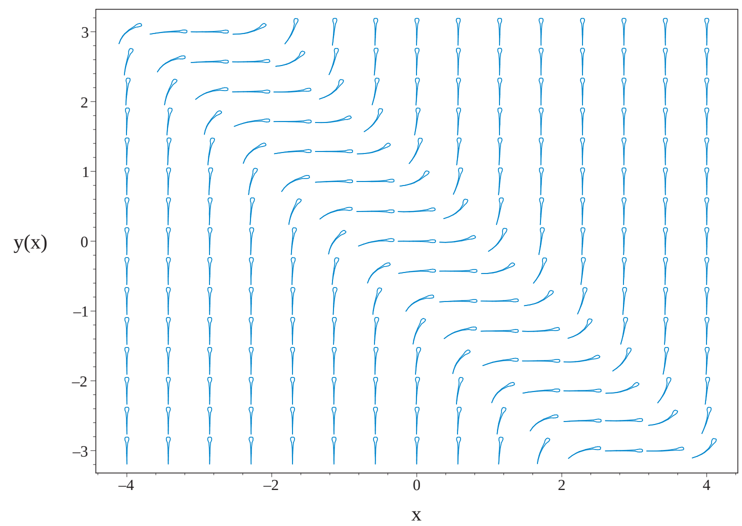


Figure 2.171: Slope field plot
$y' = (x+y)^4$

Summary of solutions found

$$-x = -\frac{\sqrt{2}\left(\ln\left(\frac{(x+y)^2+(x+y)\sqrt{2}+1}{(x+y)^2-(x+y)\sqrt{2}+1}\right)+2\arctan\left((x+y)\sqrt{2}+1\right)+2\arctan\left((x+y)\sqrt{2}-1\right)\right)}{8} + c_2$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = (x+y(x))^4$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = (x+y(x))^4$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.618` (sec)
Leaf size : 882

```
dsolve(diff(y(x),x) = (x+y(x))^4,y(x),singsol=all)
```

Expression too large to display

Mathematica DSolve solution

Solving time : `0.142` (sec)
Leaf size : 175

```
DSolve[{D[y[x],x] == (x + y[x])^4,{}},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\int_1^{y(x)}\left(\frac{1}{x^4+4K[2]x^3+6K[2]^2x^2+4K[2]^3x+K[2]^4+1}-\int_1^x -\frac{4K[1]^3+12K[2]K[1]^2+12K[2]^2K[1]+4K[2]^3}{\left(K[1]^4+4K[2]K[1]^3+6K[2]^2K[1]^2+4K[2]^3K[1]+K[2]^4+1\right)^2}dK[1]\right)dK[2]+\int_1^x\left(\frac{1}{K[1]^4+4y(x)K[1]^3+6y(x)^2K[1]^2+4y(x)^3K[1]+y(x)^4+1}-1\right)dK[1]=c_1,y(x)\right]$$

2.2.8 Problem 9

Internal problem ID [9131]
Book : Second order enumerated odes
Section : section 2
Problem number : 9
Date solved : `Tuesday, January 28, 2025 at 04:00:08 PM`
CAS classification :
`[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]`

Solve

$$y'' + (3 + x)\, y' + \left(3 + y^2\right) y'^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 32

```
dsolve(diff(diff(y(x),x),x)+(x+3)*diff(y(x),x)+(y(x)^2+3)*diff(y(x),x)^2 = 0,y(x),sing
```

$$c_1 \operatorname{erf}\left(\frac{\sqrt{2}\,(x+3)}{2}\right) - c_2 + \int^{y} \mathrm{e}^{\frac{\_a\left(\_a^2+9\right)}{3}} d\_a = 0$$

Mathematica DSolve solution

Solving time : `9.111` (sec)
Leaf size : 56

```
DSolve[{D[y[x],{x,2}]+(3+x)*D[y[x],x]+(3+y[x]^2)*(D[y[x],x])^2==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} e^{\frac{K[2]^3}{3}+3K[2]} dK[2]\&\right]\left[\int_1^x -e^{-\frac{1}{2}K[3](K[3]+6)} c_1 dK[3] + c_2\right]$$

2.2.9 Problem 10

Internal problem ID [9132]
Book : Second order enumerated odes
Section : section 2
Problem number : 10
Date solved : `Monday, January 27, 2025 at 05:48:36 PM`
CAS classification : `[_Liouville, [_2nd_order, _reducible, _mu_xy]]`

Solve

$$y'' + xy' + yy'^2 = 0$$

Solved as second nonlinear ode solved by Mainardi Lioville method

Time used: 0.211 (sec)

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$\begin{aligned} f(x) &= x \\ g(y) &= y \end{aligned}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = y$ and $f = x$, then

$$\begin{aligned}\int -g dy &= \int -y dy \\ &= -\frac{y^2}{2} \\ \int -f dx &= \int -x dx \\ &= -\frac{x^2}{2}\end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = c_2 \,\mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{-\frac{x^2}{2}}$$

Which is now solved as first order separable ode. The ode

$$y' = c_2 \,\mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{-\frac{x^2}{2}} \tag{2.14}$$

is separable as it can be written as

$$\begin{aligned}y' &= c_2 \,\mathrm{e}^{-\frac{y^2}{2}} \mathrm{e}^{-\frac{x^2}{2}} \\ &= f(x)g(y)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= \mathrm{e}^{-\frac{x^2}{2}} c_2 \\ g(y) &= \mathrm{e}^{-\frac{y^2}{2}}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int \mathrm{e}^{\frac{y^2}{2}}\, dy &= \int \mathrm{e}^{-\frac{x^2}{2}} c_2\, dx\end{aligned}$$

$$-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,y}{2}\right)}{2} = \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} + c_3$$

Will add steps showing solving for IC soon.

Solving for y from the above solution(s) gives (after possible removing of solutions that do not verify)

$$y = -i\,\mathrm{RootOf}\left(i\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) c_2 + i\sqrt{2}\,c_3 - \mathrm{erf}\left(\_Z\right)\sqrt{\pi}\right)\sqrt{2}$$

Summary of solutions found

$$y = -i\,\mathrm{RootOf}\left(i\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) c_2 + i\sqrt{2}\,c_3 - \mathrm{erf}\left(\_Z\right)\sqrt{\pi}\right)\sqrt{2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+y(x)*diff(y(x),x)^2 = 0,y(x),singsol=all)
```

$$y = -i \operatorname{RootOf}\left(i\sqrt{\pi}\, \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) c_1 + i\sqrt{2}\, c_2 - \operatorname{erf}\left(\_Z\right)\sqrt{\pi}\right)\sqrt{2}$$

Mathematica DSolve solution

Solving time : 1.341 (sec)
Leaf size : 44

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+y[x]*(D[y[x],x])^2==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to -i\sqrt{2}\operatorname{erf}^{-1}\left(i\left(\sqrt{\frac{2}{\pi}}c_2 - c_1\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)\right)$$

2.2.10 Problem 11

Internal problem ID [9133]
Book : Second order enumerated odes
Section : section 2
Problem number : 11
Date solved : `Monday, January 27, 2025 at 05:48:37 PM`
CAS classification :
`[[_2nd_order, _missing_y], _Liouville, [_2nd_order, _reducible, _mu_xy]]`

Solve

$$y'' + \sin(x)\, y' + y'^2 = 0$$

Solved as second order missing y ode

Time used: 0.751 (sec)

This is second order ode with missing dependent variable y. Let

$$u(x) = y'$$

Then

$$u'(x) = y''$$

Hence the ode becomes

$$u'(x) + \sin(x)\, u(x) + u(x)^2 = 0$$

Which is now solved for $u(x)$ as first order ode.

In canonical form, the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= -\sin(x)\, u - u^2 \end{aligned}$$

This is a Bernoulli ODE.

$$u' = (-\sin(x))\, u(x) + (-1)\, u^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$u' = f_0(x)u + f_1(x)u^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= -\sin(x) \\ f_1 &= -1 \end{aligned}$$

The first step is to divide the above equation by u^n which gives

$$\frac{u'}{u^n} = f_0(x)u^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $v = u^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $u(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\sin(x) \\
f_1(x) &= -1 \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $u^n = u^2$ gives

$$
u' \frac{1}{u^2} = -\frac{\sin(x)}{u} - 1 \tag{4}
$$

Let

$$
\begin{aligned}
v &= u^{1-n} \\
&= \frac{1}{u}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
v' = -\frac{1}{u^2} u' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-v'(x) &= -\sin(x)\, v(x) - 1 \\
v' &= \sin(x)\, v + 1
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $v(x)$ which is now solved.

In canonical form a linear first order is

$$
v'(x) + q(x)v(x) = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\sin(x) \\
p(x) &= 1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int -\sin(x) dx} \\
&= \mathrm{e}^{\cos(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= \mu \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(v\, \mathrm{e}^{\cos(x)}\right) &= \mathrm{e}^{\cos(x)} \\
\mathrm{d}\left(v\, \mathrm{e}^{\cos(x)}\right) &= \mathrm{e}^{\cos(x)} \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
v\, \mathrm{e}^{\cos(x)} &= \int \mathrm{e}^{\cos(x)}\, dx \\
&= \int \mathrm{e}^{\cos(x)} dx + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor $e^{\cos(x)}$ gives the final solution

$$v(x) = e^{-\cos(x)}\left(\int e^{\cos(x)}dx + c_1\right)$$

The substitution $v = u^{1-n}$ is now used to convert the above solution back to $u(x)$ which results in

$$\frac{1}{u(x)} = e^{-\cos(x)}\left(\int e^{\cos(x)}dx + c_1\right)$$

Solving for $u(x)$ gives

$$u(x) = \frac{e^{\cos(x)}}{\int e^{\cos(x)}dx + c_1}$$

In summary, these are the solution found for $u(x)$

$$u(x) = \frac{e^{\cos(x)}}{\int e^{\cos(x)}dx + c_1}$$

For solution $u(x) = \frac{e^{\cos(x)}}{\int e^{\cos(x)}dx+c_1}$, since $u = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{e^{\cos(x)}}{\int e^{\cos(x)}dx + c_1}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{e^{\cos(x)}}{\int e^{\cos(x)}dx + c_1}dx$$
$$y = \ln\left(\int e^{\cos(x)}dx + c_1\right) + c_2$$

$$y = \int \frac{e^{\cos(x)}}{\int e^{\cos(x)}dx + c_1}dx + c_2$$

In summary, these are the solution found for (y)

$$y = \int \frac{e^{\cos(x)}}{\int e^{\cos(x)}dx + c_1}dx + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \int \frac{e^{\cos(x)}}{\int e^{\cos(x)}dx + c_1}dx + c_2$$

Solved as second nonlinear ode solved by Mainardi Lioville method

Time used: 0.273 (sec)

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = \sin(x)$$
$$g(y) = 1$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx} \ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = 1$ and $f = \sin(x)$, then

$$\begin{aligned} \int -g dy &= \int (-1)\, dy \\ &= -y \\ \int -f dx &= \int -\sin(x)\, dx \\ &= \cos(x) \end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = c_2\, e^{-y} e^{\cos(x)}$$

Which is now solved as first order separable ode. The ode

$$y' = c_2\, e^{-y} e^{\cos(x)} \tag{2.15}$$

is separable as it can be written as

$$\begin{aligned} y' &= c_2\, e^{-y} e^{\cos(x)} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= e^{\cos(x)} c_2 \\ g(y) &= e^{-y} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int e^{y}\, dy &= \int e^{\cos(x)} c_2\, dx \end{aligned}$$

$$e^{y} = \int e^{\cos(x)} c_2 dx + 2c_3$$

Solving for y gives

$$y = \ln\left(c_2\left(\int e^{\cos(x)} dx\right) + 2c_3\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \ln\left(c_2\left(\int e^{\cos(x)} dx\right) + 2c_3\right)$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)+sin(x)*diff(y(x),x)+diff(y(x),x)^2 = 0,y(x),singsol=all)
```

$$y = \ln\left(c_1\left(\int e^{\cos(x)} dx\right) + c_2\right)$$

Mathematica DSolve solution

Solving time : 3.314 (sec)
Leaf size : 63

```
DSolve[{D[y[x],{x,2}]+Sin[x]*D[y[x],x]+(D[y[x],x])^2==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \int_1^x \frac{\exp\left(\int_1^{K[3]} -\sin(K[1])dK[1]\right)}{c_1 - \int_1^{K[3]} -\exp\left(\int_1^{K[2]} -\sin(K[1])dK[1]\right) dK[2]} dK[3] + c_2$$

2.2.11 Problem 12

Internal problem ID [9134]
Book : Second order enumerated odes
Section : section 2
Problem number : 12
Date solved : `Monday, January 27, 2025 at 05:48:38 PM`
CAS classification :
`[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]`

Solve

$$3y'' + \cos(x)\, y' + \sin(y)\, y'^2 = 0$$

Solved as second nonlinear ode solved by Mainardi Lioville method

Time used: 0.369 (sec)

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = \frac{\cos(x)}{3}$$
$$g(y) = \frac{\sin(y)}{3}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy}\, e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = \frac{\sin(y)}{3}$ and $f = \frac{\cos(x)}{3}$, then

$$\begin{aligned}
\int -g dy &= \int -\frac{\sin(y)}{3} dy \\
&= \frac{\cos(y)}{3} \\
\int -f dx &= \int -\frac{\cos(x)}{3} dx \\
&= -\frac{\sin(x)}{3}
\end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = c_2 \, \mathrm{e}^{\frac{\cos(y)}{3}} \mathrm{e}^{-\frac{\sin(x)}{3}}$$

Which is now solved as first order separable ode. The ode

$$y' = c_2 \, \mathrm{e}^{\frac{\cos(y)}{3}} \mathrm{e}^{-\frac{\sin(x)}{3}} \tag{2.16}$$

is separable as it can be written as

$$\begin{aligned}
y' &= c_2 \, \mathrm{e}^{\frac{\cos(y)}{3}} \mathrm{e}^{-\frac{\sin(x)}{3}} \\
&= f(x) g(y)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= \mathrm{e}^{-\frac{\sin(x)}{3}} c_2 \\
g(y) &= \mathrm{e}^{\frac{\cos(y)}{3}}
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(y)} \, dy &= \int f(x) \, dx \\
\int \mathrm{e}^{-\frac{\cos(y)}{3}} \, dy &= \int \mathrm{e}^{-\frac{\sin(x)}{3}} c_2 \, dx
\end{aligned}$$

$$\int^{y} \mathrm{e}^{-\frac{\cos(\tau)}{3}} d\tau = \int \mathrm{e}^{-\frac{\sin(x)}{3}} c_2 dx + 2c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\int^{y} \mathrm{e}^{-\frac{\cos(\tau)}{3}} d\tau = \int \mathrm{e}^{-\frac{\sin(x)}{3}} c_2 dx + 2c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 27

```
dsolve(3*diff(diff(y(x),x),x)+cos(x)*diff(y(x),x)+sin(y(x))*diff(y(x),x)^2 = 0,y(x),sing
```

$$\int^{y} e^{-\frac{\cos(\_a)}{3}} d\_a - c_1 \left(\int e^{-\frac{\sin(x)}{3}} dx\right) - c_2 = 0$$

Mathematica DSolve solution

Solving time : 1.388 (sec)
Leaf size : 67

```
DSolve[{3*D[y[x],{x,2}]+Cos[x]*D[y[x],x]+Sin[y[x]]*(D[y[x],x])^2==0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \exp\left(-\int_1^{K[3]} -\frac{1}{3}\sin(K[1])dK[1]\right) dK[3]\&\right]\left[\int_1^x -\exp\left(-\int_1^{K[4]} \frac{1}{3}\cos(K[2])dK[2]\right) c_1 dK[4] + c_2\right]$$

2.2.12 Problem 13

Internal problem ID [9135]
Book : Second order enumerated odes
Section : section 2
Problem number : 13
Date solved : `Monday, January 27, 2025 at 05:48:39 PM`
CAS classification :
`[_Liouville, [_2nd_order, _with_linear_symmetries], [_2nd_order, _reducible, _mu_x_y1]`

Solve

$$10y'' + x^2y' + \frac{3{y'}^2}{y} = 0$$

Solved as second nonlinear ode solved by Mainardi Lioville method

Time used: 0.222 (sec)

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y){y'}^2 = 0 \tag{1A}$$

Where in this problem

$$\begin{aligned} f(x) &= \frac{x^2}{10} \\ g(y) &= \frac{3}{10y} \end{aligned}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = \frac{3}{10y}$ and $f = \frac{x^2}{10}$, then

$$\begin{aligned}
\int -g dy &= \int -\frac{3}{10y} dy \\
&= -\frac{3\ln(y)}{10} \\
\int -f dx &= \int -\frac{x^2}{10} dx \\
&= -\frac{x^3}{30}
\end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = \frac{c_2\, e^{-\frac{x^3}{30}}}{y^{3/10}}$$

Which is now solved as first order separable ode. The ode

$$y' = \frac{c_2\, e^{-\frac{x^3}{30}}}{y^{3/10}} \tag{2.17}$$

is separable as it can be written as

$$\begin{aligned}
y' &= \frac{c_2\, e^{-\frac{x^3}{30}}}{y^{3/10}} \\
&= f(x)g(y)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= c_2\, e^{-\frac{x^3}{30}} \\
g(y) &= \frac{1}{y^{3/10}}
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\
\int y^{3/10}\, dy &= \int c_2\, e^{-\frac{x^3}{30}}\, dx
\end{aligned}$$

$$\frac{10y^{13/10}}{13} = \int c_2\, e^{-\frac{x^3}{30}} dx + 2c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\frac{10y^{13/10}}{13} = \int c_2\, e^{-\frac{x^3}{30}} dx + 2c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.014 (sec)
Leaf size : 55

```
dsolve(10*diff(diff(y(x),x),x)+diff(y(x),x)*x^2+3*diff(y(x),x)^2/y(x) = 0,y(x),singsol
```

$$-\frac{3\left(x c_1 \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{x^3}{30}\right) \mathrm{e}^{-\frac{x^3}{60}} 30^{1/6}+\frac{4\left(x^3\right)^{1/6}\left(c_1 x\,\mathrm{e}^{-\frac{x^3}{30}}+c_2-\frac{10 y^{13/10}}{13}\right)}{3}\right)}{4\left(x^3\right)^{1/6}}=0$$

Mathematica DSolve solution

Solving time : 66.264 (sec)
Leaf size : 73

```
DSolve[{10*D[y[x],{x,2}]+x^2*D[y[x],x]+3/y[x]*(D[y[x],x])^2==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to c_2 \exp\left(\int_1^x \frac{30 e^{-\frac{1}{30}K[1]^3} \sqrt[3]{K[1]^3}}{30 c_1 \sqrt[3]{K[1]^3}-13\sqrt[3]{30}\Gamma\left(\frac{1}{3}, \frac{K[1]^3}{30}\right) K[1]} dK[1]\right)$$

2.2.13 Problem 14

Internal problem ID [9136]
Book : Second order enumerated odes
Section : section 2
Problem number : 14
Date solved : `Tuesday, January 28, 2025 at 04:00:09 PM`
CAS classification :
`[_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]`

Solve

$$10y'' + (e^x + 3x)\, y' + \frac{3\, e^y {y'}^2}{\sin(y)} = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 38

```
dsolve(10*diff(diff(y(x),x),x)+(exp(x)+3*x)*diff(y(x),x)+3/sin(y(x))*exp(y(x))*diff(y(x)
```

$$\int^{y} e^{\frac{3\left(\int \csc(\_b) e^{\_b} d\_b\right)}{10}} d\_b - c_1 \left(\int e^{-\frac{3x^2}{20} - \frac{e^x}{10}} dx\right) - c_2 = 0$$

Mathematica DSolve solution

Solving time : `33.212` (sec)
Leaf size : 71

```
DSolve[{10*D[y[x],{x,2}]+(Exp[x]+3*x)*D[y[x],x]+3/Sin[y[x]]*Exp[y[x]]*(D[y[x],x])^2==0,{}},y[x]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \exp\left(-\int_1^{K[2]} -\frac{3}{10} e^{K[1]} \csc(K[1]) dK[1]\right) dK[2] \&\right] \left[\int_1^x -e^{\frac{1}{20}\left(-3K[3]^2 - 2e^{K[3]}\right)} c_1 dK[3] + c_2\right]$$

2.2.14 Problem 15

Internal problem ID [9137]
Book : Second order enumerated odes
Section : section 2
Problem number : 15
Date solved : `Monday, January 27, 2025 at 05:48:41 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - \frac{2y}{x^2} = x\,\mathrm{e}^{-\sqrt{x}}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.193 (sec)

Writing the ode as

$$y'' - \frac{2y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.70: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}}\,dx \\ &= \frac{1}{x}\left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - \frac{2y}{x^2} = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2\, x^2}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{x^2}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^2}{3} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x^2}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^2}{3} \\ -\frac{1}{x^2} & \frac{2x}{3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{2x}{3}\right) - \left(\frac{x^2}{3}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3 \mathrm{e}^{-\sqrt{x}}}{3}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{x^3 \mathrm{e}^{-\sqrt{x}}}{3} dx$$

Hence

$$u_1 = \frac{2x^{7/2}\mathrm{e}^{-\sqrt{x}}}{3} + \frac{14x^3\mathrm{e}^{-\sqrt{x}}}{3} + 28x^{5/2}\mathrm{e}^{-\sqrt{x}} + 140x^2\mathrm{e}^{-\sqrt{x}} + 560x^{3/2}\mathrm{e}^{-\sqrt{x}} + 1680x\,\mathrm{e}^{-\sqrt{x}} + 3360\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} + 3360\,\mathrm{e}^{-\sqrt{x}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-\sqrt{x}}}{1} dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{-\sqrt{x}} dx$$

Hence

$$u_2 = -2\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} - 2\,\mathrm{e}^{-\sqrt{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{2x^{7/2}\mathrm{e}^{-\sqrt{x}}}{3} + \frac{14x^3\mathrm{e}^{-\sqrt{x}}}{3} + 28x^{5/2}\mathrm{e}^{-\sqrt{x}} + 140x^2\mathrm{e}^{-\sqrt{x}} + 560x^{3/2}\mathrm{e}^{-\sqrt{x}} + 1680x\,\mathrm{e}^{-\sqrt{x}} + 3360\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} + 3360\,\mathrm{e}^{-\sqrt{x}}}{x} + \frac{x^2\left(-2\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} - 2\,\mathrm{e}^{-\sqrt{x}}\right)}{3}$$

Which simplifies to

$$y_p(x) = \frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{5/2} + 140x^{3/2} + x^3 + 35x^2 + 840\sqrt{x} + 420x + 840\right)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2\,x^2}{3}\right) + \left(\frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{5/2} + 140x^{3/2} + x^3 + 35x^2 + 840\sqrt{x} + 420x + 840\right)}{x}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1}{x} + \frac{c_2\,x^2}{3} + \frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{5/2} + 140x^{3/2} + x^3 + 35x^2 + 840\sqrt{x} + 420x + 840\right)}{x}$$

Solved as second order ode adjoint method

Time used: 0.898 (sec)

In normal form the ode

$$y'' - \frac{2y}{x^2} = x\,\mathrm{e}^{-\sqrt{x}} \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{2}{x^2} \\ r(x) &= x\,\mathrm{e}^{-\sqrt{x}} \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (0)' + \left(-\frac{2\xi(x)}{x^2}\right) &= 0 \\ \xi''(x) - \frac{2\xi(x)}{x^2} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' - \frac{2\xi}{x^2} = 0 \tag{1}$$

$$A\xi'' + B\xi' + C\xi = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \xi e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.71: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\,dx}\\ &= e^{\int -\frac{1}{x}dx}\\ &= \frac{1}{x}\end{aligned}$$

The first solution to the original ode in ξ is found from

$$\xi_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}\xi_1 &= z_1\\ &= \frac{1}{x}\end{aligned}$$

Which simplifies to

$$\xi_1 = \frac{1}{x}$$

The second solution ξ_2 to the original ode is found using reduction of order

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{\xi_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}\xi_2 &= \xi_1 \int \frac{1}{\xi_1^2}\,dx\\ &= \frac{1}{x}\int \frac{1}{\frac{1}{x^2}}\,dx\\ &= \frac{1}{x}\left(\frac{x^3}{3}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}\xi &= c_1\xi_1 + c_2\xi_2\\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y = \int \xi(x)\, r(x)\, dx$$

$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi\left(x\right)}\right) = \frac{\int \xi(x)\, r(x)\, dx}{\xi\left(x\right)}$$

Or

$$y' - \frac{y\left(-\frac{c_1}{x^2} + \frac{2c_2 x}{3}\right)}{\frac{c_1}{x} + \frac{c_2\, x^2}{3}} = \frac{\frac{2c_2\left(-x^{7/2}\mathrm{e}^{-\sqrt{x}} - 7x^3\mathrm{e}^{-\sqrt{x}} - 42x^{5/2}\mathrm{e}^{-\sqrt{x}} - 210x^2\mathrm{e}^{-\sqrt{x}} - 840x^{3/2}\mathrm{e}^{-\sqrt{x}} - 2520x\,\mathrm{e}^{-\sqrt{x}} - 5040\sqrt{x}\,\mathrm{e}^{-\sqrt{x}} - 5040\,\mathrm{e}^{-\sqrt{x}}\right)}{3}}{\frac{c_1}{x} + \frac{c_2\, x^2}{3}}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{2c_2\, x^3 - 3c_1}{x\,(c_2\, x^3 + 3c_1)}$$

$$p(x) = -\frac{14x\,\mathrm{e}^{-\sqrt{x}}\left(120c_2\, x^{3/2} + 6x^{5/2}c_2 + \frac{x^{7/2}c_2}{7} + \left(\frac{3c_1}{7} + 720c_2\right)\sqrt{x} + (x^3 + 30x^2 + 360x + 720)\, c_2 + \frac{3c_1}{7}\right)}{c_2\, x^3 + 3c_1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx}\\ &= \mathrm{e}^{\int -\frac{2c_2\, x^3 - 3c_1}{x\left(c_2\, x^3 + 3c_1\right)}dx}\\ &= \frac{x}{c_2\, x^3 + 3c_1}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(-\frac{14x\,\mathrm{e}^{-\sqrt{x}}\left(120c_2\, x^{3/2} + 6x^{5/2}c_2 + \frac{x^{7/2}c_2}{7} + \left(\frac{3c_1}{7} + 720c_2\right)\sqrt{x} + (x^3 + 30x^2 + 360x + 720)\, c_2 + \frac{3c_1}{7}\right)}{c_2\, x^3 + 3c_1}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yx}{c_2\, x^3 + 3c_1}\right) = \left(\frac{x}{c_2\, x^3 + 3c_1}\right)\left(-\frac{14x\,\mathrm{e}^{-\sqrt{x}}\left(120c_2\, x^{3/2} + 6x^{5/2}c_2 + \frac{x^{7/2}c_2}{7} + \left(\frac{3c_1}{7} + 720c_2\right)\sqrt{x} + (x^3 + 30x^2 + 360x + 720)\, c_2 + \frac{3c_1}{7}\right)}{c_2\, x^3 + 3c_1}\right)$$

$$\mathrm{d}\left(\frac{yx}{c_2\, x^3 + 3c_1}\right) = \left(-\frac{14x^2\mathrm{e}^{-\sqrt{x}}\left(120c_2\, x^{3/2} + 6x^{5/2}c_2 + \frac{x^{7/2}c_2}{7} + \left(\frac{3c_1}{7} + 720c_2\right)\sqrt{x} + (x^3 + 30x^2 + 360x + 720)\, c_2 + \frac{3c_1}{7}\right)}{\left(c_2\, x^3 + 3c_1\right)^2}\right)$$

Integrating gives

$$\begin{aligned}\frac{yx}{c_2\, x^3 + 3c_1} &= \int -\frac{14x^2\mathrm{e}^{-\sqrt{x}}\left(120c_2\, x^{3/2} + 6x^{5/2}c_2 + \frac{x^{7/2}c_2}{7} + \left(\frac{3c_1}{7} + 720c_2\right)\sqrt{x} + (x^3 + 30x^2 + 360x + 720\right.}{\left(c_2\, x^3 + 3c_1\right)^2}\\ &= \int -\frac{14x^2\mathrm{e}^{-\sqrt{x}}\left(120c_2\, x^{3/2} + 6x^{5/2}c_2 + \frac{x^{7/2}c_2}{7} + \left(\frac{3c_1}{7} + 720c_2\right)\sqrt{x} + (x^3 + 30x^2 + 360x + 720\right.}{\left(c_2\, x^3 + 3c_1\right)^2}\end{aligned}$$

Dividing throughout by the integrating factor $\frac{x}{c_2\,x^3+3c_1}$ gives the final solution

$$y = \frac{(c_2\,x^3+3c_1)\left(\int -\frac{14x^2 \mathrm{e}^{-\sqrt{x}}\left(120c_2\,x^{3/2}+6x^{5/2}c_2+\frac{x^{7/2}c_2}{7}+\left(\frac{3c_1}{7}+720c_2\right)\sqrt{x}+(x^3+30x^2+360x+720)\,c_2+\frac{3c_1}{7}\right)}{(c_2\,x^3+3c_1)^2}dx+c_3\right)}{x}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(c_2\,x^3+3c_1)\left(\int -\frac{14x^2 \mathrm{e}^{-\sqrt{x}}\left(120c_2\,x^{3/2}+6x^{5/2}c_2+\frac{x^{7/2}c_2}{7}+\left(\frac{3c_1}{7}+720c_2\right)\sqrt{x}+(x^3+30x^2+360x+720)\,c_2+\frac{3c_1}{7}\right)}{(c_2\,x^3+3c_1)^2}dx+c_3\right)}{x}$$

The constants can be merged to give

$$y = \frac{(c_2\,x^3+3c_1)\left(\int -\frac{14x^2 \mathrm{e}^{-\sqrt{x}}\left(120c_2\,x^{3/2}+6x^{5/2}c_2+\frac{x^{7/2}c_2}{7}+\left(\frac{3c_1}{7}+720c_2\right)\sqrt{x}+(x^3+30x^2+360x+720)\,c_2+\frac{3c_1}{7}\right)}{(c_2\,x^3+3c_1)^2}dx+1\right)}{x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(c_2\,x^3+3c_1)\left(\int -\frac{14x^2 \mathrm{e}^{-\sqrt{x}}\left(120c_2\,x^{3/2}+6x^{5/2}c_2+\frac{x^{7/2}c_2}{7}+\left(\frac{3c_1}{7}+720c_2\right)\sqrt{x}+(x^3+30x^2+360x+720)\,c_2+\frac{3c_1}{7}\right)}{(c_2\,x^3+3c_1)^2}dx+1\right)}{x}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-2/x^2*y(x) = x*exp(-x^(1/2)),y(x),singsol=all)
```

$$y = \frac{4\,\mathrm{e}^{-\sqrt{x}}\left(7x^{5/2}+140x^{3/2}+x^3+35x^2+840\sqrt{x}+420x+840\right)+c_1x^3+c_2}{x}$$

Mathematica DSolve solution

Solving time : `0.046` (sec)
Leaf size : 57

```
DSolve[{D[y[x],{x,2}]-2/x^2*y[x] == x*Exp[-x^(1/2)],{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to x^2 \int_1^x \frac{1}{3} e^{-\sqrt{K[1]}} dK[1] + c_2 x^2 + \frac{2\Gamma\left(8, \sqrt{x}\right)}{3x} + \frac{c_1}{x}$$

2.2.15 Problem 16

Internal problem ID [9138]
Book : Second order enumerated odes
Section : section 2
Problem number : 16
Date solved : `Monday, January 27, 2025 at 05:48:43 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - \frac{y'}{\sqrt{x}} + \frac{\left(x+\sqrt{x}-8\right) y}{4x^2} = x$$

Solved as second order ode using Kovacic algorithm

Time used: 0.290 (sec)

Writing the ode as

$$y'' - \frac{y'}{\sqrt{x}} + \left(\frac{1}{4x} + \frac{1}{4x^{3/2}} - \frac{2}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{1}{\sqrt{x}} \\ C &= \frac{1}{4x} + \frac{1}{4x^{3/2}} - \frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.72: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= e^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{1}{\sqrt{x}}}{1}\, dx} \\
&= z_1 e^{\sqrt{x}} \\
&= z_1\left(e^{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\sqrt{x}}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{1}{\sqrt{x}}}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\sqrt{x}}}{(y_1)^2}\, dx \\
&= y_1\left(\frac{x^3 e^{2\sqrt{x}} e^{-2\sqrt{x}}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{\sqrt{x}}}{x}\right) + c_2\left(\frac{e^{\sqrt{x}}}{x}\left(\frac{x^3 e^{2\sqrt{x}} e^{-2\sqrt{x}}}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-\frac{y'}{\sqrt{x}}+\left(\frac{1}{4x}+\frac{1}{4x^{3/2}}-\frac{2}{x^2}\right)y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\frac{c_1\,\mathrm{e}^{\sqrt{x}}}{x}+\frac{c_2\,x^2\mathrm{e}^{\sqrt{x}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\frac{\mathrm{e}^{\sqrt{x}}}{x}$$

$$y_2=\frac{x^2\mathrm{e}^{\sqrt{x}}}{3}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}\frac{\mathrm{e}^{\sqrt{x}}}{x} & \frac{x^2\mathrm{e}^{\sqrt{x}}}{3}\\ \frac{d}{dx}\left(\frac{\mathrm{e}^{\sqrt{x}}}{x}\right) & \frac{d}{dx}\left(\frac{x^2\mathrm{e}^{\sqrt{x}}}{3}\right)\end{vmatrix}$$

Which gives

$$W=\begin{vmatrix}\frac{\mathrm{e}^{\sqrt{x}}}{x} & \frac{x^2\mathrm{e}^{\sqrt{x}}}{3}\\ \frac{\mathrm{e}^{\sqrt{x}}}{2x^{3/2}}-\frac{\mathrm{e}^{\sqrt{x}}}{x^2} & \frac{2x\,\mathrm{e}^{\sqrt{x}}}{3}+\frac{x^{3/2}\mathrm{e}^{\sqrt{x}}}{6}\end{vmatrix}$$

Therefore

$$W=\left(\frac{\mathrm{e}^{\sqrt{x}}}{x}\right)\left(\frac{2x\,\mathrm{e}^{\sqrt{x}}}{3}+\frac{x^{3/2}\mathrm{e}^{\sqrt{x}}}{6}\right)-\left(\frac{x^2\mathrm{e}^{\sqrt{x}}}{3}\right)\left(\frac{\mathrm{e}^{\sqrt{x}}}{2x^{3/2}}-\frac{\mathrm{e}^{\sqrt{x}}}{x^2}\right)$$

Which simplifies to

$$W=\mathrm{e}^{2\sqrt{x}}$$

Which simplifies to

$$W=\mathrm{e}^{2\sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3 e^{\sqrt{x}}}{3}}{e^{2\sqrt{x}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^3 e^{-\sqrt{x}}}{3} dx$$

Hence

$$u_1 = \frac{2x^{7/2} e^{-\sqrt{x}}}{3} + \frac{14x^3 e^{-\sqrt{x}}}{3} + 28x^{5/2} e^{-\sqrt{x}} + 140x^2 e^{-\sqrt{x}} + 560x^{3/2} e^{-\sqrt{x}} + 1680x\, e^{-\sqrt{x}} + 3360\, e^{-\sqrt{x}}\sqrt{x} + 3360\, e^{-\sqrt{x}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{x}}}{e^{2\sqrt{x}}}\, dx$$

Which simplifies to

$$u_2 = \int e^{-\sqrt{x}} dx$$

Hence

$$u_2 = -2\, e^{-\sqrt{x}}\sqrt{x} - 2\, e^{-\sqrt{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\frac{2x^{7/2} e^{-\sqrt{x}}}{3} + \frac{14x^3 e^{-\sqrt{x}}}{3} + 28x^{5/2} e^{-\sqrt{x}} + 140x^2 e^{-\sqrt{x}} + 560x^{3/2} e^{-\sqrt{x}} + 1680x\, e^{-\sqrt{x}} + 3360\, e^{-\sqrt{x}}\sqrt{x} + 3360\, e\right.}{x} + \frac{x^2 e^{\sqrt{x}} \left(-2\, e^{-\sqrt{x}}\sqrt{x} - 2\, e^{-\sqrt{x}}\right)}{3}$$

Which simplifies to

$$y_p(x) = \frac{28x^{5/2} + 560x^{3/2} + 4x^3 + 140x^2 + 3360\sqrt{x} + 1680x + 3360}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\, e^{\sqrt{x}}}{x} + \frac{c_2\, x^2 e^{\sqrt{x}}}{3}\right) + \left(\frac{28x^{5/2} + 560x^{3/2} + 4x^3 + 140x^2 + 3360\sqrt{x} + 1680x + 3360}{x}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1\, e^{\sqrt{x}}}{x} + \frac{c_2\, x^2 e^{\sqrt{x}}}{3} + \frac{28x^{5/2} + 560x^{3/2} + 4x^3 + 140x^2 + 3360\sqrt{x} + 1680x + 3360}{x}$$

Solved as second order ode adjoint method

Time used: 1.281 (sec)

In normal form the ode

$$y'' - \frac{y'}{\sqrt{x}} + \frac{\left(x+\sqrt{x}-8\right) y}{4x^2} = x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{\sqrt{x}} \\ q(x) &= \frac{1}{4x} + \frac{1}{4x^{3/2}} - \frac{2}{x^2} \\ r(x) &= x \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(-\frac{\xi(x)}{\sqrt{x}}\right)' + \left(\left(\frac{1}{4x} + \frac{1}{4x^{3/2}} - \frac{2}{x^2}\right)\xi(x)\right) &= 0 \\ \xi''(x) + \frac{\xi'(x)}{\sqrt{x}} - \frac{\left(\sqrt{x}-x+8\right)\xi(x)}{4x^2} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' + \frac{\xi'}{\sqrt{x}} + \left(-\frac{1}{4x^{3/2}} + \frac{1}{4x} - \frac{2}{x^2}\right)\xi = 0 \tag{1}$$

$$A\xi'' + B\xi' + C\xi = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{\sqrt{x}} \\ C &= -\frac{1}{4x^{3/2}} + \frac{1}{4x} - \frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \xi e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.73: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in ξ is found from

$$\begin{aligned} \xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{\frac{1}{\sqrt{x}}}{1} \, dx} \\ &= z_1 e^{-\sqrt{x}} \\ &= z_1 \left(e^{-\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$\xi_1 = \frac{e^{-\sqrt{x}}}{x}$$

The second solution ξ_2 to the original ode is found using reduction of order

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{\xi_1^2} \, dx$$

Substituting gives

$$\begin{aligned} \xi_2 &= \xi_1 \int \frac{e^{\int -\frac{\frac{1}{\sqrt{x}}}{1} \, dx}}{(\xi_1)^2} \, dx \\ &= \xi_1 \int \frac{e^{-2\sqrt{x}}}{(\xi_1)^2} \, dx \\ &= \xi_1 \left(\frac{x^3 e^{2\sqrt{x}} e^{-2\sqrt{x}}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} \xi &= c_1 \xi_1 + c_2 \xi_2 \\ &= c_1 \left(\frac{e^{-\sqrt{x}}}{x}\right) + c_2 \left(\frac{e^{-\sqrt{x}}}{x} \left(\frac{x^3 e^{2\sqrt{x}} e^{-2\sqrt{x}}}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(-\frac{1}{\sqrt{x}} - \frac{-\frac{c_1\,\mathrm{e}^{-\sqrt{x}}}{2x^{3/2}} - \frac{c_1\,\mathrm{e}^{-\sqrt{x}}}{x^2} + \frac{2c_2 x\,\mathrm{e}^{-\sqrt{x}}}{3} - \frac{c_2\,x^{3/2}\mathrm{e}^{-\sqrt{x}}}{6}}{\frac{c_1\,\mathrm{e}^{-\sqrt{x}}}{x} + \frac{c_2\,x^2\mathrm{e}^{-\sqrt{x}}}{3}}\right) = \frac{2c_2\left(-x^{7/2}\mathrm{e}^{-\sqrt{x}} - 7x^3\mathrm{e}^{-\sqrt{x}} - 42x^{5/2}\mathrm{e}^{-\sqrt{x}} - 210x^2\mathrm{e}^{-\sqrt{x}} - 840x\right.}{3}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{4x^{7/2}c_2 - 6c_1\sqrt{x} + x(c_2\,x^3 + 3c_1)}{x^{3/2}\,(2c_2\,x^3 + 6c_1)}$$
$$p(x) = -\frac{2\sqrt{x}\,(2520c_2\,x^{3/2} + 210c_2\,x^{5/2} + 7x^{7/2}c_2 + (3c_1 + 5040c_2)\,\sqrt{x} + x((x^3 + 42x^2 + 840x + 5040)\,c_2 + 3}{c_2\,x^3 + 3c_1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\frac{4x^{7/2}c_2 - 6c_1\sqrt{x} + x\left(c_2\,x^3 + 3c_1\right)}{x^{3/2}\left(2c_2\,x^3 + 6c_1\right)}dx} \\ &= \frac{x\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3 + 3c_1}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(-\frac{2\sqrt{x}\,(2520c_2\,x^{3/2} + 210c_2\,x^{5/2} + 7x^{7/2}c_2 + (3c_1 + 5040c_2)\,\sqrt{x} + x((x^3 + 42x^2 + 840x + 5040)\,c_2 + 3}{c_2\,x^3 + 3c_1}\right.$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yx\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3 + 3c_1}\right) = \left(\frac{x\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3 + 3c_1}\right)\left(-\frac{2\sqrt{x}\,(2520c_2\,x^{3/2} + 210c_2\,x^{5/2} + 7x^{7/2}c_2 + (3c_1 + 5040c_2)\,\sqrt{x} + x((x^3 + 42x^2 + 840x +}{c_2\,x^3 + 3c_1}\right.$$
$$\mathrm{d}\left(\frac{yx\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3 + 3c_1}\right) = \left(-\frac{2x^{3/2}(2520c_2\,x^{3/2} + 210c_2\,x^{5/2} + 7x^{7/2}c_2 + (3c_1 + 5040c_2)\,\sqrt{x} + x((x^3 + 42x^2 + 840x + 5040)\,c_2 + 3c_1}{(c_2\,x^3 + 3c_1)^2}\right.$$

Integrating gives

$$\begin{aligned}\frac{yx\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3 + 3c_1} &= \int -\frac{2x^{3/2}(2520c_2\,x^{3/2} + 210c_2\,x^{5/2} + 7x^{7/2}c_2 + (3c_1 + 5040c_2)\,\sqrt{x} + x((x^3 + 42x^2 + 840x + 5}{(c_2\,x^3 + 3c_1)^2} \\ &= \int -\frac{2x^{3/2}(2520c_2\,x^{3/2} + 210c_2\,x^{5/2} + 7x^{7/2}c_2 + (3c_1 + 5040c_2)\,\sqrt{x} + x((x^3 + 42x^2 + 840x + 5}{(c_2\,x^3 + 3c_1)^2}\end{aligned}$$

Dividing throughout by the integrating factor $\frac{x\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3+3c_1}$ gives the final solution

$$y = \frac{(c_2\,x^3+3c_1)\,\mathrm{e}^{\sqrt{x}}\left(\int -\frac{2x^{3/2}(2520c_2\,x^{3/2}+210c_2\,x^{5/2}+7x^{7/2}c_2+(3c_1+5040c_2)\sqrt{x}+x((x^3+42x^2+840x+5040)c_2+3c_1))\mathrm{e}^{-\sqrt{x}}}{(c_2\,x^3+3c_1)^2}dx -}{x}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(c_2\,x^3+3c_1)\,\mathrm{e}^{\sqrt{x}}\left(\int -\frac{2x^{3/2}(2520c_2\,x^{3/2}+210c_2\,x^{5/2}+7x^{7/2}c_2+(3c_1+5040c_2)\sqrt{x}+x((x^3+42x^2+840x+5040)c_2+3c_1))\mathrm{e}^{-\sqrt{x}}}{(c_2\,x^3+3c_1)^2}dx + c}{x}$$

The constants can be merged to give

$$y = \frac{(c_2\,x^3+3c_1)\,\mathrm{e}^{\sqrt{x}}\left(\int -\frac{2x^{3/2}(2520c_2\,x^{3/2}+210c_2\,x^{5/2}+7x^{7/2}c_2+(3c_1+5040c_2)\sqrt{x}+x((x^3+42x^2+840x+5040)c_2+3c_1))\mathrm{e}^{-\sqrt{x}}}{(c_2\,x^3+3c_1)^2}dx +}{x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(c_2\,x^3+3c_1)\,\mathrm{e}^{\sqrt{x}}\left(\int -\frac{2x^{3/2}(2520c_2\,x^{3/2}+210c_2\,x^{5/2}+7x^{7/2}c_2+(3c_1+5040c_2)\sqrt{x}+x((x^3+42x^2+840x+5040)c_2+3c_1))\mathrm{e}^{-\sqrt{x}}}{(c_2\,x^3+3c_1)^2}dx +}{x}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 50

```
dsolve(diff(diff(y(x),x),x)-1/x^(1/2)*diff(y(x),x)+1/4/x^2*(x+x^(1/2)-8)*y(x) = x,y(x
```

$$y = \frac{560x^{3/2}+28x^{5/2}+(c_1x^3+c_2)\,\mathrm{e}^{\sqrt{x}}+4x^3+140x^2+1680x+3360\sqrt{x}+3360}{x}$$

Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 60

```
DSolve[{D[y[x],{x,2}]-1/Sqrt[x]*D[y[x],x]+1/(4*x^2)*(x+Sqrt[x]-8)*y[x]==x,{}},y[x],x,IncludeSi
```

$$y(x) \to \frac{e^{\sqrt{x}}\left(x^3 \int_1^x e^{-\sqrt{K[1]}} dK[1] + c_2 x^3 + 2\Gamma\left(8, \sqrt{x}\right) + 3c_1\right)}{3x}$$

2.2.16 Problem 17

Internal problem ID [9139]
Book : Second order enumerated odes
Section : section 2
Problem number : 17
Date solved : `Monday, January 27, 2025 at 05:48:45 PM`
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]`

Solve

$$y'' + \frac{2y'}{x} + \frac{a^2 y}{x^4} = 0$$

Solved as second order ode using change of variable on x method 2

Time used: 0.382 (sec)

In normal form the ode

$$y'' + \frac{2y'}{x} + \frac{a^2 y}{x^4} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{a^2}{x^4}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\int p(x)dx} dx \\
&= \int \mathrm{e}^{-\int \frac{2}{x}dx} dx \\
&= \int e^{-2\ln(x)}\, dx \\
&= \int \frac{1}{x^2} dx \\
&= -\frac{1}{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{a^2}{x^4}}{\frac{1}{x^4}} \\
&= a^2
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) = 0 \\
\frac{d^2}{d\tau^2} y(\tau) + a^2 y(\tau) = 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = a^2$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\tau\lambda} + a^2 \mathrm{e}^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$a^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = a^2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(a^2)} \\
&= \pm\sqrt{-a^2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +\sqrt{-a^2} \\
\lambda_2 &= -\sqrt{-a^2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= ia \\
\lambda_2 &= -ia
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = a$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha \tau}(c_1 \cos(\beta \tau) + c_2 \sin(\beta \tau))$$

Which becomes

$$y(\tau) = e^{0}(c_1 \cos(a\tau) + c_2 \sin(a\tau))$$

Or

$$y(\tau) = c_1 \cos(a\tau) + c_2 \sin(a\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\frac{a}{x}\right) - c_2 \sin\left(\frac{a}{x}\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(\frac{a}{x}\right) - c_2 \sin\left(\frac{a}{x}\right)$$

