Differential Equations Algorithms

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Contents

Introduction

This gives detailed description of all supported differential equations in my step-by-step ode solver. Whenever possible, each ode type algorithm is described using flow chart.

Each ode type is given an internal code name. This internal code is used internally by the solver to determine which solver to call to solve the specific ode.

The following is the top level chart of supported solvers.

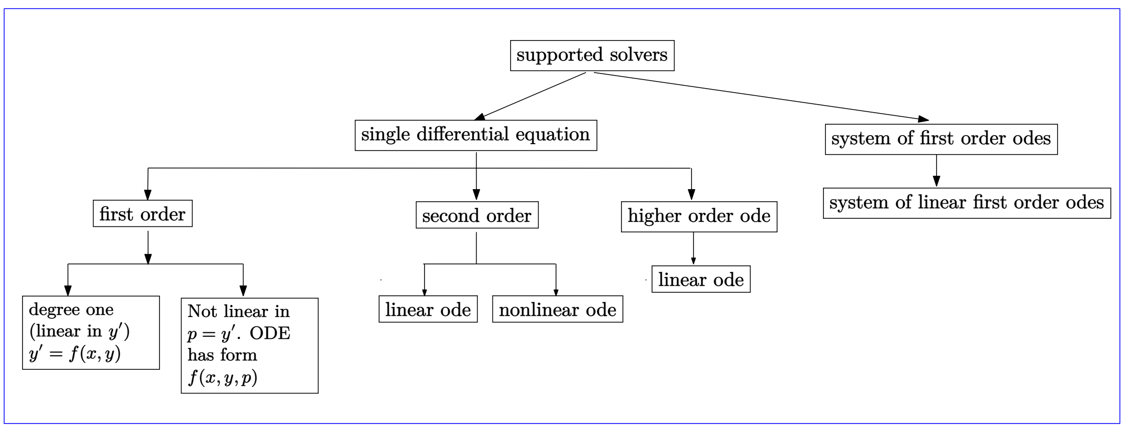

Figure 1: Top level flow chart for ode solver

This diagram illustrate some of the plots generated for direction field and phase plots.

