Small note on solving $x^{\frac{n}{m}} = a$

Nasser M. Abbasi

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We want to solve

$$x^{\frac{n}{m}} = a$$

Where n, m are integers. n is called the power and m is called the root. We start by writing the above as

$$\left(x^{\frac{1}{m}}\right)^n = a$$

Let $x^{\frac{1}{m}} = y$. The above becomes

$$y^n = a$$

This is solved using De Moivre's formula.

$$\begin{aligned} y &= a^{\frac{1}{n}} \\ &= (a \times 1)^{\frac{1}{n}} \\ &= \left(ae^{2i\pi}\right)^{\frac{1}{n}} \end{aligned}$$

Since $1 = e^{2\pi i}$. Using Euler formula $1 = \cos(2\pi) + i\sin(2\pi)$. Hence

$$y = a^{\frac{1}{n}}(\cos(2\pi) + i\sin(2\pi))^{\frac{1}{n}}$$

But by De Moivre's formula

$$(\cos(2\pi) + i\sin(2\pi))^{\frac{1}{n}} = \cos\left(\frac{2\pi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n} + k\frac{2\pi}{n}\right) \qquad k = 0, 1, \cdots n-1$$

Therefore

$$y = a^{\frac{1}{n}}\left(\cos\left(\frac{2\pi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n} + k\frac{2\pi}{n}\right)\right) \qquad k = 0, 1, \cdots n-1$$

For example, let $n = 3$ then we have 3 solutions

$$y = \begin{cases} a^{\frac{1}{3}}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \\ a^{\frac{1}{3}}\left(\cos\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right)\right) \\ a^{\frac{1}{3}}\left(\cos\left(\frac{2\pi}{3} + \frac{4\pi}{3}\right) + i\sin\left(\frac{2\pi}{3} + \frac{4\pi}{3}\right)\right) \end{cases}$$

Which simplifies to

$$y = \begin{cases} a^{\frac{1}{3}}\left(\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right) \\ a^{\frac{1}{3}}\left(-\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right) \\ a^{\frac{1}{3}} \end{cases}$$

Now we need to replace y back to $x^{\frac{1}{m}}$ and the above becomes

$$x^{\frac{1}{m}} = \begin{cases} a^{\frac{1}{3}}\left(\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right) \\ a^{\frac{1}{3}}\left(-\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right) \\ a^{\frac{1}{3}} \end{cases}$$

Since the exponent now is a root, then

$$x = \begin{cases} \left(a^{\frac{1}{3}}\left(\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right)\right)^m \\ \left(a^{\frac{1}{3}}\left(-\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right)\right)^m \\ a^{\frac{m}{3}} \end{cases}$$

For example, if $m = 2$

$$x = \begin{cases} \left(a^{\frac{1}{3}}\left(\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right)\right)^2 \\ \left(a^{\frac{1}{3}}\left(-\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right)\right)^2 \\ a^{\frac{2}{3}} \end{cases}$$

Notice that if the solution x is meant to be *real*, then the above reduces to

$$x = a^{\frac{2}{3}}$$

And for $m = 4$

$$\begin{aligned} x &= \begin{cases} a^{\frac{4}{3}}\left(\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right)^4 \\ a^{\frac{4}{3}}\left(-\frac{1}{2}i\sqrt{3} - \frac{1}{2}\right)^4 \\ a^{\frac{4}{3}} \end{cases} \\ &= \begin{cases} a^{\frac{4}{3}}\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) \\ a^{\frac{4}{3}}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\ a^{\frac{4}{3}} \end{cases} \end{aligned}$$

Notice that if the solution x is meant to be *real*, then the above reduces to

$$x = a^{\frac{4}{3}}$$

For $a \geq 0$. And so on. For the case of power n being negative integer, for example,

$$x^{\frac{-3}{2}} = a$$

Then let $n = 3$ and move the negative sign to the denominator to become $x^{\frac{3}{-2}}$. This way we can now use De Moivre's formula for positive n.