Solved as second order ode using change of variable on x method 1

Time used: 0.087 (sec)

In normal form the ode

$$y'' + \frac{2y'}{x} + \frac{a^2 y}{x^4} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{a^2}{x^4}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{a^2}{x^4}}}{c} \\ \tau'' &= -\frac{2a^2}{c\sqrt{\frac{a^2}{x^4}}\, x^5}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2a^2}{c\sqrt{\frac{a^2}{x^4}}\, x^5} + \frac{2}{x}\frac{\sqrt{\frac{a^2}{x^4}}}{c}}{\left(\frac{\sqrt{\frac{a^2}{x^4}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\ &= \frac{\int \sqrt{\frac{a^2}{x^4}}\, dx}{c} \\ &= -\frac{x\sqrt{\frac{a^2}{x^4}}}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(x\sqrt{\frac{a^2}{x^4}}\right) - c_2 \sin\left(x\sqrt{\frac{a^2}{x^4}}\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(x\sqrt{\frac{a^2}{x^4}}\right) - c_2 \sin\left(x\sqrt{\frac{a^2}{x^4}}\right)$$

Solved as second order Bessel ode

Time used: 0.142 (sec)

Writing the ode as

$$x^2y'' + 2y'x + \frac{a^2y}{x^2} = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}\left(n, \beta\, x^\gamma\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^\gamma\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -\frac{1}{2} \\
\beta &= a \\
n &= \frac{1}{2} \\
\gamma &= -1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\sin\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}} - \frac{c_2\sqrt{2}\,\cos\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1\sqrt{2}\,\sin\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}} - \frac{c_2\sqrt{2}\,\cos\left(\frac{a}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{a}{x}}}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.143 (sec)

Writing the ode as

$$y'' + \frac{2y'}{x} + \frac{a^2y}{x^4} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= 1 \\
B &= \frac{2}{x} \\
C &= \frac{a^2}{x^4}
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a^2 \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{a^2}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.74: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{a^2}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{ia}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{ia}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = ia$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{ia}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{0}{ia} + 2\right) &&= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{0}{ia} + 2\right) &&= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{a^2}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{ia}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{ia}{x^2} + \frac{1}{x} + (-)\,(0) \\
&= -\frac{ia}{x^2} + \frac{1}{x} \\
&= \frac{-ia + x}{x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{ia}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2ia}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{ia}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{a^2}{x^4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{ia}{x^2} + \frac{1}{x}\right) dx} \\
&= x\, e^{\frac{ia}{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{ia}{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i\mathrm{e}^{-\frac{2ia}{x}}}{2a}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\frac{ia}{x}}\right) + c_2 \left(\mathrm{e}^{\frac{ia}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2ia}{x}}}{2a}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \mathrm{e}^{\frac{ia}{x}} - \frac{ic_2 \mathrm{e}^{-\frac{ia}{x}}}{2a}$$

Solved as second order ode adjoint method

Time used: 0.498 (sec)

In normal form the ode

$$y'' + \frac{2y'}{x} + \frac{a^2 y}{x^4} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{a^2}{x^4} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\xi'' - (\xi\, p)' + \xi q = 0$$
$$\xi'' - \left(\frac{2\xi(x)}{x}\right)' + \left(\frac{a^2\xi(x)}{x^4}\right) = 0$$
$$\xi''(x) - \frac{2\xi'(x)}{x} + \frac{(a^2+2x^2)\,\xi(x)}{x^4} = 0$$

Which is solved for $\xi(x)$. Writing the ode as

$$x^2\xi'' - 2\xi' x + \left(2 + \frac{a^2}{x^2}\right)\xi = 0 \tag{1}$$

Bessel ode has the form

$$x^2\xi'' + \xi' x + \left(-n^2 + x^2\right)\xi = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2\xi'' + (1-2\alpha)\, x\xi' + \left(\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)\xi = 0 \tag{3}$$

With the standard solution

$$\xi = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{3}{2}$$
$$\beta = a$$
$$n = -\frac{1}{2}$$
$$\gamma = -1$$

Substituting all the above into (4) gives the solution as

$$\xi = \frac{c_1\, x^{3/2}\sqrt{2}\,\cos\left(\frac{a}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{a}{x}}} + \frac{c_2\, x^{3/2}\sqrt{2}\,\sin\left(\frac{a}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{a}{x}}}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y = \int \xi(x)\, r(x)\, dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}$$

Or

$$y' + y\left(\frac{2}{x} - \frac{\frac{3c_1\sqrt{x}\sqrt{2}\cos\left(\frac{a}{x}\right)}{2\sqrt{\pi}\sqrt{\frac{a}{x}}} + \frac{c_1\sqrt{2}\cos\left(\frac{a}{x}\right)a}{2\sqrt{x}\sqrt{\pi}\left(\frac{a}{x}\right)^{3/2}} + \frac{c_1\sqrt{2}\,a\sin\left(\frac{a}{x}\right)}{\sqrt{x}\sqrt{\pi}\sqrt{\frac{a}{x}}} + \frac{3c_2\sqrt{x}\sqrt{2}\sin\left(\frac{a}{x}\right)}{2\sqrt{\pi}\sqrt{\frac{a}{x}}} + \frac{c_2\sqrt{2}\sin\left(\frac{a}{x}\right)a}{2\sqrt{x}\sqrt{\pi}\left(\frac{a}{x}\right)^{3/2}} - \frac{c_2\sqrt{2}\,a\cos\left(\frac{a}{x}\right)}{\sqrt{x}\sqrt{\pi}\sqrt{\frac{a}{x}}}}{\frac{c_1\, x^{3/2}\sqrt{2}\cos\left(\frac{a}{x}\right)}{\sqrt{\pi}\sqrt{\frac{a}{x}}} + \frac{c_2\, x^{3/2}\sqrt{2}\sin\left(\frac{a}{x}\right)}{\sqrt{\pi}\sqrt{\frac{a}{x}}}}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{\left(-\cos\left(\frac{a}{x}\right)c_2 + \sin\left(\frac{a}{x}\right)c_1\right)a}{\left(c_1\cos\left(\frac{a}{x}\right) + c_2\sin\left(\frac{a}{x}\right)\right)x^2}$$
$$p(x) = 0$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= e^{\int -\frac{\left(-\cos\left(\frac{a}{x}\right)c_2+\sin\left(\frac{a}{x}\right)c_1\right)a}{\left(c_1\cos\left(\frac{a}{x}\right)+c_2\sin\left(\frac{a}{x}\right)\right)x^2}dx} \\
&= \frac{1}{c_1\cos\left(\frac{a}{x}\right)+c_2\sin\left(\frac{a}{x}\right)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1\cos\left(\frac{a}{x}\right)+c_2\sin\left(\frac{a}{x}\right)}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1\cos\left(\frac{a}{x}\right)+c_2\sin\left(\frac{a}{x}\right)} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1\cos\left(\frac{a}{x}\right)+c_2\sin\left(\frac{a}{x}\right)}$ gives the final solution

$$
y = \left(c_1\cos\left(\frac{a}{x}\right)+c_2\sin\left(\frac{a}{x}\right)\right)c_3
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \left(c_1\cos\left(\frac{a}{x}\right)+c_2\sin\left(\frac{a}{x}\right)\right)c_3
$$

The constants can be merged to give

$$
y = c_1\cos\left(\frac{a}{x}\right)+c_2\sin\left(\frac{a}{x}\right)
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1\cos\left(\frac{a}{x}\right)+c_2\sin\left(\frac{a}{x}\right)
$$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x)+\frac{2\left(\frac{d}{dx}y(x)\right)}{x}+\frac{a^2y(x)}{x^4}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Multiply by denominators of the ODE

  $\left(\frac{d^2}{dx^2}y(x)\right)x^4+2\left(\frac{d}{dx}y(x)\right)x^3+a^2y(x)=0$

- Make a change of variables

  $t=\ln(x)$

- □ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x)=\left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

- Compute derivative

  $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^4 + 2\left(\frac{d}{dt}y(t)\right)x^2 + a^2y(t) = 0$

- Simplify

  $x^2\left(\frac{d^2}{dt^2}y(t)\right) + \left(\frac{d}{dt}y(t)\right)x^2 + a^2y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{a^2y(t)}{x^2} - \frac{d}{dt}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + \frac{a^2y(t)}{x^2} = 0$

- Characteristic polynomial of ODE

  $r^2 + r + \frac{a^2}{x^2} = 0$

- Factor the characteristic polynomial

  $\frac{r^2x^2 + r\,x^2 + a^2}{x^2} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{-\frac{x}{2} + \frac{\sqrt{-4a^2+x^2}}{2}}{x}, \frac{-\frac{x}{2} - \frac{\sqrt{-4a^2+x^2}}{2}}{x}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{\left(-\frac{x}{2} + \frac{\sqrt{-4a^2+x^2}}{2}\right)t}{x}}$

- 2nd solution of the ODE

  $y_2(t) = e^{\frac{\left(-\frac{x}{2} - \frac{\sqrt{-4a^2+x^2}}{2}\right)t}{x}}$

- General solution of the ODE

  $y(t) = C1y_1(t) + C2y_2(t)$

- Substitute in solutions

  $y(t) = C1\,e^{\frac{\left(-\frac{x}{2} + \frac{\sqrt{-4a^2+x^2}}{2}\right)t}{x}} + C2\,e^{\frac{\left(-\frac{x}{2} - \frac{\sqrt{-4a^2+x^2}}{2}\right)t}{x}}$

- Change variables back using $t = \ln(x)$

  $y(x) = C1\,e^{\frac{\left(-\frac{x}{2} + \frac{\sqrt{-4a^2+x^2}}{2}\right)\ln(x)}{x}} + C2\,e^{\frac{\left(-\frac{x}{2} - \frac{\sqrt{-4a^2+x^2}}{2}\right)\ln(x)}{x}}$

- Simplify

  $y(x) = C1\,x^{\frac{-x+\sqrt{-4a^2+x^2}}{2x}} + C2\,x^{-\frac{x+\sqrt{-4a^2+x^2}}{2x}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 21

```
dsolve(diff(diff(y(x),x),x)+2/x*diff(y(x),x)+a^2/x^4*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \sin\left(\frac{a}{x}\right) + c_2 \cos\left(\frac{a}{x}\right)$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 25

```
DSolve[{D[y[x],{x,2}]+2/x*D[y[x],x]+a^2/x^4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to c_1 \cos\left(\frac{a}{x}\right) - c_2 \sin\left(\frac{a}{x}\right)$$

2.2.17 Problem 18

Internal problem ID [9140]
Book : Second order enumerated odes
Section : section 2
Problem number : 18
Date solved : `Monday, January 27, 2025 at 05:48:47 PM`
CAS classification :
`[_Gegenbauer, [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]`

Solve

$$\left(-x^{2}+1\right) y''-x y'-c^{2} y=0$$

Solved as second order ode using change of variable on x method 2

Time used: 0.593 (sec)

In normal form the ode

$$\left(-x^{2}+1\right) y''-x y'-c^{2} y=0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y=0 \tag{2}$$

Where

$$p(x)=\frac{x}{x^{2}-1}$$
$$q(x)=\frac{c^{2}}{x^{2}-1}$$

Applying change of variables $\tau=g(x)$ to (2) gives

$$\frac{d^{2}}{d\tau^{2}}y(\tau)+p_{1}\left(\frac{d}{d\tau}y(\tau)\right)+q_{1}y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_{1}(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^{2}} \tag{4}$$
$$q_{1}(\tau)=\frac{q(x)}{\tau'(x)^{2}} \tag{5}$$

Let $p_1=0$. Eq (4) simplifies to

$$\tau''(x)+p(x)\,\tau'(x)=0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\int p(x)dx} dx \\ &= \int \mathrm{e}^{-\int \frac{x}{x^2-1}dx} dx \\ &= \int e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} dx \\ &= \int \frac{1}{\sqrt{x-1}\,\sqrt{x+1}} dx \\ &= \frac{\sqrt{(x-1)\,(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{c^2}{x^2-1}}{\frac{1}{(x-1)(x+1)}} \\ &= c^2 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = c^2$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\tau\lambda} + c^2 \mathrm{e}^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$c^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = c^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(c^2)} \\ &= \pm\sqrt{-c^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-c^2} \\ \lambda_2 &= -\sqrt{-c^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= ic \\ \lambda_2 &= -ic\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = c$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^{0}(c_1 \cos(c\tau) + c_2 \sin(c\tau))$$

Or

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\frac{c\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}}\right) + c_2 \sin\left(\frac{c\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}}\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(\frac{c\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}}\right) + c_2 \sin\left(\frac{c\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}}\right)$$

Solved as second order ode using change of variable on x method 1

Time used: 0.145 (sec)

In normal form the ode

$$\left(-x^2+1\right) y'' - xy' - c^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{x}{x^2-1}$$
$$q(x) = \frac{c^2}{x^2-1}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{\frac{c^2}{x^2-1}}}{c} \qquad (6)\\ \tau'' &= -\frac{c^2 x}{c\sqrt{\frac{c^2}{x^2-1}}\,(x^2-1)^2}\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{-\frac{c^2 x}{c\sqrt{\frac{c^2}{x^2-1}}\,(x^2-1)^2} + \frac{x}{x^2-1}\frac{\sqrt{\frac{c^2}{x^2-1}}}{c}}{\left(\frac{\sqrt{\frac{c^2}{x^2-1}}}{c}\right)^2}\\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \qquad (7)\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx\\ &= \frac{\int \sqrt{\frac{c^2}{x^2-1}}dx}{c}\\ &= \frac{\sqrt{\frac{c^2}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$\begin{aligned}y &= c_1 \cos\left(\sqrt{\frac{c^2}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)\\ &\quad + c_2 \sin\left(\sqrt{\frac{c^2}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y = {} & c_1 \cos\left(\sqrt{\frac{c^2}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
& + c_2 \sin\left(\sqrt{\frac{c^2}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Solved as second order ode using Kovacic algorithm

Time used: 0.326 (sec)

Writing the ode as

$$
\begin{aligned}
\left(-x^2+1\right) y'' - x y' - c^2 y &= 0 && (1)\\
A y'' + B y' + C y &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= -x^2+1 \\
B &= -x \\
C &= -c^2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4c^2x^2 + 4c^2 - x^2 - 2}{4\left(x^2-1\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4c^2x^2 + 4c^2 - x^2 - 2 \\
t &= 4\left(x^2-1\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-4c^2x^2 + 4c^2 - x^2 - 2}{4\left(x^2-1\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.76: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{3}{16\,(x+1)^2} + \frac{-\frac{1}{16}+\frac{c^2}{2}}{x+1} - \frac{3}{16\,(x-1)^2} + \frac{-\frac{c^2}{2}+\frac{1}{16}}{x-1}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\
&= \{1, 2, 3\}
\end{aligned}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\
&= \{1, 2, 3\}
\end{aligned}
$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-4c^2x^2 + 4c^2 - x^2 - 2}{4\,(x^2-1)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=-1$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\
&= \{2\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$
e_1 = 1,\ e_2 = 1,\ e_\infty = 2
$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\
&= \frac{1}{2}(2-(1+(1)))\\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\
&= \frac{1}{2}\left(\frac{1}{(x-(1))}+\frac{1}{(x-(-1))}\right)\\
&= \frac{1}{2x-2}+\frac{1}{2x+2}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d=0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p}\\
&= \frac{1}{2x-2}+\frac{1}{2x+2}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) w + \frac{4c^2x^2 - 4c^2 + x^2}{4\left(x^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2c\sqrt{-x^2+1}}{2\left(x-1\right)\left(x+1\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{x+2c\sqrt{-x^2+1}}{2(x-1)(x+1)}dx} \\
&= \left(x^2-1\right)^{1/4} \mathrm{e}^{-c\arcsin(x)}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{-x^2+1}\, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{4} - \frac{\ln(x+1)}{4}} \\
&= z_1 \left(\frac{1}{(x-1)^{1/4}\,(x+1)^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2-1\right)^{1/4} \mathrm{e}^{-c\arcsin(x)}}{(x-1)^{1/4}\,(x+1)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{2c\arcsin(x)}}{\sqrt{x^2-1}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2-1\right)^{1/4} \mathrm{e}^{-c\arcsin(x)}}{(x-1)^{1/4}\,(x+1)^{1/4}}\right) + c_2 \left(\frac{\left(x^2-1\right)^{1/4} \mathrm{e}^{-c\arcsin(x)}}{(x-1)^{1/4}\,(x+1)^{1/4}}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{2c\arcsin(x)}}{\sqrt{x^2-1}} d
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1\left(x^2-1\right)^{1/4} \mathrm{e}^{-c\arcsin(x)}}{(x-1)^{1/4}\,(x+1)^{1/4}} + \frac{c_2\left(x^2-1\right)^{1/4} \mathrm{e}^{-c\arcsin(x)} \left(\int \frac{\mathrm{e}^{2c\arcsin(x)}}{\sqrt{x^2-1}} dx\right)}{(x-1)^{1/4}\,(x+1)^{1/4}}$$

Solved as second order ode adjoint method

Time used: 1.010 (sec)

In normal form the ode

$$(-x^2+1)\,y''-xy'-c^2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{2}$$

Where

$$\begin{aligned} p(x)&=\frac{x}{x^2-1}\\ q(x)&=\frac{c^2}{x^2-1}\\ r(x)&=0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi''-(\xi\,p)'+\xi q&=0\\ \xi''-\left(\frac{x\xi(x)}{x^2-1}\right)'+\left(\frac{c^2\xi(x)}{x^2-1}\right)&=0\\ \xi''(x)-\frac{x\xi'(x)}{x^2-1}+\frac{\xi(x)\,(x^2+1+c^2(x^2-1))}{(x^2-1)^2}&=0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi''-\frac{x\xi'}{x^2-1}+\frac{\xi(x^2+1+c^2(x^2-1))}{(x^2-1)^2}=0 \tag{1}$$

$$A\xi''+B\xi'+C\xi=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=-\frac{x}{x^2-1}\\ C&=\frac{x^2+1+c^2(x^2-1)}{(x^2-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=\xi e^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-4c^2x^2+4c^2-x^2-2}{4\,(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4c^2x^2 + 4c^2 - x^2 - 2 \\
t &= 4(x^2 - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-4c^2x^2 + 4c^2 - x^2 - 2}{4(x^2 - 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$
\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.77: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{3}{16(x+1)^2} + \frac{-\frac{1}{16} + \frac{c^2}{2}}{x+1} - \frac{3}{16(x-1)^2} + \frac{-\frac{c^2}{2} + \frac{1}{16}}{x-1}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4c^2x^2 + 4c^2 - x^2 - 2}{4\left(x^2-1\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1, 2, 3\}$
-1	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_2 = 1, \; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(1))} + \frac{1}{(x-(-1))}\right) \\ &= \frac{1}{2x-2} + \frac{1}{2x+2} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x-2} + \frac{1}{2x+2}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) w + \frac{4c^2x^2 - 4c^2 + x^2}{4\left(x^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2c\sqrt{-x^2+1}}{2\left(x-1\right)\left(x+1\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{x+2c\sqrt{-x^2+1}}{2(x-1)(x+1)} dx} \\ &= \left(x^2-1\right)^{1/4} \mathrm{e}^{-c \arcsin(x)}\end{aligned}$$

The first solution to the original ode in ξ is found from

$$\begin{aligned}\xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{x}{x^2-1}}{1} dx} \\ &= z_1 e^{\frac{\ln(x-1)}{4} + \frac{\ln(x+1)}{4}} \\ &= z_1 \left(\left(x-1\right)^{1/4} \left(x+1\right)^{1/4}\right)\end{aligned}$$

Which simplifies to

$$\xi_1 = \left(x-1\right)^{1/4} \left(x+1\right)^{1/4} \left(x^2-1\right)^{1/4} \mathrm{e}^{-c \arcsin(x)}$$

The second solution ξ_2 to the original ode is found using reduction of order

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A} dx}}{\xi_1^2} dx$$

Substituting gives

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{-\frac{x}{x^2-1}}{1}\,dx}}{(\xi_1)^2}\,dx$$

$$= \xi_1 \int \frac{e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}}{(\xi_1)^2}\,dx$$

$$= \xi_1 \left(\int \frac{e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}} e^{2c\arcsin(x)}}{\sqrt{x-1}\,\sqrt{x+1}\,\sqrt{x^2-1}} dx \right)$$

Therefore the solution is

$$\xi = c_1\xi_1 + c_2\xi_2$$
$$= c_1\Big((x$$
$$-1)^{1/4}(x+1)^{1/4}(x^2-1)^{1/4} e^{-c\arcsin(x)}\Big) + c_2\left((x-1)^{1/4}(x+1)^{1/4}(x^2-1)^{1/4} e^{-c\arcsin(x)}\left(\int \frac{e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}} e}{\sqrt{x-1}\,\sqrt{x+}}\right.\right.$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$

$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(\frac{x}{x^2-1} - \frac{\frac{c_1(x+1)^{1/4}(x^2-1)^{1/4}e^{-c\arcsin(x)}}{4(x-1)^{3/4}} + \frac{c_1(x-1)^{1/4}(x^2-1)^{1/4}e^{-c\arcsin(x)}}{4(x+1)^{3/4}} + \frac{c_1(x-1)^{1/4}(x+1)^{1/4}e^{-c\arcsin(x)}x}{2(x^2-1)^{3/4}} - c_1(x-1}{}\right.$$

Which is now a first order ode. This is now solved for y. The ode

$$y' = -\frac{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right)c_2 c\,x^2 - c_2\,e^{2c\arcsin(x)}\sqrt{x^2-1}\,\sqrt{-x^2+1} + c_1 c\,x^2 - c_2 c\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) - c_1 c\right)y}{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right)c_2 + c_1\right)\sqrt{-x^2+1}\,(x-1)(x+1)} \tag{2.18}$$

is separable as it can be written as

$$y' = -\frac{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right)c_2 c\,x^2 - c_2\,e^{2c\arcsin(x)}\sqrt{x^2-1}\,\sqrt{-x^2+1} + c_1 c\,x^2 - c_2 c\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) - c_1 c\right)y}{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right)c_2 + c_1\right)\sqrt{-x^2+1}\,(x-1)(x+1)}$$
$$= f(x)g(y)$$

Where

$$f(x) = -\frac{\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right)c_2 c\,x^2 - c_2\,e^{2c\arcsin(x)}\sqrt{x^2-1}\,\sqrt{-x^2+1} + c_1 c\,x^2 - c_2 c\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) - c_1 c}{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right)c_2 + c_1\right)\sqrt{-x^2+1}\,(x-1)(x+1)}$$

$$g(y) = y$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

$$\int \frac{1}{y}dy = \int -\frac{\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 c\,x^2 - c_2\,e^{2c\arcsin(x)}\sqrt{x^2-1}\,\sqrt{-x^2+1} + c_1 c\,x^2 - c_2 c\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) - c_1 c}{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 + c_1\right)\sqrt{-x^2+1}\,(x-1)\,(x+1)}dx$$

$$\ln(y) = \int -\frac{\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 c\,x^2 - c_2\,e^{2c\arcsin(x)}\sqrt{x^2-1}\,\sqrt{-x^2+1} + c_1 c\,x^2 - c_2 c\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) - c_1 c}{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 + c_1\right)\sqrt{-x^2+1}\,(x-1)\,(x+1)}dx + 2c_3$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or

$$y = 0$$

for y gives

$$y = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(y) = \int -\frac{\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 c\,x^2 - c_2\,e^{2c\arcsin(x)}\sqrt{x^2-1}\,\sqrt{-x^2+1} + c_1 c\,x^2 - c_2 c\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) - c_1 c}{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 + c_1\right)\sqrt{-x^2+1}\,(x-1)\,(x+1)}dx + 2c_3$$

$$y = 0$$

Hence, the solution found using Lagrange adjoint equation method is

$$\ln(y) = \int -\frac{\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 c\,x^2 - c_2\,e^{2c\arcsin(x)}\sqrt{x^2-1}\,\sqrt{-x^2+1} + c_1 c\,x^2 - c_2 c\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) - c_1 c}{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 + c_1\right)\sqrt{-x^2+1}\,(x-1)\,(x+1)}dx + 2c_3$$

$$y = 0$$

The constants can be merged to give

$$\ln(y) = \int -\frac{\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 c\,x^2 - c_2\,e^{2c\arcsin(x)}\sqrt{x^2-1}\,\sqrt{-x^2+1} + c_1 c\,x^2 - c_2 c\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) - c_1 c}{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 + c_1\right)\sqrt{-x^2+1}\,(x-1)\,(x+1)}dx + 2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\ln(y) = \int -\frac{\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 c\, x^2 - c_2\, e^{2c\arcsin(x)}\sqrt{x^2-1}\,\sqrt{-x^2+1} + c_1 c\, x^2 - c_2 c\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) - c_1 c}{\left(\left(\int \frac{e^{2c\arcsin(x)}}{\sqrt{x^2-1}}dx\right) c_2 + c_1\right)\sqrt{-x^2+1}\,(x-1)(x+1)} dx + 2$$

$$y = 0$$

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x\left(\frac{d}{dx}y(x)\right) - c^2 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{c^2 y(x)}{x^2-1} - \frac{\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\left(\frac{d}{dx}y(x)\right)x}{x^2-1} + \frac{c^2 y(x)}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x}{x^2-1}, P_3(x) = \frac{c^2}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{1}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + c^2 y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) + c^2 y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+1+2r) + a_k(c^2+k^2+2kr+r^2)\right) u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, \frac{1}{2}\}$
- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+\frac{1}{2}+r\right)a_{k+1} + a_k(c^2+k^2+2kr+r^2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(c^2+k^2+2kr+r^2)}{(k+1+r)(2k+1+2r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(c^2+k^2)}{(k+1)(2k+1)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(c^2+k^2)}{(k+1)(2k+1)}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(c^2+k^2)}{(k+1)(2k+1)}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k(c^2+k^2+k+\frac{1}{4})}{(k+\frac{3}{2})(2k+2)}$
- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k(c^2+k^2+k+\frac{1}{4})}{(k+\frac{3}{2})(2k+2)}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k(c^2+k^2+k+\frac{1}{4})}{(k+\frac{3}{2})(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k(c^2+k^2)}{(k+1)(2k+1)}, b_{k+1} = \frac{b_k(c^2+k^2+k+\frac{1}{4})}{(k+\frac{3}{2})(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 35

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-diff(y(x),x)*x-c^2*y(x) = 0,y(x),singsol=all)
```

$$y = c_1\left(x+\sqrt{x^2-1}\right)^{ic}+c_2\left(x+\sqrt{x^2-1}\right)^{-ic}$$

Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 42

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-x*D[y[x],x]-c^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to c_1 \cos\left(c \log\left(\sqrt{x^2-1}+x\right)\right)+c_2 \sin\left(c \log\left(\sqrt{x^2-1}+x\right)\right)$$

2.2.18 Problem 19

Internal problem ID [9141]
Book : Second order enumerated odes
Section : section 2
Problem number : 19
Date solved : `Monday, January 27, 2025 at 05:48:50 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$x^6y'' + 3x^5y' + a^2y = \frac{1}{x^2}$$

Solved as second order ode using change of variable on x method 2

Time used: 0.570 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^6y'' + 3x^5y' + a^2y = 0$$

In normal form the ode

$$x^6y'' + 3x^5y' + a^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{a^2}{x^6}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\int p(x)dx} dx \\
&= \int e^{-\int \frac{3}{x}dx} dx \\
&= \int e^{-3\ln(x)}\, dx \\
&= \int \frac{1}{x^3} dx \\
&= -\frac{1}{2x^2} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{a^2}{x^6}}{\frac{1}{x^6}} \\
&= a^2 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + a^2 y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = a^2$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\tau\lambda} + a^2 e^{\tau\lambda} = 0 \qquad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$a^2 + \lambda^2 = 0 \qquad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = a^2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(a^2)} \\
&= \pm\sqrt{-a^2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +\sqrt{-a^2} \\
\lambda_2 &= -\sqrt{-a^2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = ia$$
$$\lambda_2 = -ia$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = a$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1\cos(\beta\tau) + c_2\sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1\cos(a\tau) + c_2\sin(a\tau))$$

Or

$$y(\tau) = c_1\cos(a\tau) + c_2\sin(a\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1\cos\left(\frac{a}{2x^2}\right) - c_2\sin\left(\frac{a}{2x^2}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\cos\left(\frac{a}{2x^2}\right) - c_2\sin\left(\frac{a}{2x^2}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{a}{2x^2}\right)$$
$$y_2 = \sin\left(\frac{a}{2x^2}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{a}{2x^2}\right) & \sin\left(\frac{a}{2x^2}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{a}{2x^2}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{a}{2x^2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{a}{2x^2}\right) & \sin\left(\frac{a}{2x^2}\right) \\ \frac{a\sin\left(\frac{a}{2x^2}\right)}{x^3} & -\frac{a\cos\left(\frac{a}{2x^2}\right)}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{a}{2x^2}\right)\right)\left(-\frac{a\cos\left(\frac{a}{2x^2}\right)}{x^3}\right) - \left(\sin\left(\frac{a}{2x^2}\right)\right)\left(\frac{a\sin\left(\frac{a}{2x^2}\right)}{x^3}\right)$$

Which simplifies to

$$W = -\frac{a\left(\cos\left(\frac{a}{2x^2}\right)^2 + \sin\left(\frac{a}{2x^2}\right)^2\right)}{x^3}$$

Which simplifies to

$$W = -\frac{a}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin\left(\frac{a}{2x^2}\right)}{x^2}}{-a\,x^3}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{a}{2x^2}\right)}{x^5 a}dx$$

Hence

$$u_1 = \frac{\frac{\cos\left(\frac{a}{2x^2}\right)}{a\,x^2} - \frac{2\sin\left(\frac{a}{2x^2}\right)}{a^2}}{a}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\frac{a}{2x^2}\right)}{x^2}}{-a\,x^3}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{a}{2x^2}\right)}{x^5 a}dx$$

Hence

$$u_2 = -\frac{-\frac{\sin\left(\frac{a}{2x^2}\right)}{x^2 a} - \frac{2\cos\left(\frac{a}{2x^2}\right)}{a^2}}{a}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\frac{\cos\left(\frac{a}{2x^2}\right)}{a\,x^2} - \frac{2\sin\left(\frac{a}{2x^2}\right)}{a^2}\right)\cos\left(\frac{a}{2x^2}\right)}{a} - \frac{\sin\left(\frac{a}{2x^2}\right)\left(-\frac{\sin\left(\frac{a}{2x^2}\right)}{x^2 a} - \frac{2\cos\left(\frac{a}{2x^2}\right)}{a^2}\right)}{a}$$

Which simplifies to

$$y_p(x) = \frac{1}{a^2 x^2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right)\right) + \left(\frac{1}{a^2x^2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{1}{a^2x^2} + c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right)
$$

Solved as second order ode using change of variable on x method 1

Time used: 0.855 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^6, B = 3x^5, C = a^2, f(x) = \frac{1}{x^2}$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$
x^6y'' + 3x^5y' + a^2y = 0
$$

In normal form the ode

$$
x^6y'' + 3x^5y' + a^2y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{3}{x} \\
q(x) &= \frac{a^2}{x^6}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{a^2}{x^6}}}{c} \qquad (6) \\
\tau'' &= -\frac{3a^2}{c\sqrt{\frac{a^2}{x^6}}\,x^7}
\end{aligned}
$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3a^2}{c\sqrt{\frac{a^2}{x^6}}\,x^7} + \frac{3}{x}\frac{\sqrt{\frac{a^2}{x^6}}}{c}}{\left(\frac{\sqrt{\frac{a^2}{x^6}}}{c}\right)^2} \\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{a^2}{x^6}}\,dx}{c} \\
&= -\frac{x\sqrt{\frac{a^2}{x^6}}}{2c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) - c_2 \sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \qquad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)$$

$$y_2 = -\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \qquad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \qquad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) & -\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\right) & \frac{d}{dx}\left(-\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) & -\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) \\ -\left(\frac{\sqrt{\frac{a^2}{x^6}}}{2} - \frac{3a^2}{2x^6\sqrt{\frac{a^2}{x^6}}}\right)\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) & -\left(\frac{\sqrt{\frac{a^2}{x^6}}}{2} - \frac{3a^2}{2x^6\sqrt{\frac{a^2}{x^6}}}\right)\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) \end{vmatrix}$$

Therefore

$$\begin{aligned} W = &\left(\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\right)\left(-\left(\frac{\sqrt{\frac{a^2}{x^6}}}{2} - \frac{3a^2}{2x^6\sqrt{\frac{a^2}{x^6}}}\right)\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\right) \\ &- \left(-\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\right)\left(-\left(\frac{\sqrt{\frac{a^2}{x^6}}}{2} - \frac{3a^2}{2x^6\sqrt{\frac{a^2}{x^6}}}\right)\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\right) \end{aligned}$$

Which simplifies to

$$W = \frac{a^2\left(\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)^2 + \sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)^2\right)}{x^6\sqrt{\frac{a^2}{x^6}}}$$

Which simplifies to

$$W = \frac{a^2}{x^6\sqrt{\frac{a^2}{x^6}}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)}{x^2}}{\frac{a^2}{\sqrt{\frac{a^2}{x^6}}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\sqrt{\frac{a^2}{x^6}}}{x^2a^2}dx$$

Hence

$$u_1 = -\frac{4x^3\sqrt{\pi}\,\sqrt{\frac{a^2}{x^6}}\left(-\frac{x^7\left(\frac{a^2}{x^6}\right)^{3/2}\cos\left(\frac{a}{2x^2}\right)}{4\sqrt{\pi}\,a^2} + \frac{x^9\left(\frac{a^2}{x^6}\right)^{3/2}\sin\left(\frac{a}{2x^2}\right)}{2\sqrt{\pi}\,a^3}\right)}{a^4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)}{\frac{x^2}{\frac{a^2}{\sqrt{\frac{a^2}{x^6}}}}} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\sqrt{\frac{a^2}{x^6}}}{x^2 a^2} dx$$

Hence

$$u_2 = -\frac{4\sqrt{\pi}\,\sqrt{\frac{a^2}{x^6}}\,x^3\left(-\frac{1}{2\sqrt{\pi}} + \frac{\cos\left(\frac{a}{2x^2}\right)}{2\sqrt{\pi}} + \frac{a\sin\left(\frac{a}{2x^2}\right)}{4\sqrt{\pi}\,x^2}\right)}{a^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{4x^3\sqrt{\pi}\,\sqrt{\frac{a^2}{x^6}}\left(-\frac{x^7\left(\frac{a^2}{x^6}\right)^{3/2}\cos\left(\frac{a}{2x^2}\right)}{4\sqrt{\pi}\,a^2} + \frac{x^9\left(\frac{a^2}{x^6}\right)^{3/2}\sin\left(\frac{a}{2x^2}\right)}{2\sqrt{\pi}\,a^3}\right)\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)}{a^4} + \frac{4\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\sqrt{\pi}\,\sqrt{\frac{a^2}{x^6}}\,x^3\left(-\frac{1}{2\sqrt{\pi}} + \frac{\cos\left(\frac{a}{2x^2}\right)}{2\sqrt{\pi}} + \frac{a\sin\left(\frac{a}{2x^2}\right)}{4\sqrt{\pi}\,x^2}\right)}{a^4}$$

Which simplifies to

$$y_p(x) = \frac{x^3\sqrt{\frac{a^2}{x^6}}\left(2x^2\cos\left(\frac{a}{2x^2}\right) + a\sin\left(\frac{a}{2x^2}\right) - 2x^2\right)\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) + a\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\left(-2x^2\sin\left(\frac{a}{2x^2}\right) + a\cos\left(\frac{a}{2x^2}\right)\right)}{x^2 a^4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) - c_2\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\right) \\ &\quad + \left(\frac{x^3\sqrt{\frac{a^2}{x^6}}\left(2x^2\cos\left(\frac{a}{2x^2}\right) + a\sin\left(\frac{a}{2x^2}\right) - 2x^2\right)\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) + a\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\left(-2x^2\sin\left(\frac{a}{2x^2}\right) + a\cos\left(\frac{a}{2x^2}\right)\right)}{x^2 a^4}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) - c_2\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) + \frac{x^3\sqrt{\frac{a^2}{x^6}}\left(2x^2\cos\left(\frac{a}{2x^2}\right) + a\sin\left(\frac{a}{2x^2}\right) - 2x^2\right)\sin\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right) + a\cos\left(\frac{x\sqrt{\frac{a^2}{x^6}}}{2}\right)\left(-2x^2\sin\left(\frac{a}{2x^2}\right) + a\cos\left(\frac{a}{2x^2}\right)\right)}{x^2 a^4}$$

Solved as second order Bessel ode

Time used: 0.288 (sec)

Writing the ode as

$$x^2y'' + 3y'x + \frac{a^2y}{x^4} = \frac{1}{x^6} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= -1\\ \beta &= \frac{a}{2}\\ n &= \frac{1}{2}\\ \gamma &= -2\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{2c_1 \sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2c_2 \cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{2c_1 \sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2c_2 \cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{2 \sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}$$

$$y_2 = -\frac{2 \cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{2\sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}} & -\frac{2\cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}} \\ \frac{d}{dx}\left(\frac{2\sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}}\right) & \frac{d}{dx}\left(-\frac{2\cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{2\sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}} & -\frac{2\cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}} \\ -\frac{2\sin\left(\frac{a}{2x^2}\right)}{x^2\sqrt{\pi}\sqrt{\frac{a}{x^2}}} + \frac{2\sin\left(\frac{a}{2x^2}\right)a}{x^4\sqrt{\pi}\left(\frac{a}{x^2}\right)^{3/2}} - \frac{2a\cos\left(\frac{a}{2x^2}\right)}{x^4\sqrt{\pi}\sqrt{\frac{a}{x^2}}} & \frac{2\cos\left(\frac{a}{2x^2}\right)}{x^2\sqrt{\pi}\sqrt{\frac{a}{x^2}}} - \frac{2\cos\left(\frac{a}{2x^2}\right)a}{x^4\sqrt{\pi}\left(\frac{a}{x^2}\right)^{3/2}} - \frac{2a\sin\left(\frac{a}{2x^2}\right)}{x^4\sqrt{\pi}\sqrt{\frac{a}{x^2}}} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\frac{2\sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}}\right)\left(\frac{2\cos\left(\frac{a}{2x^2}\right)}{x^2\sqrt{\pi}\sqrt{\frac{a}{x^2}}} - \frac{2\cos\left(\frac{a}{2x^2}\right)a}{x^4\sqrt{\pi}\left(\frac{a}{x^2}\right)^{3/2}} - \frac{2a\sin\left(\frac{a}{2x^2}\right)}{x^4\sqrt{\pi}\sqrt{\frac{a}{x^2}}}\right) \\ & - \left(-\frac{2\cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}}\right)\left(-\frac{2\sin\left(\frac{a}{2x^2}\right)}{x^2\sqrt{\pi}\sqrt{\frac{a}{x^2}}} + \frac{2\sin\left(\frac{a}{2x^2}\right)a}{x^4\sqrt{\pi}\left(\frac{a}{x^2}\right)^{3/2}} - \frac{2a\cos\left(\frac{a}{2x^2}\right)}{x^4\sqrt{\pi}\sqrt{\frac{a}{x^2}}}\right) \end{aligned}$$

Which simplifies to

$$W = -\frac{4\left(\cos\left(\frac{a}{2x^2}\right)^2 + \sin\left(\frac{a}{2x^2}\right)^2\right)}{x^3\pi}$$

Which simplifies to

$$W = -\frac{4}{x^3\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{2\cos\left(\frac{a}{2x^2}\right)}{x^7\sqrt{\pi}\sqrt{\frac{a}{x^2}}}}{-\frac{4}{x\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{\pi}\cos\left(\frac{a}{2x^2}\right)}{2x^6\sqrt{\frac{a}{x^2}}}dx$$

Hence

$$u_1 = \frac{\sqrt{\pi}\cos\left(\frac{a}{2x^2}\right)}{x\,a^2\sqrt{\frac{a}{x^2}}} + \frac{\sqrt{\pi}\sin\left(\frac{a}{2x^2}\right)}{2a\,x^3\sqrt{\frac{a}{x^2}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\sin\left(\frac{a}{2x^2}\right)}{x^7\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}}{-\frac{4}{x\pi}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\sqrt{\pi}\,\sin\left(\frac{a}{2x^2}\right)}{2x^6\sqrt{\frac{a}{x^2}}}dx$$

Hence

$$u_2 = -\frac{\sqrt{\pi}\,\cos\left(\frac{a}{2x^2}\right)}{2a\,x^3\sqrt{\frac{a}{x^2}}} + \frac{\sqrt{\pi}\,\sin\left(\frac{a}{2x^2}\right)}{x\,a^2\sqrt{\frac{a}{x^2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2\left(\frac{\sqrt{\pi}\,\cos\left(\frac{a}{2x^2}\right)}{x\,a^2\sqrt{\frac{a}{x^2}}} + \frac{\sqrt{\pi}\,\sin\left(\frac{a}{2x^2}\right)}{2a\,x^3\sqrt{\frac{a}{x^2}}}\right)\sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2\cos\left(\frac{a}{2x^2}\right)\left(-\frac{\sqrt{\pi}\,\cos\left(\frac{a}{2x^2}\right)}{2a\,x^3\sqrt{\frac{a}{x^2}}} + \frac{\sqrt{\pi}\,\sin\left(\frac{a}{2x^2}\right)}{x\,a^2\sqrt{\frac{a}{x^2}}}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}$$

Which simplifies to

$$y_p(x) = \frac{1}{a^2x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{2c_1\sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2c_2\cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}\right) + \left(\frac{1}{a^2x^2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{2c_1\sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} - \frac{2c_2\cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} + \frac{1}{a^2x^2}$$

Solved as second order ode adjoint method

Time used: 2.421 (sec)

In normal form the ode

$$x^6y'' + 3x^5y' + a^2y = \frac{1}{x^2} \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{a^2}{x^6} \\ r(x) &= \frac{1}{x^8} \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{3\xi(x)}{x}\right)' + \left(\frac{a^2\xi(x)}{x^6}\right) &= 0 \\
\xi''(x) - \frac{3\xi'(x)}{x} + \frac{(3x^4 + a^2)\,\xi(x)}{x^6} &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$x^2\xi'' - 3\xi' x + \left(3 + \frac{a^2}{x^4}\right)\xi = 0 \tag{1}$$

Bessel ode has the form

$$x^2\xi'' + \xi' x + \left(-n^2 + x^2\right)\xi = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2\xi'' + (1 - 2\alpha)\, x\xi' + \left(\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)\xi = 0 \tag{3}$$

With the standard solution

$$\xi = x^{\alpha}(c_3 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_4 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= 2 \\
\beta &= \frac{a}{2} \\
n &= -\frac{1}{2} \\
\gamma &= -2
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$\xi = \frac{2c_3\, x^2 \cos\left(\frac{a}{2x^2}\right)}{\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}} + \frac{2c_4\, x^2 \sin\left(\frac{a}{2x^2}\right)}{\sqrt{\pi}\,\sqrt{\frac{a}{x^2}}}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' + y\left(\frac{3}{x} - \frac{\frac{4c_3 x \cos\left(\frac{a}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{a}{x^2}}} + \frac{2c_3 \cos\left(\frac{a}{2x^2}\right) a}{x\sqrt{\pi}\left(\frac{a}{x^2}\right)^{3/2}} + \frac{2c_3 a \sin\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}} + \frac{4c_4 x \sin\left(\frac{a}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{a}{x^2}}} + \frac{2c_4 \sin\left(\frac{a}{2x^2}\right) a}{x\sqrt{\pi}\left(\frac{a}{x^2}\right)^{3/2}} - \frac{2c_4 a \cos\left(\frac{a}{2x^2}\right)}{x\sqrt{\pi}\sqrt{\frac{a}{x^2}}}}{\frac{2c_3\, x^2 \cos\left(\frac{a}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{a}{x^2}}} + \frac{2c_4\, x^2 \sin\left(\frac{a}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{a}{x^2}}}}\right) = \frac{\frac{2(-2c_3\, x^2 \cdots}{\sqrt{\pi}\, \cdots}}{2 \cdots}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{a\left(-\cos\left(\frac{a}{2x^2}\right) c_4 + \sin\left(\frac{a}{2x^2}\right) c_3\right)}{x^3\left(c_3 \cos\left(\frac{a}{2x^2}\right) + c_4 \sin\left(\frac{a}{2x^2}\right)\right)} \\
p(x) &= \frac{(-2c_4\, x^2 - c_3 a)\sin\left(\frac{a}{2x^2}\right) + (-2c_3\, x^2 + c_4 a)\cos\left(\frac{a}{2x^2}\right)}{a^2 x^5\left(c_3 \cos\left(\frac{a}{2x^2}\right) + c_4 \sin\left(\frac{a}{2x^2}\right)\right)}
\end{aligned}$$

The integrating factor μ is

$$\mu = e^{\int q\,dx} = e^{\int -\frac{a\left(-\cos\left(\frac{a}{2x^2}\right)c_4+\sin\left(\frac{a}{2x^2}\right)c_3\right)}{x^3\left(c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)\right)}dx} = \frac{1}{c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{(-2c_4\,x^2-c_3a)\sin\left(\frac{a}{2x^2}\right)+(-2c_3\,x^2+c_4a)\cos\left(\frac{a}{2x^2}\right)}{a^2x^5\left(c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)\right)}\right)$$

$$\frac{d}{dx}\left(\frac{y}{c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)}\right) = \left(\frac{1}{c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)}\right)\left(\frac{(-2c_4\,x^2-c_3a)\sin\left(\frac{a}{2x^2}\right)+(-2c_3\,x^2+c_4a)\cos\left(\frac{a}{2x^2}\right)}{a^2x^5\left(c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)\right)}\right)$$

$$d\left(\frac{y}{c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)}\right) = \left(\frac{(-2c_4\,x^2-c_3a)\sin\left(\frac{a}{2x^2}\right)+(-2c_3\,x^2+c_4a)\cos\left(\frac{a}{2x^2}\right)}{a^2x^5\left(c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)\right)^2}\right)dx$$

Integrating gives

$$\frac{y}{c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)} = \int \frac{(-2c_4\,x^2-c_3a)\sin\left(\frac{a}{2x^2}\right)+(-2c_3\,x^2+c_4a)\cos\left(\frac{a}{2x^2}\right)}{a^2x^5\left(c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)\right)^2}\,dx = \frac{2\,e^{\frac{ia}{2x^2}}}{a^2x^2\left(-ic_4\,e^{\frac{ia}{x^2}}+e^{\frac{ia}{x^2}}c_3+ic_4+c_3\right)}+c_5$$

Dividing throughout by the integrating factor $\frac{1}{c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)}$ gives the final solution

$$y = \frac{\left(c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)\right)\left(-2\,e^{\frac{ia}{2x^2}}+a^2c_5\,x^2\left((ic_4-c_3)\,e^{\frac{ia}{x^2}}-ic_4-c_3\right)\right)}{a^2x^2\left((ic_4-c_3)\,e^{\frac{ia}{x^2}}-ic_4-c_3\right)}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)\right)\left(-2\,e^{\frac{ia}{2x^2}}+a^2c_5\,x^2\left((ic_4-c_3)\,e^{\frac{ia}{x^2}}-ic_4-c_3\right)\right)}{a^2x^2\left((ic_4-c_3)\,e^{\frac{ia}{x^2}}-ic_4-c_3\right)}$$

The constants can be merged to give

$$y = \frac{\left(c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)\right)\left(-2\,e^{\frac{ia}{2x^2}}+a^2x^2\left((ic_4-c_3)\,e^{\frac{ia}{x^2}}-ic_4-c_3\right)\right)}{a^2x^2\left((ic_4-c_3)\,e^{\frac{ia}{x^2}}-ic_4-c_3\right)}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_3\cos\left(\frac{a}{2x^2}\right)+c_4\sin\left(\frac{a}{2x^2}\right)\right)\left(-2\,e^{\frac{ia}{2x^2}}+a^2x^2\left((ic_4-c_3)\,e^{\frac{ia}{x^2}}-ic_4-c_3\right)\right)}{a^2x^2\left((ic_4-c_3)\,e^{\frac{ia}{x^2}}-ic_4-c_3\right)}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x)*x^6+3*x^5*diff(y(x),x)+a^2*y(x) = 1/x^2,y(x),singsol=all)
```

$$y = \sin\left(\frac{a}{2x^2}\right) c_2 + \cos\left(\frac{a}{2x^2}\right) c_1 + \frac{1}{a^2x^2}$$