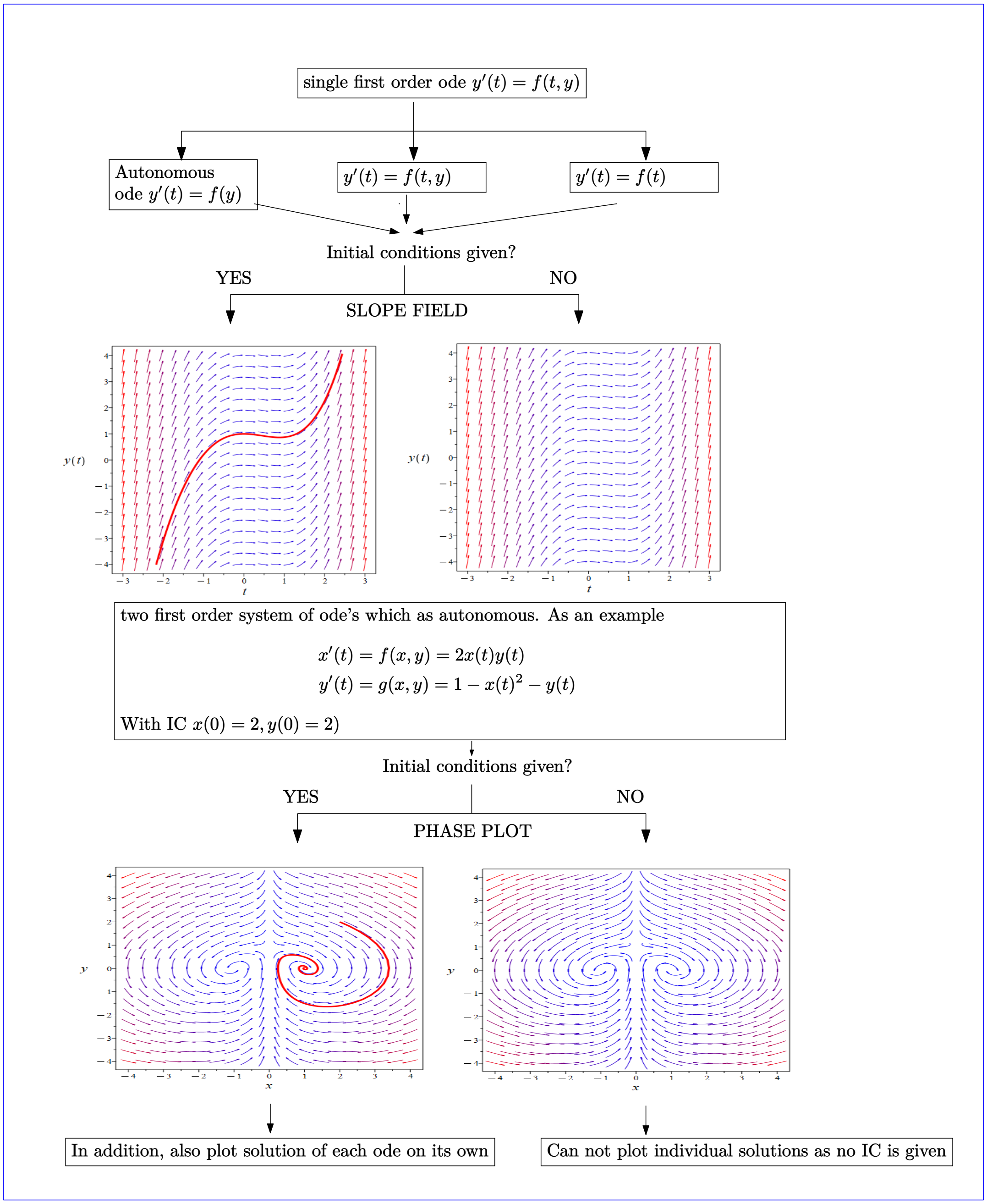

Figure 2: Direction and slope fields generated

CHAPTER 1

FIRST ORDER ODE

$$F(x, y, y') = 0$$

1.1 Existence and uniqueness for first order ode

There are two theorems that we will be using. One is for first order ode which is linear in y and one for first order ode which is not linear in y.

1.1.1 Existence and uniqueness for non linear first order ode in y

Given a first order ode $y' = f(x,y)$ (where y enters the ode as nonlinear, for example y^2 or $\frac{1}{y}$) and with initial conditions $y(x_0) = y_0$ then we say a solution exists somewhere in vicinity of initial point (x_0, y_0) if $f(x,y)$ is continuous at (x_0, y_0). But we do not know yet if there is only one solution or infinite number of solutions. If $f(x,y)$ is not continuous at (x_0, y_0) then we say the theory does not apply and we do not do the next check. Solution could still exist and even be unique, but theory does not say anything about this.

If we found that $f(x,y)$ is continuous at (x_0, y_0) then now we check if $f_y(x,y)$ is also continuous at (x_0, y_0). If it is, then we say there is only one solution curve (i.e. a unique solution) that passes through the initial point (x_0, y_0) and in some region around it.

If $f_y(x,y)$ turns out not to be continuous at (x_0, y_0) then theory does not guarantee uniqueness. Solution could still be unique but theory does not say anything about this. We have to solve the ode to find out.

1.1.1.1 Example 1

$$\begin{aligned} y' &= 2\sqrt{y} \\ y(0) &= 0 \end{aligned}$$

First we find the region where solution exists and is unique. Domain of $f(x,y) = 2\sqrt{y}$ is $y \geq 0$ (since we do not want complex numbers). Since $y_0 = 0$ is inside this domain, then we know solution exists. The domain of $f_y = \frac{1}{\sqrt{y}}$ is $y > 0$. We see that the region is all x and $y > 0$. i.e. the top half of the plane not including x-axis.

Since the point given is $(0,0)$ then the theory do not apply. The point x_0, y_0 have to be inside the region and not on the edge.