Mathematica DSolve solution

Solving time : 0.071 (sec)
Leaf size : 104

```
DSolve[{x^6*D[y[x],{x,2}]+3*x^5*D[y[x],x]+a^2*y[x]==1/x^2,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to -\sin\left(\frac{a}{2x^2}\right) \int_1^x \frac{\cos\left(\frac{a}{2K[1]^2}\right)}{aK[1]^5} dK[1] + \frac{-2a^3c_2x^2 \sin\left(\frac{a}{2x^2}\right) - 2x^2 \sin\left(\frac{a}{x^2}\right) + a\cos\left(\frac{a}{x^2}\right) + a}{2a^3x^2} + c_1 \cos\left(\frac{a}{2x^2}\right)$$

2.2.19 Problem 20

Internal problem ID [9142]
Book : Second order enumerated odes
Section : section 2
Problem number : 20
Date solved : `Monday, January 27, 2025 at 05:48:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 3xy' + 3y = 2x^3 - x^2$$

Solved as second order Euler type ode

Time used: 0.140 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = 2x^3 - x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 3 = 0$$

Or

$$r^2 - 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2\, x^3 + c_1 x$$

Next, we find the particular solution to the ODE

$$x^2 y'' - 3xy' + 3y = 2x^3 - x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(2x^3 - x^2)}{2x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \left(x - \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 + \frac{1}{2}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^3 - x^2)}{2x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2x - 1}{2x^2} dx$$

Hence

$$u_2 = \frac{1}{2x} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 + \frac{1}{2}x\right) x + x^3\left(\frac{1}{2x} + \ln(x)\right)$$

Which simplifies to

$$y_p(x) = x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_2\, x^3 + c_1 x + x^3 \ln(x) - \frac{x^3}{2} + x^2 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2\, x^3 + c_1 x + x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Solved as second order ode using change of variable on x method 2

Time used: 0.696 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\int p(x)dx}dx \\ &= \int \mathrm{e}^{-\int -\frac{3}{x}dx}dx \\ &= \int e^{3\ln(x)}\, dx \\ &= \int x^3dx \\ &= \frac{x^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{3}{x^2}}{x^6} \\ &= \frac{3}{x^8} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{x^8} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{3}{x^8} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. Writing the ode as

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0 \tag{1}$$

$$A\frac{d^2}{d\tau^2}y(\tau) + B\frac{d}{d\tau}y(\tau) + Cy(\tau) = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{3}{16\tau^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(\tau) = y(\tau)\, e^{\int \frac{B}{2A}\, d\tau}$$

Then (2) becomes

$$z''(\tau) = rz(\tau) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16\tau^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 16\tau^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(\tau) = \left(-\frac{3}{16\tau^2}\right) z(\tau) \tag{7}$$

Equation (7) is now solved. After finding $z(\tau)$ then $y(\tau)$ is found using the inverse transformation

$$y(\tau) = z(\tau)\, e^{-\int \frac{B}{2A}\, d\tau}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.79: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16\tau^2$. There is a pole at $\tau = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{3}{16\tau^2}
$$

For the pole at $\tau = 0$ let b be the coefficient of $\frac{1}{\tau^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{\tau^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{3}{16\tau^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16\tau^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{\tau - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{\tau - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4\tau} + (-)\,(0) \\ &= \frac{1}{4\tau} \\ &= \frac{1}{4\tau} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(\tau)$ of degree $d = 0$ to solve the ode. The polynomial $p(\tau)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(\tau) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{4\tau}\right)(0) + \left(\left(-\frac{1}{4\tau^2}\right) + \left(\frac{1}{4\tau}\right)^2 - \left(-\frac{3}{16\tau^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(\tau) &= p e^{\int \omega \, d\tau} \\ &= e^{\int \frac{1}{4\tau} d\tau} \\ &= \tau^{1/4} \end{aligned}$$

The first solution to the original ode in $y(\tau)$ is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,d\tau}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \tau^{1/4} \end{aligned}$$

Which simplifies to

$$y_1 = \tau^{1/4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,d\tau}}{y_1^2}\,d\tau$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,d\tau \\ &= \tau^{1/4} \int \frac{1}{\sqrt{\tau}}\,d\tau \\ &= \tau^{1/4}\left(2\sqrt{\tau}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(\tau) &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\tau^{1/4}\right) + c_2\left(\tau^{1/4}\left(2\sqrt{\tau}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 4^{3/4} \left(x^4\right)^{1/4}}{4} + \frac{c_2 4^{1/4} \left(x^4\right)^{3/4}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 4^{3/4} \left(x^4\right)^{1/4}}{4} + \frac{c_2 4^{1/4} \left(x^4\right)^{3/4}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \left(x^4\right)^{1/4} \\ y_2 &= \left(x^4\right)^{3/4} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^4)^{1/4} & (x^4)^{3/4} \\ \frac{d}{dx}\left((x^4)^{1/4}\right) & \frac{d}{dx}\left((x^4)^{3/4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^4)^{1/4} & (x^4)^{3/4} \\ \frac{x^3}{(x^4)^{3/4}} & \frac{3x^3}{(x^4)^{1/4}} \end{vmatrix}$$

Therefore

$$W = \left((x^4)^{1/4}\right)\left(\frac{3x^3}{(x^4)^{1/4}}\right) - \left((x^4)^{3/4}\right)\left(\frac{x^3}{(x^4)^{3/4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^4)^{3/4}(2x^3 - x^2)}{2x^5}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^4)^{3/4}(2x - 1)}{2x^3}dx$$

Hence

$$u_1 = -\frac{(x - 1)(x^4)^{3/4}}{2x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^4)^{1/4}(2x^3 - x^2)}{2x^5}\,dx$$

Which simplifies to

$$u_2 = \int \frac{(x^4)^{1/4}(2x - 1)}{2x^3}dx$$

Hence

$$u_2 = \frac{(x^4)^{1/4}}{2x^2} + \frac{(x^4)^{1/4}\ln(x)}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(x-1)}{2} + \left(x^4\right)^{3/4}\left(\frac{\left(x^4\right)^{1/4}}{2x^2} + \frac{\left(x^4\right)^{1/4}\ln(x)}{x}\right)$$

Which simplifies to

$$y_p(x) = x^3\ln(x) - \frac{x^3}{2} + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 4^{3/4}\left(x^4\right)^{1/4}}{4} + \frac{c_2 4^{1/4}\left(x^4\right)^{3/4}}{2}\right) + \left(x^3\ln(x) - \frac{x^3}{2} + x^2\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^3\ln(x) - \frac{x^3}{2} + x^2 + \frac{c_1 4^{3/4}\left(x^4\right)^{1/4}}{4} + \frac{c_2 4^{1/4}\left(x^4\right)^{3/4}}{2}$$

Solved as second order ode using change of variable on x method 1

Time used: 0.247 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = 2x^3 - x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2}\\ &= -\frac{4c\sqrt{3}}{3}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{3}\left(\frac{d}{d\tau}y(\tau)\right)}{3} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{2\sqrt{3}\,c\tau}{3}}\left(c_1\cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + ic_2\sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right)\right)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx\\ &= \frac{\int \sqrt{3}\,\sqrt{\frac{1}{x^2}}dx}{c}\\ &= \frac{\sqrt{3}\,\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^2\left(c_1\left(\frac{x}{2} + \frac{1}{2x}\right) + ic_2\left(\frac{x}{2} - \frac{1}{2x}\right)\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2\left(\frac{x}{2} + \frac{1}{2x}\right)$$
$$y_2 = i x^2\left(\frac{x}{2} - \frac{1}{2x}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2\left(\frac{x}{2} + \frac{1}{2x}\right) & i x^2\left(\frac{x}{2} - \frac{1}{2x}\right) \\ \frac{d}{dx}\left(x^2\left(\frac{x}{2} + \frac{1}{2x}\right)\right) & \frac{d}{dx}\left(i x^2\left(\frac{x}{2} - \frac{1}{2x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2\left(\frac{x}{2} + \frac{1}{2x}\right) & i x^2\left(\frac{x}{2} - \frac{1}{2x}\right) \\ 2x\left(\frac{x}{2} + \frac{1}{2x}\right) + x^2\left(\frac{1}{2} - \frac{1}{2x^2}\right) & 2ix\left(\frac{x}{2} - \frac{1}{2x}\right) + i x^2\left(\frac{1}{2} + \frac{1}{2x^2}\right) \end{vmatrix}$$

Therefore

$$W = \left(x^2\left(\frac{x}{2} + \frac{1}{2x}\right)\right)\left(2ix\left(\frac{x}{2} - \frac{1}{2x}\right) + i x^2\left(\frac{1}{2} + \frac{1}{2x^2}\right)\right) - \left(i x^2\left(\frac{x}{2} - \frac{1}{2x}\right)\right)\left(2x\left(\frac{x}{2} + \frac{1}{2x}\right) + x^2\left(\frac{1}{2} - \frac{1}{2x^2}\right)\right)$$

Which simplifies to

$$W = i x^3$$

Which simplifies to

$$W = i x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{i x^2\left(\frac{x}{2} - \frac{1}{2x}\right)(2x^3 - x^2)}{i x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^2 - 1)(2x - 1)}{2x^2} dx$$

Hence

$$u_1 = -\frac{x^2}{2} + \frac{x}{2} + \frac{1}{2x} + \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2\left(\frac{x}{2}+\frac{1}{2x}\right)(2x^3-x^2)}{i x^5}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{i(x^2+1)(2x-1)}{2x^2}dx$$

Hence

$$u_2 = -\frac{i\left(x^2-x+\frac{1}{x}+2\ln(x)\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x^2}{2}+\frac{x}{2}+\frac{1}{2x}+\ln(x)\right)x^2\left(\frac{x}{2}+\frac{1}{2x}\right)+\frac{x^2\left(\frac{x}{2}-\frac{1}{2x}\right)\left(x^2-x+\frac{1}{x}+2\ln(x)\right)}{2}$$

Which simplifies to

$$y_p(x) = x^3\ln(x)-\frac{x^3}{2}+x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^2\left(c_1\left(\frac{x}{2}+\frac{1}{2x}\right)+ic_2\left(\frac{x}{2}-\frac{1}{2x}\right)\right)\right)+\left(x^3\ln(x)-\frac{x^3}{2}+x^2\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^2\left(c_1\left(\frac{x}{2}+\frac{1}{2x}\right)+ic_2\left(\frac{x}{2}-\frac{1}{2x}\right)\right)+x^3\ln(x)-\frac{x^3}{2}+x^2$$

Solved as second order ode using change of variable on y method 2

Time used: 0.302 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x^2, B=-3x, C=3, f(x)=2x^3-x^2$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''-3xy'+3y=0$$

In normal form the ode

$$x^2y''-3xy'+3y=0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{3}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int \frac{3}{x}dx} \\ &= x^3 \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu u = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(u\, x^3\right) = 0$$

Integrating gives

$$\begin{aligned} u\, x^3 &= \int 0\, dx + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor x^3 gives the final solution

$$u(x) = \frac{c_1}{x^3}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x^3 \\ &= c_2\, x^3 - \frac{1}{2} c_1 x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3xy' + 3y = 2x^3 - x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(2x^3 - x^2)}{2x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \left(x - \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 + \frac{1}{2}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^3 - x^2)}{2x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2x - 1}{2x^2} dx$$

Hence

$$u_2 = \frac{1}{2x} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 + \frac{1}{2}x\right) x + x^3 \left(\frac{1}{2x} + \ln(x)\right)$$

Which simplifies to

$$y_p(x) = x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x^3\right) + \left(x^3 \ln(x) - \frac{x^3}{2} + x^2\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x^3 + x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Solved as second order ode using non constant coeff transformation on B method

Time used: 0.134 (sec)

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 3 \\ F &= 2x^3 - x^2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (-3x)(-3) + (3)(-3x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-3x^3v'' + (3x^2)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$-3x^2(u'(x)\,x - u(x)) = 0$$

Which is now solved for u. In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{1}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{x}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{u}{x} &= \int 0\,dx + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{x}$ gives the final solution

$$u(x) = c_1 x$$

The ode for v now becomes

$$v'(x) = c_1 x$$

Which is now solved for v. Since the ode has the form $v'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dv &= \int c_1 x\,dx \\ v(x) &= \frac{c_1\,x^2}{2} + c_2 \end{aligned}$$

Replacing $v(x)$ above by $-\frac{y}{3x}$, then the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= -\frac{3(c_1\,x^2 + 2c_2)\,x}{2} \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(2x^3 - x^2)}{2x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \left(x - \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 + \frac{1}{2}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^3 - x^2)}{2x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2x-1}{2x^2} dx$$

Hence

$$u_2 = \frac{1}{2x} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 + \frac{1}{2}x\right)x + x^3\left(\frac{1}{2x} + \ln(x)\right)$$

Which simplifies to

$$y_p(x) = x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{3(c_1 x^2 + 2c_2)\, x}{2}\right) + \left(x^3 \ln(x) - \frac{x^3}{2} + x^2\right) \\ &= x^3 \ln(x) + \frac{(-3c_1 - 1)\, x^3}{2} + x^2 - 3c_2 x \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^3 \ln(x) + \frac{(-3c_1 - 1)\, x^3}{2} + x^2 - 3c_2 x$$

Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$\begin{aligned} x^2 y'' - 3xy' + 3y &= 0 && (1) \\ Ay'' + By' + Cy &= 0 && (2) \end{aligned}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 3 \end{aligned} \qquad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \qquad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \qquad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.80: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2}} \\ &= z_1 \left(x^{3/2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{x^2}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2\left(x\left(\frac{x^2}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3xy' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1x + \frac{1}{2}c_2\, x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{x^3}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{x^3}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ 1 & \frac{3x^2}{2} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{3x^2}{2}\right) - \left(\frac{x^3}{2}\right)(1)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3(2x^3-x^2)}{2}}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \left(x - \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 + \frac{1}{2}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^3 - x^2)}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2x - 1}{x^2} dx$$

Hence

$$u_2 = \frac{1}{x} + 2\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 + \frac{1}{2}x\right) x + \frac{x^3\left(\frac{1}{x} + 2\ln(x)\right)}{2}$$

Which simplifies to

$$y_p(x) = x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x + \frac{1}{2}c_2\, x^3\right) + \left(x^3 \ln(x) - \frac{x^3}{2} + x^2\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + \frac{c_2\, x^3}{2} + x^3 \ln(x) - \frac{x^3}{2} + x^2$$

Solved as second order ode adjoint method

Time used: 0.239 (sec)

In normal form the ode

$$x^2y'' - 3xy' + 3y = 2x^3 - x^2 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \\ r(x) &= 2x - 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(-\frac{3\xi(x)}{x}\right)' + \left(\frac{3\xi(x)}{x^2}\right) &= 0 \\ \xi''(x) + \frac{3\xi'(x)}{x} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order ode with missing dependent variable ξ. Let

$$u(x) = \xi'$$

Then

$$u'(x) = \xi''$$

Hence the ode becomes

$$u'(x) + \frac{3u(x)}{x} = 0$$

Which is now solved for $u(x)$ as first order ode.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{3}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= e^{\int \frac{3}{x} dx} \\ &= x^3 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(u\, x^3\right) &= 0 \end{aligned}$$

Integrating gives

$$u\,x^3 = \int 0\,dx + c_1$$
$$= c_1$$

Dividing throughout by the integrating factor x^3 gives the final solution

$$u(x) = \frac{c_1}{x^3}$$

In summary, these are the solution found for $u(x)$

$$u(x) = \frac{c_1}{x^3}$$

For solution $u(x) = \frac{c_1}{x^3}$, since $u = \xi'$ then we now have a new first order ode to solve which is

$$\xi' = \frac{c_1}{x^3}$$

Since the ode has the form $\xi' = f(x)$, then we only need to integrate $f(x)$.

$$\int d\xi = \int \frac{c_1}{x^3}\,dx$$
$$\xi = -\frac{c_1}{2x^2} + c_2$$

In summary, these are the solution found for (ξ)

$$\xi = -\frac{c_1}{2x^2} + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}$$

Or

$$y' + y\left(-\frac{3}{x} - \frac{c_1}{x^3\left(-\frac{c_1}{2x^2} + c_2\right)}\right) = \frac{c_2\,x^2 - c_2 x - \frac{c_1}{2x} - c_1 \ln(x)}{-\frac{c_1}{2x^2} + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{-6c_2\,x^2 + c_1}{x\,(-2c_2\,x^2 + c_1)}$$
$$p(x) = \frac{x(-2c_2\,x^3 + 2c_1 \ln(x)\,x + 2c_2\,x^2 + c_1)}{-2c_2\,x^2 + c_1}$$

The integrating factor μ is

$$\mu = e^{\int q\,dx}$$
$$= \mathrm{e}^{\int -\frac{-6c_2\,x^2 + c_1}{x\left(-2c_2\,x^2 + c_1\right)}dx}$$
$$= -\frac{1}{x\,(-2c_2\,x^2 + c_1)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x(-2c_2\,x^3+2c_1\ln(x)\,x+2c_2\,x^2+c_1)}{-2c_2\,x^2+c_1}\right)$$

$$\frac{d}{dx}\left(-\frac{y}{x\,(-2c_2\,x^2+c_1)}\right) = \left(-\frac{1}{x\,(-2c_2\,x^2+c_1)}\right)\left(\frac{x(-2c_2\,x^3+2c_1\ln(x)\,x+2c_2\,x^2+c_1)}{-2c_2\,x^2+c_1}\right)$$

$$d\left(-\frac{y}{x\,(-2c_2\,x^2+c_1)}\right) = \left(-\frac{-2c_2\,x^3+2c_1\ln(x)\,x+2c_2\,x^2+c_1}{(-2c_2\,x^2+c_1)^2}\right)dx$$

Integrating gives

$$-\frac{y}{x\,(-2c_2\,x^2+c_1)} = \int -\frac{-2c_2\,x^3+2c_1\ln(x)\,x+2c_2\,x^2+c_1}{(-2c_2\,x^2+c_1)^2}\,dx$$

$$= \frac{\frac{x}{2}-\frac{c_1}{8c_2}}{-\frac{c_1}{2}+c_2\,x^2}+\frac{\ln(x)\,x^2}{2c_2\,x^2-c_1}+c_3$$

Dividing throughout by the integrating factor $-\frac{1}{x(-2c_2\,x^2+c_1)}$ gives the final solution

$$y = \frac{x(8c_2^2c_3\,x^2+4x^2\ln(x)\,c_2-4c_1c_2c_3+4c_2x-c_1)}{4c_2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{x(8c_2^2c_3\,x^2+4x^2\ln(x)\,c_2-4c_1c_2c_3+4c_2x-c_1)}{4c_2}$$

The constants can be merged to give

$$y = \frac{x(8c_2^2x^2+4x^2\ln(x)\,c_2-4c_1c_2+4c_2x-c_1)}{4c_2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x(8c_2^2x^2+4x^2\ln(x)\,c_2-4c_1c_2+4c_2x-c_1)}{4c_2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 28

```
dsolve(x^2*diff(diff(y(x),x),x)-3*diff(y(x),x)*x+3*y(x) = 2*x^3-x^2,y(x),singsol=all)
```

$$y = \frac{x(2\ln(x)\, x^2 + (c_1 - 1)\, x^2 + 2x + 2c_2)}{2}$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 27

```
DSolve[{x^2*D[y[x],{x,2}]-3*x*D[y[x],x]+3*y[x]==2*x^3-x^2,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to x\left(x^2 \log(x) + \left(-\frac{3}{2} + c_2\right) x^2 + x + c_1\right)$$

2.2.20 Problem 21

Internal problem ID [9143]
Book : Second order enumerated odes
Section : section 2
Problem number : 21
Date solved : `Monday, January 27, 2025 at 05:48:57 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']`

Solve

$$y'' + \cot(x) y' + 4y \csc(x)^2 = 0$$

Solved as second order ode using change of variable on x method 2

Time used: 0.534 (sec)

In normal form the ode

$$y'' + \cot(x) y' + 4y \csc(x)^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \cot(x) \\ q(x) &= 4\csc(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\int p(x)dx} dx \\
&= \int \mathrm{e}^{-\int \cot(x)dx} dx \\
&= \int e^{-\ln(\sin(x))} \, dx \\
&= \int \csc(x) \, dx \\
&= -\ln(\csc(x) + \cot(x))
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{4\csc(x)^2}{\csc(x)^2} \\
&= 4
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + 4y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\tau\lambda} + 4\,\mathrm{e}^{\tau\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 + 4 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\
&= \pm 2i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2i \\
\lambda_2 &= -2i
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^{0}(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(2\ln(\csc(x) + \cot(x))) - c_2 \sin(2\ln(\csc(x) + \cot(x)))$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos(2\ln(\csc(x) + \cot(x))) - c_2 \sin(2\ln(\csc(x) + \cot(x)))$$

Solved as second order ode using change of variable on x method 1

Time used: 0.398 (sec)

In normal form the ode

$$y'' + \cot(x)\, y' + 4y\csc(x)^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \cot(x) \\ q(x) &= 4\csc(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\csc(x)^2}}{c} \qquad (6) \\ \tau'' &= -\frac{2\csc(x)^2 \cot(x)}{c\sqrt{\csc(x)^2}} \end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2\csc(x)^2\cot(x)}{c\sqrt{\csc(x)^2}} + \cot(x)\,\frac{2\sqrt{\csc(x)^2}}{c}}{\left(\frac{2\sqrt{\csc(x)^2}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1\cos(c\tau) + c_2\sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 2\sqrt{\csc(x)^2}dx}{c} \\ &= \frac{2\ln(\csc(x) - \cot(x))}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1\cos(2\ln(\csc(x) - \cot(x))) + c_2\sin(2\ln(\csc(x) - \cot(x)))$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\cos(2\ln(\csc(x) - \cot(x))) + c_2\sin(2\ln(\csc(x) - \cot(x)))$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 23

```
dsolve(diff(diff(y(x),x),x)+cot(x)*diff(y(x),x)+4*y(x)*csc(x)^2 = 0,y(x),singsol=all)
```

$$y = c_1(\csc(x) + \cot(x))^{-2i} + c_2(\csc(x) + \cot(x))^{2i}$$

Mathematica DSolve solution

Solving time : `0.048` (sec)
Leaf size : 25

```
DSolve[{D[y[x],{x,2}]+Cot[x]*D[y[x],x]+4*y[x]*Csc[x]^2==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to c_1 \cos(2\text{arctanh}(\cos(x))) - c_2 \sin(2\text{arctanh}(\cos(x)))$$

2.2.21 Problem 22

Internal problem ID [9144]
Book : Second order enumerated odes
Section : section 2
Problem number : 22
Date solved : `Tuesday, January 28, 2025 at 04:00:10 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$\left(x^2+1\right) y''+(1+x) y'+y=4\cos\left(\ln\left(1+x\right)\right)$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.077 (sec)
Leaf size : 280

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)+y(x) = 4*cos(ln(x+1)),y(x),singso
```

$$y = \text{hypergeom}\left([i,-i],\left[\frac{1}{2}+\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)c_2 + (x+i)^{\frac{1}{2}-\frac{i}{2}}\,\text{hypergeom}\left(\left[\frac{1}{2}+\frac{i}{2},\frac{1}{2}-\frac{3i}{2}\right],\left[\frac{3}{2}-\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)c_1$$

$$-80\left(\int \frac{}{(x^2+1)\left(10\,\text{hypergeom}\left(\left[\frac{1}{2}+\frac{i}{2},\frac{1}{2}-\frac{3i}{2}\right],\left[\frac{3}{2}-\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)\left((-1-i+(-1+i)x)\,\text{hypergeom}\left([1\right.\right.\right.}\right.$$

$$\left.-i],\left[\frac{1}{2}+\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)$$

$$-80\left(\int \frac{\text{hypergeom}\left([i,-\right.}{7\left(\frac{10((1-i+(-1-i)x)\,\text{hypergeom}([1-i,1+i],[\frac{3}{2}+\frac{i}{2}],\frac{1}{2}-\frac{ix}{2})+(-1+i)\,\text{hypergeom}([i,-i],[\frac{1}{2}+\frac{i}{2}],\frac{1}{2}-\frac{ix}{2}))\,\text{hypergeom}([\frac{1}{2}+\frac{i}{2},\frac{1}{2}-\frac{3i}{2}],}{7}\right.}\right.$$

$$\left.+i)^{\frac{1}{2}-\frac{i}{2}}\,\text{hypergeom}\left(\left[\frac{1}{2}+\frac{i}{2},\frac{1}{2}-\frac{3i}{2}\right],\left[\frac{3}{2}-\frac{i}{2}\right],\frac{1}{2}-\frac{ix}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+(1+x)*D[y[x],x]+y[x]==4*Cos[Log[1+x]],{}},y[x],x,IncludeSingular
```

Not solved

2.2.22 Problem 23

Internal problem ID [9145]
Book : Second order enumerated odes
Section : section 2
Problem number : 23
Date solved : `Monday, January 27, 2025 at 05:49:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \tan(x)\, y' + \cos(x)^2\, y = 0$$

Solved as second order ode using change of variable on x method 2

Time used: 0.437 (sec)

In normal form the ode

$$y'' + \tan(x)\, y' + \cos(x)^2\, y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= \cos(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\int p(x)dx} dx \\ &= \int e^{-\int \tan(x)dx} dx \\ &= \int e^{\ln(\cos(x))}\, dx \\ &= \int \cos(x)\, dx \\ &= \sin(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\cos(x)^2}{\cos(x)^2} \\ &= 1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\tau\lambda} + e^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i \\ \lambda_2 &= -i \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x))$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x))$$

Solved as second order ode using change of variable on x method 1

Time used: 0.171 (sec)

In normal form the ode

$$y'' + \tan(x)\, y' + \cos(x)^2\, y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= \cos(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{\cos(2x)}{2} + \frac{1}{2}}}{c} \\ \tau'' &= -\frac{2\cos(x)\sin(x)}{c\sqrt{2\cos(2x) + 2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2\cos(x)\sin(x)}{c\sqrt{2\cos(2x)+2}} + \tan(x)\frac{\sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}}{c}}{\left(\frac{\sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{\frac{\cos(2x)}{2} + \frac{1}{2}} dx}{c} \\
&= \frac{\sin(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x))
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x))
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
   Change of variables used:
      [x = arcsin(t)]
   Linear ODE actually solved:
      (-2*t^2+2)*u(t)+(-2*t^2+2)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

Maple dsolve solution

Solving time : 0.085 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x)+tan(x)*diff(y(x),x)+cos(x)^2*y(x) = 0,y(x),singsol=all)
```

$$
y = c_1 \sin(\sin(x)) + c_2 \cos(\sin(x))
$$

Mathematica DSolve solution

Solving time : 1.833 (sec)
Leaf size : 37

```
DSolve[{D[y[x],{x,2}]+Tan[x]*D[y[x],x]+Cos[x]^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to c_1 \cosh\left(\sqrt{-\sin^2(x)}\right) + ic_2 \sinh\left(\sqrt{-\sin^2(x)}\right)$$

2.2.23 Problem 24

Internal problem ID [9146]
Book : Second order enumerated odes
Section : section 2
Problem number : 24
Date solved : `Monday, January 27, 2025 at 05:49:06 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$xy'' - y' + 4x^3y = 8x^3 \sin(x)^2$$

Solved as second order ode using change of variable on x method 2

Time used: 10.686 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' - y' + 4x^3y = 0$$

In normal form the ode

$$xy'' - y' + 4x^3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = 4x^2$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\int p(x)dx} dx \\
&= \int \mathrm{e}^{-\int -\frac{1}{x}dx} dx \\
&= \int e^{\ln(x)}\, dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{4x^2}{x^2} \\
&= 4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) = 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\tau\lambda} + 4\,\mathrm{e}^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\
&= \pm 2i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha \tau}(c_1 \cos(\beta \tau) + c_2 \sin(\beta \tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x^2) + c_2 \sin(x^2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x^2)$$
$$y_2 = \sin(x^2)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ \frac{d}{dx}(\cos(x^2)) & \frac{d}{dx}(\sin(x^2)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ -2x\sin(x^2) & 2x\cos(x^2) \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(x^2\right)\right)\left(2x\cos\left(x^2\right)\right) - \left(\sin\left(x^2\right)\right)\left(-2x\sin\left(x^2\right)\right)$$

Which simplifies to

$$W = 2\cos\left(x^2\right)^2 x + 2\sin\left(x^2\right)^2 x$$

Which simplifies to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\sin(x^2)\,x^3\sin(x)^2}{2x^2}\,dx$$

Which simplifies to

$$u_1 = -\int 4\sin\left(x^2\right)x\sin(x)^2\,dx$$

Hence

$$\begin{aligned}u_1 = {}& \cos\left(x^2\right) - \frac{\cos(x^2-2x)}{2} \\ &+ \frac{\sqrt{2}\sqrt{\pi}\left(\cos(1)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) - \sin(1)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} - \frac{\cos(x^2+2x)}{2} \\ &- \frac{\sqrt{2}\sqrt{\pi}\left(\cos(1)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) - \sin(1)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\right)}{2}\end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\cos(x^2)\,x^3\sin(x)^2}{2x^2}\,dx$$

Which simplifies to

$$u_2 = \int 4\cos\left(x^2\right)x\sin(x)^2\,dx$$

Hence

$$\begin{aligned}u_2 = {}& \sin\left(x^2\right) - \frac{\sin(x^2-2x)}{2} \\ &- \frac{\sqrt{2}\sqrt{\pi}\left(\cos(1)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) + \sin(1)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} - \frac{\sin(x^2+2x)}{2} \\ &+ \frac{\sqrt{2}\sqrt{\pi}\left(\cos(1)\operatorname{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) + \sin(1)\operatorname{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\right)}{2}\end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\cos\left(x^2\right) - \frac{\cos\left(x^2-2x\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\left(\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) - \sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} - \frac{\cos\left(x^2+2x\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\left(\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) - \sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\right)}{2}\right)\cos\left(x^2\right) + \sin\left(x^2\right)\left(\sin\left(x^2\right) - \frac{\sin\left(x^2-2x\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\left(\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) + \sin\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} - \frac{\sin\left(x^2+2x\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\left(\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) + \sin\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\right)}{2}\right)$$

Which simplifies to

$$y_p(x) = 1 - \cos\left(2x\right) + \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\sin\left(x^2+1\right)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)\right) + \left(1 - \cos\left(2x\right) + \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\sin\left(x^2+1\right)}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 1-\cos(2x)+\frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\cos(x^2+1)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\cos(x^2+1)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin(x^2+1)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\sin(x^2+1)}{2} + c_1\cos\left(x^2\right)+c_2\sin\left(x^2\right)$$

Solved as second order ode using change of variable on x method 1

Time used: 10.152 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x, B=-1, C=4x^3, f(x)=8x^3\sin(x)^2$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$xy''-y'+4x^3y=0$$

In normal form the ode

$$xy''-y'+4x^3y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=-\frac{1}{x}\\ q(x)&=4x^2\end{aligned}$$

Applying change of variables $\tau=g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau)=\frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1=c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau'&=\frac{1}{c}\sqrt{q}\\ &=\frac{2\sqrt{x^2}}{c} \qquad (6)\\ \tau''&=\frac{2x}{c\sqrt{x^2}}\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2x}{c\sqrt{x^2}} - \frac{1}{x}\frac{2\sqrt{x^2}}{c}}{\left(\frac{2\sqrt{x^2}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 2\sqrt{x^2}dx}{c} \\ &= \frac{x\sqrt{x^2}}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(x\sqrt{x^2}\right) + c_2 \sin\left(x\sqrt{x^2}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos\left(x\sqrt{x^2}\right) \\ y_2 &= \sin\left(x\sqrt{x^2}\right) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(x\sqrt{x^2}\right) & \sin\left(x\sqrt{x^2}\right) \\ \frac{d}{dx}\left(\cos\left(x\sqrt{x^2}\right)\right) & \frac{d}{dx}\left(\sin\left(x\sqrt{x^2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(x\sqrt{x^2}\right) & \sin\left(x\sqrt{x^2}\right) \\ -\left(\sqrt{x^2}+\frac{x^2}{\sqrt{x^2}}\right)\sin\left(x\sqrt{x^2}\right) & \left(\sqrt{x^2}+\frac{x^2}{\sqrt{x^2}}\right)\cos\left(x\sqrt{x^2}\right) \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\cos\left(x\sqrt{x^2}\right)\right)\left(\left(\sqrt{x^2}+\frac{x^2}{\sqrt{x^2}}\right)\cos\left(x\sqrt{x^2}\right)\right) \\ & - \left(\sin\left(x\sqrt{x^2}\right)\right)\left(-\left(\sqrt{x^2}+\frac{x^2}{\sqrt{x^2}}\right)\sin\left(x\sqrt{x^2}\right)\right) \end{aligned}$$

Which simplifies to

$$W = \frac{2x^2\left(\cos\left(x\sqrt{x^2}\right)^2 + \sin\left(x\sqrt{x^2}\right)^2\right)}{\sqrt{x^2}}$$

Which simplifies for $0 < x$ to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\sin\left(x\sqrt{x^2}\right)x^3\sin(x)^2}{2x^2}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int 4\sin\left(x^2\right)x\sin(x)^2\,dx$$

Hence

$$\begin{aligned} u_1 = {} & \cos\left(x^2\right) - \frac{\cos\left(x^2-2x\right)}{2} \\ & + \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) - \sin(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} - \frac{\cos\left(x^2+2x\right)}{2} \\ & - \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) - \sin(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\right)}{2} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\cos\left(x\sqrt{x^2}\right)x^3\sin(x)^2}{2x^2}\,dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int 4\cos\left(x^2\right)x\sin(x)^2\,dx$$

Hence

$$\begin{aligned} u_2 = {} & \sin\left(x^2\right) - \frac{\sin\left(x^2-2x\right)}{2} \\ & - \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) + \sin(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{2} - \frac{\sin\left(x^2+2x\right)}{2} \\ & + \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) + \sin(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\right)}{2} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\cos\left(x^2\right) - \frac{\cos\left(x^2-2x\right)}{2} + \frac{\sqrt{2}\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}(x-1)}{\sqrt{\pi}}\right) - \sin(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}(x-1)}{\sqrt{\pi}}\right)\right)}{2} - \frac{\cos\left(x^2+2x\right)}{2} - \frac{\sqrt{2}\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}(x+1)}{\sqrt{\pi}}\right) - \sin(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}(x+1)}{\sqrt{\pi}}\right)\right)}{2}\right)\cos\left(x\sqrt{x^2}\right) + \sin\left(x\sqrt{x^2}\right)\left(\sin\left(x^2\right) - \frac{\sin\left(x^2-2x\right)}{2} - \frac{\sqrt{2}\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}(x-1)}{\sqrt{\pi}}\right) + \sin(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}(x-1)}{\sqrt{\pi}}\right)\right)}{2} - \frac{\sin\left(x^2+2x\right)}{2} + \frac{\sqrt{2}\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}(x+1)}{\sqrt{\pi}}\right) + \sin(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}(x+1)}{\sqrt{\pi}}\right)\right)}{2}\right)$$

Which simplifies to

$$\begin{aligned} y_p(x) = {} & -\frac{\sqrt{2}\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}(x-1)}{\sqrt{\pi}}\right)\left(\sin\left(x^2\right)\cos(1) + \cos\left(x^2\right)\sin(1)\right)}{2} \\ & + \frac{\sqrt{2}\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}(x+1)}{\sqrt{\pi}}\right)\left(\sin\left(x^2\right)\cos(1) + \cos\left(x^2\right)\sin(1)\right)}{2} \\ & - \frac{\sqrt{2}\sqrt{\pi}\left(-\cos\left(x^2\right)\cos(1) + \sin\left(x^2\right)\sin(1)\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}(x-1)}{\sqrt{\pi}}\right)}{2} \\ & + \frac{\sqrt{2}\sqrt{\pi}\left(-\cos\left(x^2\right)\cos(1) + \sin\left(x^2\right)\sin(1)\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}(x+1)}{\sqrt{\pi}}\right)}{2} \\ & - \left(\cos\left(x^2\right)^2 + \sin\left(x^2\right)^2\right)\left(-1 + \cos(2x)\right) \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(x\sqrt{x^2}\right) + c_2\sin\left(x\sqrt{x^2}\right)\right) \\ &\quad + \left(-\frac{\sqrt{2}\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}(x-1)}{\sqrt{\pi}}\right)\left(\sin\left(x^2\right)\cos(1) + \cos\left(x^2\right)\sin(1)\right)}{2}\right. \\ &\quad + \frac{\sqrt{2}\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}(x+1)}{\sqrt{\pi}}\right)\left(\sin\left(x^2\right)\cos(1) + \cos\left(x^2\right)\sin(1)\right)}{2} \\ &\quad - \frac{\sqrt{2}\sqrt{\pi}\left(-\cos\left(x^2\right)\cos(1) + \sin\left(x^2\right)\sin(1)\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}(x-1)}{\sqrt{\pi}}\right)}{2} \\ &\quad + \frac{\sqrt{2}\sqrt{\pi}\left(-\cos\left(x^2\right)\cos(1) + \sin\left(x^2\right)\sin(1)\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}(x+1)}{\sqrt{\pi}}\right)}{2} \\ &\quad \left. - \left(\cos\left(x^2\right)^2 + \sin\left(x^2\right)^2\right)\left(-1 + \cos(2x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y = {} & c_1 \cos\left(x\sqrt{x^2}\right) + c_2 \sin\left(x\sqrt{x^2}\right) \\
& - \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))}{2} \\
& + \frac{\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)(\sin(x^2)\cos(1)+\cos(x^2)\sin(1))}{2} \\
& - \frac{\sqrt{2}\,\sqrt{\pi}\,(-\cos(x^2)\cos(1)+\sin(x^2)\sin(1))\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)}{2} \\
& + \frac{\sqrt{2}\,\sqrt{\pi}\,(-\cos(x^2)\cos(1)+\sin(x^2)\sin(1))\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)}{2} \\
& - \left(\cos(x^2)^2+\sin(x^2)^2\right)(-1+\cos(2x))
\end{aligned}
$$

Solved as second order Bessel ode

Time used: 0.264 (sec)

Writing the ode as

$$
x^2y'' - y'x + 4x^4y = 8x^4\sin(x)^2 \tag{1}
$$

Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$
x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}
$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$
x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}
$$

With the standard solution

$$
y = x^{\alpha}(c_1\,\mathrm{BesselJ}(n, \beta\,x^{\gamma}) + c_2\,\mathrm{BesselY}(n, \beta\,x^{\gamma})) \tag{4}
$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 1 \\
\beta &= 1 \\
n &= \frac{1}{2} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = \frac{c_1 x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2 x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}
$$

Therefore the homogeneous solution y_h is

$$
y_h = \frac{c_1 x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2 x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{x\sqrt{2}\,\sin\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}}$$

$$y_2 = -\frac{x\sqrt{2}\,\cos\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} & -\frac{x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} \\ \frac{d}{dx}\left(\frac{x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}\right) & \frac{d}{dx}\left(-\frac{x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} & -\frac{x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} \\ \frac{\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{x^2\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,(x^2)^{3/2}} + \frac{2x^2\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} & -\frac{\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} + \frac{x^2\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,(x^2)^{3/2}} + \frac{2x^2\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\frac{x\sqrt{2}\,\sin\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}}\right)\left(-\frac{\sqrt{2}\,\cos\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}} + \frac{x^2\sqrt{2}\,\cos\left(x^2\right)}{\sqrt{\pi}\,\left(x^2\right)^{3/2}} + \frac{2x^2\sqrt{2}\,\sin\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}}\right) \\ & - \left(-\frac{x\sqrt{2}\,\cos\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}}\right)\left(\frac{\sqrt{2}\,\sin\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{x^2\sqrt{2}\,\sin\left(x^2\right)}{\sqrt{\pi}\,\left(x^2\right)^{3/2}} + \frac{2x^2\sqrt{2}\,\cos\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}}\right) \end{aligned}$$

Which simplifies to

$$W = \frac{4x\left(\cos\left(x^2\right)^2 + \sin\left(x^2\right)^2\right)}{\pi}$$

Which simplifies to

$$W = \frac{4x}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{8x^5\sqrt{2}\,\cos(x^2)\sin(x)^2}{\sqrt{\pi}\,\sqrt{x^2}}}{\frac{4x^3}{\pi}}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -2x\sqrt{2}\,\sqrt{\pi}\,\cos\left(x^2\right)\sin\left(x\right)^2 dx$$

Hence

$$u_1 = 2\sqrt{2}\,\sqrt{\pi}\left(\frac{\sin\left(x^2\right)}{4} - \frac{\sin\left(x^2-2x\right)}{8} - \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) + \sin\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{8} - \frac{\sin\left(x^2+2x\right)}{8} + \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) + \sin\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\right)}{8}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{8x^5\sqrt{2}\,\sin(x^2)\sin(x)^2}{\sqrt{\pi}\,\sqrt{x^2}}}{\frac{4x^3}{\pi}}\,dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int 2x\sqrt{2}\,\sqrt{\pi}\,\sin\left(x^2\right)\sin\left(x\right)^2 dx$$

Hence

$$u_2 = 2\sqrt{2}\,\sqrt{\pi}\left(-\frac{\cos\left(x^2\right)}{4} + \frac{\cos\left(x^2-2x\right)}{8} - \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) - \sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{8} + \frac{\cos\left(x^2+2x\right)}{8} + \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos\left(1\right)\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) - \sin\left(1\right)\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\right)}{8}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{4\left(\frac{\sin(x^2)}{4} - \frac{\sin(x^2-2x)}{8} - \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)+\sin(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{8} - \frac{\sin(x^2+2x)}{8} + \cdots\right.}{\sqrt{x^2}} - \frac{4x\cos\left(x^2\right)\left(-\frac{\cos(x^2)}{4} + \frac{\cos(x^2-2x)}{8} - \frac{\sqrt{2}\,\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)-\sin(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\right)}{8} + \frac{\cos(x^2+2x)}{8} + \cdots\right.}{\sqrt{x^2}}$$

Which simplifies to

$$y_p(x) = 1 - \cos\left(2x\right) + \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right)\sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right)\sin\left(x^2+1\right)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1 x\sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x\sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}}\right) + \left(1 - \cos\left(2x\right)\right. \\
&\quad + \frac{\sqrt{2}\, \sqrt{\pi}\, \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\, \sqrt{\pi}\, \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} \\
&\quad \left. - \frac{\sqrt{2}\, \sqrt{\pi}\, \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\, \sqrt{\pi}\, \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y = {} & \frac{c_1 x\sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x\sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} + 1 - \cos\left(2x\right) \\
& + \frac{\sqrt{2}\, \sqrt{\pi}\, \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} - \frac{\sqrt{2}\, \sqrt{\pi}\, \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) \cos\left(x^2+1\right)}{2} \\
& - \frac{\sqrt{2}\, \sqrt{\pi}\, \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2} + \frac{\sqrt{2}\, \sqrt{\pi}\, \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) \sin\left(x^2+1\right)}{2}
\end{aligned}
$$

Solved as second order ode using Kovacic algorithm

Time used: 0.818 (sec)

Writing the ode as

$$
x y'' - y' + 4x^3 y = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x \\
B &= -1 \\
C &= 4x^3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16x^4 + 3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.81: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(2ix\right) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= e^{\int (-\frac{1}{2x} - 2ix) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{x} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-ix^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{i e^{2ix^2}}{4} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-ix^2} \right) + c_2 \left(e^{-ix^2} \left(-\frac{i e^{2ix^2}}{4} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' - y' + 4x^3 y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-ix^2} - \frac{i c_2 e^{ix^2}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-i x^2}$$
$$y_2 = -\frac{i e^{i x^2}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-i x^2} & -\frac{i e^{i x^2}}{4} \\ \frac{d}{dx}\left(e^{-i x^2}\right) & \frac{d}{dx}\left(-\frac{i e^{i x^2}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-i x^2} & -\frac{i e^{i x^2}}{4} \\ -2 i x\, e^{-i x^2} & \frac{x\, e^{i x^2}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-i x^2}\right)\left(\frac{x\, e^{i x^2}}{2}\right) - \left(-\frac{i e^{i x^2}}{4}\right)\left(-2 i x\, e^{-i x^2}\right)$$

Which simplifies to

$$W = e^{-i x^2} x\, e^{i x^2}$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2 i e^{i x^2} x^3 \sin(x)^2}{x^2}\, dx$$

Which simplifies to

$$u_1 = -\int -2 i e^{i x^2} x \sin(x)^2\, dx$$

Hence

$$u_1 = -\frac{e^{ix(2+x)}}{4} + \frac{i\sqrt{\pi}\, e^{-i} \operatorname{erf}\left(\sqrt{-i}\, x - \frac{i}{\sqrt{-i}}\right)}{4\sqrt{-i}} - \frac{e^{ix(-2+x)}}{4} - \frac{i\sqrt{\pi}\, e^{-i} \operatorname{erf}\left(\sqrt{-i}\, x + \frac{i}{\sqrt{-i}}\right)}{4\sqrt{-i}} + \frac{e^{ix^2}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\, e^{-ix^2} x^3 \sin(x)^2}{x^2}\, dx$$

Which simplifies to

$$u_2 = \int 8\, e^{-ix^2} x \sin(x)^2\, dx$$

Hence

$$\begin{aligned} u_2 = -i e^{-ix(-2+x)} + \sqrt{\pi}\, e^{i} (-1)^{3/4} \operatorname{erf}\left((-1)^{1/4} x \right. \\ \left. - (-1)^{1/4}\right) - i e^{-ix(2+x)} - \sqrt{\pi}\, e^{i} (-1)^{3/4} \operatorname{erf}\left((-1)^{1/4} x + (-1)^{1/4}\right) + 2i e^{-ix^2} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = \left(-\frac{e^{ix(2+x)}}{4} + \frac{i\sqrt{\pi}\, e^{-i} \operatorname{erf}\left(\sqrt{-i}\, x - \frac{i}{\sqrt{-i}}\right)}{4\sqrt{-i}} - \frac{e^{ix(-2+x)}}{4} \right. \\ \left. - \frac{i\sqrt{\pi}\, e^{-i} \operatorname{erf}\left(\sqrt{-i}\, x + \frac{i}{\sqrt{-i}}\right)}{4\sqrt{-i}} + \frac{e^{ix^2}}{2} \right) e^{-ix^2} \\ - \frac{i e^{ix^2} \left(-i e^{-ix(-2+x)} + \sqrt{\pi}\, e^{i} (-1)^{3/4} \operatorname{erf}\left((-1)^{1/4} x - (-1)^{1/4}\right) - i e^{-ix(2+x)} - \sqrt{\pi}\, e^{i} (-1)^{3/4} \operatorname{erf}\left((-1)^{1/4} x\right.\right.}{4} \end{aligned}$$

Which simplifies to

$$\begin{aligned} y_p(x) = \left(\frac{1}{8}\right. \\ \left. + \frac{i}{8}\right) \sqrt{2} \left(i \left(\operatorname{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right) \sqrt{2}\,(x+1)\right) - \operatorname{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right)(x-1)\sqrt{2}\right)\right) \sqrt{\pi}\, e^{-ix^2 - i} \right. \\ - \sqrt{\pi} \left(\operatorname{erf}\left(\left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\,(x+1)\right) - \operatorname{erf}\left(\left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\,(x-1)\right)\right) e^{ix^2 + i} \\ \left. + (-2+2i)(-1+\cos(2x))\sqrt{2} \right) \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-ix^2} - \frac{i c_2 e^{ix^2}}{4}\right) + \left(\left(\frac{1}{8}\right.\right. \\ &\quad \left. + \frac{i}{8}\right) \sqrt{2} \left(i \left(\operatorname{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right) \sqrt{2}\,(x+1)\right) - \operatorname{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right)(x-1)\sqrt{2}\right)\right) \sqrt{\pi}\, e^{-ix^2 - i} \right. \\ &\quad - \sqrt{\pi} \left(\operatorname{erf}\left(\left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\,(x+1)\right) - \operatorname{erf}\left(\left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\,(x-1)\right)\right) e^{ix^2 + i} \\ &\quad \left.\left. + (-2+2i)(-1+\cos(2x))\sqrt{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y = {} & c_1 \,\mathrm{e}^{-i x^2} - \frac{i c_2 \,\mathrm{e}^{i x^2}}{4} \\
& + \left(\frac{1}{8}+\frac{i}{8}\right)\sqrt{2}\left(i\left(\operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{2}\,(x+1)\right)-\operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right)\right)\sqrt{\pi}\,\mathrm{e}^{-i x^2-i}\right. \\
& \quad - \sqrt{\pi}\left(\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\,(x+1)\right)-\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\,(x-1)\right)\right)\mathrm{e}^{i x^2+i} \\
& \quad \left. + (-2+2i)\left(-1+\cos\left(2x\right)\right)\sqrt{2}\right)
\end{aligned}
$$

Solved as second order ode adjoint method

Time used: 8.671 (sec)

In normal form the ode

$$
x y'' - y' + 4x^3 y = 8x^3 \sin\left(x\right)^2 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 4x^2 \\
r(x) &= 8x^2 \sin\left(x\right)^2
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(-\frac{\xi(x)}{x}\right)' + \left(4x^2 \xi(x)\right) &= 0 \\
\xi''(x) + \frac{\xi'(x)}{x} + \frac{\left(4x^4-1\right)\xi(x)}{x^2} &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. Writing the ode as

$$
\xi'' x^2 + \xi' x + \left(4x^4 - 1\right)\xi = 0 \tag{1}
$$

Bessel ode has the form

$$
\xi'' x^2 + \xi' x + \left(-n^2 + x^2\right)\xi = 0 \tag{2}
$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$
\xi'' x^2 + (1-2\alpha)\, x \xi' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right)\xi = 0 \tag{3}
$$

With the standard solution

$$
\xi = x^{\alpha}\left(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)\right) \tag{4}
$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= 1 \\
n &= -\frac{1}{2} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$\xi = \frac{c_1\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} + \frac{c_2\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(-\frac{1}{x} - \frac{-\frac{c_1\sqrt{2}\,\cos(x^2)x}{\sqrt{\pi}\,(x^2)^{3/2}} - \frac{2c_1x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2\sqrt{2}\,\sin(x^2)x}{\sqrt{\pi}\,(x^2)^{3/2}} + \frac{2c_2x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}}{\frac{c_1\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} + \frac{c_2\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}}\right) = -\frac{\sqrt{2}\,\sqrt{x^2}\,c_1\,\mathrm{e}^{-i}\,\mathrm{erf}\left(\sqrt{-i}\,x + \frac{i}{\sqrt{-i}}\right)}{2x\sqrt{-i}} + \cdots$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{2(c_2\cos(x^2) - c_1\sin(x^2))\,x}{c_1\cos(x^2) + c_2\sin(x^2)}$$
$$p(x) = \frac{(1-i)\,x\sqrt{2}\left(-\sqrt{\pi}\,\mathrm{e}^{-i}\,\mathrm{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1 + c_2) - \sqrt{\pi}\,\mathrm{e}^{i}\,\mathrm{erf}\left(\left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\,(x-1)\right)(ic_2 + c_1) \cdots\right.}{\cdots}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= \mathrm{e}^{\int -\frac{2\left(c_2\cos\left(x^2\right) - c_1\sin\left(x^2\right)\right)x}{c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)}dx}\\ &= \frac{1}{c_1\cos(x^2) + c_2\sin(x^2)}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{(1-i)\,x\sqrt{2}\left(-\sqrt{\pi}\,\mathrm{e}^{-i}\,\mathrm{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1 + c_2) - \sqrt{\pi}\,\mathrm{e}^{i}\,\mathrm{erf}\left(\left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\,(x-1)\right)(ic_2 + c_1) \cdots\right.}{\cdots}\right.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1\cos(x^2) + c_2\sin(x^2)}\right) = \left(\frac{1}{c_1\cos(x^2) + c_2\sin(x^2)}\right)\left(\frac{(1-i)\,x\sqrt{2}\left(-\sqrt{\pi}\,\mathrm{e}^{-i}\,\mathrm{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1 + c_2) - \sqrt{\pi}\,\mathrm{e}^{i}\,\mathrm{erf}\left(\left(\frac{1}{2} + \frac{i}{2}\right) \cdots\right.\right.}{\cdots}\right.$$

$$\mathrm{d}\left(\frac{y}{c_1\cos(x^2) + c_2\sin(x^2)}\right) = \left(\frac{(1-i)\,x\sqrt{2}\left(-\sqrt{\pi}\,\mathrm{e}^{-i}\,\mathrm{erf}\left(\left(\frac{1}{2} - \frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1 + c_2) - \sqrt{\pi}\,\mathrm{e}^{i}\,\mathrm{erf}\left(\left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\,(x-1)\right)(ic_2 + c_1) + \sqrt{\cdots}\right.}{\cdots}\right.$$

Integrating gives

$$\frac{y}{c_1 \cos(x^2) + c_2 \sin(x^2)} = \int \frac{(1-i)\, x\sqrt{2}\left(-\sqrt{\pi}\, \mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1+c_2) - \sqrt{\pi}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right.\right.\right.}{}$$

$$= \int \frac{(1-i)\, x\sqrt{2}\left(-\sqrt{\pi}\, \mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1+c_2) - \sqrt{\pi}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right.\right.\right.}{}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x^2)+c_2 \sin(x^2)}$ gives the final solution

$$y = \left(c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)\right)\left(\int \frac{(1-i)\, x\sqrt{2}\left(-\sqrt{\pi}\, \mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1+c_2) - \sqrt{\pi}\, \mathrm{e}^{i} \operatorname{erf}\left(\right.\right.}{}\right.$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)\right)\left(\int \frac{(1-i)\, x\sqrt{2}\left(-\sqrt{\pi}\, \mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1+c_2) - \sqrt{\pi}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\,(x\right.\right.}{} + c_3\right)$$

The constants can be merged to give

$$y = \left(c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)\right)\left(\int \frac{(1-i)\, x\sqrt{2}\left(-\sqrt{\pi}\, \mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1+c_2) - \sqrt{\pi}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\,(x\right.\right.}{} + 1\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)\right)\left(\int \frac{(1-i)\, x\sqrt{2}\left(-\sqrt{\pi}\, \mathrm{e}^{-i} \operatorname{erf}\left(\left(\frac{1}{2}-\frac{i}{2}\right)(x-1)\sqrt{2}\right)(ic_1+c_2) - \sqrt{\pi}\, \mathrm{e}^{i} \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\,(x\right.\right.}{} + 1\right)$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 124

```
dsolve(x*diff(diff(y(x),x),x)-diff(y(x),x)+4*x^3*y(x) = 8*sin(x)^2*x^3,y(x),singsol=all)
```

$$y = \sin\left(x^2\right) c_2 + \cos\left(x^2\right) c_1 + 1 - \cos\left(2x\right) - \frac{\sqrt{\pi}\, \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \sqrt{2}\, \sin\left(x^2+1\right)}{2} + \frac{\sqrt{\pi}\, \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x-1)}{\sqrt{\pi}}\right) \sqrt{2}\, \cos\left(x^2+1\right)}{2} + \frac{\sqrt{\pi}\, \mathrm{FresnelC}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) \sqrt{2}\, \sin\left(x^2+1\right)}{2} - \frac{\sqrt{\pi}\, \mathrm{FresnelS}\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{\pi}}\right) \sqrt{2}\, \cos\left(x^2+1\right)}{2}$$

Mathematica DSolve solution

Solving time : `0.417` (sec)
Leaf size : 72

```
DSolve[{x*D[y[x],{x,2}]-D[y[x],x]+4*x^3*y[x]==8*x^3*Sin[x]^2,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \cos\left(x^2\right) \int_1^x -4K[1] \sin^2(K[1]) \sin\left(K[1]^2\right) dK[1] + \sin\left(x^2\right) \int_1^x 4 \cos\left(K[2]^2\right) K[2] \sin^2(K[2]) dK[2] + c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)$$

2.2.24 Problem 25

Internal problem ID [9147]
Book : Second order enumerated odes
Section : section 2
Problem number : 25
Date solved : `Monday, January 27, 2025 at 05:49:37 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$xy'' - y' + 4x^3y = x^5$$

Solved as second order ode using change of variable on x method 2

Time used: 0.391 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' - y' + 4x^3y = 0$$

In normal form the ode

$$xy'' - y' + 4x^3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = 4x^2$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\int p(x)dx}dx \\ &= \int e^{-\int -\frac{1}{x}dx}dx \\ &= \int e^{\ln(x)}\,dx \\ &= \int x dx \\ &= \frac{x^2}{2}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{4x^2}{x^2} \\ &= 4\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) &= 0\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\tau\lambda} + 4\,e^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^{0}(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x^2) + c_2 \sin(x^2)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^5$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3, x^4, x^5\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x^2), \sin(x^2)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_6x^5 + A_5x^4 + A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$x(20x^3A_6 + 12x^2A_5 + 6xA_4 + 2A_3) - 5x^4A_6 - 4x^3A_5 - 3x^2A_4 \\ - 2xA_3 - A_2 + 4x^3(A_6x^5 + A_5x^4 + A_4x^3 + A_3x^2 + A_2x + A_1) = x^5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{4}, A_4 = 0, A_5 = 0, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)\right) + \left(\frac{x^2}{4}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{4} + c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)$$

Solved as second order ode using change of variable on x method 1

Time used: 0.233 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x, B = -1, C = 4x^3, f(x) = x^5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$xy'' - y' + 4x^3 y = 0$$

In normal form the ode

$$xy'' - y' + 4x^3 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{x^2}}{c} \\ \tau'' &= \frac{2x}{c\sqrt{x^2}}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2x}{c\sqrt{x^2}} - \frac{1}{x}\frac{2\sqrt{x^2}}{c}}{\left(\frac{2\sqrt{x^2}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 2\sqrt{x^2}dx}{c} \\ &= \frac{x\sqrt{x^2}}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(x\sqrt{x^2}\right) + c_2 \sin\left(x\sqrt{x^2}\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^5$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3, x^4, x^5\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(x\sqrt{x^2}\right), \sin\left(x\sqrt{x^2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_6x^5 + A_5x^4 + A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$x\left(20x^3A_6 + 12x^2A_5 + 6xA_4 + 2A_3\right) - 5x^4A_6 - 4x^3A_5 - 3x^2A_4 \\ - 2xA_3 - A_2 + 4x^3\left(A_6x^5 + A_5x^4 + A_4x^3 + A_3x^2 + A_2x + A_1\right) = x^5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{4}, A_4 = 0, A_5 = 0, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(x\sqrt{x^2}\right) + c_2 \sin\left(x\sqrt{x^2}\right)\right) + \left(\frac{x^2}{4}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(x\sqrt{x^2}\right) + c_2 \sin\left(x\sqrt{x^2}\right) + \frac{x^2}{4}$$

Solved as second order Bessel ode

Time used: 0.388 (sec)

Writing the ode as

$$x^2y'' - y'x + 4x^4y = x^6 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= 1 \\ \beta &= 1 \\ n &= \frac{1}{2} \\ \gamma &= 2 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2 x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2 x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}$$

$$y_2 = -\frac{x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{x\sqrt{2}\sin(x^2)}{\sqrt{\pi}\sqrt{x^2}} & -\frac{x\sqrt{2}\cos(x^2)}{\sqrt{\pi}\sqrt{x^2}} \\ \frac{d}{dx}\left(\frac{x\sqrt{2}\sin(x^2)}{\sqrt{\pi}\sqrt{x^2}}\right) & \frac{d}{dx}\left(-\frac{x\sqrt{2}\cos(x^2)}{\sqrt{\pi}\sqrt{x^2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{x\sqrt{2}\sin(x^2)}{\sqrt{\pi}\sqrt{x^2}} & -\frac{x\sqrt{2}\cos(x^2)}{\sqrt{\pi}\sqrt{x^2}} \\ \frac{\sqrt{2}\sin(x^2)}{\sqrt{\pi}\sqrt{x^2}} - \frac{x^2\sqrt{2}\sin(x^2)}{\sqrt{\pi}(x^2)^{3/2}} + \frac{2x^2\sqrt{2}\cos(x^2)}{\sqrt{\pi}\sqrt{x^2}} & -\frac{\sqrt{2}\cos(x^2)}{\sqrt{\pi}\sqrt{x^2}} + \frac{x^2\sqrt{2}\cos(x^2)}{\sqrt{\pi}(x^2)^{3/2}} + \frac{2x^2\sqrt{2}\sin(x^2)}{\sqrt{\pi}\sqrt{x^2}} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\frac{x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}\right)\left(-\frac{\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} + \frac{x^2\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,(x^2)^{3/2}} + \frac{2x^2\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}\right) \\ & - \left(-\frac{x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}\right)\left(\frac{\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{x^2\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,(x^2)^{3/2}} + \frac{2x^2\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}\right) \end{aligned}$$

Which simplifies to

$$W = \frac{4x\left(\sin(x^2)^2 + \cos(x^2)^2\right)}{\pi}$$

Which simplifies to

$$W = \frac{4x}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{x^7\sqrt{2}\cos(x^2)}{\sqrt{\pi}\sqrt{x^2}}}{\frac{4x^3}{\pi}}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -\frac{x^3\sqrt{2}\,\sqrt{\pi}\,\cos(x^2)}{4}dx$$

Hence

$$u_1 = \frac{\sqrt{2}\,\sqrt{\pi}\left(\frac{\cos(x^2)}{2} + \frac{x^2\sin(x^2)}{2}\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^7\sqrt{2}\sin(x^2)}{\sqrt{\pi}\sqrt{x^2}}}{\frac{4x^3}{\pi}}\,dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{x^3\sqrt{2}\,\sqrt{\pi}\,\sin(x^2)}{4}dx$$

Hence

$$u_2 = \frac{\sqrt{2}\,\sqrt{\pi}\left(\frac{\sin(x^2)}{2} - \frac{x^2\cos(x^2)}{2}\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\frac{\cos(x^2)}{2} + \frac{x^2\sin(x^2)}{2}\right)x\sin(x^2)}{2\sqrt{x^2}} - \frac{x\cos(x^2)\left(\frac{\sin(x^2)}{2} - \frac{x^2\cos(x^2)}{2}\right)}{2\sqrt{x^2}}$$

Which simplifies to

$$y_p(x) = \frac{x\sqrt{x^2}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2 x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}\right) + \left(\frac{x\sqrt{x^2}}{4}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2 x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} + \frac{x\sqrt{x^2}}{4}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.354 (sec)

Writing the ode as

$$xy'' - y' + 4x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.82: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= 2ix
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{\mathrm{e}^{-ix^2}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-ix^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{i\,\mathrm{e}^{2ix^2}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-ix^2}\right) + c_2 \left(\mathrm{e}^{-ix^2} \left(-\frac{i\,\mathrm{e}^{2ix^2}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' - y' + 4x^3 y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\,\mathrm{e}^{-ix^2} - \frac{i c_2\,\mathrm{e}^{ix^2}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-ix^2} \\ y_2 &= -\frac{i\,\mathrm{e}^{ix^2}}{4} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-ix^2} & -\frac{ie^{ix^2}}{4} \\ \frac{d}{dx}\left(e^{-ix^2}\right) & \frac{d}{dx}\left(-\frac{ie^{ix^2}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-ix^2} & -\frac{ie^{ix^2}}{4} \\ -2ix\,e^{-ix^2} & \frac{x\,e^{ix^2}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-ix^2}\right)\left(\frac{x\,e^{ix^2}}{2}\right) - \left(-\frac{ie^{ix^2}}{4}\right)\left(-2ix\,e^{-ix^2}\right)$$

Which simplifies to

$$W = e^{-ix^2} x\,e^{ix^2}$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{ie^{ix^2} x^5}{4}}{x^2}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{ie^{ix^2} x^3}{4} dx$$

Hence

$$u_1 = -\frac{i(ix^2 - 1)\,e^{ix^2}}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-ix^2} x^5}{x^2}\,dx$$

Which simplifies to

$$u_2 = \int e^{-ix^2} x^3 dx$$

Hence

$$u_2 = \frac{(ix^2 + 1)\,e^{-ix^2}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{i(ix^2 - 1)\,e^{ix^2} e^{-ix^2}}{8} - \frac{ie^{ix^2}(ix^2 + 1)\,e^{-ix^2}}{8}$$

Which simplifies to

$$y_p(x) = \frac{x^2}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \,\mathrm{e}^{-i x^2} - \frac{i c_2 \,\mathrm{e}^{i x^2}}{4}\right) + \left(\frac{x^2}{4}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1 \,\mathrm{e}^{-i x^2} - \frac{i c_2 \,\mathrm{e}^{i x^2}}{4} + \frac{x^2}{4}
$$

Solved as second order ode adjoint method

Time used: 2.402 (sec)

In normal form the ode

$$
x y'' - y' + 4x^3 y = x^5 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 4x^2 \\
r(x) &= x^4
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(-\frac{\xi(x)}{x}\right)' + \left(4x^2 \xi(x)\right) &= 0 \\
\xi''(x) + \frac{\xi'(x)}{x} + \frac{(4x^4 - 1)\, \xi(x)}{x^2} &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. Writing the ode as

$$
\xi'' x^2 + \xi' x + \left(4x^4 - 1\right) \xi = 0 \tag{1}
$$

Bessel ode has the form

$$
\xi'' x^2 + \xi' x + \left(-n^2 + x^2\right) \xi = 0 \tag{2}
$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$
\xi'' x^2 + (1 - 2\alpha)\, x \xi' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) \xi = 0 \tag{3}
$$

With the standard solution

$$
\xi = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}
$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= 1 \\
n &= -\frac{1}{2} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$\xi = \frac{c_1\sqrt{2}\,\cos\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}} + \frac{c_2\sqrt{2}\,\sin\left(x^2\right)}{\sqrt{\pi}\,\sqrt{x^2}}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}
\end{aligned}$$

Or

$$y' + y\left(-\frac{1}{x} - \frac{-\frac{c_1\sqrt{2}\,\cos(x^2)x}{\sqrt{\pi}\,(x^2)^{3/2}} - \frac{2c_1x\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2\sqrt{2}\,\sin(x^2)x}{\sqrt{\pi}\,(x^2)^{3/2}} + \frac{2c_2x\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}}{\frac{c_1\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} + \frac{c_2\sqrt{2}\,\sin(x^2)}{\sqrt{\pi}\,\sqrt{x^2}}}\right) = \frac{\sqrt{2}\,\mathrm{csgn}\,(x)\left(\frac{c_1\left(\cos(x^2)+x^2\sin(x^2\right.}{2}\right.}{\left(\frac{c_1\sqrt{2}\,\cos(x^2)}{\sqrt{\pi}\,\sqrt{x^2}} + c_2\sqrt{\right.}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{2x(-c_1\sin\left(x^2\right) + c_2\cos\left(x^2\right))}{c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)} \\
p(x) &= \frac{\left(\left(-c_2\,x^2 + c_1\right)\cos\left(x^2\right) + \sin\left(x^2\right)\left(c_1\,x^2 + c_2\right)\right)x}{2c_1\cos\left(x^2\right) + 2c_2\sin\left(x^2\right)}
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{2x\left(-c_1\sin\left(x^2\right)+c_2\cos\left(x^2\right)\right)}{c_1\cos\left(x^2\right)+c_2\sin\left(x^2\right)}dx} \\
&= \frac{1}{c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\left(\left(-c_2\,x^2 + c_1\right)\cos\left(x^2\right) + \sin\left(x^2\right)\left(c_1\,x^2 + c_2\right)\right)x}{2c_1\cos\left(x^2\right) + 2c_2\sin\left(x^2\right)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)}\right) & \\
&= \left(\frac{1}{c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)}\right)\left(\frac{\left(\left(-c_2\,x^2 + c_1\right)\cos\left(x^2\right) + \sin\left(x^2\right)\left(c_1\,x^2 + c_2\right)\right)x}{2c_1\cos\left(x^2\right) + 2c_2\sin\left(x^2\right)}\right) \\
\mathrm{d}\left(\frac{y}{c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)}\right) &= \left(\frac{\left(\left(-c_2\,x^2 + c_1\right)\cos\left(x^2\right) + \sin\left(x^2\right)\left(c_1\,x^2 + c_2\right)\right)x}{\left(2c_1\cos\left(x^2\right) + 2c_2\sin\left(x^2\right)\right)\left(c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)\right)}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)} &= \int \frac{\left(\left(-c_2\,x^2 + c_1\right)\cos\left(x^2\right) + \sin\left(x^2\right)\left(c_1\,x^2 + c_2\right)\right)x}{\left(2c_1\cos\left(x^2\right) + 2c_2\sin\left(x^2\right)\right)\left(c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)\right)}\,dx \\
&= \frac{-\frac{x^2}{4} - \frac{x^2\tan\left(\frac{x^2}{2}\right)^2}{2} - \frac{x^2\tan\left(\frac{x^2}{2}\right)^4}{4}}{\left(1 + \tan\left(\frac{x^2}{2}\right)^2\right)\left(\tan\left(\frac{x^2}{2}\right)^2 c_1 - 2\tan\left(\frac{x^2}{2}\right)c_2 - c_1\right)} + c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_1\cos(x^2)+c_2\sin(x^2)}$ gives the final solution

$$y = c_1c_3\cos\left(x^2\right) + c_2c_3\sin\left(x^2\right) + \frac{x^2}{4}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_1c_3\cos\left(x^2\right) + c_2c_3\sin\left(x^2\right) + \frac{x^2}{4}$$

The constants can be merged to give

$$y = \frac{x^2}{4} + c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{x^2}{4} + c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 22

```
dsolve(x*diff(diff(y(x),x),x)-diff(y(x),x)+4*x^3*y(x) = x^5,y(x),singsol=all)
```

$$y = \sin\left(x^2\right)c_2 + \cos\left(x^2\right)c_1 + \frac{x^2}{4}$$

Mathematica DSolve solution

Solving time : 0.057 (sec)
Leaf size : 68

```
DSolve[{x*D[y[x],{x,2}]-D[y[x],x]+4*x^3*y[x]==x^5,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sin\left(x^2\right)\int_1^x \frac{1}{2}\cos\left(K[1]^2\right)K[1]^3dK[1] - \frac{1}{8}\sin\left(2x^2\right) + \frac{1}{4}x^2\cos^2\left(x^2\right) + c_1\cos\left(x^2\right) + c_2\sin\left(x^2\right)$$

2.2.25 Problem 25

Internal problem ID [9148]
Book : Second order enumerated odes
Section : section 2
Problem number : 25
Date solved : `Monday, January 27, 2025 at 05:49:41 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$\cos(x)\, y'' + \sin(x)\, y' - 2y\cos(x)^3 = 2\cos(x)^5$$

Solved as second order ode using change of variable on x method 2

Time used: 0.924 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$\cos(x)\, y'' + \sin(x)\, y' - 2y\cos(x)^3 = 0$$

In normal form the ode

$$\cos(x)\, y'' + \sin(x)\, y' - 2y\cos(x)^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= -2\cos(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\int p(x)dx} dx \\ &= \int \mathrm{e}^{-\int \tan(x)dx} dx \\ &= \int e^{\ln(\cos(x))}\, dx \\ &= \int \cos(x)\, dx \\ &= \sin(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-2\cos(x)^2}{\cos(x)^2} \\ &= -2 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2y(\tau) = 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -2$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\tau\lambda} - 2\,\mathrm{e}^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-2)} \\ &= \pm\sqrt{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +\sqrt{2} \\ \lambda_2 &= -\sqrt{2} \end{aligned}$$

Which simplifies to

$$\lambda_1 = \sqrt{2}$$
$$\lambda_2 = -\sqrt{2}$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{\left(\sqrt{2}\right)\tau} + c_2 e^{\left(-\sqrt{2}\right)\tau}$$

Or

$$y(\tau) = c_1 \,\mathrm{e}^{\tau\sqrt{2}} + c_2 \,\mathrm{e}^{-\tau\sqrt{2}}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \,\mathrm{e}^{\sin(x)\sqrt{2}} + c_2 \,\mathrm{e}^{-\sin(x)\sqrt{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \,\mathrm{e}^{\sin(x)\sqrt{2}} + c_2 \,\mathrm{e}^{-\sin(x)\sqrt{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-\sin(x)\sqrt{2}}$$
$$y_2 = \mathrm{e}^{\sin(x)\sqrt{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-\sin(x)\sqrt{2}} & \mathrm{e}^{\sin(x)\sqrt{2}} \\ \frac{d}{dx}\left(\mathrm{e}^{-\sin(x)\sqrt{2}}\right) & \frac{d}{dx}\left(\mathrm{e}^{\sin(x)\sqrt{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-\sin(x)\sqrt{2}} & \mathrm{e}^{\sin(x)\sqrt{2}} \\ -\mathrm{e}^{-\sin(x)\sqrt{2}} \cos(x)\sqrt{2} & \cos(x)\sqrt{2}\,\mathrm{e}^{\sin(x)\sqrt{2}} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-\sin(x)\sqrt{2}}\right)\left(\cos\left(x\right)\sqrt{2}\,\mathrm{e}^{\sin(x)\sqrt{2}}\right) - \left(\mathrm{e}^{\sin(x)\sqrt{2}}\right)\left(-\mathrm{e}^{-\sin(x)\sqrt{2}}\cos\left(x\right)\sqrt{2}\right)$$

Which simplifies to

$$W = 2\cos\left(x\right)\sqrt{2}$$

Which simplifies to

$$W = 2\cos\left(x\right)\sqrt{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,\mathrm{e}^{\sin(x)\sqrt{2}}\cos\left(x\right)^5}{2\cos\left(x\right)^2\sqrt{2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{\sin(x)\sqrt{2}}\cos\left(x\right)^3\sqrt{2}}{2}dx$$

Hence

$$u_1 = \frac{\mathrm{e}^{\sin(x)\sqrt{2}}\sin\left(x\right)^2}{2} - \frac{\sin\left(x\right)\sqrt{2}\,\mathrm{e}^{\sin(x)\sqrt{2}}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,\mathrm{e}^{-\sin(x)\sqrt{2}}\cos\left(x\right)^5}{2\cos\left(x\right)^2\sqrt{2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\mathrm{e}^{-\sin(x)\sqrt{2}}\cos\left(x\right)^3\sqrt{2}}{2}dx$$

Hence

$$u_2 = \frac{\mathrm{e}^{-\sin(x)\sqrt{2}}\sin\left(x\right)^2}{2} + \frac{\sin\left(x\right)\sqrt{2}\,\mathrm{e}^{-\sin(x)\sqrt{2}}}{2}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \left(\frac{\mathrm{e}^{\sin(x)\sqrt{2}}\sin\left(x\right)^2}{2} - \frac{\sin\left(x\right)\sqrt{2}\,\mathrm{e}^{\sin(x)\sqrt{2}}}{2}\right)\mathrm{e}^{-\sin(x)\sqrt{2}} \\ & + \mathrm{e}^{\sin(x)\sqrt{2}}\left(\frac{\mathrm{e}^{-\sin(x)\sqrt{2}}\sin\left(x\right)^2}{2} + \frac{\sin\left(x\right)\sqrt{2}\,\mathrm{e}^{-\sin(x)\sqrt{2}}}{2}\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \sin\left(x\right)^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,\mathrm{e}^{\sin(x)\sqrt{2}} + c_2\,\mathrm{e}^{-\sin(x)\sqrt{2}}\right) + \left(\sin\left(x\right)^2\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \sin\left(x\right)^2 + c_1\,\mathrm{e}^{\sin(x)\sqrt{2}} + c_2\,\mathrm{e}^{-\sin(x)\sqrt{2}}$$

Solved as second order ode using change of variable on x method 1

Time used: 1.759 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = \cos(x), B = \sin(x), C = -2\cos(x)^3, f(x) = 2\cos(x)^5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$\cos(x)\, y'' + \sin(x)\, y' - 2y\cos(x)^3 = 0$$

In normal form the ode

$$\cos(x)\, y'' + \sin(x)\, y' - 2y\cos(x)^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= -2\cos(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-2\cos(x)^2}}{c} \\ \tau'' &= \frac{2\sin(x)\cos(x)}{c\sqrt{-2\cos(x)^2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2\sin(x)\cos(x)}{c\sqrt{-2\cos(x)^2}} + \tan(x)\frac{\sqrt{-2\cos(x)^2}}{c}}{\left(\frac{\sqrt{-2\cos(x)^2}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{-2\cos(x)^2} dx}{c} \\
&= -\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}\, c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right) - c_2 \sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
y_1 = \cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)
$$

$$
y_2 = -\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right) & -\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\right) & \frac{d}{dx}\left(-\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right) & -\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right) \\ -\left(\frac{2\cos(2x)}{\sqrt{-2\cos(x)^2}} - \frac{2\sin(2x)\sin(x)\cos(x)}{\left(-2\cos(x)^2\right)^{3/2}}\right)\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right) & -\left(\frac{2\cos(2x)}{\sqrt{-2\cos(x)^2}} - \frac{2\sin(2x)\sin(x)\cos(x)}{\left(-2\cos(x)^2\right)^{3/2}}\right)\cos\left(\frac{\sin(2x}{\sqrt{-2\cos(}\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = &\left(\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\right)\left(-\left(\frac{2\cos(2x)}{\sqrt{-2\cos(x)^2}} \right.\right. \\ &\left.\left. - \frac{2\sin(2x)\sin(x)\cos(x)}{\left(-2\cos(x)^2\right)^{3/2}}\right)\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\right) \\ &- \left(-\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\right)\left(-\left(\frac{2\cos(2x)}{\sqrt{-2\cos(x)^2}} - \frac{2\sin(2x)\sin(x)\cos(x)}{\left(-2\cos(x)^2\right)^{3/2}}\right)\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} &W \\ &= \frac{2\cos(x)\left(\sin(2x)\sin(x) + 2\cos(2x)\cos(x)\right)\left(\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)^2 + \sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)^2\right)}{\left(-2\cos(x)^2\right)^{3/2}} \end{aligned}$$

Which simplifies to

$$W = -\frac{\cos(x)^2\sqrt{2}}{\sqrt{-\cos(x)^2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\cos(x)^5}{-\frac{\cos(x)^3\sqrt{2}}{\sqrt{-\cos(x)^2}}}\,dx$$

Which simplifies to

$$u_1 = -\int \sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\cos(x)^2\sqrt{2}\sqrt{-\cos(x)^2}dx$$

Hence

$$u_1 = -\int_0^x \sin\left(\frac{\sin(2\alpha)}{\sqrt{-2\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{2}\sqrt{-\cos(\alpha)^2}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\cos(x)^5}{-\frac{\cos(x)^3\sqrt{2}}{\sqrt{-\cos(x)^2}}}\,dx$$

Which simplifies to

$$u_2 = \int -\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\cos(x)^2\sqrt{2}\sqrt{-\cos(x)^2}dx$$

Hence

$$u_2 = \int_0^x -\cos\left(\frac{\sin(2\alpha)}{\sqrt{-2\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{2}\sqrt{-\cos(\alpha)^2}d\alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}y_p(x) = &-\left(\int_0^x \sin\left(\frac{\sin(2\alpha)}{\sqrt{-2\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{2}\sqrt{-\cos(\alpha)^2}d\alpha\right)\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\\ &-\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\left(\int_0^x -\cos\left(\frac{\sin(2\alpha)}{\sqrt{-2\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{2}\sqrt{-\cos(\alpha)^2}d\alpha\right)\end{aligned}$$

Which simplifies to

$$\begin{aligned}y_p(x) = \sqrt{2}\Bigg(&-\left(\int_0^x \sin\left(\frac{\sin(2\alpha)\sqrt{2}}{2\sqrt{-\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{-\cos(\alpha)^2}d\alpha\right)\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\\ &+\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\left(\int_0^x \cos\left(\frac{\sin(2\alpha)\sqrt{2}}{2\sqrt{-\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{-\cos(\alpha)^2}d\alpha\right)\Bigg)\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(c_1\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right) - c_2\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\right)\\ &\quad+\Bigg(\sqrt{2}\Bigg(-\left(\int_0^x \sin\left(\frac{\sin(2\alpha)\sqrt{2}}{2\sqrt{-\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{-\cos(\alpha)^2}d\alpha\right)\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\\ &\quad+\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\left(\int_0^x \cos\left(\frac{\sin(2\alpha)\sqrt{2}}{2\sqrt{-\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{-\cos(\alpha)^2}d\alpha\right)\Bigg)\Bigg)\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned}y = &\;c_1\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right) - c_2\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\\ &+\sqrt{2}\Bigg(-\left(\int_0^x \sin\left(\frac{\sin(2\alpha)\sqrt{2}}{2\sqrt{-\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{-\cos(\alpha)^2}d\alpha\right)\cos\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\\ &+\sin\left(\frac{\sin(2x)}{\sqrt{-2\cos(x)^2}}\right)\left(\int_0^x \cos\left(\frac{\sin(2\alpha)\sqrt{2}}{2\sqrt{-\cos(\alpha)^2}}\right)\cos(\alpha)^2\sqrt{-\cos(\alpha)^2}d\alpha\right)\Bigg)\end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.059 (sec)
Leaf size : 30

```
dsolve(cos(x)*diff(diff(y(x),x),x)+sin(x)*diff(y(x),x)-2*y(x)*cos(x)^3 = 2*cos(x)^5,y(x)
```

$$y = \sinh\left(\sin\left(x\right)\sqrt{2}\right) c_2 + \cosh\left(\sin\left(x\right)\sqrt{2}\right) c_1 + \frac{1}{2} - \frac{\cos\left(2x\right)}{2}$$

Mathematica DSolve solution

Solving time : 11.572 (sec)
Leaf size : 90

```
DSolve[{Cos[x]*D[y[x],{x,2}]+Sin[x]*D[y[x],x]-2*y[x]*Cos[x]^3==2*Cos[x]^5,{}},y[x],x,IncludeSi
```

$$y(x) \to \frac{1}{2} e^{-i\sqrt{\cos(2x)-1}} \left(\cos\left(\sqrt{\cos(2x)-1}\right) + i\sin\left(\sqrt{\cos(2x)-1}\right)\right) \left(-\cos(2x) + 2c_1\cos\left(\sqrt{\cos(2x)-1}\right) + 2c_2\sin\left(\sqrt{\cos(2x)-1}\right) + 1\right)$$

2.2.26 Problem 26

Internal problem ID [9149]
Book : Second order enumerated odes
Section : section 2
Problem number : 26
Date solved : `Monday, January 27, 2025 at 05:49:46 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + \left(1-\frac{1}{x}\right) y' + 4x^2 y\,\mathrm{e}^{-2x} = 4\left(x^3+x^2\right)\mathrm{e}^{-3x}$$

Solved as second order ode using change of variable on x method 2

Time used: 2.070 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y'' + \frac{(x-1)\,y'}{x} + 4x^2 y\,\mathrm{e}^{-2x} = 0$$

In normal form the ode

$$y'' + \frac{(x-1)\,y'}{x} + 4x^2 y\,\mathrm{e}^{-2x} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{x-1}{x}$$
$$q(x) = 4x^2\mathrm{e}^{-2x}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\int p(x)dx} dx \\
&= \int e^{-\int \frac{x-1}{x} dx} dx \\
&= \int e^{-x+\ln(x)}\, dx \\
&= \int x\, e^{-x} dx \\
&= -(x+1)\, e^{-x} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{4x^2 e^{-2x}}{x^2 e^{-2x}} \\
&= 4 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
&\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) = 0 \\
&\frac{d^2}{d\tau^2} y(\tau) + 4y(\tau) = 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\tau\lambda} + 4\, e^{\tau\lambda} = 0 \qquad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \qquad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\
&= \pm 2i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(2(x+1)\,\mathrm{e}^{-x}\right) - c_2 \sin\left(2(x+1)\,\mathrm{e}^{-x}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(2(x+1)\,\mathrm{e}^{-x}\right) - c_2 \sin\left(2(x+1)\,\mathrm{e}^{-x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 = &-4\cos\left(x\,\mathrm{e}^{-x}\right)^2 \sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right) - 4\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)^2 \\ &+ 2\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right) + 2\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right) \end{aligned}$$

$$\begin{aligned} y_2 = &\,4\cos\left(x\,\mathrm{e}^{-x}\right)^2\cos\left(\mathrm{e}^{-x}\right)^2 - 2\cos\left(x\,\mathrm{e}^{-x}\right)^2 - 2\cos\left(\mathrm{e}^{-x}\right)^2 \\ &+ 1 - 4\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right)\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -4\cos(x\,e^{-x})^2 \sin(e^{-x})\cos(e^{-x}) - 4\sin(x\,e^{-x})\cos(x\,e^{-x})\cos(e^{-x})^2 + 2\sin(x\,e^{-x})\cos(x\,e^{-x}) + 2 \\ \frac{d}{dx}\left(-4\cos(x\,e^{-x})^2 \sin(e^{-x})\cos(e^{-x}) - 4\sin(x\,e^{-x})\cos(x\,e^{-x})\cos(e^{-x})^2 + 2\sin(x\,e^{-x})\cos(x\,e^{-x}) + 2\right. \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \\ 8\cos(x\,e^{-x})\sin(e^{-x})\cos(e^{-x})(e^{-x} - x\,e^{-x})\sin(x\,e^{-x}) + 4\cos(x\,e^{-x})^2 e^{-x}\cos(e^{-x})^2 - 4\cos(x\,e^{-x})^2 s \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(-4\cos\left(x\,e^{-x}\right)^2 \sin\left(e^{-x}\right)\cos\left(e^{-x}\right) - 4\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)\cos\left(e^{-x}\right)^2 \right. \\ & + 2\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right) \\ & \left. + 2\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\right)\left(-8\cos\left(x\,e^{-x}\right)\cos\left(e^{-x}\right)^2\left(e^{-x} - x\,e^{-x}\right)\sin\left(x\,e^{-x}\right)\right. \\ & + 8\cos\left(x\,e^{-x}\right)^2\cos\left(e^{-x}\right)e^{-x}\sin\left(e^{-x}\right) + 4\cos\left(x\,e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right)\sin\left(x\,e^{-x}\right) \\ & - 4\cos\left(e^{-x}\right)e^{-x}\sin\left(e^{-x}\right) - 4\left(e^{-x} - x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)^2\sin\left(e^{-x}\right)\cos\left(e^{-x}\right) \\ & + 4\sin\left(x\,e^{-x}\right)^2\left(e^{-x} - x\,e^{-x}\right)\sin\left(e^{-x}\right)\cos\left(e^{-x}\right) \\ & \left. + 4\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)e^{-x}\cos\left(e^{-x}\right)^2 - 4\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)\sin\left(e^{-x}\right)^2 e^{-x}\right) \\ & - \left(4\cos\left(x\,e^{-x}\right)^2\cos\left(e^{-x}\right)^2 - 2\cos\left(x\,e^{-x}\right)^2 - 2\cos\left(e^{-x}\right)^2 + 1\right. \\ & \left. - 4\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\right)\left(8\cos\left(x\,e^{-x}\right)\sin\left(e^{-x}\right)\cos\left(e^{-x}\right)\left(e^{-x}\right.\right. \\ & \left. - x\,e^{-x}\right)\sin\left(x\,e^{-x}\right) + 4\cos\left(x\,e^{-x}\right)^2 e^{-x}\cos\left(e^{-x}\right)^2 - 4\cos\left(x\,e^{-x}\right)^2\sin\left(e^{-x}\right)^2 e^{-x} \\ & - 4\left(e^{-x} - x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)^2\cos\left(e^{-x}\right)^2 + 4\sin\left(x\,e^{-x}\right)^2\left(e^{-x} - x\,e^{-x}\right)\cos\left(e^{-x}\right)^2 \\ & - 8\sin\left(x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)\cos\left(e^{-x}\right)e^{-x}\sin\left(e^{-x}\right) + 2\left(e^{-x} - x\,e^{-x}\right)\cos\left(x\,e^{-x}\right)^2 \\ & \left. - 2\sin\left(x\,e^{-x}\right)^2\left(e^{-x} - x\,e^{-x}\right) - 2\,e^{-x}\cos\left(e^{-x}\right)^2 + 2\sin\left(e^{-x}\right)^2 e^{-x}\right) \end{aligned}$$

Which simplifies to

$$W = \text{Expression too large to display}$$

Which simplifies to

$$W = -2x\,e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(4\cos(x\,e^{-x})^2\cos(e^{-x})^2 - 2\cos(x\,e^{-x})^2 - 2\cos(e^{-x})^2 + 1 - 4\sin(x\,e^{-x})\cos(x\,e^{-x})\sin(e^{-x})\cos(e\right.}{-2x\,e^{-x}}$$

Which simplifies to

$$u_1 = -\int 2x(x+1)\,e^{-2x}\left(\sin\left(2x\,e^{-x}\right)\sin\left(2\,e^{-x}\right) - \cos\left(2\,e^{-x}\right)\cos\left(2x\,e^{-x}\right)\right)dx$$

Hence

$$u_1 = \frac{i(2x\,e^{-x} + i + 2\,e^{-x})\,e^{2ie^{-x}}e^{2ix\,e^{-x}}}{4} - \frac{i(2x\,e^{-x} - i + 2\,e^{-x})\,e^{-2ie^{-x}}e^{-2ix\,e^{-x}}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(-4\cos\left(x\,\mathrm{e}^{-x}\right)^2 \sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right) - 4\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)^2 + 2\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right) + \cdots\right.}{-2x\,\mathrm{e}^{-x}}$$

Which simplifies to

$$u_2 = \int 2x(x+1)\,\mathrm{e}^{-2x}\left(\sin\left(2\,\mathrm{e}^{-x}\right)\cos\left(2x\,\mathrm{e}^{-x}\right) + \cos\left(2\,\mathrm{e}^{-x}\right)\sin\left(2x\,\mathrm{e}^{-x}\right)\right)dx$$

Hence

$$u_2 = \frac{\left(2x\,\mathrm{e}^{-x} + i + 2\,\mathrm{e}^{-x}\right)\mathrm{e}^{2i\mathrm{e}^{-x}}\,\mathrm{e}^{2ix\,\mathrm{e}^{-x}}}{4} + \frac{\left(2x\,\mathrm{e}^{-x} - i + 2\,\mathrm{e}^{-x}\right)\mathrm{e}^{-2i\mathrm{e}^{-x}}\,\mathrm{e}^{-2ix\,\mathrm{e}^{-x}}}{4}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(\frac{i\left(2x\,\mathrm{e}^{-x} + i + 2\,\mathrm{e}^{-x}\right)\mathrm{e}^{2i\mathrm{e}^{-x}}\,\mathrm{e}^{2ix\,\mathrm{e}^{-x}}}{4} - \frac{i\left(2x\,\mathrm{e}^{-x} - i + 2\,\mathrm{e}^{-x}\right)\mathrm{e}^{-2i\mathrm{e}^{-x}}\,\mathrm{e}^{-2ix\,\mathrm{e}^{-x}}}{4}\right)\left(-4\cos\left(x\,\mathrm{e}^{-x}\right)^2\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)\right. \\ &- 4\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)^2 + 2\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right) \\ &\left. + 2\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)\right) \\ &+ \left(4\cos\left(x\,\mathrm{e}^{-x}\right)^2\cos\left(\mathrm{e}^{-x}\right)^2 - 2\cos\left(x\,\mathrm{e}^{-x}\right)^2 - 2\cos\left(\mathrm{e}^{-x}\right)^2 + 1\right. \\ &\left. - 4\sin\left(x\,\mathrm{e}^{-x}\right)\cos\left(x\,\mathrm{e}^{-x}\right)\sin\left(\mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x}\right)\right)\left(\frac{\left(2x\,\mathrm{e}^{-x} + i + 2\,\mathrm{e}^{-x}\right)\mathrm{e}^{2i\mathrm{e}^{-x}}\,\mathrm{e}^{2ix\,\mathrm{e}^{-x}}}{4}\right. \\ &\left. + \frac{\left(2x\,\mathrm{e}^{-x} - i + 2\,\mathrm{e}^{-x}\right)\mathrm{e}^{-2i\mathrm{e}^{-x}}\,\mathrm{e}^{-2ix\,\mathrm{e}^{-x}}}{4}\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = (x+1)\,\mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(2(x+1)\,\mathrm{e}^{-x}\right) - c_2\sin\left(2(x+1)\,\mathrm{e}^{-x}\right)\right) + \left((x+1)\,\mathrm{e}^{-x}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = (x+1)\,\mathrm{e}^{-x} + c_1\cos\left(2(x+1)\,\mathrm{e}^{-x}\right) - c_2\sin\left(2(x+1)\,\mathrm{e}^{-x}\right)$$

Solved as second order ode using change of variable on x method 1

Time used: 0.737 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = \frac{x-1}{x}, C = 4x^2 e^{-2x}, f(x) = 4x^2(x+1)\,e^{-3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$y'' + \frac{(x-1)\,y'}{x} + 4x^2 y\,e^{-2x} = 0$$

In normal form the ode

$$y'' + \frac{(x-1)\,y'}{x} + 4x^2 y\,e^{-2x} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x-1}{x} \\ q(x) &= 4x^2 e^{-2x} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{x^2 e^{-2x}}}{c} \qquad (6) \\ \tau'' &= \frac{2x\,e^{-2x} - 2x^2 e^{-2x}}{c\sqrt{x^2 e^{-2x}}} \end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2x\,e^{-2x} - 2x^2 e^{-2x}}{c\sqrt{x^2 e^{-2x}}} + \frac{x-1}{x}\frac{2\sqrt{x^2 e^{-2x}}}{c}}{\left(\frac{2\sqrt{x^2 e^{-2x}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int 2\sqrt{x^2 e^{-2x}} dx}{c} \\
&= -\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{cx}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right) - c_2 \sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right) \\
y_2 &= -\sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right) & -\sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)\right) & \frac{d}{dx}\left(-\sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right) & -\sin \\ -\left(\frac{2\sqrt{x^2 e^{-2x}}}{x} + \frac{(x+1)(2x\,e^{-2x}-2x^2 e^{-2x})}{\sqrt{x^2 e^{-2x}}\,x} - \frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x^2}\right)\sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right) & -\left(\frac{2\sqrt{x^2 e^{-2x}}}{x} + \frac{(x+1)(2x\,e^{-2x}-2x^2 e}{\sqrt{x^2 e^{-2x}}\,x}\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)\right)\left(-\left(\frac{2\sqrt{x^2 e^{-2x}}}{x} + \frac{(x+1)\left(2x\,e^{-2x} - 2x^2 e^{-2x}\right)}{\sqrt{x^2 e^{-2x}}\,x}\right.\right. \\ & \left.\left. - \frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x^2}\right)\cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)\right) \\ & - \left(-\sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)\right)\left(-\left(\frac{2\sqrt{x^2 e^{-2x}}}{x} + \frac{(x+1)\left(2x\,e^{-2x} - 2x^2 e^{-2x}\right)}{\sqrt{x^2 e^{-2x}}\,x}\right.\right. \\ & \left.\left. - \frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x^2}\right)\sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)\right) \end{aligned}$$

Which simplifies to

$$W = \frac{2x^2 e^{-2x}\left(\cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)^2 + \sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)^2\right)}{\sqrt{x^2 e^{-2x}}}$$

Which simplifies to

$$W = \frac{2x^2 e^{-2x}}{\sqrt{x^2 e^{-2x}}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-4\sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)x^2(x+1)\,e^{-3x}}{\frac{2x^2 e^{-2x}}{\sqrt{x^2 e^{-2x}}}}\,dx$$

Which simplifies to

$$u_1 = -\int -2(x+1)\,e^{-x}\sqrt{x^2 e^{-2x}}\,\sin\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)dx$$

Hence

$$u_1 = -\int_0^x -2(\alpha+1)\,e^{-\alpha}\sqrt{\alpha^2 e^{-2\alpha}}\,\sin\left(\frac{2(\alpha+1)\sqrt{\alpha^2 e^{-2\alpha}}}{\alpha}\right)d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)x^2(x+1)\,e^{-3x}}{\frac{2x^2 e^{-2x}}{\sqrt{x^2 e^{-2x}}}}\,dx$$

Which simplifies to

$$u_2 = \int 2(x+1)\,e^{-x}\sqrt{x^2 e^{-2x}}\,\cos\left(\frac{2(x+1)\sqrt{x^2 e^{-2x}}}{x}\right)dx$$

Hence

$$u_2 = \int_0^x 2(\alpha+1)\,\mathrm{e}^{-\alpha}\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}\cos\left(\frac{2(\alpha+1)\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}}{\alpha}\right)d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x -2(\alpha+1)\,\mathrm{e}^{-\alpha}\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}\sin\left(\frac{2(\alpha+1)\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}}{\alpha}\right)d\alpha\right)\cos\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right) - \sin\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right)\left(\int_0^x 2(\alpha+1)\,\mathrm{e}^{-\alpha}\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}\cos\left(\frac{2(\alpha+1)\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}}{\alpha}\right)d\alpha\right)$$

Which simplifies to

$$y_p(x) = 2\left(\int_0^x (\alpha+1)\,\mathrm{e}^{-\alpha}\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}\sin\left(\frac{2(\alpha+1)\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}}{\alpha}\right)d\alpha\right)\cos\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right) - 2\sin\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right)\left(\int_0^x (\alpha+1)\,\mathrm{e}^{-\alpha}\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}\cos\left(\frac{2(\alpha+1)\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}}{\alpha}\right)d\alpha\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right) - c_2\sin\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right)\right) \\ &\quad + \left(2\left(\int_0^x (\alpha+1)\,\mathrm{e}^{-\alpha}\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}\sin\left(\frac{2(\alpha+1)\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}}{\alpha}\right)d\alpha\right)\cos\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right) \right. \\ &\quad \left. -2\sin\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right)\left(\int_0^x (\alpha+1)\,\mathrm{e}^{-\alpha}\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}\cos\left(\frac{2(\alpha+1)\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}}{\alpha}\right)d\alpha\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned} y &= c_1\cos\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right) - c_2\sin\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right) \\ &\quad + 2\left(\int_0^x (\alpha+1)\,\mathrm{e}^{-\alpha}\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}\sin\left(\frac{2(\alpha+1)\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}}{\alpha}\right)d\alpha\right)\cos\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right) \\ &\quad - 2\sin\left(\frac{2(x+1)\sqrt{x^2\mathrm{e}^{-2x}}}{x}\right)\left(\int_0^x (\alpha+1)\,\mathrm{e}^{-\alpha}\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}\cos\left(\frac{2(\alpha+1)\sqrt{\alpha^2\mathrm{e}^{-2\alpha}}}{\alpha}\right)d\alpha\right) \end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 39

```
dsolve(diff(diff(y(x),x),x)+(1-1/x)*diff(y(x),x)+4*x^2*y(x)*exp(-2*x) = 4*(x^3+x^2)*exp(
```

$$y = \sin\left(2(x+1)\,\mathrm{e}^{-x}\right) c_2 + \cos\left(2(x+1)\,\mathrm{e}^{-x}\right) c_1 + x\,\mathrm{e}^{-x} + \mathrm{e}^{-x}$$

Mathematica DSolve solution

Solving time : 4.276 (sec)
Leaf size : 142

```
DSolve[{D[y[x],{x,2}]+(1-1/x)*D[y[x],x]+4*x^2*y[x]*Exp[-2*x]==4*(x^2+x^3)*Exp[-3*x],{}},y[x],x
```

$$\begin{aligned} y(x) \to & \cos\left(2e^{-x}(\log\left(e^{x}\right)+1)\right) \int_1^{e^x} \frac{2\log(K[1])(\log(K[1])+1)\sin\left(\frac{2(\log(K[1])+1)}{K[1]}\right)}{K[1]^3} dK[1] \\ & - \sin\left(2e^{-x}(\log\left(e^{x}\right)+1)\right) \int_1^{e^x} \frac{2\cos\left(\frac{2(\log(K[2])+1)}{K[2]}\right)\log(K[2])(\log(K[2])+1)}{K[2]^3} dK[2] \\ & + c_1\cos\left(2e^{-x}(\log\left(e^{x}\right)+1)\right) - c_2\sin\left(2e^{-x}(\log\left(e^{x}\right)+1)\right) \end{aligned}$$

2.2.27 Problem 27

Internal problem ID [9150]
Book : Second order enumerated odes
Section : section 2
Problem number : 27
Date solved : `Monday, January 27, 2025 at 05:49:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2y' + xy = x^{m+1}$$

Solved as second order ode using change of variable on y method 2

Time used: 1.028 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -x^2, C = x, f(x) = x^{m+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$y'' - x^2y' + xy = 0$$

In normal form the ode

$$y'' - x^2y' + xy = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -x^2$$
$$q(x) = x$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - nx + x = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - x^2\right) v'(x) &= 0 \\ v''(x) + \frac{(-x^3+2)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-x^3+2)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{x^3-2}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= e^{\int -\frac{x^3-2}{x} dx} \\ &= x^2 e^{-\frac{x^3}{3}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu u &= 0 \\ \frac{d}{dx}\left(u\, x^2 e^{-\frac{x^3}{3}}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} u\, x^2 e^{-\frac{x^3}{3}} &= \int 0\, dx + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor $x^2 e^{-\frac{x^3}{3}}$ gives the final solution

$$u(x) = \frac{e^{\frac{x^3}{3}} c_1}{x^2}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= \frac{3^{2/3}(-1)^{1/3} c_1 \left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\,3^{1/3}(-1)^{2/3}e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{3^{2/3}(-1)^{1/3} c_1 \left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\,3^{1/3}(-1)^{2/3}e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9} + c_2\right) x \\
&= \frac{\left(3c_2x - 3\,e^{\frac{x^3}{3}} c_1\right)(-x^3)^{2/3} + x^3 c_1 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\,(-x^3)^{2/3}}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
y'' - x^2y' + xy = x^{m+1}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}\right)}{3\,(-x^3)^{2/3}} - \frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\,(-x^3)^{2/3}} - e^{\frac{x^3}{3}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & \frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{2/3}} - \frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{2/3}} - e^{\frac{x^3}{3}} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{2/3}} - \frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{2/3}} - e^{\frac{x^3}{3}}\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x & \frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{2/3}} - \frac{3^{2/3}x^3\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{3(-x^3)^{2/3}} - \mathrm{e}^{\frac{x^3}{3}} \\ 1 & \frac{3^{2/3}x^2\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{2\,3^{2/3}x^5\Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{5/3}} - \frac{3^{2/3}x^2\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{(-x^3)^{2/3}} - \frac{2\,3^{2/3}x^5\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{3(-x^3)^{5/3}} - \frac{3^{2/3}x^5\mathrm{e}^{\frac{x^3}{3}}}{3(-x^3)^{2/3}\left(-\frac{x^3}{3}\right)^{1/3}} - x^2\mathrm{e}^{\frac{x^3}{3}} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{3^{2/3}x^2\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{2\,3^{2/3}x^5\Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{5/3}} - \frac{3^{2/3}x^2\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{(-x^3)^{2/3}} - \frac{2\,3^{2/3}x^5\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{3\left(-x^3\right)^{5/3}} - \frac{3^{2/3}x^5\mathrm{e}^{\frac{x^3}{3}}}{3\left(-x^3\right)^{2/3}\left(-\frac{x^3}{3}\right)^{1/3}} - x^2\mathrm{e}^{\frac{x^3}{3}}\right) - \left(\frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{2/3}} - \frac{3^{2/3}x^3\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{3\left(-x^3\right)^{2/3}} - \mathrm{e}^{\frac{x^3}{3}}\right)(1)$$

Which simplifies to

$$W = \frac{\mathrm{e}^{\frac{x^3}{3}}\left(3^{2/3}x^9 - 3(-x^3)^{5/3}\left(-\frac{x^3}{3}\right)^{1/3}x^3 + 3(-x^3)^{5/3}\left(-\frac{x^3}{3}\right)^{1/3}\right)}{3\left(-x^3\right)^{5/3}\left(-\frac{x^3}{3}\right)^{1/3}}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^3}{3}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{2/3}} - \frac{3^{2/3}x^3\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{3(-x^3)^{2/3}} - \mathrm{e}^{\frac{x^3}{3}}\right)x^{m+1}}{\mathrm{e}^{\frac{x^3}{3}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{m+1}\left(-3(-x^3)^{2/3} + x^3 3^{2/3}\mathrm{e}^{-\frac{x^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{2/3}}dx$$

Hence

$$u_1 = -\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3} + \alpha^3 3^{2/3}\mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{3\left(-\alpha^3\right)^{2/3}}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\,x^{m+1}}{\mathrm{e}^{\frac{x^3}{3}}}\,dx$$

Which simplifies to

$$u_2 = \int x^{m+2}\mathrm{e}^{-\frac{x^3}{3}}dx$$

Hence

$$u_2 = \frac{3^{\frac{m}{6}+1}x^m(x^3)^{-\frac{m}{6}}\,\mathrm{e}^{-\frac{x^3}{6}}\,\mathrm{WhittakerM}\left(\frac{m}{6},\frac{m}{6}+\frac{1}{2},\frac{x^3}{3}\right)}{m+3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) =$$

$$-\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3}+\alpha^3 3^{2/3}\mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{3\left(-\alpha^3\right)^{2/3}}d\alpha\right)x+\frac{\left(\frac{3^{2/3}x^3\Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{2/3}}-\frac{3^{2/3}x^3\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{3(-x^3)^{2/3}}-\right.}{}$$

Which simplifies to

$$y_p(x) =$$

$$-\frac{\left(-x^3\right)^{2/3}x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3}+\alpha^3 3^{2/3}\mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{2/3}}d\alpha\right)+9(x^3)^{-\frac{m}{6}}\,\text{WhittakerM}\left(\frac{m}{6},\right.}{\left(-x^3\right)^{2/3}(3m+9)}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$=\left(\left(\frac{3^{2/3}(-1)^{1/3}c_1\left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}}+\frac{3\,3^{1/3}(-1)^{2/3}\mathrm{e}^{\frac{x^3}{3}}}{x}+\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9}\right.\right.$$

$$\left.\left.+c_2\right)x\right)+\left(-\frac{\left(-x^3\right)^{2/3}x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3}+\alpha^3 3^{2/3}\mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{2/3}}d\alpha\right)+9(x^3)^{-\frac{m}{6}}\,\mathrm{W}}{\left(-x^3\right)^{2/3}(3}\right.$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y=\left(\frac{3^{2/3}(-1)^{1/3}c_1\left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}}+\frac{3\,3^{1/3}(-1)^{2/3}\mathrm{e}^{\frac{x^3}{3}}}{x}+\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9}\right.$$

$$\left.+c_2\right)x-\frac{\left(-x^3\right)^{2/3}x(m+3)\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3}+\alpha^3 3^{2/3}\mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{2/3}}d\alpha\right)+9(x^3)^{-\frac{m}{6}}\,\text{Whittake}}{\left(-x^3\right)^{2/3}(3m+9)}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.563 (sec)

Writing the ode as

$$y'' - x^2y' + xy = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 - 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 - 8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 - 8)}{4} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.83: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} - \frac{2}{x} - \frac{4}{x^4} - \frac{16}{x^7} - \frac{80}{x^{10}} - \frac{448}{x^{13}} - \frac{2688}{x^{16}} - \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3 - 8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 - 2x\right) + (0) \\ &= \frac{1}{4}x^4 - 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 2\right) &= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 2\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x(x^3-8)}{4}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x^2}{2}\right) \\
&= -\frac{x^2}{2} \\
&= -\frac{x^2}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{x^2}{2}\right)(1) + \left((-x) + \left(-\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3-8)}{4}\right)\right) &= 0 \\
x a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, e^{\int -\frac{x^2}{2} dx} \\ &= (x) \, e^{-\frac{x^3}{6}} \\ &= x \, e^{-\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(e^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{3^{2/3}(-1)^{1/3} \left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\, 3^{1/3}(-1)^{2/3} e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}} \right)}{9} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2 \left(x \left(\frac{3^{2/3}(-1)^{1/3} \left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\, 3^{1/3}(-1)^{2/3} e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}} \right)}{9} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-x^2y'+xy=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x + \frac{c_2\left(-3(-x^3)^{2/3}\,\mathrm{e}^{\frac{x^3}{3}}+x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{2/3}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x\\
y_2 &= \frac{-3(-x^3)^{2/3}\,\mathrm{e}^{\frac{x^3}{3}}+x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)}{3\left(-x^3\right)^{2/3}}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{-3(-x^3)^{2/3}\mathrm{e}^{\frac{x^3}{3}}+x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)}{3(-x^3)^{2/3}} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{-3(-x^3)^{2/3}\mathrm{e}^{\frac{x^3}{3}}+x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)}{3(-x^3)^{2/3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{-3(-x^3)^{2/3}\mathrm{e}^{\frac{x^3}{3}}+x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)}{3(-x^3)^{2/3}} \\ 1 & \frac{2\left(-3(-x^3)^{2/3}\mathrm{e}^{\frac{x^3}{3}}+x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)\right)x^2}{3(-x^3)^{5/3}}+\frac{\frac{6\,\mathrm{e}^{\frac{x^3}{3}}x^2}{\left(-x^3\right)^{1/3}}-3(-x^3)^{2/3}x^2\mathrm{e}^{\frac{x^3}{3}}+3x^2 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)\right)-\frac{x^5 3^{2/3}\mathrm{e}^{\frac{x^3}{3}}}{\left(-\frac{x^3}{3}\right)^{1/3}}}{3(-x^3)^{2/3}} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{2\left(-3(-x^3)^{2/3}\,\mathrm{e}^{\frac{x^3}{3}} + x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)x^2}{3\left(-x^3\right)^{5/3}} + \frac{\frac{6\,\mathrm{e}^{\frac{x^3}{3}}\,x^2}{(-x^3)^{1/3}} - 3(-x^3)^{2/3}\,x^2\mathrm{e}^{\frac{x^3}{3}} + 3x^2 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right) - \frac{x^5 3^{2/3}\mathrm{e}^{\frac{x^3}{3}}}{\left(-\frac{x^3}{3}\right)^{1/3}}}{3\left(-x^3\right)^{2/3}}\right) - \left(\frac{-3(-x^3)^{2/3}\,\mathrm{e}^{\frac{x^3}{3}} + x^3 3^{2/3}}{3\left(-x^3\right.}\right.$$

Which simplifies to

$$W = -\frac{\mathrm{e}^{\frac{x^3}{3}}\left(-3^{2/3}x^9 + 3(-x^3)^{5/3}\left(-\frac{x^3}{3}\right)^{1/3}x^3 - 3(-x^3)^{5/3}\left(-\frac{x^3}{3}\right)^{1/3}\right)}{3\left(-x^3\right)^{5/3}\left(-\frac{x^3}{3}\right)^{1/3}}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^3}{3}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\left(-3(-x^3)^{2/3}\mathrm{e}^{\frac{x^3}{3}} + x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)x^{m+1}}{3(-x^3)^{2/3}}}{\mathrm{e}^{\frac{x^3}{3}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{m+1}\left(-3(-x^3)^{2/3} + x^3 3^{2/3}\mathrm{e}^{-\frac{x^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{2/3}}dx$$

Hence

$$u_1 = -\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3} + \alpha^3 3^{2/3}\mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{3\left(-\alpha^3\right)^{2/3}}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\,x^{m+1}}{\mathrm{e}^{\frac{x^3}{3}}}\,dx$$

Which simplifies to

$$u_2 = \int x^{m+2}\mathrm{e}^{-\frac{x^3}{3}}\,dx$$

Hence

$$u_2 = \frac{3^{\frac{m}{6}+1}x^m(x^3)^{-\frac{m}{6}}\,\mathrm{e}^{-\frac{x^3}{6}}\,\mathrm{WhittakerM}\left(\frac{m}{6}, \frac{m}{6} + \frac{1}{2}, \frac{x^3}{3}\right)}{m+3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3} + \alpha^3 3^{2/3}\mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{3\left(-\alpha^3\right)^{2/3}}d\alpha\right)x + \frac{\left(-3(-x^3)^{2/3}\,\mathrm{e}^{\frac{x^3}{3}} + x^3 3^{2/3}\left(\Gamma\right.\right.}{}$$

Which simplifies to

$$y_p(x) = -\frac{(-x^3)^{2/3} x(m+3) \left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3}+\alpha^3 3^{2/3} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{2/3}} d\alpha\right) + 9(x^3)^{-\frac{m}{6}} \operatorname{WhittakerM}\left(\frac{m}{6}, \frac{m}{6}\right.}{(-x^3)^{2/3}(3m+9)}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(c_1 x + \frac{c_2\left(-3(-x^3)^{2/3} e^{\frac{x^3}{3}} + x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3(-x^3)^{2/3}}\right) + \left(-\frac{(-x^3)^{2/3} x(m+3) \left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3}+\alpha^3 3\right.}{(}\right.}{}\right.$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + \frac{c_2\left(-3(-x^3)^{2/3} e^{\frac{x^3}{3}} + x^3 3^{2/3}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3(-x^3)^{2/3}} - \frac{(-x^3)^{2/3} x(m+3) \left(\int_0^x \frac{\alpha^{m+1}\left(-3(-\alpha^3)^{2/3}+\alpha^3 3^{2/3} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{2/3}} d\alpha\right) + 9(x^3)^{-\frac{m}{6}} \operatorname{WhittakerM}\left(\frac{m}{6}, \frac{m}{6}\right.}{(-x^3)^{2/3}(3m+9)}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.026` (sec)
Leaf size : 192

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2+x*y(x) = x^(m+1),y(x),singsol=all)
```

$$y = \frac{\left(\text{WhittakerM}\left(\frac{m}{6}, \frac{m}{6}+\frac{1}{2}, \frac{x^3}{3}\right) \mathrm{e}^{-\frac{x^3}{6}} 3^{\frac{5}{3}+\frac{m}{6}} \left(x^3\right)^{-\frac{m}{6}} x^m - 3\, 3^{2/3} c_1 (m+3)\right) \left(\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) - \Gamma\left(\frac{2}{3}\right)\right) \left(-x^3\right)^{1/3}}{}$$

Mathematica DSolve solution

Solving time : `0.277` (sec)
Leaf size : 144

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]+x*y[x]==x^(m+1),{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to x \int_1^x \frac{e^{-\frac{1}{3}K[1]^3} \Gamma\left(-\frac{1}{3}, -\frac{1}{3}K[1]^3\right) K[1]^{m+1} \sqrt[3]{-K[1]^3}}{3\sqrt[3]{3}} dK[1] - \frac{\sqrt[3]{-x^3} \left(x^3\right)^{-m/3} \Gamma\left(-\frac{1}{3}, -\frac{x^3}{3}\right) \left(-3^{m/3} x^m \Gamma\left(\frac{m+3}{3}, \frac{x^3}{3}\right) + c_2 \left(x^3\right)^{m/3}\right)}{3\sqrt[3]{3}} + c_1 x$$

2.2.28 Problem 28

Internal problem ID [9151]
Book : Second order enumerated odes
Section : section 2
Problem number : 28
Date solved : `Monday, January 27, 2025 at 05:49:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{y'}{\sqrt{x}} + \frac{y\left(-8+\sqrt{x}+x\right)}{4x^2} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.116 (sec)

Writing the ode as

$$y'' - \frac{y'}{\sqrt{x}} + \left(-\frac{2}{x^2} + \frac{1}{4x^{3/2}} + \frac{1}{4x}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{1}{\sqrt{x}} \\ C &= -\frac{2}{x^2} + \frac{1}{4x^{3/2}} + \frac{1}{4x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.84: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{1}{\sqrt{x}}}{1} \, dx} \\ &= z_1 e^{\sqrt{x}} \\ &= z_1 \left(\mathrm{e}^{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{x}}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{1}{\sqrt{x}}}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2\sqrt{x}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3 \mathrm{e}^{2\sqrt{x}} \mathrm{e}^{-2\sqrt{x}}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{\sqrt{x}}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{x}}}{x} \left(\frac{x^3 \mathrm{e}^{2\sqrt{x}} \mathrm{e}^{-2\sqrt{x}}}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 \, \mathrm{e}^{\sqrt{x}}}{x} + \frac{c_2 \, x^2 \mathrm{e}^{\sqrt{x}}}{3}$$

Solved as second order ode adjoint method

Time used: 1.090 (sec)

In normal form the ode

$$y'' - \frac{y'}{\sqrt{x}} + \frac{y(-8+\sqrt{x}+x)}{4x^2} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{\sqrt{x}} \\ q(x) &= -\frac{2}{x^2} + \frac{1}{4x^{3/2}} + \frac{1}{4x} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(-\frac{\xi(x)}{\sqrt{x}}\right)' + \left(\left(-\frac{2}{x^2} + \frac{1}{4x^{3/2}} + \frac{1}{4x}\right)\xi(x)\right) &= 0 \\ \xi''(x) + \frac{\xi'(x)}{\sqrt{x}} - \frac{(\sqrt{x} - x + 8)\,\xi(x)}{4x^2} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' + \frac{\xi'}{\sqrt{x}} + \left(-\frac{1}{4x^{3/2}} - \frac{2}{x^2} + \frac{1}{4x}\right)\xi = 0 \tag{1}$$

$$A\xi'' + B\xi' + C\xi = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{\sqrt{x}} \\ C &= -\frac{1}{4x^{3/2}} - \frac{2}{x^2} + \frac{1}{4x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \xi e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.85: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in ξ is found from

$$\begin{aligned} \xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{\sqrt{x}}}{1} \, dx} \\ &= z_1 e^{-\sqrt{x}} \\ &= z_1 \left(e^{-\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$\xi_1 = \frac{e^{-\sqrt{x}}}{x}$$

The second solution ξ_2 to the original ode is found using reduction of order

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{\xi_1^2} \, dx$$

Substituting gives

$$\begin{aligned} \xi_2 &= \xi_1 \int \frac{e^{\int -\frac{\frac{1}{\sqrt{x}}}{1} \, dx}}{(\xi_1)^2} \, dx \\ &= \xi_1 \int \frac{e^{-2\sqrt{x}}}{(\xi_1)^2} \, dx \\ &= \xi_1 \left(\frac{x^3 e^{2\sqrt{x}} e^{-2\sqrt{x}}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} \xi &= c_1 \xi_1 + c_2 \xi_2 \\ &= c_1 \left(\frac{e^{-\sqrt{x}}}{x}\right) + c_2 \left(\frac{e^{-\sqrt{x}}}{x} \left(\frac{x^3 e^{2\sqrt{x}} e^{-2\sqrt{x}}}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(-\frac{1}{\sqrt{x}} - \frac{-\frac{c_1\,\mathrm{e}^{-\sqrt{x}}}{2x^{3/2}} - \frac{c_1\,\mathrm{e}^{-\sqrt{x}}}{x^2} + \frac{2c_2 x\,\mathrm{e}^{-\sqrt{x}}}{3} - \frac{c_2\,x^{3/2}\mathrm{e}^{-\sqrt{x}}}{6}}{\frac{c_1\,\mathrm{e}^{-\sqrt{x}}}{x} + \frac{c_2\,x^2\mathrm{e}^{-\sqrt{x}}}{3}}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{4c_2\,x^{7/2} - 6c_1\sqrt{x} + x(c_2\,x^3 + 3c_1)}{x^{3/2}\,(2c_2\,x^3 + 6c_1)}$$
$$p(x) = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= \mathrm{e}^{\int -\frac{4c_2\,x^{7/2}-6c_1\sqrt{x}+x\left(c_2\,x^3+3c_1\right)}{x^{3/2}\left(2c_2\,x^3+6c_1\right)}dx}\\ &= \frac{x\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3 + 3c_1}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yx\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3 + 3c_1}\right) = 0$$

Integrating gives

$$\begin{aligned}\frac{yx\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3 + 3c_1} &= \int 0\,dx + c_3\\ &= c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{x\,\mathrm{e}^{-\sqrt{x}}}{c_2\,x^3+3c_1}$ gives the final solution

$$y = \frac{(c_2\,x^3 + 3c_1)\,\mathrm{e}^{\sqrt{x}}c_3}{x}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(c_2\,x^3 + 3c_1)\,\mathrm{e}^{\sqrt{x}}c_3}{x}$$

The constants can be merged to give

$$y = \frac{(c_2\,x^3 + 3c_1)\,\mathrm{e}^{\sqrt{x}}}{x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(c_2\,x^3 + 3c_1)\,\mathrm{e}^{\sqrt{x}}}{x}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 19

```
dsolve(diff(diff(y(x),x),x)-1/x^(1/2)*diff(y(x),x)+1/4/x^2*(x+x^(1/2)-8)*y(x) = 0,y(x)
```

$$y = \frac{\mathrm{e}^{\sqrt{x}}(c_2 x^3 + c_1)}{x}$$

Mathematica DSolve solution

Solving time : 0.039 (sec)
Leaf size : 30

```
DSolve[{D[y[x],{x,2}]-1/x^(1/2)*D[y[x],x]+y[x]/(4*x^2)*(-8+x^(1/2)+x)==0,{}},y[x],x,IncludeS
```

$$y(x) \to \frac{e^{\sqrt{x}}(c_2 x^3 + 3c_1)}{3x}$$

2.2.29 Problem 29

Internal problem ID [9152]
Book : Second order enumerated odes
Section : section 2
Problem number : 29
Date solved : `Monday, January 27, 2025 at 05:49:54 PM`
CAS classification : `[_Lienard]`

Solve

$$\cos(x)^2 y'' - 2\cos(x)\sin(x)\, y' + y\cos(x)^2 = 0$$

Solved as second order ode using change of variable on y method 1

Time used: 0.551 (sec)

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{\sin(2x)}{\cos(x)^2} \\ q(x) &= 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 1 - \frac{\left(-\frac{\sin(2x)}{\cos(x)^2}\right)'}{2} - \frac{\left(-\frac{\sin(2x)}{\cos(x)^2}\right)^2}{4} \\ &= 1 - \frac{\left(-\frac{2\cos(2x)}{\cos(x)^2} - \frac{2\sin(2x)\sin(x)}{\cos(x)^3}\right)}{2} - \frac{\left(\frac{\sin(2x)^2}{\cos(x)^4}\right)}{4} \\ &= 1 - \left(-\frac{\cos(2x)}{\cos(x)^2} - \frac{\sin(2x)\sin(x)}{\cos(x)^3}\right) - \frac{\sin(2x)^2}{4\cos(x)^4} \\ &= 2 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= e^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{-\frac{\sin(2x)}{\cos(x)^2}}{2}} \\ &= \sec(x) \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x) \sec(x) \tag{4}$$

Applying this change of variable to the original ode results in

$$\cos(x)(2v(x) + v''(x)) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$2v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + 2\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0\left(c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)\right)$$

Or

$$v(x) = c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sec(x)$$

Hence (7) becomes

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) \sec(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) \sec(x)$$

Solved as second order ode adjoint method

Time used: 2.455 (sec)

In normal form the ode

$$\cos(x)^2 y'' - 2\cos(x)\sin(x)\, y' + y\cos(x)^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -2\tan(x) \\ q(x) &= 1 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q = 0 \\ \xi'' - (-2\tan(x)\, \xi(x))' + (\xi(x)) = 0 \\ 2\tan(x)\, \xi'(x) + \xi(x) + \xi''(x) + 2\sec(x)^2\, \xi(x) = 0 \end{aligned}$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$\xi'' + p(x)\, \xi' + q(x)\, \xi = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2\tan(x) \\ q(x) &= 2\sec(x)^2 + 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 2\sec(x)^2 + 1 - \frac{(2\tan(x))'}{2} - \frac{(2\tan(x))^2}{4} \\ &= 2\sec(x)^2 + 1 - \frac{\left(2 + 2\tan(x)^2\right)}{2} - \frac{\left(4\tan(x)^2\right)}{4} \\ &= 2\sec(x)^2 + 1 - \left(1 + \tan(x)^2\right) - \tan(x)^2 \\ &= 2 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$\xi = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{2\tan(x)}{2}} \\ &= \cos(x) \end{aligned} \tag{5}$$

Hence (3) becomes

$$\xi = v(x)\cos(x) \tag{4}$$

Applying this change of variable to the original ode results in

$$(2v(x) + v''(x))\cos(x) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$2v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + 2\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)} \\ &= \pm i\sqrt{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2} \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^{0}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

Or

$$v(x) = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned}\xi &= v(x)\, z(x) \\ &= \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)(z(x)) \qquad (7)\end{aligned}$$

But from (5)

$$z(x) = \cos(x)$$

Hence (7) becomes

$$\xi = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)\cos(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' + y\left(-2\tan(x) - \frac{\left(-c_1\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + c_2\sqrt{2}\, \cos\left(\sqrt{2}\, x\right)\right)\cos(x) - \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)\sin(x}{\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)\cos(x)}\right.$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= -\frac{\left(-c_1\sqrt{2} + \tan(x)\, c_2\right)\sin\left(\sqrt{2}\, x\right) + \left(\tan(x)\, c_1 + c_2\sqrt{2}\right)\cos\left(\sqrt{2}\, x\right)}{c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)} \\ p(x) &= 0\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= \mathrm{e}^{\int -\frac{\left(-c_1\sqrt{2}+\tan(x)c_2\right)\sin\left(\sqrt{2}\, x\right)+\left(\tan(x)c_1+c_2\sqrt{2}\right)\cos\left(\sqrt{2}\, x\right)}{c_1\cos\left(\sqrt{2}\, x\right)+c_2\sin\left(\sqrt{2}\, x\right)}dx} \\ &= \mathrm{e}^{-\frac{\ln\left(1+\tan(x)^2\right)}{2}-\ln\left(c_1\tan\left(\frac{\sqrt{2}\, x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\, x}{2}\right)-c_1\right)+\ln\left(1+\tan\left(\frac{\sqrt{2}\, x}{2}\right)^2\right)}\end{aligned}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$

$$\frac{d}{dx}\left(y\,e^{-\frac{\ln\left(1+\tan(x)^2\right)}{2}-\ln\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right)-c_1\right)+\ln\left(1+\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2\right)}\right) = 0$$

Integrating gives

$$\begin{aligned} y\,e^{-\frac{\ln\left(1+\tan(x)^2\right)}{2}-\ln\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right)-c_1\right)+\ln\left(1+\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2\right)} &= \int 0\,dx + c_3 \\ &= c_3 \end{aligned}$$

Dividing throughout by the integrating factor $e^{-\frac{\ln\left(1+\tan(x)^2\right)}{2}-\ln\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right)-c_1\right)+\ln\left(1+\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2\right)}$ gives the final solution

$$y = \frac{\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right)-c_1\right)\sqrt{1+\tan(x)^2}\,c_3}{1+\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right)-c_1\right)\sqrt{1+\tan(x)^2}\,c_3}{1+\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2}$$

The constants can be merged to give

$$y = \frac{\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right)-c_1\right)\sqrt{1+\tan(x)^2}}{1+\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right)-c_1\right)\sqrt{1+\tan(x)^2}}{1+\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 24

```
dsolve(cos(x)^2*diff(diff(y(x),x),x)-2*cos(x)*sin(x)*diff(y(x),x)+cos(x)^2*y(x) = 0,y(x)
```

$$y = \sec\left(x\right)\left(c_1 \sin\left(\sqrt{2}\, x\right)+c_2 \cos\left(\sqrt{2}\, x\right)\right)$$

Mathematica DSolve solution

Solving time : `0.062` (sec)
Leaf size : 51

```
DSolve[{Cos[x]^2*D[y[x],{x,2}]-2*Cos[x]*Sin[x]*D[y[x],x]+y[x]*Cos[x]^2==0,{}},y[x],x,IncludeSi
```

$$y(x)\to \frac{1}{4}e^{-i\sqrt{2}x}\left(4c_1-i\sqrt{2}c_2e^{2i\sqrt{2}x}\right)\sec(x)$$

2.2.30 Problem 30

Internal problem ID [9153]
Book : Second order enumerated odes
Section : section 2
Problem number : 30
Date solved : `Monday, January 27, 2025 at 05:49:58 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 4xy' + \left(4x^2 - 1\right) y = -3\,\mathrm{e}^{x^2} \sin\left(x\right)$$

Solved as second order ode using change of variable on y method 1

Time used: 0.536 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4xy' + \left(4x^2 - 1\right) y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -4x \\ q(x) &= 4x^2 - 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 4x^2 - 1 - \frac{(-4x)'}{2} - \frac{(-4x)^2}{4} \\ &= 4x^2 - 1 - \frac{(-4)}{2} - \frac{(16x^2)}{4} \\ &= 4x^2 - 1 - (-2) - 4x^2 \\ &= 1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{-4x}{2}} \\ &= \mathrm{e}^{x^2} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\,\mathrm{e}^{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$\mathrm{e}^{x^2}(v(x) + v''(x)) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i \\ \lambda_2 &= -i \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x))\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = e^{x^2}$$

Hence (7) becomes

$$y = (c_1 \cos(x) + c_2 \sin(x))\, e^{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = (c_1 \cos(x) + c_2 \sin(x))\, e^{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(x)\, e^{x^2} \\ y_2 &= e^{x^2} \sin(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\, e^{x^2} & e^{x^2} \sin(x) \\ \frac{d}{dx}\left(\cos(x)\, e^{x^2}\right) & \frac{d}{dx}\left(e^{x^2} \sin(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\, e^{x^2} & e^{x^2} \sin(x) \\ -e^{x^2} \sin(x) + 2\cos(x)\, x\, e^{x^2} & 2x\, e^{x^2} \sin(x) + \cos(x)\, e^{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x)\,e^{x^2}\right)\left(2x\,e^{x^2}\sin(x)+\cos(x)\,e^{x^2}\right)-\left(e^{x^2}\sin(x)\right)\left(-e^{x^2}\sin(x)+2\cos(x)\,x\,e^{x^2}\right)$$

Which simplifies to

$$W = \cos(x)^2\,e^{2x^2}+\sin(x)^2\,e^{2x^2}$$

Which simplifies to

$$W = e^{2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-3\sin(x)^2\,e^{2x^2}}{e^{2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int -3\sin(x)^2\,dx$$

Hence

$$u_1 = -\frac{3\sin(x)\cos(x)}{2}+\frac{3x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-3\cos(x)\,e^{2x^2}\sin(x)}{e^{2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{3\sin(2x)}{2}\,dx$$

Hence

$$u_2 = \frac{3\cos(2x)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{3\sin(x)\cos(x)}{2}+\frac{3x}{2}\right)\cos(x)\,e^{x^2}+\frac{3\,e^{x^2}\sin(x)\cos(2x)}{4}$$

Which simplifies to

$$y_p(x) = \frac{3\,e^{x^2}(-\sin(x)+2\cos(x)\,x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1\cos(x)+c_2\sin(x))\,e^{x^2}\right)+\left(\frac{3\,e^{x^2}(-\sin(x)+2\cos(x)\,x)}{4}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = (c_1\cos(x)+c_2\sin(x))\,e^{x^2}+\frac{3\,e^{x^2}(-\sin(x)+2\cos(x)\,x)}{4}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.287 (sec)

Writing the ode as

$$y'' - 4xy' + (4x^2 - 1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.86: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\ &= z_1 e^{x^2} \\ &= z_1\left(\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\,\mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos(x)\,\mathrm{e}^{x^2}\right) + c_2\left(\cos(x)\,\mathrm{e}^{x^2}(\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4xy' + (4x^2 - 1)\,y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) e^{x^2} + c_2 e^{x^2} \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x) e^{x^2}$$
$$y_2 = e^{x^2} \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) e^{x^2} & e^{x^2} \sin(x) \\ \frac{d}{dx}\left(\cos(x) e^{x^2}\right) & \frac{d}{dx}\left(e^{x^2} \sin(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) e^{x^2} & e^{x^2} \sin(x) \\ -e^{x^2} \sin(x) + 2\cos(x) x e^{x^2} & 2x e^{x^2} \sin(x) + \cos(x) e^{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x) e^{x^2}\right)\left(2x e^{x^2} \sin(x) + \cos(x) e^{x^2}\right) - \left(e^{x^2} \sin(x)\right)\left(-e^{x^2} \sin(x) + 2\cos(x) x e^{x^2}\right)$$

Which simplifies to

$$W = \cos(x)^2 e^{2x^2} + \sin(x)^2 e^{2x^2}$$

Which simplifies to

$$W = e^{2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-3\sin(x)^2 e^{2x^2}}{e^{2x^2}} \, dx$$

Which simplifies to

$$u_1 = -\int -3\sin(x)^2 \, dx$$

Hence

$$u_1 = -\frac{3\sin(x)\cos(x)}{2} + \frac{3x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-3\cos(x)\,\mathrm{e}^{2x^2}\sin(x)}{\mathrm{e}^{2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{3\sin(2x)}{2}\,dx$$

Hence

$$u_2 = \frac{3\cos(2x)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{3\sin(x)\cos(x)}{2} + \frac{3x}{2}\right)\cos(x)\,\mathrm{e}^{x^2} + \frac{3\,\mathrm{e}^{x^2}\sin(x)\cos(2x)}{4}$$

Which simplifies to

$$y_p(x) = \frac{3\,\mathrm{e}^{x^2}(-\sin(x) + 2\cos(x)\,x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(x)\,\mathrm{e}^{x^2} + c_2\,\mathrm{e}^{x^2}\sin(x)\right) + \left(\frac{3\,\mathrm{e}^{x^2}(-\sin(x) + 2\cos(x)\,x)}{4}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\cos(x)\,\mathrm{e}^{x^2} + c_2\,\mathrm{e}^{x^2}\sin(x) + \frac{3\,\mathrm{e}^{x^2}(-\sin(x) + 2\cos(x)\,x)}{4}$$

Solved as second order ode adjoint method

Time used: 5.453 (sec)

In normal form the ode

$$y'' - 4xy' + (4x^2 - 1)\,y = -3\,\mathrm{e}^{x^2}\sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -4x \\ q(x) &= 4x^2 - 1 \\ r(x) &= -3\,\mathrm{e}^{x^2}\sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (-4x\xi(x))' + \left(\left(4x^2-1\right)\xi(x)\right) &= 0 \\
4\xi(x)\, x^2 + 4x\xi'(x) + \xi''(x) + 3\xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$
\xi'' + p(x)\,\xi' + q(x)\,\xi = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= 4x \\
q(x) &= 4x^2 + 3
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 4x^2 + 3 - \frac{(4x)'}{2} - \frac{(4x)^2}{4} \\
&= 4x^2 + 3 - \frac{(4)}{2} - \frac{(16x^2)}{4} \\
&= 4x^2 + 3 - (2) - 4x^2 \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
\xi = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\
&= e^{-\int \frac{4x}{2}} \\
&= \mathrm{e}^{-x^2} \qquad (5)
\end{aligned}
$$

Hence (3) becomes

$$
\xi = v(x)\, \mathrm{e}^{-x^2} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
\mathrm{e}^{-x^2} \left(v''(x) + v(x)\right) = 0
$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$
v''(x) + v(x) = 0
$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned}\xi &= v(x)\, z(x)\\ &= (c_1 \cos(x) + c_2 \sin(x))\,(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{-x^2}$$

Hence (7) becomes

$$\xi = (c_1 \cos(x) + c_2 \sin(x))\,\mathrm{e}^{-x^2}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' + y\left(-4x - \frac{\left(\left(-c_1 \sin(x) + c_2 \cos(x)\right) e^{-x^2} - 2\left(c_1 \cos(x) + c_2 \sin(x)\right) x\, e^{-x^2}\right) e^{x^2}}{c_1 \cos(x) + c_2 \sin(x)}\right) = \frac{e^{x^2} \left(\frac{3\cos(x)^2 c_1}{2} - \right.}{c_1 \cos(}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{\sin(x)\left(2c_2 x - c_1\right) + 2\cos(x)\left(c_1 x + \frac{c_2}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)} \\ p(x) &= \frac{3\left(\cos(x)^2 c_1 + c_2 \sin(x)\cos(x) - c_2 x\right) e^{x^2}}{2c_1 \cos(x) + 2c_2 \sin(x)} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= e^{\int -\frac{\sin(x)(2c_2 x - c_1) + 2\cos(x)\left(c_1 x + \frac{c_2}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\ &= \frac{\sec\left(\frac{x}{2}\right)^2 e^{-x^2}}{c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{3\left(\cos(x)^2 c_1 + c_2 \sin(x)\cos(x) - c_2 x\right) e^{x^2}}{2c_1 \cos(x) + 2c_2 \sin(x)}\right)$$

$$\frac{d}{dx}\left(\frac{y \sec\left(\frac{x}{2}\right)^2 e^{-x^2}}{c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1}\right) = \left(\frac{\sec\left(\frac{x}{2}\right)^2 e^{-x^2}}{c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1}\right)\left(\frac{3\left(\cos(x)^2 c_1 + c_2 \sin(x)\cos(x) - c_2 x\right) e^{x^2}}{2c_1 \cos(x) + 2c_2 \sin(x)}\right)$$

$$d\left(\frac{y \sec\left(\frac{x}{2}\right)^2 e^{-x^2}}{c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1}\right) = \left(\frac{3\left(\cos(x)^2 c_1 + c_2 \sin(x)\cos(x) - c_2 x\right) e^{x^2} \sec\left(\frac{x}{2}\right)^2 e^{-x^2}}{\left(2c_1 \cos(x) + 2c_2 \sin(x)\right)\left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1\right)}\right) dx$$

Integrating gives

$$\begin{aligned} \frac{y \sec\left(\frac{x}{2}\right)^2 e^{-x^2}}{c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1} &= \int \frac{3\left(\cos(x)^2 c_1 + c_2 \sin(x)\cos(x) - c_2 x\right) e^{x^2} \sec\left(\frac{x}{2}\right)^2 e^{-x^2}}{\left(2c_1 \cos(x) + 2c_2 \sin(x)\right)\left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1\right)} dx \\ &= \frac{6x}{4ic_2 - 4c_1} - \frac{3ic_2 x}{(ic_1 + c_2)\left(e^{2ix} c_2 + ic_1 e^{2ix} - c_2 + ic_1\right)} + c_3 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{\sec\left(\frac{x}{2}\right)^2 e^{-x^2}}{c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1}$ gives the final solution

$$y = -\frac{\left(c_1 \cos(x) + c_2 \sin(x)\right) e^{x^2} \left(-2 e^{2ix} c_1^2 c_3 + 2 e^{2ix} c_2^2 c_3 + 3 e^{2ix} c_1 x - 2c_1^2 c_3 - 2c_2^2 c_3 + 4i e^{2ix} c_1 c_2 c_3 + 3c_1 x\right.}{2\left(ic_1 + c_2\right)\left(e^{2ix} c_2 + ic_1 e^{2ix} - c_2 + ic_1\right)}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = -\frac{(c_1 \cos(x) + c_2 \sin(x)) e^{x^2} (-2 e^{2ix} c_1^2 c_3 + 2 e^{2ix} c_2^2 c_3 + 3 e^{2ix} c_1 x - 2c_1^2 c_3 - 2c_2^2 c_3 + 4i e^{2ix} c_1 c_2 c_3 + 3c_1 x - 3i}{2 (ic_1 + c_2) (e^{2ix} c_2 + ic_1 e^{2ix} - c_2 + ic_1)}$$

The constants can be merged to give

$$y = -\frac{(c_1 \cos(x) + c_2 \sin(x)) e^{x^2} (-2 e^{2ix} c_1^2 + 2 e^{2ix} c_2^2 + 3 e^{2ix} c_1 x - 2c_1^2 - 2c_2^2 + 4i e^{2ix} c_1 c_2 + 3c_1 x - 3i e^{2ix} c_2 x -}{2 (ic_1 + c_2) (e^{2ix} c_2 + ic_1 e^{2ix} - c_2 + ic_1)}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{(c_1 \cos(x) + c_2 \sin(x)) e^{x^2} (-2 e^{2ix} c_1^2 + 2 e^{2ix} c_2^2 + 3 e^{2ix} c_1 x - 2c_1^2 - 2c_2^2 + 4i e^{2ix} c_1 c_2 + 3c_1 x - 3i e^{2ix} c_2 x -}{2 (ic_1 + c_2) (e^{2ix} c_2 + ic_1 e^{2ix} - c_2 + ic_1)}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 29

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2-1)*y(x) = -3*sin(x)*exp(x^2),y(x),si
```

$$y = \frac{((2c_2 + 3x) \cos(x) + \sin(x) (2c_1 - 3)) e^{x^2}}{2}$$

Mathematica DSolve solution

Solving time : 0.09 (sec)
Leaf size : 50

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-1)*y[x]==-3*Exp[x^2]*Sin[x],{}},y[x],x,IncludeSin
```

$$y(x) \to \frac{1}{8} e^{x(x-i)} \left(6x + e^{2ix}(6x + 3i - 4ic_2) - 3i + 8c_1\right)$$

2.2.31 Problem 31

Internal problem ID [9154]
Book : Second order enumerated odes
Section : section 2
Problem number : 31
Date solved : `Monday, January 27, 2025 at 05:50:05 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 2bxy' + b^2x^2y = x$$

Solved as second order ode using change of variable on y method 1

Time used: 6.645 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2bxy' + b^2x^2y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -2xb \\ q(x) &= x^2b^2 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= x^2b^2 - \frac{(-2xb)'}{2} - \frac{(-2xb)^2}{4} \\ &= x^2b^2 - \frac{(-2b)}{2} - \frac{(4x^2b^2)}{4} \\ &= x^2b^2 - (-b) - x^2b^2 \\ &= b \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{-2xb}{2}} \\ &= \mathrm{e}^{\frac{b\,x^2}{2}} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\,\mathrm{e}^{\frac{b\,x^2}{2}} \tag{4}$$

Applying this change of variable to the original ode results in

$$\mathrm{e}^{\frac{b\,x^2}{2}}(bv(x) + v''(x)) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$bv(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = b$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + b\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + b = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = b$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(b)} \\ &= \pm\sqrt{-b} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +\sqrt{-b} \\ \lambda_2 &= -\sqrt{-b} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= \frac{\left((-1+i)\,\sqrt{\operatorname{signum}(b)} + 1 + i\right)\sqrt{b}}{2} \\ \lambda_2 &= -\frac{\left((-1+i)\,\sqrt{\operatorname{signum}(b)} + 1 + i\right)\sqrt{b}}{2} \end{aligned}$$

The roots are complex but they are not conjugate of each others. Hence simplification using Euler relation is not possible here. Therefore the final solution is

$$\begin{aligned} v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ &= c_1 e^{\frac{x\left((-1+i)\sqrt{\operatorname{signum}(b)}+1+i\right)\sqrt{b}}{2}} + c_2 e^{-\frac{x\left((-1+i)\sqrt{\operatorname{signum}(b)}+1+i\right)\sqrt{b}}{2}} \end{aligned}$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \, e^{\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}} + c_2 \, e^{-\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}}\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = e^{\frac{b\,x^2}{2}}$$

Hence (7) becomes

$$y = \left(c_1 \, e^{\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}} + c_2 \, e^{-\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}}\right) e^{\frac{b\,x^2}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 \, e^{\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}} + c_2 \, e^{-\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}}\right) e^{\frac{b\,x^2}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}} \\ y_2 &= e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}} & e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}} \\ \frac{d}{dx}\left(e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}\right) & \frac{d}{dx}\left(e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \left| -\frac{\sqrt{b}\,e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2} - \frac{ie^{-\frac{x\sqrt{b}}{2}} \sqrt{b}\,e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2} + e^{-\frac{x\sqrt{b}}{2}} e^{-} \right.$$

Therefore

$$
\begin{aligned}
W &= \left(e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}\right) \left(\frac{\sqrt{b}\,e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2}\right. \\
&\quad + \frac{i e^{\frac{x\sqrt{b}}{2}} \sqrt{b}\,e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2} \\
&\quad - \frac{e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} \sqrt{b}\,\sqrt{\text{signum}\,(b)}\,e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2} \\
&\quad + \frac{i e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} \sqrt{b}\,\sqrt{\text{signum}\,(b)}\,e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2} \\
&\quad \left. + e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} x b\,e^{\frac{b\,x^2}{2}}\right) \\
&\quad - \left(e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}\right) \left(-\frac{\sqrt{b}\,e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2}\right. \\
&\quad - \frac{i e^{-\frac{x\sqrt{b}}{2}} \sqrt{b}\,e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2} \\
&\quad + \frac{e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} \sqrt{b}\,\sqrt{\text{signum}\,(b)}\,e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2} \\
&\quad - \frac{i e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} \sqrt{b}\,\sqrt{\text{signum}\,(b)}\,e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2} \\
&\quad \left. + e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} x b\,e^{\frac{b\,x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
W &= i e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{b\,x^2} e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} \sqrt{b}\,\sqrt{\text{signum}\,(b)}\,e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} \\
&\quad + i e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{b\,x^2} e^{\frac{x\sqrt{b}}{2}} \sqrt{b}\,e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} \\
&\quad - e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{b\,x^2} e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} \sqrt{b}\,\sqrt{\text{signum}\,(b)}\,e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} \\
&\quad + e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{b\,x^2} \sqrt{b}\,e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}}
\end{aligned}
$$

Which simplifies for $0 < x$ to

$$W = 2i\sqrt{b}\,e^{b\,x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\text{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}} x}{2i\sqrt{b}\,e^{b\,x^2}}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -\frac{ix\,e^{\frac{\left(2i\sqrt{b}-xb\right)x}{2}}}{2\sqrt{b}}\,dx$$

Hence

$$u_1 = \frac{i\left(-\frac{e^{-\frac{b\,x^2}{2}+ix\sqrt{b}}}{b} + \frac{i\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\text{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2} - \frac{i\sqrt{2}}{2}\right)}{2b}\right)}{2\sqrt{b}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}} x}{2i\sqrt{b}\,e^{b\,x^2}} dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int -\frac{ix\,e^{-\frac{\left(2i\sqrt{b}+xb\right)x}{2}}}{2\sqrt{b}} dx$$

Hence

$$u_2 = -\frac{i\left(-\frac{e^{-\frac{b\,x^2}{2}-ix\sqrt{b}}}{b} - \frac{i\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}+\frac{i\sqrt{2}}{2}\right)}{2b}\right)}{2\sqrt{b}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{i\left(-\frac{e^{-\frac{b\,x^2}{2}+ix\sqrt{b}}}{b} + \frac{i\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}-\frac{i\sqrt{2}}{2}\right)}{2b}\right) e^{-\frac{x\sqrt{b}}{2}} e^{-\frac{ix\sqrt{b}}{2}} e^{\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{-\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}}{2\sqrt{b}} - \frac{i e^{\frac{x\sqrt{b}}{2}} e^{\frac{ix\sqrt{b}}{2}} e^{-\frac{x\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{ix\sqrt{b}\,\sqrt{\mathrm{signum}(b)}}{2}} e^{\frac{b\,x^2}{2}}\left(-\frac{e^{-\frac{b\,x^2}{2}-ix\sqrt{b}}}{b} - \frac{i\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}+\frac{i\sqrt{2}}{2}\right)}{2b}\right)}{2\sqrt{b}}$$

Which simplifies to

$$y_p(x) = \frac{\left(\sqrt{2}\,\sqrt{\pi}\,e^{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{b}\,x\sqrt{\mathrm{signum}(b)}-\frac{1}{2}+\left(-\frac{1}{2}-\frac{i}{2}\right)x\sqrt{b}}\,\mathrm{erf}\left(\frac{\sqrt{2}\left(-x\sqrt{b}+i\right)}{2}\right) - \sqrt{2}\,\sqrt{\pi}\,e^{\left(-\frac{1}{2}+\frac{i}{2}\right)\sqrt{b}\,x\sqrt{\mathrm{signum}(b)}-\frac{1}{2}+\left(\frac{1}{2}+\frac{i}{2}\right)x\sqrt{b}}\,\mathrm{erf}}{4b^{3/2}}$$

Therefore the general solution is

$$y = y_h + y_p = \left(\left(c_1\,e^{\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}} + c_2\,e^{-\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}}\right) e^{\frac{b\,x^2}{2}}\right) + \left(\frac{\left(\sqrt{2}\,\sqrt{\pi}\,e^{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{b}\,x\sqrt{\mathrm{signum}(b)}-\frac{1}{2}+\left(-\frac{1}{2}-\frac{i}{2}\right)x\sqrt{b}}\,\mathrm{erf}\left(\frac{\sqrt{2}\left(-x\sqrt{b}+i\right)}{2}\right) - \sqrt{2}\,\sqrt{\pi}\,e^{\left(-\frac{1}{2}+\frac{i}{2}\right)\sqrt{b}\,x\sqrt{\mathrm{signum}(b)}-\frac{1}{2}+\left(\frac{1}{2}+\frac{i}{2}\right)x}}{4b^{3/2}}\right.$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(c_1\,e^{\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}} + c_2\,e^{-\frac{x\left((-1+i)\sqrt{\mathrm{signum}(b)}+1+i\right)\sqrt{b}}{2}}\right) e^{\frac{b\,x^2}{2}} + \frac{\left(\sqrt{2}\,\sqrt{\pi}\,e^{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{b}\,x\sqrt{\mathrm{signum}(b)}-\frac{1}{2}+\left(-\frac{1}{2}-\frac{i}{2}\right)x\sqrt{b}}\,\mathrm{erf}\left(\frac{\sqrt{2}\left(-x\sqrt{b}+i\right)}{2}\right) - \sqrt{2}\,\sqrt{\pi}\,e^{\left(-\frac{1}{2}+\frac{i}{2}\right)\sqrt{b}\,x\sqrt{\mathrm{signum}(b)}-\frac{1}{2}+\left(\frac{1}{2}+\frac{i}{2}\right)x\sqrt{b}}}{4b^{3/2}}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.278 (sec)

Writing the ode as

$$y'' - 2bxy' + b^2x^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2xb \\ C &= x^2b^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-b}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -b \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-b)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.87: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -b$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{\sqrt{-b}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2xb}{1}\, dx}\\ &= z_1 e^{\frac{b\,x^2}{2}}\\ &= z_1\left(e^{\frac{b\,x^2}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x(xb+2\sqrt{-b})}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2xb}{1}\, dx}}{(y_1)^2}\, dx\\ &= y_1 \int \frac{e^{b\,x^2}}{(y_1)^2}\, dx\\ &= y_1\left(-\frac{e^{b\,x^2} e^{-x(xb+2\sqrt{-b})}}{2\sqrt{-b}}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(e^{\frac{x(xb+2\sqrt{-b})}{2}}\right) + c_2\left(e^{\frac{x(xb+2\sqrt{-b})}{2}}\left(-\frac{e^{b\,x^2} e^{-x(xb+2\sqrt{-b})}}{2\sqrt{-b}}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2bxy' + b^2x^2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \,\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} - \frac{c_2 \,\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}}$$

$$y_2 = -\frac{\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} & -\frac{\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}} \\ \frac{d}{dx}\left(\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}}\right) & \frac{d}{dx}\left(-\frac{\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} & -\frac{\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}} \\ \left(xb+\sqrt{-b}\right)\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} & -\frac{(xb-\sqrt{-b})\,\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}} \end{vmatrix}$$

Therefore

$$\begin{aligned} W &= \left(\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}}\right)\left(-\frac{\left(xb-\sqrt{-b}\right)\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}\right) \\ &\quad - \left(-\frac{\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}\right)\left(\left(xb+\sqrt{-b}\right)\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}}\right) \end{aligned}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} \mathrm{e}^{-\frac{x(-xb+2\sqrt{-b})}{2}}$$

Which simplifies to

$$W = \mathrm{e}^{b\,x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}} x}{2\sqrt{-b}}}{\mathrm{e}^{b\,x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{x\,\mathrm{e}^{-\frac{x(xb+2\sqrt{-b})}{2}}}{2\sqrt{-b}}\,dx$$

Hence

$$u_1 = \frac{-\frac{\mathrm{e}^{-\frac{b\,x^2}{2}-\sqrt{-b}\,x}}{b} - \frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}+\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{3/2}}}{2\sqrt{-b}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} x}{\mathrm{e}^{b\,x^2}}\,dx$$

Which simplifies to

$$u_2 = \int x\,\mathrm{e}^{-\frac{x(xb-2\sqrt{-b})}{2}}\,dx$$

Hence

$$u_2 = -\frac{\mathrm{e}^{-\frac{b\,x^2}{2}+\sqrt{-b}\,x}}{b} + \frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}-\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{3/2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(-\frac{\mathrm{e}^{-\frac{b\,x^2}{2}-\sqrt{-b}\,x}}{b} - \frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}+\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{3/2}}\right)\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}}}{2\sqrt{-b}} - \frac{\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}\left(-\frac{\mathrm{e}^{-\frac{b\,x^2}{2}+\sqrt{-b}\,x}}{b} + \frac{\sqrt{-b}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{b}\,x}{2}-\frac{\sqrt{-b}\,\sqrt{2}}{2\sqrt{b}}\right)}{2b^{3/2}}\right)}{2\sqrt{-b}}$$

Which simplifies to

$$y_p(x) = -\frac{\sqrt{\pi}\,\sqrt{2}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right)\mathrm{e}^{-\frac{1}{2}+\frac{b\,x^2}{2}+\sqrt{-b}\,x} - \operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right)\mathrm{e}^{-\frac{1}{2}+\frac{b\,x^2}{2}-\sqrt{-b}\,x}\right)}{4b^{3/2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} - \frac{c_2\,\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}}\right) \\ &\quad + \left(-\frac{\sqrt{\pi}\,\sqrt{2}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right)\mathrm{e}^{-\frac{1}{2}+\frac{b\,x^2}{2}+\sqrt{-b}\,x} - \operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right)\mathrm{e}^{-\frac{1}{2}+\frac{b\,x^2}{2}-\sqrt{-b}\,x}\right)}{4b^{3/2}}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{\frac{x(xb+2\sqrt{-b})}{2}} - \frac{c_2 \,\mathrm{e}^{\frac{x(xb-2\sqrt{-b})}{2}}}{2\sqrt{-b}} - \frac{\sqrt{\pi}\,\sqrt{2}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,(xb+\sqrt{-b})}{2\sqrt{b}}\right)\mathrm{e}^{-\frac{1}{2}+\frac{b\,x^2}{2}+\sqrt{-b}\,x} - \operatorname{erf}\left(\frac{\sqrt{2}\,(-xb+\sqrt{-b})}{2\sqrt{b}}\right)\mathrm{e}^{-\frac{1}{2}+\frac{b\,x^2}{2}-\sqrt{-b}\,x}\right)}{4b^{3/2}}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 137

```
dsolve(diff(diff(y(x),x),x)-2*b*x*diff(y(x),x)+b^2*x^2*y(x) = x,y(x),singsol=all)
```

$$y = \frac{4\,\mathrm{e}^{\frac{x(bx-2\sqrt{-b})}{2}} c_1 b^{3/2} + 4\,\mathrm{e}^{\frac{x(bx+2\sqrt{-b})}{2}} c_2 b^{3/2} - \sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,(bx+\sqrt{-b})}{2\sqrt{b}}\right)\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{2}+\frac{b\,x^2}{2}+x\sqrt{-b}} + \sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2}\,(-bx+\sqrt{-b}}{2\sqrt{b}}\right.}{4b^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.461 (sec)
Leaf size : 162

```
DSolve[{D[y[x],{x,2}]-2*b*x*D[y[x],x]+b^2*x^2*y[x]==x,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{e^{\frac{bx^2}{2}-i\sqrt{b}x}\left(2\sqrt{b}\int_1^x \frac{ie^{i\sqrt{b}K[1]-\frac{1}{2}bK[1]^2}K[1]}{2\sqrt{b}}dK[1] - ie^{2i\sqrt{b}x}\int_1^x e^{-\frac{1}{2}bK[2]^2-i\sqrt{b}K[2]}K[2]dK[2] - ic_2e^{2i\sqrt{b}x} + 2\sqrt{b}c_1\right)}{2\sqrt{b}}$$

2.2.32 Problem 32

Internal problem ID [9155]
Book : Second order enumerated odes
Section : section 2
Problem number : 32
Date solved : `Monday, January 27, 2025 at 05:50:57 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 4xy' + \left(4x^2 - 3\right) y = e^{x^2}$$

Solved as second order ode using change of variable on y method 1

Time used: 0.427 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4xy' + \left(4x^2 - 3\right) y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -4x \\ q(x) &= 4x^2 - 3 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 4x^2 - 3 - \frac{(-4x)'}{2} - \frac{(-4x)^2}{4} \\ &= 4x^2 - 3 - \frac{(-4)}{2} - \frac{(16x^2)}{4} \\ &= 4x^2 - 3 - (-2) - 4x^2 \\ &= -1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\
&= e^{-\int \frac{-4x}{2}} \\
&= \mathrm{e}^{x^2}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\,\mathrm{e}^{x^2} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
\mathrm{e}^{x^2}\left(v''(x) - v(x)\right) = 0
$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$
v''(x) - v(x) = 0
$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{x\lambda} - \mathrm{e}^{x\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - 1 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)} \\
&= \pm 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +1 \\
\lambda_2 &= -1
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -1
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}
\end{aligned}
$$

Or

$$
v(x) = c_1\,\mathrm{e}^{x} + c_2\,\mathrm{e}^{-x}
$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\, z(x) \\
&= \left(c_1\, e^x + c_2\, e^{-x}\right) (z(x)) \qquad (7)
\end{aligned}
$$

But from (5)

$$z(x) = e^{x^2}$$

Hence (7) becomes

$$y = \left(c_1\, e^x + c_2\, e^{-x}\right) e^{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1\, e^x + c_2\, e^{-x}\right) e^{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \qquad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= e^{-x} e^{x^2} \\
y_2 &= e^{x} e^{x^2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \qquad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \qquad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} e^{x^2} & e^{x} e^{x^2} \\ \frac{d}{dx}\left(e^{-x} e^{x^2}\right) & \frac{d}{dx}\left(e^{x} e^{x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} e^{x^2} & e^{x} e^{x^2} \\ -e^{-x} e^{x^2} + 2\, e^{-x} x\, e^{x^2} & e^{x} e^{x^2} + 2\, e^{x} x\, e^{x^2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x} e^{x^2}\right)\left(e^{x} e^{x^2} + 2\, e^{x} x\, e^{x^2}\right) - \left(e^{x} e^{x^2}\right)\left(-e^{-x} e^{x^2} + 2\, e^{-x} x\, e^{x^2}\right)$$

Which simplifies to

$$W = 2\, e^{2x^2}$$

Which simplifies to

$$W = 2\,e^{2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^x e^{2x^2}}{2\,e^{2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^x}{2} dx$$

Hence

$$u_1 = -\frac{e^x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x} e^{2x^2}}{2\,e^{2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-x}}{2} dx$$

Hence

$$u_2 = -\frac{e^{-x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{x^2}}{2} - \frac{e^x e^{x^2} e^{-x}}{2}$$

Which simplifies to

$$y_p(x) = -e^{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1\,e^x + c_2\,e^{-x}\right) e^{x^2}\right) + \left(-e^{x^2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(c_1\,e^x + c_2\,e^{-x}\right) e^{x^2} - e^{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$y'' - 4xy' + (4x^2 - 3)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.88: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\ &= z_1 e^{x^2} \\ &= z_1\left(\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x(x-1)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{\mathrm{e}^{2x^2}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x^2}\mathrm{e}^{-2x(x-1)}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{x(x-1)}\right) + c_2\left(\mathrm{e}^{x(x-1)}\left(\frac{\mathrm{e}^{2x^2}\mathrm{e}^{-2x(x-1)}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4xy' + \left(4x^2 - 3\right) y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \,\mathrm{e}^{x(x-1)} + \frac{c_2 \,\mathrm{e}^{x(x+1)}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{x(x-1)}$$
$$y_2 = \frac{\mathrm{e}^{x(x+1)}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{x(x-1)} & \frac{\mathrm{e}^{x(x+1)}}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{x(x-1)}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x(x+1)}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{x(x-1)} & \frac{\mathrm{e}^{x(x+1)}}{2} \\ (2x-1)\,\mathrm{e}^{x(x-1)} & \frac{(2x+1)\mathrm{e}^{x(x+1)}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{x(x-1)}\right)\left(\frac{(2x+1)\,\mathrm{e}^{x(x+1)}}{2}\right) - \left(\frac{\mathrm{e}^{x(x+1)}}{2}\right)\left((2x-1)\,\mathrm{e}^{x(x-1)}\right)$$

Which simplifies to

$$W = \mathrm{e}^{x(x-1)}\mathrm{e}^{x(x+1)}$$

Which simplifies to

$$W = \mathrm{e}^{2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{x(x+1)} e^{x^2}}{2}}{e^{2x^2}} dx$$

Which simplifies to

$$u_1 = -\int \frac{e^x}{2} dx$$

Hence

$$u_1 = -\frac{e^x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{x(x-1)} e^{x^2}}{e^{2x^2}} dx$$

Which simplifies to

$$u_2 = \int e^{-x} dx$$

Hence

$$u_2 = -e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^x e^{x(x-1)}}{2} - \frac{e^{x(x+1)} e^{-x}}{2}$$

Which simplifies to

$$y_p(x) = -e^{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x(x-1)} + \frac{c_2 e^{x(x+1)}}{2}\right) + \left(-e^{x^2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{x(x-1)} + \frac{c_2 e^{x(x+1)}}{2} - e^{x^2}$$

Solved as second order ode adjoint method

Time used: 0.460 (sec)

In normal form the ode

$$y'' - 4xy' + (4x^2 - 3)\, y = e^{x^2} \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -4x \\ q(x) &= 4x^2 - 3 \\ r(x) &= e^{x^2} \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (-4x\xi(x))' + \left((4x^2 - 3)\, \xi(x)\right) &= 0 \\ 4\xi(x)\, x^2 + 4x\xi'(x) + \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$\xi'' + p(x)\, \xi' + q(x)\, \xi = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 4x \\ q(x) &= 4x^2 + 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 4x^2 + 1 - \frac{(4x)'}{2} - \frac{(4x)^2}{4} \\ &= 4x^2 + 1 - \frac{(4)}{2} - \frac{(16x^2)}{4} \\ &= 4x^2 + 1 - (2) - 4x^2 \\ &= -1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$\xi = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= e^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{4x}{2}} \\ &= e^{-x^2} \end{aligned} \tag{5}$$

Hence (3) becomes

$$\xi = v(x)\, e^{-x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$e^{-x^2}(-v(x) + v''(x)) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$-v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} - e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$v(x) = c_1 e^x + c_2 e^{-x}$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned}\xi &= v(x)\, z(x)\\ &= \left(c_1 e^x + c_2 e^{-x}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = e^{-x^2}$$

Hence (7) becomes

$$\xi = \left(c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}\right) \mathrm{e}^{-x^2}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' + y\left(-4x - \frac{\left(\left(c_1 \,\mathrm{e}^{x} - c_2 \,\mathrm{e}^{-x}\right) \mathrm{e}^{-x^2} - 2\left(c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}\right) x \,\mathrm{e}^{-x^2}\right) \mathrm{e}^{x^2}}{c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}}\right) = \frac{\mathrm{e}^{x^2}\left(c_1 \,\mathrm{e}^{x} - c_2 \,\mathrm{e}^{-x}\right)}{c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= -\frac{c_1(2x+1)\,\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} \\ p(x) &= \frac{\mathrm{e}^{x^2}\left(c_1 \,\mathrm{e}^{2x} - c_2\right)}{c_1 \,\mathrm{e}^{2x} + c_2}\end{aligned}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{c_1(2x+1)\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} dx}$$

Therefore the solution is

$$y = \left(\int \frac{\mathrm{e}^{x^2}\left(c_1 \,\mathrm{e}^{2x} - c_2\right) \mathrm{e}^{\int -\frac{c_1(2x+1)\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} dx}}{c_1 \,\mathrm{e}^{2x} + c_2} dx + c_3\right) \mathrm{e}^{-\int -\frac{c_1(2x+1)\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} dx}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(\int \frac{\mathrm{e}^{x^2}\left(c_1 \,\mathrm{e}^{2x} - c_2\right) \mathrm{e}^{\int -\frac{c_1(2x+1)\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} dx}}{c_1 \,\mathrm{e}^{2x} + c_2} dx + c_3\right) \mathrm{e}^{-\int -\frac{c_1(2x+1)\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} dx}$$

The constants can be merged to give

$$y = \left(\int \frac{\mathrm{e}^{x^2}\left(c_1 \,\mathrm{e}^{2x} - c_2\right) \mathrm{e}^{\int -\frac{c_1(2x+1)\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} dx}}{c_1 \,\mathrm{e}^{2x} + c_2} dx + 1\right) \mathrm{e}^{-\int -\frac{c_1(2x+1)\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} dx}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(\int \frac{\mathrm{e}^{x^2}\left(c_1 \,\mathrm{e}^{2x} - c_2\right) \mathrm{e}^{\int -\frac{c_1(2x+1)\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} dx}}{c_1 \,\mathrm{e}^{2x} + c_2} dx + 1\right) \mathrm{e}^{-\int -\frac{c_1(2x+1)\mathrm{e}^{2x} + c_2(2x-1)}{c_1 \,\mathrm{e}^{2x} + c_2} dx}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2-3)*y(x) = exp(x^2),y(x),singsol=al
```

$$y = e^{x(x+1)} c_2 + e^{(x-1)x} c_1 - e^{x^2}$$

Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-3)*y[x]==Exp[x^2],{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{2} e^{(x-1)x} \left(-2e^x + c_2 e^{2x} + 2c_1\right)$$

2.2.33 Problem 33

Internal problem ID [9156]
Book : Second order enumerated odes
Section : section 2
Problem number : 33
Date solved : `Monday, January 27, 2025 at 05:50:59 PM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 2\tan\left(x\right)y' + 5y = \mathrm{e}^{x^2}\sec\left(x\right)$$

Solved as second order ode using change of variable on y method 1

Time used: 0.851 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2\tan\left(x\right)y' + 5y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -2\tan\left(x\right) \\ q(x) &= 5 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 5 - \frac{\left(-2\tan\left(x\right)\right)'}{2} - \frac{\left(-2\tan\left(x\right)\right)^2}{4} \\ &= 5 - \frac{\left(-2 - 2\tan\left(x\right)^2\right)}{2} - \frac{\left(4\tan\left(x\right)^2\right)}{4} \\ &= 5 - \left(-1 - \tan\left(x\right)^2\right) - \tan\left(x\right)^2 \\ &= 6 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\
&= e^{-\int \frac{-2\tan(x)}{2}} \\
&= \sec(x) \qquad (5)
\end{aligned}
$$

Hence (3) becomes

$$y = v(x)\sec(x) \tag{4}$$

Applying this change of variable to the original ode results in

$$\sec(x)\left(v''(x) + 6v(x)\right) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$v''(x) + 6v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 6$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + 6\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 6$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(6)} \\
&= \pm i\sqrt{6}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +i\sqrt{6} \\
\lambda_2 &= -i\sqrt{6}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= i\sqrt{6} \\
\lambda_2 &= -i\sqrt{6}
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{6}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$$

Which becomes

$$v(x) = e^{0}\left(c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)\right)$$

Or

$$v(x) = c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sec(x)$$

Hence (7) becomes

$$y = \left(c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)\right)\sec(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)\right)\sec(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos\left(\sqrt{6}\, x\right)\sec(x) \\ y_2 &= \sin\left(\sqrt{6}\, x\right)\sec(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\, x\right)\sec(x) & \sin\left(\sqrt{6}\, x\right)\sec(x) \\ \frac{d}{dx}\left(\cos\left(\sqrt{6}\, x\right)\sec(x)\right) & \frac{d}{dx}\left(\sin\left(\sqrt{6}\, x\right)\sec(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\,x\right)\sec(x) & \sin\left(\sqrt{6}\,x\right)\sec(x) \\ -\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)+\cos\left(\sqrt{6}\,x\right)\sec(x)\tan(x) & \sqrt{6}\,\cos\left(\sqrt{6}\,x\right)\sec(x)+\sin\left(\sqrt{6}\,x\right)\sec(x)\tan \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\cos\left(\sqrt{6}\,x\right)\sec(x)\right)\left(\sqrt{6}\,\cos\left(\sqrt{6}\,x\right)\sec(x)+\sin\left(\sqrt{6}\,x\right)\sec(x)\tan(x)\right) \\ & - \left(\sin\left(\sqrt{6}\,x\right)\sec(x)\right)\left(-\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)+\cos\left(\sqrt{6}\,x\right)\sec(x)\tan(x)\right) \end{aligned}$$

Which simplifies to

$$W = \sec(x)^2\sqrt{6}\,\cos\left(\sqrt{6}\,x\right)^2+\sec(x)^2\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)^2$$

Which simplifies to

$$W = \sqrt{6}\,\sec(x)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\sqrt{6}\,x\right)\sec(x)^2\,\mathrm{e}^{x^2}}{\sqrt{6}\,\sec(x)^2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(\sqrt{6}\,x\right)\mathrm{e}^{x^2}\sqrt{6}}{6}dx$$

Hence

$$u_1 = \frac{\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\,\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)}{24}-\frac{\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\,\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\sqrt{6}\,x\right)\sec(x)^2\,\mathrm{e}^{x^2}}{\sqrt{6}\,\sec(x)^2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(\sqrt{6}\,x\right)\mathrm{e}^{x^2}\sqrt{6}}{6}dx$$

Hence

$$u_2 = -\frac{i\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\,\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)}{24}-\frac{i\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\,\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)}{24}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \left(\frac{\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\,\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)}{24}-\frac{\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\,\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)}{24}\right)\cos\left(\sqrt{6}\,x\right)\sec(x) \\ & + \sin\left(\sqrt{6}\,x\right)\sec(x)\left(-\frac{i\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\,\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)}{24}-\frac{i\sqrt{6}\,\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\,\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)}{24}\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = -\frac{\sec(x)\left(\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\operatorname{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\left(i\sin\left(\sqrt{6}\,x\right)+\cos\left(\sqrt{6}\,x\right)\right)\right)\sqrt{\pi}\,\sqrt{6}}{24}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(\left(c_1\cos\left(\sqrt{6}\,x\right)+c_2\sin\left(\sqrt{6}\,x\right)\right)\sec(x)\right)\\ &\quad+\left(-\frac{\sec(x)\left(\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\operatorname{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\left(i\sin\left(\sqrt{6}\,x\right)+\cos\left(\sqrt{6}\,x\right)\right)\right)\sqrt{}}{24}\right.\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(c_1\cos\left(\sqrt{6}\,x\right)+c_2\sin\left(\sqrt{6}\,x\right)\right)\sec(x) - \frac{\sec(x)\left(\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\operatorname{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\left(i\sin\left(\sqrt{6}\,x\right)+\cos\left(\sqrt{6}\,x\right)\right)\right)\sqrt{\pi}\,\sqrt{6}}{24}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.302 (sec)

Writing the ode as

$$y'' - 2\tan(x)\,y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1\\ B &= -2\tan(x)\\ C &= 5\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-6}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -6 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -6z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.89: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -6$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{6}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2\tan(x)}{1}\, dx} \\ &= z_1 e^{-\ln(\cos(x))} \\ &= z_1(\sec(x)) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{6}\, x\right) \sec(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2\tan(x)}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(\cos(x))}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{6}\, \tan\left(\sqrt{6}\, x\right)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\cos\left(\sqrt{6}\, x\right) \sec(x)\right) + c_2 \left(\cos\left(\sqrt{6}\, x\right) \sec(x) \left(\frac{\sqrt{6}\, \tan\left(\sqrt{6}\, x\right)}{6}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2\tan(x)\, y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(\sqrt{6}\, x\right) \sec(x) + \frac{c_2 \sqrt{6}\, \sin\left(\sqrt{6}\, x\right) \sec(x)}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\sqrt{6}\, x\right) \sec(x)$$

$$y_2 = \frac{\sqrt{6}\, \sin\left(\sqrt{6}\, x\right) \sec(x)}{6}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\,x\right)\sec\left(x\right) & \frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)}{6} \\ \frac{d}{dx}\left(\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\right) & \frac{d}{dx}\left(\frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)}{6}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\sqrt{6}\,x\right)\sec\left(x\right) & \frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)}{6} \\ -\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right) + \cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan\left(x\right) & \cos\left(\sqrt{6}\,x\right)\sec\left(x\right) + \frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)\tan(x)}{6} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\right)\left(\cos\left(\sqrt{6}\,x\right)\sec\left(x\right) + \frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan\left(x\right)}{6}\right) \\ & - \left(\frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right)}{6}\right)\left(-\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec\left(x\right) + \cos\left(\sqrt{6}\,x\right)\sec\left(x\right)\tan\left(x\right)\right) \end{aligned}$$

Which simplifies to

$$W = \cos\left(\sqrt{6}\,x\right)^2 \sec\left(x\right)^2 + \sin\left(\sqrt{6}\,x\right)^2 \sec\left(x\right)^2$$

Which simplifies to

$$W = \sec\left(x\right)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{6}\,\sin\left(\sqrt{6}\,x\right)\sec(x)^2 e^{x^2}}{6}}{\sec\left(x\right)^2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(\sqrt{6}\,x\right)e^{x^2}\sqrt{6}}{6}\,dx$$

Hence

$$u_1 = \frac{\sqrt{6}\,\sqrt{\pi}\,e^{\frac{3}{2}}\operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right)}{24} - \frac{\sqrt{6}\,\sqrt{\pi}\,e^{\frac{3}{2}}\operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\sqrt{6}\,x\right)\sec\left(x\right)^2 e^{x^2}}{\sec\left(x\right)^2}\,dx$$

Which simplifies to

$$u_2 = \int \cos\left(\sqrt{6}\, x\right) e^{x^2} dx$$

Hence

$$u_2 = -\frac{i\sqrt{\pi}\, e^{\frac{3}{2}} \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)}{4} - \frac{i\sqrt{\pi}\, e^{\frac{3}{2}} \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\sqrt{6}\,\sqrt{\pi}\, e^{\frac{3}{2}} \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right)}{24} - \frac{\sqrt{6}\,\sqrt{\pi}\, e^{\frac{3}{2}} \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)}{24}\right) \cos\left(\sqrt{6}\, x\right) \sec(x) + \frac{\sqrt{6}\, \sin\left(\sqrt{6}\, x\right) \sec(x) \left(-\frac{i\sqrt{\pi}\, e^{\frac{3}{2}} \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)}{4} - \frac{i\sqrt{\pi}\, e^{\frac{3}{2}} \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right)}{4}\right)}{6}$$

Which simplifies to

$$y_p(x) = -\frac{\sec(x)\left(\left(i\sin\left(\sqrt{6}\, x\right) - \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)\left(i\sin\left(\sqrt{6}\, x\right) + \cos\left(\sqrt{6}\, x\right)\right)\right) \sqrt{\pi}\, \sqrt{6}}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{6}\, x\right) \sec(x) + \frac{c_2 \sqrt{6}\, \sin\left(\sqrt{6}\, x\right) \sec(x)}{6}\right) \\ &\quad + \left(-\frac{\sec(x)\left(\left(i\sin\left(\sqrt{6}\, x\right) - \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)\left(i\sin\left(\sqrt{6}\, x\right) + \cos\left(\sqrt{6}\, x\right)\right)\right) \sqrt{}}{24}\right. \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(\sqrt{6}\, x\right) \sec(x) + \frac{c_2 \sqrt{6}\, \sin\left(\sqrt{6}\, x\right) \sec(x)}{6} - \frac{\sec(x)\left(\left(i\sin\left(\sqrt{6}\, x\right) - \cos\left(\sqrt{6}\, x\right)\right) \operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)\left(i\sin\left(\sqrt{6}\, x\right) + \cos\left(\sqrt{6}\, x\right)\right)\right) \sqrt{\pi}\, \sqrt{6}}{24}$$

Solved as second order ode adjoint method

Time used: 16.073 (sec)

In normal form the ode

$$y'' - 2\tan(x)\, y' + 5y = e^{x^2} \sec(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -2\tan(x) \\ q(x) &= 5 \\ r(x) &= e^{x^2}\sec(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (-2\tan(x)\,\xi(x))' + (5\xi(x)) &= 0 \\ 2\tan(x)\,\xi'(x) + \xi''(x) + 5\xi(x) + 2\sec(x)^2\,\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$\xi'' + p(x)\,\xi' + q(x)\,\xi = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2\tan(x) \\ q(x) &= 5 + 2\sec(x)^2 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 5 + 2\sec(x)^2 - \frac{(2\tan(x))'}{2} - \frac{(2\tan(x))^2}{4} \\ &= 5 + 2\sec(x)^2 - \frac{(2 + 2\tan(x)^2)}{2} - \frac{(4\tan(x)^2)}{4} \\ &= 5 + 2\sec(x)^2 - (1 + \tan(x)^2) - \tan(x)^2 \\ &= 6 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$\xi = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= e^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{2\tan(x)}{2}} \\ &= \cos(x) \end{aligned} \tag{5}$$

Hence (3) becomes

$$\xi = v(x)\cos(x) \tag{4}$$

Applying this change of variable to the original ode results in

$$(6v(x) + v''(x))\cos(x) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$6v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 6$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + 6\, e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(6)}\\ &= \pm i\sqrt{6}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{6}\\ \lambda_2 &= -i\sqrt{6}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{6}\\ \lambda_2 &= -i\sqrt{6}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{6}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0\left(c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)\right)$$

Or

$$v(x) = c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned}\xi &= v(x)\, z(x)\\ &= \left(c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \cos(x)$$

Hence (7) becomes

$$\xi = \left(c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)\right)\cos(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(-2\tan(x) - \frac{\left(-c_1\sqrt{6}\,\sin\left(\sqrt{6}\,x\right) + c_2\sqrt{6}\,\cos\left(\sqrt{6}\,x\right)\right)\cos(x) - \left(c_1\cos\left(\sqrt{6}\,x\right) + c_2\sin\left(\sqrt{6}\,x\right)\right)\sin}{\left(c_1\cos\left(\sqrt{6}\,x\right) + c_2\sin\left(\sqrt{6}\,x\right)\right)\cos(x)}\right.$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{\left(-\sqrt{6}\,c_1 + \tan(x)\,c_2\right)\sin\left(\sqrt{6}\,x\right) + \left(\tan(x)\,c_1 + c_2\sqrt{6}\right)\cos\left(\sqrt{6}\,x\right)}{c_1\cos\left(\sqrt{6}\,x\right) + c_2\sin\left(\sqrt{6}\,x\right)}$$
$$p(x) = -\frac{\sqrt{\pi}\,e^{\frac{3}{2}}\left((ic_1 + c_2)\operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)(ic_1 - c_2)\right)\sec(x)}{4c_1\cos\left(\sqrt{6}\,x\right) + 4c_2\sin\left(\sqrt{6}\,x\right)}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= e^{\int -\frac{\left(-\sqrt{6}\,c_1 + \tan(x)c_2\right)\sin\left(\sqrt{6}\,x\right) + \left(\tan(x)c_1 + c_2\sqrt{6}\right)\cos\left(\sqrt{6}\,x\right)}{c_1\cos\left(\sqrt{6}\,x\right) + c_2\sin\left(\sqrt{6}\,x\right)}dx} \\
&= e^{-\frac{\ln\left(1+\tan(x)^2\right)}{2} - \ln\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2 - 2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right) - c_1\right) + \ln\left(1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2\right)}
\end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{\sqrt{\pi}\,e^{\frac{3}{2}}\left((ic_1 + c_2)\operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)(ic_1 - c_2)\right)\sec(x)}{4c_1\cos\left(\sqrt{6}\,x\right) + 4c_2\sin\left(\sqrt{6}\,x\right)}\right)$$

$$\frac{d}{dx}\left(y\,e^{-\frac{\ln\left(1+\tan(x)^2\right)}{2} - \ln\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2 - 2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right) - c_1\right) + \ln\left(1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2\right)}\right)$$
$$= \left(e^{-\frac{\ln\left(1+\tan(x)^2\right)}{2} - \ln\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2 - 2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right) - c_1\right) + \ln\left(1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2\right)}\right)\left(-\frac{\sqrt{\pi}\,e^{\frac{3}{2}}\left((ic_1 + c_2)\operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + e\right.}{4c_1\cos\left(\sqrt{6}\,x\right) + }\right.$$

$$d\left(y\,e^{-\frac{\ln\left(1+\tan(x)^2\right)}{2} - \ln\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2 - 2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right) - c_1\right) + \ln\left(1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2\right)}\right)$$
$$= \left(-\frac{\sqrt{\pi}\,e^{\frac{3}{2}}\left((ic_1 + c_2)\operatorname{erf}\left(ix - \frac{\sqrt{6}}{2}\right) + \operatorname{erf}\left(ix + \frac{\sqrt{6}}{2}\right)(ic_1 - c_2)\right)\sec(x)\,e^{-\frac{\ln\left(1+\tan(x)^2\right)}{2} - \ln\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2 - 2c_2\,t\right.}}{4c_1\cos\left(\sqrt{6}\,x\right) + 4c_2\sin\left(\sqrt{6}\,x\right)}\right.$$

Integrating gives

$$y\,\mathrm{e}^{-\frac{\ln\left(1+\tan(x)^2\right)}{2}-\ln\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right)-c_1\right)+\ln\left(1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2\right)} = \int -\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(\left(ic_1+c_2\right)\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(\right.\right.}{}$$

$$= \int -\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(\left(ic_1+c_2\right)\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(\right.\right.}{}$$

Dividing throughout by the integrating factor $\mathrm{e}^{-\frac{\ln\left(1+\tan(x)^2\right)}{2}-\ln\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right)-c_1\right)+\ln\left(1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2\right)}$ gives the final solution

$$y = \frac{\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right)-c_1\right)\sqrt{1+\tan(x)^2}\left(\int -\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(\left(ic_1+c_2\right)\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\left(ic_1-c_2\right)\right)\sec(}{4c_1\,\mathrm{c}}\right.}{1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right)-c_1\right)\sqrt{1+\tan(x)^2}\left(\int -\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(\left(ic_1+c_2\right)\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\left(ic_1-c_2\right)\right)\sec(x)\mathrm{e}}{4c_1\cos(}\right.}{1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2}$$

The constants can be merged to give

$$y = \frac{\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right)-c_1\right)\sqrt{1+\tan(x)^2}\left(\int -\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(\left(ic_1+c_2\right)\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\left(ic_1-c_2\right)\right)\sec(x)\mathrm{e}}{4c_1\cos(}\right.}{1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{6}\,x}{2}\right)-c_1\right)\sqrt{1+\tan(x)^2}\left(\int -\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{3}{2}}\left(\left(ic_1+c_2\right)\mathrm{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+\mathrm{erf}\left(ix+\frac{\sqrt{6}}{2}\right)\left(ic_1-c_2\right)\right)\sec(x)\mathrm{e}}{4c_1\cos(}\right.}{1+\tan\left(\frac{\sqrt{6}\,x}{2}\right)^2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.014` (sec)
Leaf size : 97

```
dsolve(diff(diff(y(x),x),x)-2*tan(x)*diff(y(x),x)+5*y(x) = exp(x^2)*sec(x),y(x),singso
```

$$y = -\frac{\left(e^{\frac{3}{2}}\left(i\sin\left(\sqrt{6}\,x\right)-\cos\left(\sqrt{6}\,x\right)\right)\sqrt{\pi}\,\sqrt{6}\,\operatorname{erf}\left(ix-\frac{\sqrt{6}}{2}\right)+e^{\frac{3}{2}}\left(i\sin\left(\sqrt{6}\,x\right)+\cos\left(\sqrt{6}\,x\right)\right)\sqrt{\pi}\,\sqrt{6}\,\operatorname{erf}\left(ix\right.}{24}$$

Mathematica DSolve solution

Solving time : `0.25` (sec)
Leaf size : 133

```
DSolve[{D[y[x],{x,2}]-2*Tan[x]*D[y[x],x]+5*y[x]==Exp[x^2]*Sec[x],{}},y[x],x,IncludeSingularS
```

$$y(x)\to\frac{1}{12}e^{-i\sqrt{6}x}\sec(x)\left(12\int_1^x\frac{ie^{K[1]\left(K[1]+i\sqrt{6}\right)}}{2\sqrt{6}}dK[1]-i\sqrt{6}e^{2i\sqrt{6}x}\int_1^x e^{K[2]\left(K[2]-i\sqrt{6}\right)}dK[2]\right.$$
$$\left.-i\sqrt{6}c_2e^{2i\sqrt{6}x}+12c_1\right)$$

2.2.34 Problem 34

Internal problem ID [9157]
Book : Second order enumerated odes
Section : section 2
Problem number : 34
Date solved : `Monday, January 27, 2025 at 05:51:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + 2(x^2+1)\,y = 0$$

Solved as second order ode using change of variable on y method 1

Time used: 0.391 (sec)

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2x^2+2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2x^2+2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\ &= \frac{2x^2+2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= \frac{2x^2+2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= 2 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2}dx} \\ &= e^{-\int \frac{-\frac{2}{x}}{2}} \\ &= x \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x \tag{4}$$

Applying this change of variable to the original ode results in

$$x^3 (2v(x) + v''(x)) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$2v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + 2\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(2)}\\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\lambda_1 = +i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$

Which simplifies to

$$\lambda_1 = i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0 \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

Or

$$v(x) = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) x$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) x$$

Solved as second order Bessel ode

Time used: 0.062 (sec)

Writing the ode as

$$x^2 y'' - 2xy' + \left(2x^2 + 2\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, xy' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= \frac{3}{2} \\ \beta &= \sqrt{2} \\ n &= -\frac{1}{2} \\ \gamma &= 1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\, x^{3/2} \sqrt{2}\, \cos\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}} + \frac{c_2\, x^{3/2} \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1\, x^{3/2} \sqrt{2}\, \cos\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}} + \frac{c_2\, x^{3/2} \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{\sqrt{\pi}\, \sqrt{\sqrt{2}\, x}}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$x^2y'' - 2xy' + (2x^2 + 2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.90: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{2}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, x\right) x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos\left(\sqrt{2}\, x\right) x\right) + c_2\left(\cos\left(\sqrt{2}\, x\right) x \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(\sqrt{2}\, x\right) x + \frac{c_2 \sqrt{2}\, x \sin\left(\sqrt{2}\, x\right)}{2}$$

Solved as second order ode adjoint method

Time used: 2.562 (sec)

In normal form the ode

$$x^2y'' - 2xy' + 2(x^2+1)\,y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= -\frac{2}{x}\\
q(x) &= \frac{2x^2+2}{x^2}\\
r(x) &= 0
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\,p)' + \xi q &= 0\\
\xi'' - \left(-\frac{2\xi(x)}{x}\right)' + \left(\frac{(2x^2+2)\,\xi(x)}{x^2}\right) &= 0\\
\xi''(x) + \frac{2\xi'(x)}{x} + 2\xi(x) &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$\xi'' + p(x)\,\xi' + q(x)\,\xi = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{2}{x}\\
q(x) &= 2
\end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4}\\
&= 2 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4}\\
&= 2 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4}\\
&= 2 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2}\\
&= 2
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$\xi = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2}dx}\\
&= e^{-\int \frac{\frac{2}{x}}{2}}\\
&= \frac{1}{x}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$\xi = \frac{v(x)}{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$\frac{2v(x) + v''(x)}{x} = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$2v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + 2\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0\left(c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)\right)$$

Or

$$v(x) = c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned}\xi &= v(x)\,z(x)\\ &= \left(c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$\xi = \frac{c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)}{x}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}\end{aligned}$$

Or

$$y' + y\left(-\frac{2}{x} - \frac{\left(\frac{-c_1\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)+c_2\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{x} - \frac{c_1\cos\left(\sqrt{2}\,x\right)+c_2\sin\left(\sqrt{2}\,x\right)}{x^2}\right)x}{c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= -\frac{\left(xc_2\sqrt{2}+c_1\right)\cos\left(\sqrt{2}\,x\right) - \sin\left(\sqrt{2}\,x\right)\left(xc_1\sqrt{2}-c_2\right)}{\left(c_1\cos\left(\sqrt{2}\,x\right)+c_2\sin\left(\sqrt{2}\,x\right)\right)x}\\ p(x) &= 0\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= e^{\int -\frac{\left(xc_2\sqrt{2}+c_1\right)\cos\left(\sqrt{2}\,x\right)-\sin\left(\sqrt{2}\,x\right)\left(xc_1\sqrt{2}-c_2\right)}{\left(c_1\cos\left(\sqrt{2}\,x\right)+c_2\sin\left(\sqrt{2}\,x\right)\right)x}dx}\\ &= \frac{\sec\left(\frac{\sqrt{2}\,x}{2}\right)^2}{x\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2 - 2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right) - c_1\right)}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sec\left(\frac{\sqrt{2}\,x}{2}\right)^2}{x\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2 - 2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right) - c_1\right)}\right) = 0$$

Integrating gives

$$\frac{y\sec\left(\frac{\sqrt{2}\,x}{2}\right)^2}{x\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right)-c_1\right)} = \int 0\,dx + c_3$$

$$= c_3$$

Dividing throughout by the integrating factor $\frac{\sec\left(\frac{\sqrt{2}\,x}{2}\right)^2}{x\left(c_1\tan\left(\frac{\sqrt{2}\,x}{2}\right)^2-2c_2\tan\left(\frac{\sqrt{2}\,x}{2}\right)-c_1\right)}$ gives the final solution

$$y = c_3x\left(-c_2\sin\left(\sqrt{2}\,x\right)-c_1\cos\left(\sqrt{2}\,x\right)\right)$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3x\left(-c_2\sin\left(\sqrt{2}\,x\right)-c_1\cos\left(\sqrt{2}\,x\right)\right)$$

The constants can be merged to give

$$y = x\left(-c_2\sin\left(\sqrt{2}\,x\right)-c_1\cos\left(\sqrt{2}\,x\right)\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x\left(-c_2\sin\left(\sqrt{2}\,x\right)-c_1\cos\left(\sqrt{2}\,x\right)\right)$$

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2(x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=-\frac{2(x^2+1)y(x)}{x^2}+\frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x)-\frac{2\left(\frac{d}{dx}y(x)\right)}{x}+\frac{2(x^2+1)y(x)}{x^2}=0$
- □ Check to see if $x_0=0$ is a regular singular point
  - Define functions
    $\left[P_2(x)=-\frac{2}{x}, P_3(x)=\frac{2(x^2+1)}{x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
    $\left.(x\cdot P_2(x))\right|_{x=0}=-2$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
    $\left.(x^2\cdot P_3(x))\right|_{x=0}=2$
  - $x=0$is a regular singular point
    Check to see if $x_0=0$ is a regular singular point
    $x_0=0$
- Multiply by denominators

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (2x^2+2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\,x^r + a_1 r(-1+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + 2a_{k-2}\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) + 2a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)\,(k+r) + 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{2a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{2a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{2a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{2a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{2b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*(x^2+1)*y(x) = 0,y(x),singsol=all)
```

$$y = x\left(c_1 \sin\left(\sqrt{2}\, x\right) + c_2 \cos\left(\sqrt{2}\, x\right)\right)$$

Mathematica DSolve solution

Solving time : 0.044 (sec)
Leaf size : 48

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+2*(1+x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to c_1 e^{-i\sqrt{2}x} x - \frac{i c_2 e^{i\sqrt{2}x} x}{2\sqrt{2}}$$

2.2.35 Problem 35

Internal problem ID [9158]
Book : Second order enumerated odes
Section : section 2
Problem number : 35
Date solved : `Monday, January 27, 2025 at 05:51:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4x^5y' + \left(x^8 + 6x^4 + 4\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$4x^2y'' + 4x^5y' + \left(x^8 + 6x^4 + 4\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x^5 \\ C &= x^8 + 6x^4 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.92: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-1$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{i\sqrt{3}}{2}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}-\frac{i\sqrt{3}}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-)\\
&= \frac{1}{2}-\frac{i\sqrt{3}}{2}-\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}+(-)(0)\\
&= \frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}\\
&= \frac{1-i\sqrt{3}}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)^2 - \left(-\frac{1}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} dx} \\ &= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^5}{4x^2} \, dx} \\ &= z_1 e^{-\frac{x^4}{8}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^4}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^4}{8}} x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x^5}{4x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^4}{4}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{ix\sqrt{3}\,\mathrm{e}^{-\frac{x^4}{4}}\,\mathrm{e}^{\frac{x^4}{4}}\,x^{i\sqrt{3}-1}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x^4}{8}} x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x^4}{8}} x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}\left(-\frac{ix\sqrt{3}\,\mathrm{e}^{-\frac{x^4}{4}}\,\mathrm{e}^{\frac{x^4}{4}}\,x^{i\sqrt{3}-1}}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{e}^{-\frac{x^4}{8}} x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} - \frac{ic_2\,x^{\frac{1}{2} + \frac{i\sqrt{3}}{2}}\sqrt{3}\,\mathrm{e}^{-\frac{x^4}{8}}}{3}$$

Solved as second order ode adjoint method

Time used: 1.101 (sec)

In normal form the ode

$$4x^2y'' + 4x^5y' + (x^8 + 6x^4 + 4)\, y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= x^3 \\ q(x) &= \frac{x^8 + 6x^4 + 4}{4x^2} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (x^3\xi(x))' + \left(\frac{(x^8 + 6x^4 + 4)\,\xi(x)}{4x^2}\right) &= 0 \\ \xi''(x) - x^3\xi'(x) + \frac{(x^8 - 6x^4 + 4)\,\xi(x)}{4x^2} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' - x^3\xi' + \left(\frac{x^6}{4} - \frac{3x^2}{2} + \frac{1}{x^2}\right)\xi = 0 \tag{1}$$

$$A\xi'' + B\xi' + C\xi = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^3 \\ C &= \frac{x^6}{4} - \frac{3x^2}{2} + \frac{1}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \xi e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.93: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \\
&= \frac{1 - i\sqrt{3}}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)^2 - \left(-\frac{1}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} dx} \\
&= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}
\end{aligned}$$

The first solution to the original ode in ξ is found from

$$\begin{aligned}
\xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^3}{1} \, dx} \\
&= z_1 e^{\frac{x^4}{8}} \\
&= z_1 \left(e^{\frac{x^4}{8}}\right)
\end{aligned}$$

Which simplifies to

$$\xi_1 = e^{\frac{x^4}{8}} x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}$$

The second solution ξ_2 to the original ode is found using reduction of order

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{\xi_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
\xi_2 &= \xi_1 \int \frac{e^{\int -\frac{-x^3}{1} \, dx}}{(\xi_1)^2} \, dx \\
&= \xi_1 \int \frac{e^{\frac{x^4}{4}}}{(\xi_1)^2} \, dx \\
&= \xi_1 \left(-\frac{ix\sqrt{3}\, e^{-\frac{x^4}{4}} e^{\frac{x^4}{4}} x^{i\sqrt{3}-1}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
\xi &= c_1 \xi_1 + c_2 \xi_2 \\
&= c_1 \left(e^{\frac{x^4}{8}} x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}\right) + c_2 \left(e^{\frac{x^4}{8}} x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \left(-\frac{ix\sqrt{3}\, e^{-\frac{x^4}{4}} e^{\frac{x^4}{4}} x^{i\sqrt{3}-1}}{3}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(x^3 - \frac{\frac{c_1\,x^3 \mathrm{e}^{\frac{x^4}{8}}\,x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}}{2} + \frac{c_1\,\mathrm{e}^{\frac{x^4}{8}}\,x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}{x} - \frac{ic_2\,x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\sqrt{3}\,\mathrm{e}^{\frac{x^4}{8}}}{3x} - \frac{ic_2\,x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\sqrt{3}\,x^3 \mathrm{e}^{\frac{x^4}{8}}}{6}}{c_1\,\mathrm{e}^{\frac{x^4}{8}}\,x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{ic_2\,x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\sqrt{3}\,\mathrm{e}^{\frac{x^4}{8}}}{3}}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{c_1\,x^{-\frac{i\sqrt{3}}{2}}\left(-ix^4 + i + \sqrt{3}\right) + c_2\,x^{\frac{i\sqrt{3}}{2}}\left(i + \frac{(-x^4+1)\sqrt{3}}{3}\right)}{2x\left(ix^{-\frac{i\sqrt{3}}{2}}c_1 + \frac{\sqrt{3}\,x^{\frac{i\sqrt{3}}{2}}c_2}{3}\right)}$$
$$p(x) = 0$$

The integrating factor μ is

$$\mu = e^{\int q\,dx}$$
$$= \mathrm{e}^{\int -\frac{c_1\,x^{-\frac{i\sqrt{3}}{2}}\left(-ix^4+i+\sqrt{3}\right)+c_2\,x^{\frac{i\sqrt{3}}{2}}\left(i+\frac{\left(-x^4+1\right)\sqrt{3}}{3}\right)}{2x\left(ix^{-\frac{i\sqrt{3}}{2}}c_1+\frac{\sqrt{3}\,x^{\frac{i\sqrt{3}}{2}}c_2}{3}\right)}dx}$$
$$= \frac{x^{-\frac{1}{2}-\frac{i\sqrt{3}}{2}}\,\mathrm{e}^{\frac{x^4}{8}}}{i\sqrt{3}\,c_2 - 3x^{-i\sqrt{3}}c_1}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,x^{-\frac{1}{2}-\frac{i\sqrt{3}}{2}}\,\mathrm{e}^{\frac{x^4}{8}}}{i\sqrt{3}\,c_2 - 3x^{-i\sqrt{3}}c_1}\right) = 0$$

Integrating gives

$$\frac{y\,x^{-\frac{1}{2}-\frac{i\sqrt{3}}{2}}\,\mathrm{e}^{\frac{x^4}{8}}}{i\sqrt{3}\,c_2 - 3x^{-i\sqrt{3}}c_1} = \int 0\,dx + c_3$$
$$= c_3$$

Dividing throughout by the integrating factor $\frac{x^{-\frac{1}{2}-\frac{i\sqrt{3}}{2}}\,\mathrm{e}^{\frac{x^4}{8}}}{i\sqrt{3}\,c_2-3x^{-i\sqrt{3}}c_1}$ gives the final solution

$$y = x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\,\mathrm{e}^{-\frac{x^4}{8}}\left(i\sqrt{3}\,c_2 - 3x^{-i\sqrt{3}}c_1\right)c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\,\mathrm{e}^{-\frac{x^4}{8}}\left(i\sqrt{3}\,c_2 - 3x^{-i\sqrt{3}}c_1\right)c_3$$

The constants can be merged to give

$$y = x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\,\mathrm{e}^{-\frac{x^4}{8}}\left(i\sqrt{3}\,c_2 - 3x^{-i\sqrt{3}}c_1\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\,\mathrm{e}^{-\frac{x^4}{8}}\left(i\sqrt{3}\,c_2 - 3x^{-i\sqrt{3}}c_1\right)$$

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x^5\left(\frac{d}{dx}y(x)\right)+(x^8+6x^4+4)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x^8+6x^4+4)y(x)}{4x^2}-\left(\frac{d}{dx}y(x)\right)x^3$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\left(\frac{d}{dx}y(x)\right)x^3+\frac{(x^8+6x^4+4)y(x)}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=x^3, P_3(x)=\frac{x^8+6x^4+4}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=1$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x^5\left(\frac{d}{dx}y(x)\right)+(x^8+6x^4+4)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..8$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^5\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^5\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r+4}$

  ◦ Shift index using $k->k-4$

    $x^5\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=4}^{\infty}a_{k-4}(k-4+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0(r^2-r+1)\,x^r+4a_1(r^2+r+1)\,x^{1+r}+4a_2(r^2+3r+3)\,x^{2+r}+4a_3(r^2+5r+7)\,x^{3+r}+(4a_4(r^2$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r^2-4r+4=0$$

- Values of r that satisfy the indicial equation
$$r\in\left\{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2},\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right\}$$
- The coefficients of each power of x must be 0
$$[4a_1(r^2+r+1)=0,4a_2(r^2+3r+3)=0,4a_3(r^2+5r+7)=0,4a_4(r^2+7r+13)+2a_0(3+2r$$
- Solve for the dependent coefficient(s)
$$\left\{a_1=0,a_2=0,a_3=0,a_4=-\frac{a_0(3+2r)}{2(r^2+7r+13)},a_5=0,a_6=0,a_7=0\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$4(1+k^2+(2r-1)\,k+r^2-r)\,a_k+2a_{k-4}(2k-5+2r)+a_{k-8}=0$$
- Shift index using $k->k+8$
$$4\left(1+(k+8)^2+(2r-1)\,(k+8)+r^2-r\right)a_{k+8}+2a_{k+4}(2k+11+2r)+a_k=0$$
- Recursion relation that defines series solution to ODE
$$a_{k+8}=-\frac{4ka_{k+4}+4ra_{k+4}+a_k+22a_{k+4}}{4(k^2+2kr+r^2+15k+15r+57)}$$
- Recursion relation for $r=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$
$$a_{k+8}=-\frac{4ka_{k+4}+4\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+4}+a_k+22a_{k+4}}{4\left(k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}-\frac{15\,\mathrm{I}\sqrt{3}}{2}\right)}$$
- Solution for $r=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$
$$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}},a_{k+8}=-\frac{4ka_{k+4}+4\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+4}+a_k+22a_{k+4}}{4\left(k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}-\frac{15\,\mathrm{I}\sqrt{3}}{2}\right)},a_1=0,a_2=0,a_3=0\right.$$
- Recursion relation for $r=\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}$
$$a_{k+8}=-\frac{4ka_{k+4}+4\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+4}+a_k+22a_{k+4}}{4\left(k^2+2k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}+\frac{15\,\mathrm{I}\sqrt{3}}{2}\right)}$$
- Solution for $r=\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}$
$$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}},a_{k+8}=-\frac{4ka_{k+4}+4\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+4}+a_k+22a_{k+4}}{4\left(k^2+2k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}+\frac{15\,\mathrm{I}\sqrt{3}}{2}\right)},a_1=0,a_2=0,a_3=0\right.$$
- Combine solutions and rename parameters
$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}\right),a_{k+8}=-\frac{4ka_{4+k}+4\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{4+k}+a_k+22a_{4+k}}{4\left(k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+15k+\frac{129}{2}-\frac{15\,\mathrm{I}\sqrt{3}}{2}\right.}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 33

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x^5*diff(y(x),x)+(x^8+6*x^4+4)*y(x) = 0,y(x),singsol
```

$$y = \sqrt{x}\,\mathrm{e}^{-\frac{x^4}{8}}\left(c_1 x^{\frac{i\sqrt{3}}{2}} + c_2 x^{-\frac{i\sqrt{3}}{2}}\right)$$

Mathematica DSolve solution

Solving time : 0.073 (sec)
Leaf size : 62

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x^5*D[y[x],x]+(x^8+6*x^4+4)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{1}{3} e^{-\frac{x^4}{8}} x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(3c_1 - i\sqrt{3}c_2 x^{i\sqrt{3}}\right)$$

2.2.36 Problem 36

Internal problem ID [9159]
Book : Second order enumerated odes
Section : section 2
Problem number : 36
Date solved : `Tuesday, January 28, 2025 at 04:00:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + (xy' - y)^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one inte
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
trying symmetries linear in x and y(x)
<- linear symmetries successful`
```

Maple dsolve solution

Solving time : `0.036` (sec)
Leaf size : 22

```
dsolve(x^2*diff(diff(y(x),x),x)+(diff(y(x),x)*x-y(x))^2 = 0,y(x),singsol=all)
```

$$y = \left(-\mathrm{e}^{c_1}\,\mathrm{Ei}_1\left(-\ln\left(\frac{1}{x}\right)+c_1\right)+c_2\right)x$$

Mathematica DSolve solution

Solving time : `28.572` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+(x*D[y[x],x]-y[x])^2==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(e^{c_1}\,\mathrm{ExpIntegralEi}(-c_1 - \log(x)) + c_2)$$
$$y(x) \to c_2 x$$

2.2.37 Problem 37

Internal problem ID [9160]
Book : Second order enumerated odes
Section : section 2
Problem number : 37
Date solved : `Monday, January 27, 2025 at 05:51:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + 2y' - xy = 0$$

Solved as second order ode using change of variable on y method 1

Time used: 0.315 (sec)

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= -1 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\ &= -1 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= -1 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= -1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{\frac{2}{x}}{2}} \\ &= \frac{1}{x} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) - v(x) = 0$$

Which is now solved for $v(x)$.

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$v(x) = c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x) \\ &= \left(c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$y = \frac{c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}}{x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}}{x}$$

Solved as second order Bessel ode

Time used: 0.072 (sec)

Writing the ode as

$$x^2 y'' + 2y' x - x^2 y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \,\mathrm{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \,\mathrm{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -\frac{1}{2} \\
\beta &= i \\
n &= \frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{i x}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{i x}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{i x}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{i x}}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.062 (sec)

Writing the ode as

$$x y'' + 2y' - x y = 0 \tag{1}$$

$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x \\
B &= 2 \\
C &= -x
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-x}}{x}\right) + c_2 \left(\frac{e^{-x}}{x}\left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1\,e^{-x}}{x} + \frac{c_2\,e^{x}}{2x}$$

Solved as second order ode adjoint method

Time used: 0.752 (sec)

In normal form the ode

$$xy'' + 2y' - xy = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -1 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(\frac{2\xi(x)}{x}\right)' + (-\xi(x)) &= 0 \\ \frac{\xi''(x)\, x^2 - \xi(x)\, x^2 - 2\xi'(x)\, x + 2\xi(x)}{x^2} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$\xi'' + p(x)\, \xi' + q(x)\, \xi = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} - 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} - 1 - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\ &= \frac{2}{x^2} - 1 - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= \frac{2}{x^2} - 1 - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= -1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$\xi = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= e^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{-\frac{2}{x}}{2}} \\ &= x \end{aligned} \tag{5}$$

Hence (3) becomes

$$\xi = v(x)\, x \tag{4}$$

Applying this change of variable to the original ode results in

$$-x(v(x) - v''(x)) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$-v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$v(x) = c_1\, \mathrm{e}^{x} + c_2\, \mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned}\xi &= v(x)\, z(x)\\ &= \left(c_1\, \mathrm{e}^{x} + c_2\, \mathrm{e}^{-x}\right) (z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$\xi = \left(c_1\,\mathrm{e}^{x} + c_2\,\mathrm{e}^{-x}\right) x$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}
\end{aligned}$$

Or

$$y' + y\left(\frac{2}{x} - \frac{\left(c_1\,\mathrm{e}^{x} - c_2\,\mathrm{e}^{-x}\right) x + c_1\,\mathrm{e}^{x} + c_2\,\mathrm{e}^{-x}}{\left(c_1\,\mathrm{e}^{x} + c_2\,\mathrm{e}^{-x}\right) x}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{c_1(x-1)\,\mathrm{e}^{2x} - c_2(x+1)}{\left(c_1\,\mathrm{e}^{2x} + c_2\right) x} \\
p(x) &= 0
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{c_1(x-1)\mathrm{e}^{2x} - c_2(x+1)}{\left(c_1\,\mathrm{e}^{2x} + c_2\right) x}dx} \\
&= \frac{x\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{2x} + c_2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yx\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{2x} + c_2}\right) &= 0
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{yx\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{2x} + c_2} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{x\,\mathrm{e}^{x}}{c_1\,\mathrm{e}^{2x} + c_2}$ gives the final solution

$$y = \frac{c_3\left(c_1\,\mathrm{e}^{x} + c_2\,\mathrm{e}^{-x}\right)}{x}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{c_3\left(c_1\,\mathrm{e}^{x} + c_2\,\mathrm{e}^{-x}\right)}{x}$$

The constants can be merged to give

$$y = \frac{c_1 e^x + c_2 e^{-x}}{x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 e^x + c_2 e^{-x}}{x}$$

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} - y(x) = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) - xy(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$
    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert $\frac{d}{dx}y(x)$ to series expansion
    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

$\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{a_k}{(k+2)(k+1)}, 0 = 0, b_{k+2} = \frac{b_k}{(k+3)(k+2)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)-x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{x}$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 28

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-x} + c_2 e^{x}}{2x}$$

2.2.38 Problem 38

Internal problem ID [9161]
Book : Second order enumerated odes
Section : section 2
Problem number : 38
Date solved : `Monday, January 27, 2025 at 05:51:25 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

Solved as second order ode using change of variable on y method 1

Time used: 0.318 (sec)

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 1 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\ &= 1 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= 1 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= 1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2}dx} \\ &= e^{-\int \frac{\frac{2}{x}}{2}} \\ &= \frac{1}{x} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) + v(x) = 0$$

Which is now solved for $v(x)$.

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x))\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{x}$$

Solved as second order Bessel ode

Time used: 0.064 (sec)

Writing the ode as

$$x^2 y'' + 2y'x + x^2 y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= -\frac{1}{2} \\ \beta &= 1 \\ n &= \frac{1}{2} \\ \gamma &= 1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{2}\, \sin(x)}{x \sqrt{\pi}} - \frac{c_2 \sqrt{2}\, \cos(x)}{x \sqrt{\pi}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 \sqrt{2}\, \sin(x)}{x \sqrt{\pi}} - \frac{c_2 \sqrt{2}\, \cos(x)}{x \sqrt{\pi}}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.174 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.97: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}$$

Solved as second order ode adjoint method

Time used: 0.987 (sec)

In normal form the ode

$$xy'' + 2y' + xy = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= 1 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(\frac{2\xi(x)}{x}\right)' + (\xi(x)) &= 0 \\ \frac{\xi''(x)\, x^2 + \xi(x)\, x^2 - 2\xi'(x)\, x + 2\xi(x)}{x^2} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$\xi'' + p(x)\, \xi' + q(x)\, \xi = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} + 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} + 1 - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\ &= \frac{2}{x^2} + 1 - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= \frac{2}{x^2} + 1 - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= 1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$\xi = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= e^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{-\frac{2}{x}}{2}} \\ &= x \end{aligned} \tag{5}$$

Hence (3) becomes

$$\xi = v(x)\, x \tag{4}$$

Applying this change of variable to the original ode results in

$$x(v''(x) + v(x)) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$v''(x) + v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} \xi &= v(x)\, z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x))\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$\xi = (c_1 \cos(x) + c_2 \sin(x))\, x$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)} \end{aligned}$$

Or

$$y' + y\left(\frac{2}{x} - \frac{(-c_1 \sin(x) + c_2 \cos(x))\, x + c_1 \cos(x) + c_2 \sin(x)}{(c_1 \cos(x) + c_2 \sin(x))\, x}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{(c_2 x - c_1) \cos(x) - \sin(x)\,(c_1 x + c_2)}{(c_1 \cos(x) + c_2 \sin(x))\, x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dx} \\ &= \mathrm{e}^{\int -\frac{(c_2 x - c_1)\cos(x) - \sin(x)(c_1 x + c_2)}{(c_1 \cos(x) + c_2 \sin(x))x} dx} \\ &= \frac{x \sec\left(\frac{x}{2}\right)^2}{c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y x \sec\left(\frac{x}{2}\right)^2}{c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y x \sec\left(\frac{x}{2}\right)^2}{c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1} &= \int 0\, dx + c_3 \\ &= c_3 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{x\sec\left(\frac{x}{2}\right)^{2}}{c_1\tan\left(\frac{x}{2}\right)^{2}-2c_2\tan\left(\frac{x}{2}\right)-c_1}$ gives the final solution

$$y = \frac{c_3(-c_1\cos(x) - c_2\sin(x))}{x}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{c_3(-c_1\cos(x) - c_2\sin(x))}{x}$$

The constants can be merged to give

$$y = \frac{-c_1\cos(x) - c_2\sin(x)}{x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{-c_1\cos(x) - c_2\sin(x)}{x}$$

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

$x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.2.39 Problem 39

Internal problem ID [9162]
Book : Second order enumerated odes
Section : section 2
Problem number : 39
Date solved : `Monday, January 27, 2025 at 05:51:27 PM`
CAS classification : `[_linear]`

Solve

$$y' + y\cot(x) = 2\cos(x)$$

Solved as first order linear ode

Time used: 0.088 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \cot(x) \\ p(x) &= 2\cos(x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \cot(x)dx} \\ &= \sin(x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(2\cos(x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\sin(x)) &= (\sin(x))(2\cos(x)) \\ \mathrm{d}(y\sin(x)) &= (2\cos(x)\sin(x))\ \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\sin(x) &= \int 2\cos(x)\sin(x)\ dx \\ &= \sin(x)^2 + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\sin(x)$ gives the final solution

$$y = \sin(x) + c_1\csc(x)$$

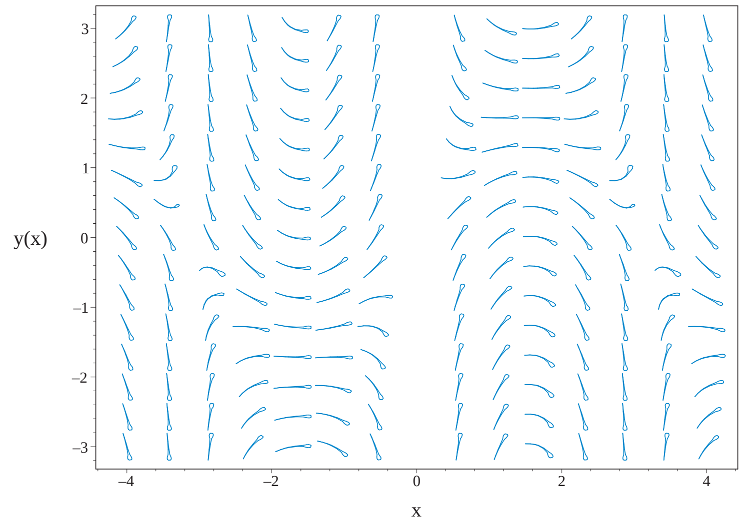


Figure 2.172: Slope field plot
$y' + y\cot(x) = 2\cos(x)$

Summary of solutions found

$$y = \sin(x) + c_1 \csc(x)$$

Solved as first order Exact ode

Time used: 0.185 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y\cot(x) + 2\cos(x))\,\mathrm{d}x$$
$$(y\cot(x) - 2\cos(x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= y\cot(x) - 2\cos(x) \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\cot(x) - 2\cos(x)) \\ &= \cot(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cot(x)) - (0)) \\ &= \cot(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cot(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\sin(x))} \\ &= \sin(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sin(x)(y\cot(x) - 2\cos(x)) \\ &= \cos(x)(-2\sin(x) + y) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sin(x)(1) \\ &= \sin(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\cos(x)(-2\sin(x) + y)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int\frac{\partial\phi}{\partial y}\,\mathrm{d}y=\int\overline{N}\,\mathrm{d}y$$
$$\int\frac{\partial\phi}{\partial y}\,\mathrm{d}y=\int\sin(x)\,\mathrm{d}y$$
$$\phi=y\sin(x)+f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x}=\cos(x)\,y+f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial x}=\cos(x)\,(-2\sin(x)+y)$. Therefore equation (4) becomes

$$\cos(x)\,(-2\sin(x)+y)=\cos(x)\,y+f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x)=-2\cos(x)\sin(x)$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x=\int(-\sin(2x))\,\mathrm{d}x$$
$$f(x)=\frac{\cos(2x)}{2}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi=y\sin(x)+\frac{\cos(2x)}{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1=y\sin(x)+\frac{\cos(2x)}{2}$$

Solving for y gives

$$y=-\frac{\cos(2x)-2c_1}{2\sin(x)}$$

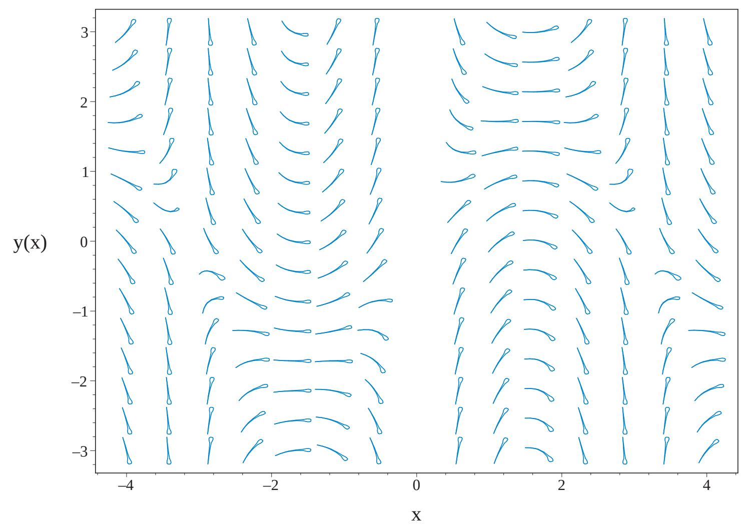


Figure 2.173: Slope field plot
$y' + y\cot(x) = 2\cos(x)$

Summary of solutions found

$$y = -\frac{\cos(2x) - 2c_1}{2\sin(x)}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) + y(x)\cot(x) = 2\cos(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = -y(x)\cot(x) + 2\cos(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + y(x)\cot(x) = 2\cos(x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\cot(x)\right) = 2\mu(x)\cos(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\cot(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \mu(x)\cot(x)$
- Solve to find the integrating factor
  $\mu(x) = \sin(x)$
- Integrate both sides with respect to x
  $\int\left(\frac{d}{dx}(y(x)\mu(x))\right)dx = \int 2\mu(x)\cos(x)\,dx + C1$
- Evaluate the integral on the lhs
  $y(x)\mu(x) = \int 2\mu(x)\cos(x)\,dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int 2\mu(x)\cos(x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \sin(x)$
  $y(x) = \frac{\int 2\sin(x)\cos(x)dx + C1}{\sin(x)}$
- Evaluate the integrals on the rhs
  $y(x) = \frac{\sin(x)^2 + C1}{\sin(x)}$

- Simplify
  $y(x) = \sin(x) + C1 \csc(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 16

```
dsolve(diff(y(x),x)+y(x)*cot(x) = 2*cos(x),y(x),singsol=all)
```

$$y = \csc(x)\left(-\cos(x)^2 + c_1 + \frac{1}{2}\right)$$

Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 23

```
DSolve[{D[y[x],x]+y[x]*Cot[x]==2*Cos[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \csc(x)\left(\int_1^x \sin(2K[1])dK[1] + c_1\right)$$

2.2.40 Problem 40

Internal problem ID [9163]
Book : Second order enumerated odes
Section : section 2
Problem number : 40
Date solved : `Monday, January 27, 2025 at 05:51:29 PM`
CAS classification : `[_rational]`

Solve

$$2xy^2 - y + \left(y^2 + x + y\right) y' = 0$$

Solved as first order Exact ode

Time used: 0.225 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y^2 + x + y\right)\mathrm{d}y &= \left(-2x\,y^2 + y\right)\mathrm{d}x\\ \left(2x\,y^2 - y\right)\mathrm{d}x + \left(y^2 + x + y\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x\,y^2 - y\\N(x,y) &= y^2 + x + y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\,y^2 - y\right)\\&= 4yx - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^2 + x + y\right)\\&= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\&= \frac{1}{y^2 + x + y}((4yx - 1) - (1))\\&= \frac{4yx - 2}{y^2 + x + y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\&= \frac{1}{2x\,y^2 - y}((1) - (4yx - 1))\\&= -\frac{2}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}y}\\&= e^{\int -\frac{2}{y} \,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)}\\&= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\&= \frac{1}{y^2}\left(2x\,y^2 - y\right)\\&= \frac{2yx - 1}{y}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(y^2 + x + y) \\
&= \frac{y^2 + x + y}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2yx - 1}{y}\right) + \left(\frac{y^2 + x + y}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{2yx - 1}{y} \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x(yx - 1)}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x^2}{y} - \frac{x(yx - 1)}{y^2} + f'(y) \tag{4}
$$

$$
= \frac{x}{y^2} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2 + x + y}{y^2}$. Therefore equation (4) becomes

$$
\frac{y^2 + x + y}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y + 1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(\frac{y + 1}{y}\right) \mathrm{d}y \\
f(y) &= y + \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(yx-1)}{y} + y + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{x(yx-1)}{y} + y + \ln(y)$$

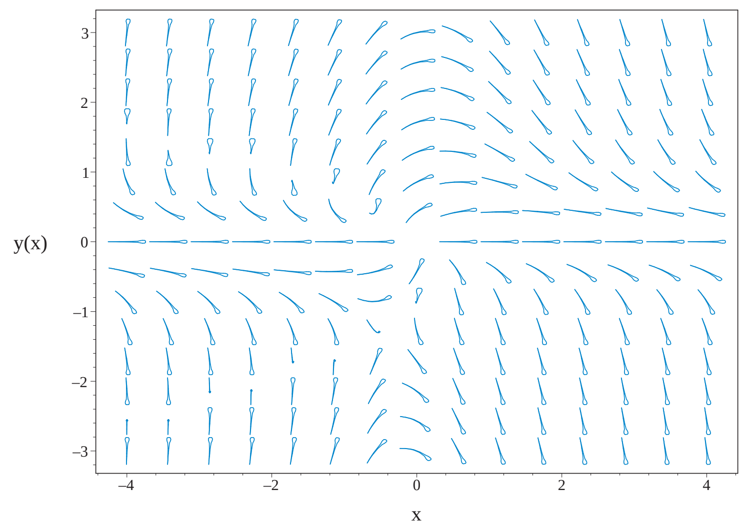


Figure 2.174: Slope field plot
$2xy^2 - y + (y^2 + x + y)\, y' = 0$

Summary of solutions found

$$\frac{x(yx-1)}{y} + y + \ln(y) = c_1$$

Maple step by step solution

Let's solve
$2xy(x)^2 - y(x) + \left(y(x)^2 + x + y(x)\right)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{-2xy(x)^2 + y(x)}{y(x)^2 + x + y(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : 28

```
dsolve(2*x*y(x)^2-y(x)+(y(x)^2+x+y(x))*diff(y(x),x) = 0,y(x),singsol=all)
```

$$y = e^{\mathrm{RootOf}\left(x^2 e^{\_Z}+e^{2\_Z}+c_1 e^{\_Z}+\_Z e^{\_Z}-x\right)}$$

Mathematica DSolve solution

Solving time : 0.183 (sec)
Leaf size : 22

```
DSolve[{(2*x*y[x]^2-y[x])+(y[x]^2+x+y[x])*D[y[x],x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$\mathrm{Solve}\left[x^2 - \frac{x}{y(x)} + y(x) + \log(y(x)) = c_1, y(x)\right]$$

2.2.41 Problem 41

Internal problem ID [9164]
Book : Second order enumerated odes
Section : section 2
Problem number : 41
Date solved : `Monday, January 27, 2025 at 05:51:31 PM`
CAS classification : `[[_Riccati, _special]]`

Solve

$$y' = x - y^2$$

Solved as first order ode of type reduced Riccati

Time used: 0.091 (sec)

This is reduced Riccati ode of the form

$$y' = a\,x^n + by^2$$

Comparing the given ode to the above shows that

$$\begin{aligned} a &= 1 \\ b &= -1 \\ n &= 1 \end{aligned}$$

Since $n \neq -2$ then the solution of the reduced Riccati ode is given by

$$\begin{aligned} w &= \sqrt{x}\begin{cases} c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) & ab > 0 \\ c_1 \operatorname{BesselI}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) & ab < 0 \end{cases} \qquad (1) \\ y &= -\frac{1}{b}\frac{w'}{w} \\ k &= 1 + \frac{n}{2} \end{aligned}$$

Since $ab < 0$ then EQ(1) gives

$$\begin{aligned} k &= \frac{3}{2} \\ w &= \sqrt{x}\left(c_1 \operatorname{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)\right) \end{aligned}$$

Therefore the solution becomes

$$y = -\frac{1}{b}\frac{w'}{w}$$

Substituting the value of b, w found above and simplyfing gives

$$y = \frac{\sqrt{x}\left(\operatorname{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right) c_1 - \operatorname{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right) c_2\right)}{c_1 \operatorname{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}$$

Letting $c_2 = 1$ the above becomes

$$y = \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right) c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1 \text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}$$

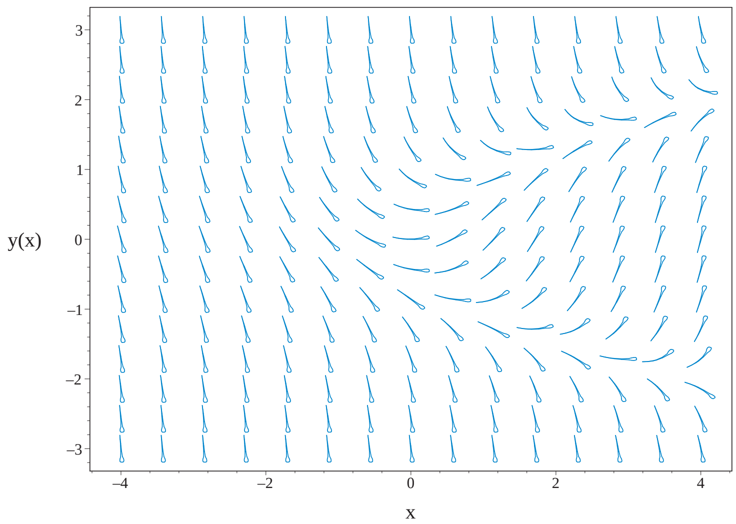


Figure 2.175: Slope field plot
$y' = x - y^2$

Summary of solutions found

$$y = \frac{\sqrt{x}\left(\text{BesselI}\left(-\frac{2}{3}, \frac{2x^{3/2}}{3}\right) c_1 - \text{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)\right)}{c_1 \text{BesselI}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselK}\left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = x - y(x)^2$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = x - y(x)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 23

```
dsolve(diff(y(x),x) = x-y(x)^2,y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{AiryAi}(1, x) + \operatorname{AiryBi}(1, x)}{c_1 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)}$$

Mathematica DSolve solution

Solving time : `0.125` (sec)
Leaf size : 223

```
DSolve[{D[y[x],x]==x-y[x]^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{-ix^{3/2}\left(2\operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2}{3}ix^{3/2}\right) + c_1\left(\operatorname{BesselJ}\left(-\frac{4}{3}, \frac{2}{3}ix^{3/2}\right) - \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2}{3}ix^{3/2}\right)\right)\right) - c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}i\right.}{2x\left(\operatorname{BesselJ}\left(\frac{1}{3}, \frac{2}{3}ix^{3/2}\right) + c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}ix^{3/2}\right)\right)}$$

$$y(x) \to \frac{ix^{3/2}\operatorname{BesselJ}\left(-\frac{4}{3}, \frac{2}{3}ix^{3/2}\right) - ix^{3/2}\operatorname{BesselJ}\left(\frac{2}{3}, \frac{2}{3}ix^{3/2}\right) + \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}ix^{3/2}\right)}{2x\operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2}{3}ix^{3/2}\right)}$$

2.2.42 Problem 42

Internal problem ID [9165]
Book : Second order enumerated odes
Section : section 2
Problem number : 42
Date solved : `Monday, January 27, 2025 at 05:51:32 PM`
CAS classification : `[[_high_order, _linear, _nonhomogeneous]]`

Solve

$$y'''' - y''' - 3y'' + 5y' - 2y = x\,\mathrm{e}^{x} + 3\,\mathrm{e}^{-2x}$$

Solved as higher order constant coeff ode

Time used: 0.131 (sec)

The characteristic equation is

$$\lambda^4 - \lambda^3 - 3\lambda^2 + 5\lambda - 2 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= 1\\ \lambda_3 &= 1\\ \lambda_4 &= 1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-2x}c_1 + \mathrm{e}^{x}c_2 + x\,\mathrm{e}^{x}c_3 + x^2\mathrm{e}^{x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-2x}\\ y_2 &= \mathrm{e}^{x}\\ y_3 &= x\,\mathrm{e}^{x}\\ y_4 &= x^2\mathrm{e}^{x}\end{aligned}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y''' - 3y'' + 5y' - 2y = 0$$

Now the particular solution to the given ODE is found

$$y'''' - y''' - 3y'' + 5y' - 2y = \left(x\,\mathrm{e}^{3x} + 3\right)\mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients.

Looking at the RHS of the ode, which is

$$\left(x\,\mathrm{e}^{3x}+3\right)\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\},\{x\,\mathrm{e}^{x},\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{x},x^{2}\mathrm{e}^{x},\mathrm{e}^{x},\mathrm{e}^{-2x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}\},\{x\,\mathrm{e}^{x},\mathrm{e}^{x}\}]$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}\},\{x\,\mathrm{e}^{x},x^{2}\mathrm{e}^{x}\}]$$

Since $x\,\mathrm{e}^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}\},\{x^{2}\mathrm{e}^{x},x^{3}\mathrm{e}^{x}\}]$$

Since $x^{2}\mathrm{e}^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}\},\{x^{3}\mathrm{e}^{x},x^{4}\mathrm{e}^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-2x}+A_2x^{3}\mathrm{e}^{x}+A_3x^{4}\mathrm{e}^{x}$$

The unknowns $\{A_1,A_2,A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$72A_3x\,\mathrm{e}^{x}-27A_1\mathrm{e}^{-2x}+18A_2\mathrm{e}^{x}+24A_3\mathrm{e}^{x}=x\,\mathrm{e}^{x}+3\,\mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-\frac{1}{9},A_2=-\frac{1}{54},A_3=\frac{1}{72}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=-\frac{x\,\mathrm{e}^{-2x}}{9}-\frac{x^{3}\mathrm{e}^{x}}{54}+\frac{x^{4}\mathrm{e}^{x}}{72}$$

Therefore the general solution is

$$\begin{aligned}y&=y_h+y_p\\&=\left(\mathrm{e}^{-2x}c_1+\mathrm{e}^{x}c_2+x\,\mathrm{e}^{x}c_3+x^{2}\mathrm{e}^{x}c_4\right)+\left(-\frac{x\,\mathrm{e}^{-2x}}{9}-\frac{x^{3}\mathrm{e}^{x}}{54}+\frac{x^{4}\mathrm{e}^{x}}{72}\right)\end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 52

```
dsolve(diff(diff(diff(diff(y(x),x),x),x),x)-diff(diff(diff(y(x),x),x),x)-3*diff(diff(y(x
```

$$y = \frac{e^{-2x}\left(\left(x^4 - \frac{4x^3}{3} + \left(72c_4 + \frac{4}{3}\right)x^2 + \left(72c_3 - \frac{8}{9}\right)x + 72c_1 + \frac{8}{27}\right)e^{3x} - 8x + 72c_2 - 8\right)}{72}$$

Mathematica DSolve solution

Solving time : `0.279` (sec)
Leaf size : 170

```
DSolve[{D[y[x],{x,4}]-D[y[x],{x,3}]-3*D[y[x],{x,2}]+5*D[y[x],x]-2*y[x]==x*Exp[x]+3*Exp[-2*x],{
```

$$\begin{aligned} y(x) \to\ & e^x x \int_1^x -\frac{1}{9}e^{-3K[3]}(3K[3]+1)\left(e^{3K[3]}K[3]+3\right)dK[3] \\ & + e^{-2x}\int_1^x \left(-\frac{1}{27}e^{3K[1]}K[1]-\frac{1}{9}\right)dK[1] \\ & + e^x \int_1^x \frac{1}{54}e^{-3K[2]}\left(e^{3K[2]}K[2]+3\right)\left(9K[2]^2+6K[2]+2\right)dK[2] \\ & + \frac{e^x x^4}{12} - \frac{1}{6}e^{-2x}x^2 + c_4 e^x x^2 + c_3 e^x x + c_1 e^{-2x} + c_2 e^x \end{aligned}$$

2.2.43 Problem 43

Internal problem ID [9166]
Book : Second order enumerated odes
Section : section 2
Problem number : 43
Date solved : `Monday, January 27, 2025 at 05:51:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(x+6)\,y' + 10y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 - 6x\right) y' + 10y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+6}{x}$$
$$q(x) = \frac{10}{x^2}$$

Table 2.102: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+6}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{10}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 - 6x\right) y' + 10y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) + \left(-x^2-6x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) + 10 \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1) \right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r} \right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1) \right) + \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r} \right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)\,(n+r-1) - 6x^{n+r} a_n (n+r) + 10 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 6x^r a_0 r + 10 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 6x^r r + 10x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)\,(r-5)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)\,(r-5) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 5$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)(r-5)x^r = 0$$

Solving for r gives the roots of the indicial equation as $[5, 2]$.

Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^5\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^2\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+5}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 6a_n(n+r) + 10a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - 7n - 7r + 10} \tag{4}$$

Which for the root $r = 5$ becomes

$$a_n = \frac{a_{n-1}(n+4)}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 5$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{r^2 - 5r + 4}$$

Which for the root $r = 5$ becomes

$$a_1 = \frac{5}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1+r}{r^3 - 8r^2 + 19r - 12}$$

Which for the root $r = 5$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24}$$

Which for the root $r = 5$ becomes

$$a_3 = \frac{7}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3+r}{(-1+r)^2 (r-2)(r-4)(r-3)}$$

Which for the root $r = 5$ becomes

$$a_4 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$
a_4	$\frac{3+r}{(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{12}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{4+r}{r(-1+r)^2 (r-2)(r-4)(r-3)}$$

Which for the root $r = 5$ becomes

$$a_5 = \frac{3}{160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$
a_4	$\frac{3+r}{(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{12}$
a_5	$\frac{4+r}{r(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{3}{160}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^5 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^5 \left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24} &= \lim_{r \to 2} \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y'' + (-x^2 - 6x)\,y' + 10y = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (-x^2 - 6x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) + 10Cy_1(x)\ln(x) + 10\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left(\left(x^2y_1''(x) + (-x^2-6x)\,y_1'(x) + 10y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(-x^2-6x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (-x^2-6x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 10\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x) + (-x^2 - 6x)\,y_1'(x) + 10y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(-x^2-6x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (-x^2-6x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 10\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)x - (x+7)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)x^2 + (-x^2-6x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) + 10\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 5$ and $r_2 = 2$ then the above becomes

$$\left(2\left(\sum_{n=0}^{\infty} x^{n+4} a_n(n+5)\right)x - (x+7)\left(\sum_{n=0}^{\infty} a_n x^{n+5}\right)\right)C + \left(\sum_{n=0}^{\infty} x^{n} b_n(n+2)(1+n)\right)x^2 + (-x^2-6x)\left(\sum_{n=0}^{\infty} x^{1+n} b_n(n+2)\right) + 10\left(\sum_{n=0}^{\infty} b_n x^{n+2}\right) = 0 \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n (n+5)\right)+\sum_{n=0}^{\infty}\left(-C\, x^{n+6} a_n\right)+\sum_{n=0}^{\infty}\left(-7C\, x^{n+5} a_n\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+2} b_n \left(n^2+3n+2\right)\right)+\sum_{n=0}^{\infty}\left(-x^{n+3} b_n (n+2)\right) \\
&+\sum_{n=0}^{\infty}\left(-6x^{n+2} b_n (n+2)\right)+\left(\sum_{n=0}^{\infty} 10 b_n x^{n+2}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+2} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n (n+5) &= \sum_{n=3}^{\infty} 2C a_{n-3} (n+2)\, x^{n+2} \\
\sum_{n=0}^{\infty}\left(-C\, x^{n+6} a_n\right) &= \sum_{n=4}^{\infty}\left(-C a_{n-4} x^{n+2}\right) \\
\sum_{n=0}^{\infty}\left(-7C\, x^{n+5} a_n\right) &= \sum_{n=3}^{\infty}\left(-7C a_{n-3} x^{n+2}\right) \\
\sum_{n=0}^{\infty}\left(-x^{n+3} b_n (n+2)\right) &= \sum_{n=1}^{\infty}\left(-b_{n-1} (1+n)\, x^{n+2}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+2$.

$$
\begin{aligned}
&\left(\sum_{n=3}^{\infty} 2C a_{n-3} (n+2)\, x^{n+2}\right)+\sum_{n=4}^{\infty}\left(-C a_{n-4} x^{n+2}\right)+\sum_{n=3}^{\infty}\left(-7C a_{n-3} x^{n+2}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+2} b_n \left(n^2+3n+2\right)\right)+\sum_{n=1}^{\infty}\left(-b_{n-1} (1+n)\, x^{n+2}\right) \\
&+\sum_{n=0}^{\infty}\left(-6x^{n+2} b_n (n+2)\right)+\left(\sum_{n=0}^{\infty} 10 b_n x^{n+2}\right)=0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-2b_1 - 2b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1 - 2 = 0$$

Solving the above for b_1 gives

$$b_1 = -1$$

For $n=2$, Eq (2B) gives

$$-2b_2 - 3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_2 + 3 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{3}{2}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$3C - 6 = 0$$

Which is solved for C. Solving for C gives

$$C = 2$$

For $n = 4$, Eq (2B) gives

$$(-a_0 + 5a_1) C - 5b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{21}{2} + 4b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{21}{8}$$

For $n = 5$, Eq (2B) gives

$$(-a_1 + 7a_2) C - 6b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{95}{4} + 10b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{19}{8}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 2$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = {} & 2\left(x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right)\right)\ln(x) \\ & + x^2\left(1 - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right) \\ &\quad + c_2\left(2\left(x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right)\right)\ln(x)\right. \\ &\quad \left. + x^2\left(1 - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} + O(x^6)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right) \\ &\quad + c_2\left(2x^5\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right)\ln(x)\right. \\ &\quad \left. + x^2\left(1 - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} + O(x^6)\right)\right) \end{aligned}$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(6+x)\left(\frac{d}{dx}y(x)\right)+10y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{10y(x)}{x^2}+\frac{(6+x)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(6+x)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{10y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{6+x}{x},P_3(x)=\frac{10}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-6$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=10$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(6+x)\left(\frac{d}{dx}y(x)\right)+10y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-5+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r-2)\,(k+r-5)-a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-5+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{2,5\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-5)-a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$a_{k+1}(k+r-1)(k-4+r)-a_k(k+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r)}{(k+r-1)(k-4+r)}$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k(k+2)}{(k+1)(k-2)}$

- Series not valid for $r=2$, division by 0 in the recursion relation at $k=2$

  $a_{k+1}=\frac{a_k(k+2)}{(k+1)(k-2)}$

- Recursion relation for $r=5$

  $a_{k+1}=\frac{a_k(k+5)}{(k+4)(k+1)}$

- Solution for $r=5$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+5}, a_{k+1}=\frac{a_k(k+5)}{(k+4)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 62

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(6+x)*diff(y(x),x)+10*y(x) = 0,y(x),series,x=0)
```

$$y=x^2\left(c_1x^3\left(1+\frac{5}{4}x+\frac{3}{4}x^2+\frac{7}{24}x^3+\frac{1}{12}x^4+\frac{3}{160}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(24x^3+30x^4+18x^5+\mathrm{O}\left(x^6\right)\right)+\left(12-12x+18x^2+26x^3+x^4-9x^5+\mathrm{O}\left(x^6\right)\right)\right)\right)$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 84

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]-x*(x+6)*D[y[x],x]+10*y[x]==0,{}},y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{1}{2}x^5(5x+4)\log(x)-\frac{1}{4}x^2\left(3x^4-6x^3-6x^2+4x-4\right)\right)+c_2\left(\frac{x^9}{12}+\frac{7x^8}{24}+\frac{3x^7}{4}+\frac{5x^6}{4}+x^5\right)$$

2.2.44 Problem 44

Internal problem ID [9167]
Book : Second order enumerated odes
Section : section 2
Problem number : 44
Date solved : `Monday, January 27, 2025 at 05:51:35 PM`
CAS classification : `[_Bessel]`

Solve

$$x^2y'' + xy' + \left(x^2 - 5\right) y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - 5\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - 5}{x^2}$$

Table 2.104: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-5}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - 5\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + (x^2 - 5) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) \right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) \right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n \right) + \sum_{n=0}^{\infty} \left(-5 a_n x^{n+r} \right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) \right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) \right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r} \right) + \sum_{n=0}^{\infty} \left(-5 a_n x^{n+r} \right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 5 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 5 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 5 x^r \right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 5 \right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 5 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \sqrt{5}$$
$$r_2 = -\sqrt{5}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 5\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[\sqrt{5}, -\sqrt{5}\right]$.

Since $r_1 - r_2 = 2\sqrt{5}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\sqrt{5}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\sqrt{5}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - 5a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 5} \tag{4}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_n = -\frac{a_{n-2}}{n\left(2\sqrt{5} + n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 4r - 1}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_2 = -\frac{1}{4 + 4\sqrt{5}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^2 + 4r - 1)(r^2 + 8r + 11)}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_4 = \frac{1}{32\left(\sqrt{5} + 1\right)\left(2 + \sqrt{5}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$
a_3	0	0
a_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$
a_3	0	0
a_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\sqrt{5}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\sqrt{5}}\left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - 5b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 - 5} \tag{4}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_n = -\frac{b_{n-2}}{n\left(-2\sqrt{5}+n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 4r - 1}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_2 = \frac{1}{-4 + 4\sqrt{5}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2+4r-1)(r^2+8r+11)}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_4 = \frac{1}{32(\sqrt{5}-1)(-2+\sqrt{5})}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$
b_3	0	0
b_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32(\sqrt{5}-1)(-2+\sqrt{5})}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$
b_3	0	0
b_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32(\sqrt{5}-1)(-2+\sqrt{5})}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\sqrt{5}}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\ &= x^{-\sqrt{5}}\left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32(\sqrt{5}-1)(-2+\sqrt{5})} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\, x^{\sqrt{5}}\left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32(\sqrt{5}+1)(2+\sqrt{5})} + O(x^6)\right) \\ &\quad + c_2\, x^{-\sqrt{5}}\left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32(\sqrt{5}-1)(-2+\sqrt{5})} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\, x^{\sqrt{5}}\left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32(\sqrt{5}+1)(2+\sqrt{5})} + O(x^6)\right) \\ &\quad + c_2\, x^{-\sqrt{5}}\left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32(\sqrt{5}-1)(-2+\sqrt{5})} + O(x^6)\right) \end{aligned}$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+(x^2-5)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x^2-5)y(x)}{x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}+\frac{(x^2-5)y(x)}{x^2}=0$

- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{1}{x},P_3(x)=\frac{x^2-5}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-5$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+(x^2-5)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(r^2-5)\,x^r+a_1(r^2+2r-4)\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k^2+2kr+r^2-5)+a_{k-2}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2-5=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\sqrt{5},-\sqrt{5}\right\}$

- Each term must be 0
  $a_1(r^2 + 2r - 4) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k^2 + 2kr + r^2 - 5) + a_{k-2} = 0$
- Shift index using $k- > k+2$
  $a_{k+2}\left((k+2)^2 + 2(k+2)\,r + r^2 - 5\right) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{k^2+2kr+r^2+4k+4r-1}$
- Recursion relation for $r = \sqrt{5}$
  $a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}$
- Solution for $r = \sqrt{5}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}, a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}, a_1 = 0\right]$
- Recursion relation for $r = -\sqrt{5}$
  $a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}}$
- Solution for $r = -\sqrt{5}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\sqrt{5}}, a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\sqrt{5}}\right), a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}, a_1 = 0, b_{k+2} = -\frac{b_k}{k^2-2k\sqrt{5}+4+4\cdots}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 89

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-5)*y(x) = 0,y(x),series,x=0)
```

$$y = c_1 x^{-\sqrt{5}}\left(1 + \frac{1}{-4+4\sqrt{5}}x^2 + \frac{1}{32}\frac{1}{\left(-2+\sqrt{5}\right)\left(\sqrt{5}-1\right)}x^4 + O\left(x^6\right)\right) + c_2 x^{\sqrt{5}}\left(1 - \frac{1}{4+4\sqrt{5}}x^2 + \frac{1}{32}\frac{1}{\left(\sqrt{5}+2\right)\left(\sqrt{5}+1\right)}x^4 + O\left(x^6\right)\right)$$

Mathematica DSolve solution

Solving time : `0.006` (sec)
Leaf size : 210

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-5)*y[x]==0,{}},y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^4}{\left(-3-\sqrt{5}+\left(1-\sqrt{5}\right)\left(2-\sqrt{5}\right)\right)\left(-1-\sqrt{5}+\left(3-\sqrt{5}\right)\left(4-\sqrt{5}\right)\right)} - \frac{x^2}{-3-\sqrt{5}+\left(1-\sqrt{5}\right)\left(2-\sqrt{5}\right)} + 1 \right) x^{-\sqrt{5}} + c_1 \left(\frac{x^4}{\left(-3+\sqrt{5}+\left(1+\sqrt{5}\right)\left(2+\sqrt{5}\right)\right)\left(-1+\sqrt{5}+\left(3+\sqrt{5}\right)\left(4+\sqrt{5}\right)\right)} - \frac{x^2}{-3+\sqrt{5}+\left(1+\sqrt{5}\right)\left(2+\sqrt{5}\right)} + 1 \right) x^{\sqrt{5}}$$

2.2.45 Problem 45

Internal problem ID [9168]
Book : Second order enumerated odes
Section : section 2
Problem number : 45
Date solved : `Monday, January 27, 2025 at 05:51:36 PM`
CAS classification : `[_Bessel]`

Solve

$$x^2y'' + xy' + \left(x^2 - 5\right) y = 0$$

Solved as second order Bessel ode

Time used: 0.070 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - 5\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 1$$
$$n = -\sqrt{5}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(-\sqrt{5}, x\right) + c_2 \operatorname{BesselY}\left(-\sqrt{5}, x\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \operatorname{BesselJ}\left(-\sqrt{5}, x\right) + c_2 \operatorname{BesselY}\left(-\sqrt{5}, x\right)$$

Solved as second order ode adjoint method

Time used: 0.984 (sec)

In normal form the ode

$$x^2y'' + xy' + (x^2 - 5) y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{x^2 - 5}{x^2} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(\frac{\xi(x)}{x}\right)' + \left(\frac{(x^2 - 5)\, \xi(x)}{x^2}\right) &= 0 \\ \frac{\xi''(x)\, x^2 + \xi(x)\, x^2 - \xi'(x)\, x - 4\xi(x)}{x^2} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' x^2 - \xi' x + (x^2 - 4) \xi = 0 \tag{1}$$

Bessel ode has the form

$$\xi'' x^2 + \xi' x + (-n^2 + x^2) \xi = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$\xi'' x^2 + (1 - 2\alpha) x\xi' + (\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2) \xi = 0 \tag{3}$$

With the standard solution

$$\xi = x^{\alpha}(c_3 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_4 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= 1 \\ \beta &= 1 \\ n &= -\sqrt{5} \\ \gamma &= 1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$\xi = c_3 x \operatorname{BesselJ}\left(-\sqrt{5}, x\right) + c_4 x \operatorname{BesselY}\left(-\sqrt{5}, x\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)} \end{aligned}$$

Or

$$y' + y\left(\frac{1}{x} - \frac{c_3 \operatorname{BesselJ}(-\sqrt{5}, x) + c_3 x\left(-\operatorname{BesselJ}(-\sqrt{5}+1, x) - \frac{\sqrt{5}\operatorname{BesselJ}(-\sqrt{5},x)}{x}\right) + c_4 \operatorname{BesselY}(-\sqrt{5}, x)}{c_3 x \operatorname{BesselJ}(-\sqrt{5}, x) + c_4 x \operatorname{BesselY}(-\sqrt{5}, x}\right.$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{-\operatorname{BesselJ}(-\sqrt{5}+1, x)\, c_3 x - \operatorname{BesselY}(-\sqrt{5}+1, x)\, c_4 x - \sqrt{5}\,(c_3 \operatorname{BesselJ}(-\sqrt{5}, x) + c_4 \operatorname{BesselY}(}{x\,(c_3 \operatorname{BesselJ}(-\sqrt{5}, x) + c_4 \operatorname{BesselY}(-\sqrt{5}, x))}$$

$$p(x) = 0$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= e^{\int -\frac{-\operatorname{BesselJ}(-\sqrt{5}+1,x)c_3x - \operatorname{BesselY}(-\sqrt{5}+1,x)c_4x - \sqrt{5}\,(c_3 \operatorname{BesselJ}(-\sqrt{5},x) + c_4 \operatorname{BesselY}(-\sqrt{5},x))}{x(c_3 \operatorname{BesselJ}(-\sqrt{5},x) + c_4 \operatorname{BesselY}(-\sqrt{5},x))}dx} \\
&= \frac{1}{c_3 \operatorname{BesselJ}(-\sqrt{5}, x) + c_4 \operatorname{BesselY}(-\sqrt{5}, x)}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_3 \operatorname{BesselJ}(-\sqrt{5}, x) + c_4 \operatorname{BesselY}(-\sqrt{5}, x)}\right) &= 0
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{c_3 \operatorname{BesselJ}(-\sqrt{5}, x) + c_4 \operatorname{BesselY}(-\sqrt{5}, x)} &= \int 0\,dx + c_5 \\
&= c_5
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_3 \operatorname{BesselJ}(-\sqrt{5},x) + c_4 \operatorname{BesselY}(-\sqrt{5},x)}$ gives the final solution

$$y = \left(c_3 \operatorname{BesselJ}\left(-\sqrt{5}, x\right) + c_4 \operatorname{BesselY}\left(-\sqrt{5}, x\right)\right) c_5$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(c_3 \operatorname{BesselJ}\left(-\sqrt{5}, x\right) + c_4 \operatorname{BesselY}\left(-\sqrt{5}, x\right)\right) c_5$$

The constants can be merged to give

$$y = c_3 \operatorname{BesselJ}\left(-\sqrt{5}, x\right) + c_4 \operatorname{BesselY}\left(-\sqrt{5}, x\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_3 \operatorname{BesselJ}\left(-\sqrt{5}, x\right) + c_4 \operatorname{BesselY}\left(-\sqrt{5}, x\right)$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\left(x^2-5\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x^2-5)y(x)}{x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}+\frac{(x^2-5)y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1}{x},P_3(x)=\frac{x^2-5}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-5$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\left(x^2-5\right)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0\left(r^2-5\right)x^r+a_1\left(r^2+2r-4\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k\left(k^2+2kr+r^2-5\right)+a_{k-2}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2-5=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\sqrt{5},-\sqrt{5}\right\}$

- Each term must be 0

  $a_1(r^2 + 2r - 4) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k^2 + 2kr + r^2 - 5) + a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}\left((k+2)^2 + 2(k+2)\,r + r^2 - 5\right) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k^2+2kr+r^2+4k+4r-1}$
- Recursion relation for $r = \sqrt{5}$

  $a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}$
- Solution for $r = \sqrt{5}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}, a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}, a_1 = 0\right]$
- Recursion relation for $r = -\sqrt{5}$

  $a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}}$
- Solution for $r = -\sqrt{5}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\sqrt{5}}, a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\sqrt{5}}\right), a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}, a_1 = 0, b_{k+2} = -\frac{b_k}{k^2-2k\sqrt{5}+4+4}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 19

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-5)*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \operatorname{BesselJ}\left(\sqrt{5}, x\right) + c_2 \operatorname{BesselY}\left(\sqrt{5}, x\right)$$

Mathematica DSolve solution

Solving time : 0.079 (sec)
Leaf size : 26

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-5)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to c_1 \text{BesselJ}\left(\sqrt{5}, x\right) + c_2 \text{BesselY}\left(\sqrt{5}, x\right)$$

2.2.46 Problem 46

Internal problem ID [9169]
Book : Second order enumerated odes
Section : section 2
Problem number : 46
Date solved : `Monday, January 27, 2025 at 05:51:38 PM`
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]`

Solve

$$x^2y'' - 4xy' + 6y = 0$$

Solved as second order Euler type ode

Time used: 0.051 (sec)

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2\, x^3 + c_1\, x^2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2\, x^3 + c_1\, x^2$$

Solved as second order solved by an integrating factor

Time used: 0.029 (sec)

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{4}{x}\,dx} \\ &= \frac{1}{x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\frac{y}{x^2}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1\, x^3 + c_2\, x^2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\, x^3 + c_2\, x^2$$

Solved as second order ode using change of variable on x method 2

Time used: 0.446 (sec)

In normal form the ode

$$x^2y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\int p(x)dx} dx \\ &= \int \mathrm{e}^{-\int -\frac{4}{x}dx} dx \\ &= \int e^{4\ln(x)}\, dx \\ &= \int x^4 dx \\ &= \frac{x^5}{5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{6}{x^2}}{x^8} \\ &= \frac{6}{x^{10}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. Writing the ode as

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0 \tag{1}$$

$$A\frac{d^2}{d\tau^2}y(\tau) + B\frac{d}{d\tau}y(\tau) + Cy(\tau) = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{6}{25\tau^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(\tau) = y(\tau)\, e^{\int \frac{B}{2A}\, d\tau}$$

Then (2) becomes

$$z''(\tau) = rz(\tau) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-6}{25\tau^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -6 \\ t &= 25\tau^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(\tau) = \left(-\frac{6}{25\tau^2}\right) z(\tau) \tag{7}$$

Equation (7) is now solved. After finding $z(\tau)$ then $y(\tau)$ is found using the inverse transformation

$$y(\tau) = z(\tau)\, e^{-\int \frac{B}{2A}\, d\tau}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s)\\
&= 2 - 0\\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 25\tau^2$. There is a pole at $\tau = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{6}{25\tau^2}
$$

For the pole at $\tau = 0$ let b be the coefficient of $\frac{1}{\tau^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{6}{25}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{5}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{2}{5}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{\tau^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{6}{25\tau^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{6}{25}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{5}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{2}{5}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{6}{25\tau^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{5}$	$\frac{2}{5}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{5}$	$\frac{2}{5}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{2}{5}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{2}{5} - \left(\frac{2}{5}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{\tau - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{\tau - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{2}{5\tau} + (-)(0) \\ &= \frac{2}{5\tau} \\ &= \frac{2}{5\tau} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(\tau)$ of degree $d = 0$ to solve the ode. The polynomial $p(\tau)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(\tau) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{2}{5\tau}\right)(0) + \left(\left(-\frac{2}{5\tau^2}\right) + \left(\frac{2}{5\tau}\right)^2 - \left(-\frac{6}{25\tau^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(\tau) &= p e^{\int \omega \, d\tau} \\ &= e^{\int \frac{2}{5\tau} d\tau} \\ &= \tau^{2/5} \end{aligned}$$

The first solution to the original ode in $y(\tau)$ is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, d\tau}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \tau^{2/5} \end{aligned}$$

Which simplifies to

$$y_1 = \tau^{2/5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, d\tau}}{y_1^2}\, d\tau$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, d\tau \\ &= \tau^{2/5} \int \frac{1}{\tau^{4/5}}\, d\tau \\ &= \tau^{2/5}\left(5\tau^{1/5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(\tau) &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\tau^{2/5}\right) + c_2\left(\tau^{2/5}\left(5\tau^{1/5}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{3/5} (x^5)^{2/5}}{5} + c_2 5^{2/5} \left(x^5\right)^{3/5}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 5^{3/5} (x^5)^{2/5}}{5} + c_2 5^{2/5} \left(x^5\right)^{3/5}$$

Solved as second order ode using change of variable on x method 1

Time used: 0.127 (sec)

In normal form the ode

$$x^2y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -\frac{4}{x}\\ q(x) &= \frac{6}{x^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \qquad (6)\\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2}\\ &= -\frac{5c\sqrt{6}}{6}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2y(\tau) &= 0 \qquad (7)\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1\cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + ic_2\sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{5/2}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = x^{5/2}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Solved as second order ode using change of variable on y method 1

Time used: 0.136 (sec)

In normal form the given ode is written as

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\,z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2}dx} \\
&= e^{-\int \frac{-\frac{4}{x}}{2}} \\
&= x^2
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4 v''(x) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$v''(x) = 0$$

Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 x + c_2)\, x^2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = (c_1 x + c_2)\, x^2$$

Solved as second order ode using change of variable on y method 2

Time used: 0.096 (sec)

In normal form the ode

$$x^2 y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{2}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \, dx} \\ &= e^{\int \frac{2}{x} dx} \\ &= x^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x} (u\, x^2) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} u\, x^2 &= \int 0 \, dx + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor x^2 gives the final solution

$$u(x) = \frac{c_1}{x^2}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^3 \\
&= (c_2 x - c_1)\, x^2
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \left(-\frac{c_1}{x} + c_2\right) x^3
$$

Solved as second order ode using Kovacic algorithm

Time used: 0.040 (sec)

Writing the ode as

$$
\begin{aligned}
x^2 y'' - 4xy' + 6y &= 0 && (1) \\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -4x \qquad (3) \\
C &= 6
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5) \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.108: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\, dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1 \left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(x)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2\, x^3 + c_1\, x^2$$

Solved as second order ode adjoint method

Time used: 0.230 (sec)

In normal form the ode

$$x^2 y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= \frac{6}{x^2} \\
r(x) &= 0
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(-\frac{4\xi(x)}{x}\right)' + \left(\frac{6\xi(x)}{x^2}\right) &= 0 \\
\xi''(x) + \frac{2\xi(x)}{x^2} + \frac{4\xi'(x)}{x} &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. This is Euler second order ODE. Let the solution be $\xi = x^r$, then $\xi' = rx^{r-1}$ and $\xi'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 4xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\, x^r + 4r\, x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)+4r+2=0$$

Or

$$r^2+3r+2=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1 &= -2\\ r_2 &= -1\end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$\xi = c_1\xi_1 + c_2\xi_2$$

Where $\xi_1 = x^{r_1}$ and $\xi_2 = x^{r_2}$. Hence

$$\xi = \frac{c_1}{x^2}+\frac{c_2}{x}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}\end{aligned}$$

Or

$$y' + y\left(-\frac{4}{x} - \frac{-\frac{2c_1}{x^3} - \frac{c_2}{x^2}}{\frac{c_1}{x^2} + \frac{c_2}{x}}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= -\frac{3c_2x+2c_1}{x\,(c_2x+c_1)}\\ p(x) &= 0\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= e^{\int -\frac{3c_2x+2c_1}{x(c_2x+c_1)}dx}\\ &= \frac{1}{x^2\,(c_2x+c_1)}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2\,(c_2x+c_1)}\right) &= 0\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2\,(c_2x+c_1)} &= \int 0\,dx + c_3\\ &= c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{x^2(c_2x+c_1)}$ gives the final solution

$$y = x^2(c_2x + c_1)\, c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = x^2(c_2x + c_1)\, c_3$$

The constants can be merged to give

$$y = x^2(c_2x + c_1)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = x^2(c_2x + c_1)$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + 6y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{6y(x)}{x^2} + \frac{4\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{4\left(\frac{d}{dx}y(x)\right)}{x} + \frac{6y(x)}{x^2} = 0$
- Multiply by denominators of the ODE

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + 6y(x) = 0$
- Make a change of variables

  $t = \ln(x)$
- ☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$
  - Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 4\frac{d}{dt}y(t) + 6y(t) = 0$
- Simplify

  $\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 6y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial
  $(r-2)(r-3)=0$
- Roots of the characteristic polynomial
  $r=(2,3)$
- 1st solution of the ODE
  $y_1(t)=e^{2t}$
- 2nd solution of the ODE
  $y_2(t)=e^{3t}$
- General solution of the ODE
  $y(t)=C1y_1(t)+C2y_2(t)$
- Substitute in solutions
  $y(t)=C1\,e^{2t}+C2\,e^{3t}$
- Change variables back using $t=\ln(x)$
  $y(x)=C2\,x^3+C1\,x^2$
- Simplify
  $y(x)=x^2(C2x+C1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 13

```
dsolve(x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+6*y(x) = 0,y(x),singsol=all)
```

$$y=x^2(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : `0.013` (sec)
Leaf size : 16

```
DSolve[{x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to x^2(c_2x+c_1)$$

2.2.47 Problem 47

Internal problem ID [9170]
Book : Second order enumerated odes
Section : section 2
Problem number : 47
Date solved : `Tuesday, January 28, 2025 at 04:00:12 PM`
CAS classification : `[[_3rd_order, _with_linear_symmetries]]`

Solve

$$y''' - xy = 0$$

Maple step by step solution

Let's solve
$\frac{d^3}{dx^3}y(x) - xy(x) = 0$

- Highest derivative means the order of the ODE is 3
  $\frac{d^3}{dx^3}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$
- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$
  - Shift index using $k- >k-1$
    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$
  - Convert $\frac{d^3}{dx^3}y(x)$ to series expansion
    $\frac{d^3}{dx^3}y(x) = \sum_{k=3}^{\infty} a_k k(k-1)(k-2)\, x^{k-3}$
  - Shift index using $k- >k+3$
    $\frac{d^3}{dx^3}y(x) = \sum_{k=0}^{\infty} a_{k+3}(k+3)(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions
  $6a_3 + \left(\sum_{k=1}^{\infty} \left(a_{k+3}(k+3)(k+2)(k+1) - a_{k-1}\right) x^k\right) = 0$
- Each term must be 0
  $6a_3 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k^3 + 6k^2 + 11k + 6)\, a_{k+3} - a_{k-1} = 0$
- Shift index using $k- >k+1$
  $\left((k+1)^3 + 6(k+1)^2 + 11k + 17\right) a_{k+4} - a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{a_k}{k^3+9k^2+26k+24}, 6a_3 = 0\right]$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
trying Louvillian solutions for 3rd order ODEs, imprimitive case
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
<- pFq successful: received ODE is equivalent to the  0F2  ODE, case  c = 0 `
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 45

```
dsolve(diff(diff(diff(y(x),x),x),x)-x*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \,\mathrm{hypergeom}\left([\,], \left[\frac{1}{2}, \frac{3}{4}\right], \frac{x^4}{64}\right) + c_2 x \,\mathrm{hypergeom}\left([\,], \left[\frac{3}{4}, \frac{5}{4}\right], \frac{x^4}{64}\right) + c_3\, x^2 \,\mathrm{hypergeom}\left([\,], \left[\frac{5}{4}, \frac{3}{2}\right], \frac{x^4}{64}\right)$$

Mathematica DSolve solution

Solving time : 0.008 (sec)
Leaf size : 76

```
DSolve[{D[y[x],{x,3}]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\, {}_0F_2\left(;\frac{1}{2},\frac{3}{4};\frac{x^4}{64}\right) + \frac{1}{8}x\left((2+2i)c_2\, {}_0F_2\left(;\frac{3}{4},\frac{5}{4};\frac{x^4}{64}\right) + ic_3 x\, {}_0F_2\left(;\frac{5}{4},\frac{3}{2};\frac{x^4}{64}\right)\right)$$

2.2.48 Problem 48

Internal problem ID [9171]
Book : Second order enumerated odes
Section : section 2
Problem number : 48
Date solved : `Monday, January 27, 2025 at 05:51:40 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = y^{1/3}$$

With initial conditions

$$y(0) = 0$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^{1/3} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^{1/3}\right) \\ &= \frac{1}{3y^{2/3}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{0 < y\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

Solved as first order autonomous ode

Time used: 0.077 (sec)

Since the ode has the form $y' = f(y)$ and initial conditions (x_0, y_0) are given such that they satisfy the ode itself, then we can write

$$0 = f(y)|_{y=y_0}$$
$$0 = 0$$

And the solution is immediately written as

$$y = y_0$$
$$y = 0$$

Singular solutions are found by solving

$$y^{1/3} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

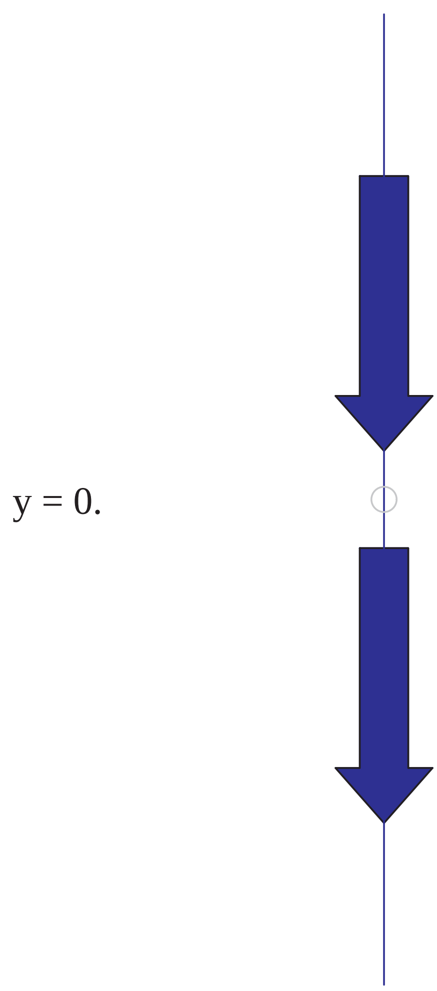
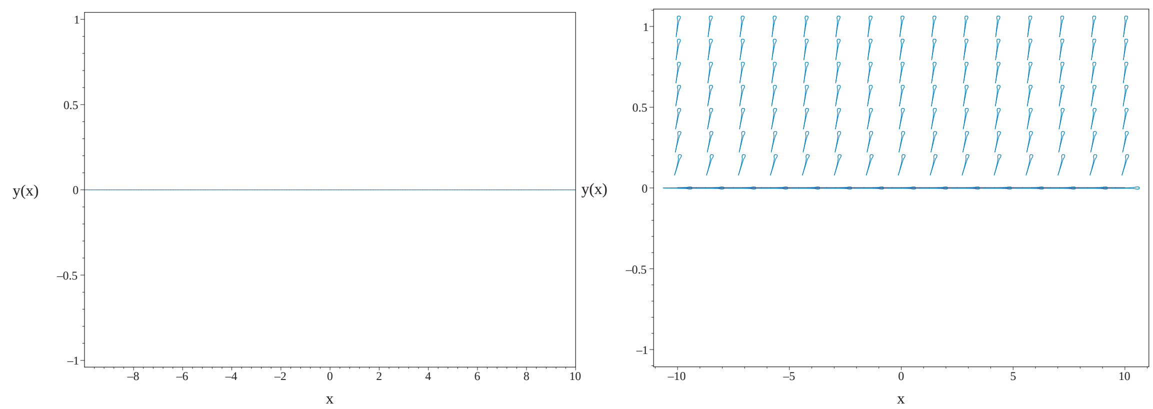


Figure 2.176: Phase line diagram



(a) Solution plot
$y = 0$

(b) Slope field plot
$y' = y^{1/3}$

Summary of solutions found

$$y = 0$$

Solved as first order Bernoulli ode

Time used: 0.306 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^{1/3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = (1)\, y^{1/3} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \end{aligned}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= 0 \\ f_1(x) &= 1 \\ n &= \frac{1}{3} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{1/3}$ gives

$$y' \frac{1}{y^{1/3}} = 0 + 1 \tag{4}$$

Let

$$\begin{aligned} v &= y^{1-n} \\ &= y^{2/3} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = \frac{2}{3y^{1/3}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{3v'(x)}{2} &= 1 \\ v' &= \frac{2}{3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

Since the ode has the form $v'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dv &= \int \frac{2}{3}\, dx \\ v(x) &= \frac{2x}{3} + c_1 \end{aligned}$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

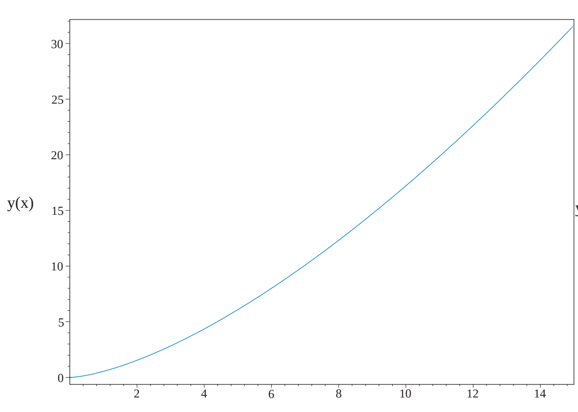
$$y^{2/3} = \frac{2x}{3} + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

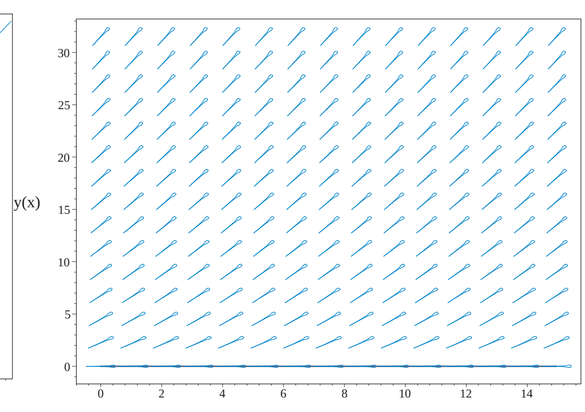
$$y^{2/3} = \frac{2x}{3}$$

Solving for y gives

$$y = \frac{2\sqrt{2}\,\sqrt{3}\,x^{3/2}}{9}$$



(a) Solution plot
$y = \frac{2\sqrt{2}\,\sqrt{3}\,x^{3/2}}{9}$



(b) Slope field plot
$y' = y^{1/3}$

Summary of solutions found

$$y = \frac{2\sqrt{2}\,\sqrt{3}\,x^{3/2}}{9}$$

Solved as first order Exact ode

Time used: 0.148 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(y^{1/3}\right)\mathrm{d}x\\ \left(-y^{1/3}\right)\mathrm{d}x+\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y^{1/3}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-y^{1/3}\right)\\ &=-\frac{1}{3y^{2/3}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{1}{3y^{2/3}}\right) - (0)\right) \\ &= -\frac{1}{3y^{2/3}} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{y^{1/3}}\left((0) - \left(-\frac{1}{3y^{2/3}}\right)\right) \\ &= -\frac{1}{3y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int -\frac{1}{3y} \,\mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{\ln(y)}{3}} \\ &= \frac{1}{y^{1/3}} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^{1/3}}\left(-y^{1/3}\right) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^{1/3}}(1) \\ &= \frac{1}{y^{1/3}} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (-1) + \left(\frac{1}{y^{1/3}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -1 \mathrm{d}x$$
$$\phi = -x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^{1/3}}$. Therefore equation (4) becomes

$$\frac{1}{y^{1/3}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^{1/3}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{1}{y^{1/3}}\right) \mathrm{d}y$$
$$f(y) = \frac{3y^{2/3}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \frac{3y^{2/3}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

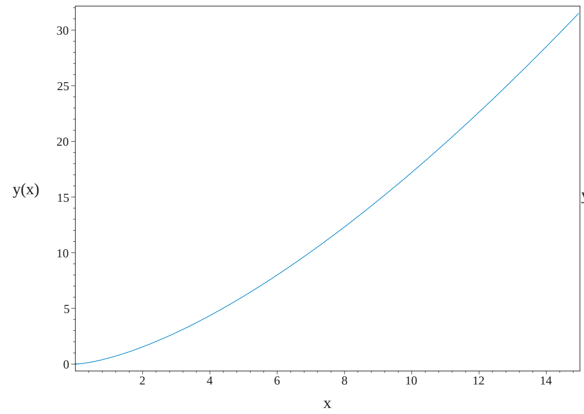
$$c_1 = -x + \frac{3y^{2/3}}{2}$$

Solving for the constant of integration from initial conditions, the solution becomes

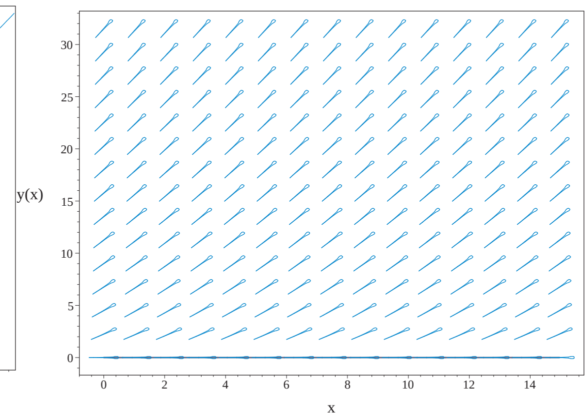
$$-x + \frac{3y^{2/3}}{2} = 0$$

Solving for y gives

$$y = \frac{2\sqrt{2}\,\sqrt{3}\,x^{3/2}}{9}$$



(a) Solution plot
$y = \frac{2\sqrt{2}\,\sqrt{3}\,x^{3/2}}{9}$



(b) Slope field plot
$y' = y^{1/3}$

Summary of solutions found

$$y = \frac{2\sqrt{2}\,\sqrt{3}\,x^{3/2}}{9}$$

Solved using Lie symmetry for first order ode

Time used: 0.544 (sec)

Writing the ode as

$$\begin{aligned} y' &= y^{1/3} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + y^{1/3}(b_3 - a_2) - y^{2/3} a_3 - \frac{x b_2 + y b_3 + b_1}{3 y^{2/3}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3y^{4/3}a_3 + 3y a_2 - 2y b_3 - 3b_2 y^{2/3} + x b_2 + b_1}{3y^{2/3}} = 0$$

Setting the numerator to zero gives

$$-3y^{4/3}a_3 + 3b_2 y^{2/3} - x b_2 - 3y a_2 + 2y b_3 - b_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, y^{2/3}, y^{4/3}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, y^{2/3} = v_3, y^{4/3} = v_4\}$$

The above PDE (6E) now becomes

$$-3v_2a_2 - 3v_4a_3 - v_1b_2 + 3b_2v_3 + 2v_2b_3 - b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-v_1b_2 + (-3a_2 + 2b_3)\, v_2 + 3b_2v_3 - 3v_4a_3 - b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -3a_3 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ 3b_2 &= 0 \\ -3a_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= \frac{3a_2}{2} \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 0 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 0 - \left(y^{1/3}\right)(1) \\ &= -y^{1/3} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-y^{1/3}} dy \end{aligned}$$

Which results in

$$S = -\frac{3y^{2/3}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y^{1/3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= -\frac{1}{y^{1/3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int -1\, dR \\ S(R) &= -R + c_2 \end{aligned}$$

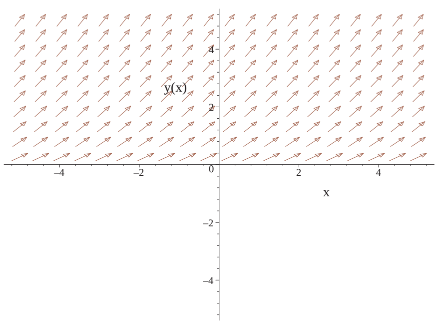
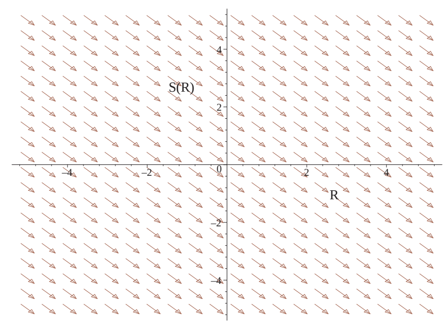
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\frac{3y^{2/3}}{2} = -x + c_2$$

Which gives

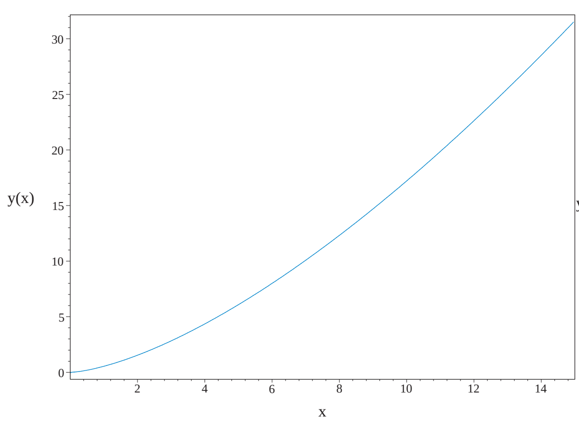
$$y = \frac{(6x - 6c_2)^{3/2}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

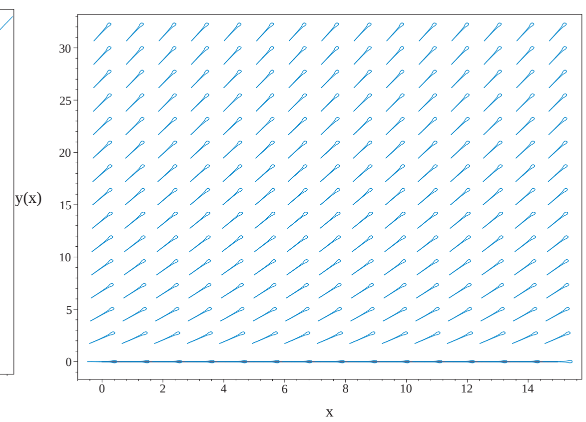
Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y^{1/3}$ 	$R = x$ $S = -\frac{3y^{2/3}}{2}$	$\frac{dS}{dR} = -1$ 

Solving for the constant of integration from initial conditions, the solution becomes

$$y = \frac{2x^{3/2}\sqrt{6}}{9}$$



(a) Solution plot
$y = \frac{2x^{3/2}\sqrt{6}}{9}$



(b) Slope field plot
$y' = y^{1/3}$

<u>Summary of solutions found</u>

$$y = \frac{2x^{3/2}\sqrt{6}}{9}$$

Maple step by step solution

Let's solve
$\left[\frac{d}{dx}y(x) = y(x)^{1/3}, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = y(x)^{1/3}$
- Separate variables

$$\frac{\frac{d}{dx}y(x)}{y(x)^{1/3}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}y(x)}{y(x)^{1/3}}dx = \int 1dx + C1$$

- Evaluate integral

$$\frac{3y(x)^{2/3}}{2} = x + C1$$

- Solve for $y(x)$

$$y(x) = \frac{(6x+6C1)^{3/2}}{27}$$

- Use initial condition $y(0) = 0$

$$0 = \frac{2\sqrt{6}\,C1^{3/2}}{9}$$

- Solve for $\_C1$

$$C1 = 0$$

- Substitute $\_C1 = 0$ into general solution and simplify

$$y(x) = \frac{2\sqrt{6}\,x^{3/2}}{9}$$

- Solution to the IVP

$$y(x) = \frac{2\sqrt{6}\,x^{3/2}}{9}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 5

```
dsolve([diff(y(x),x) = y(x)^(1/3),op([y(0) = 0])],y(x),singsol=all)
```

$$y = 0$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
DSolve[{D[y[x],x]==y[x]^(1/3),{y[0]==0}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2}{3}\sqrt{\frac{2}{3}}x^{3/2}$$

2.2.49 Problem 49

Internal problem ID [9172]
Book : Second order enumerated odes
Section : section 2
Problem number : 49
Date solved : `Monday, January 27, 2025 at 05:51:42 PM`
CAS classification : `system_of_ODEs`

$$\begin{aligned} x' &= 3x + y \\ y' &= -x + y \end{aligned}$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{2t}(1+t) & t\,\mathrm{e}^{2t} \\ -t\,\mathrm{e}^{2t} & \mathrm{e}^{2t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \mathrm{e}^{2t}(1+t) & t\,\mathrm{e}^{2t} \\ -t\,\mathrm{e}^{2t} & \mathrm{e}^{2t}(1-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{2t}(1+t)\,c_1 + t\,\mathrm{e}^{2t}c_2 \\ -t\,\mathrm{e}^{2t}c_1 + \mathrm{e}^{2t}(1-t)\,c_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{2t}(tc_1 + c_2t + c_1) \\ -\mathrm{e}^{2t}((-1+t)\,c_2 + tc_1) \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

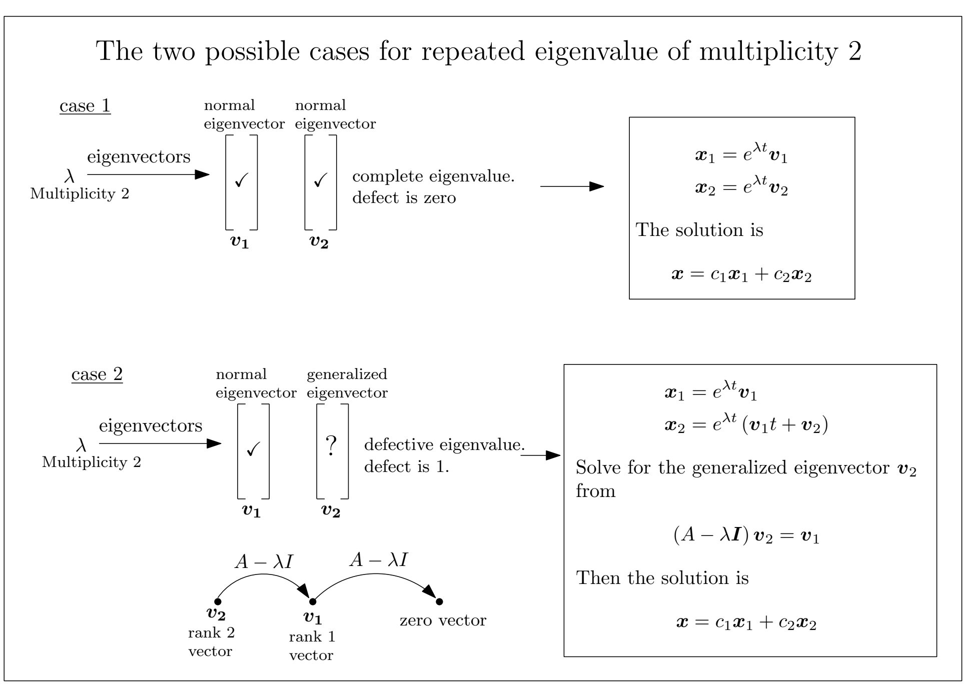


Figure 2.181: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) \mathrm{e}^{2t} \\ &= \begin{bmatrix} -\mathrm{e}^{2t}(t+2) \\ \mathrm{e}^{2t}(1+t) \end{bmatrix} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t}(-t-2) \\ \mathrm{e}^{2t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -((t+2)\, c_2 + c_1)\, \mathrm{e}^{2t} \\ \mathrm{e}^{2t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

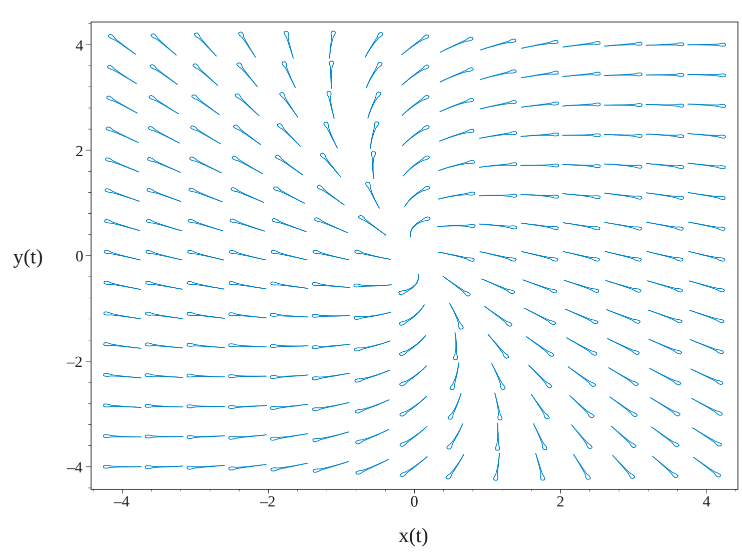


Figure 2.182: Phase plot

Maple step by step solution

Let's solve

$\left[\frac{d}{dt}x(t) = 3x(t) + y(t), \frac{d}{dt}y(t) = -x(t) + y(t)\right]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- System to solve

$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$

- Rewrite the system as

$\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$\left[\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$

- First solution from eigenvalue 2

$\vec{x}_1(t) = e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

$\vec{x}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{x}_2(t) = e^{2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = C1\vec{x}_1(t) + C2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = C1\,e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C2\,e^{2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(-C2t - C1 - C2) \\ e^{2t}(C2t + C1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = e^{2t}(-C2t - C1 - C2), y(t) = e^{2t}(C2t + C1)\}$$

Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 31

```
dsolve([diff(x(t),t) = 3*x(t)+y(t), diff(y(t),t) = -x(t)+y(t)],{op([x(t), y(t)])})
```

$$x(t) = e^{2t}(c_2t + c_1)$$
$$y(t) = -e^{2t}(c_2t + c_1 - c_2)$$

Mathematica DSolve solution

Solving time : `0.002` (sec)
Leaf size : 42

```
DSolve[{{D[x[t],t]==3*x[t]+y[t],D[y[t],t]==-x[t]+y[t]},{}},{x[t],y[t]},t,IncludeSingularSoluti
```

$$x(t) \to e^{2t}(c_1(t+1) + c_2t)$$
$$y(t) \to e^{2t}(c_2 - (c_1 + c_2)t)$$