There is no guarantee that solution will be unique. Solving this ode gives

$$\begin{aligned} 2\sqrt{y} &= 2x + c \\ \sqrt{y} &= x + c_1 \end{aligned}$$

At IC

$$0 = c_1$$

Hence solution is

$$\begin{aligned}\sqrt{y} &= x\\ y &= x^2\end{aligned}$$

But $y = 0$ is another solution. Notice that $y = 0$ can not be obtained from $\sqrt{y} = x + c_1$ for any choice of c_1. So it is a singular solution and not trivial solution. This shows that solution exists but is not unique. In this example, theory predicted that solution exists but did not say anything about uniqueness. Only by solving it, we found the solution is not unique.

1.1.1.2 Example 2

$$\begin{aligned}y' &= y^{\frac{1}{3}}\\ y(0) &= 0\end{aligned}$$

First we find the region where solution exists and is unique. $f(x,y) = y^{\frac{1}{3}}$. The domain of $y^{\frac{1}{3}}$ is $y \geq 0$ since we do not want complex values. Hence solution exists. The domain of $f_y = \frac{1}{3}\frac{1}{y^{\frac{2}{3}}}$ is $y > 0$. Hence the region is all x and $y > 0$. i.e. the top half of the plane not including x-axis. Since the point given is $(0,0)$ on the x-axis, then the theory do not apply. There is no guarantee solution is unique. Only way to find out is to try to solve the ode and find out. Solving the ode gives

$$\begin{aligned}\int \frac{dy}{y^{\frac{1}{3}}} &= \int dx\\ \frac{3}{2}y^{\frac{2}{3}} &= x + C\end{aligned}$$

Applying IC gives $C = 0$. Hence solution is

$$\frac{3}{2}y^{\frac{2}{3}} = x$$

Solving for y

$$y^2 = \left(\frac{2}{3}x\right)^3$$

Taking the square root of both sides gives

$$\begin{aligned}y &= \pm\sqrt{\left(\frac{2}{3}x\right)^3}\\ &= \pm\left(\frac{2}{3}x\right)^{\frac{3}{2}}\end{aligned}$$

So there are two solutions. There is also a trivial solution $y = 0$. We see that the solution exists but not unique.

1.1.1.3 Example 3

$$\begin{aligned} y' &= x\sqrt{y-3} \\ y(4) &= 3 \end{aligned}$$

First we find the region where solution exists and is unique. Domain of $f(x,y) = x\sqrt{y-3}$ is $y-3 \geq 0$ or $y \geq 3$ since we do not want complex numbers and all x values. This shows solution exists. Domain of $f_y = \frac{x}{2\sqrt{y-3}}$ is $y > 3$. Since point $(4,3)$ is not inside this domain (it can not be on the edge, it has to be fully inside), then theory do not apply. No guarantee that unique solution exist. Solving this gives

$$2\sqrt{y-3} = \frac{1}{2}x^2 + c$$

At initial conditions

$$0 = 8 + c$$

Hence $c = -8$ and the solution becomes

$$\begin{aligned} 2\sqrt{y-3} &= \frac{1}{2}x^2 - 8 \\ \sqrt{y-3} &= \frac{1}{4}x^2 - 4 \\ y - 3 &= \left(\frac{1}{4}x^2 - 4\right)^2 \\ y &= \left(\frac{1}{4}x^2 - 4\right)^2 + 3 \end{aligned}$$

Is this the only solution? Is this solution unique? No. By inspection we see that $y = 3$ is also a solution. Hence the solution exist but is not unique.

1.1.1.4 Example 4

$$\begin{aligned} y' &= \frac{-1}{1+x}y^2 + \frac{1}{x-1} \\ y(0) &= 0 \end{aligned}$$

$f(x,y) = \frac{-1}{1+x}y^2 + \frac{1}{x-1}$ is continuous in x everywhere except at $x = -1$ and $x = 1$. And $f_y = \frac{-2}{1+x}y$ is continuous except at $x = -1$. Since initial conditions at $x_0 = 0, y_0 = 0$

then there is a unique solution in some rectangle inside the rectangle $-1 < x < 1$ and for all y. Solving the ode gives

$$2\sqrt{y} = \int_0^x \frac{\sqrt{y \sin \tau}}{\sqrt{y}} + c_1$$

At $x = 0, y = 0$ the above gives

$$0 = c_1$$

Hence the solution is

$$2\sqrt{y} = \int_0^x \frac{\sqrt{y \sin \tau}}{\sqrt{y}}$$

1.1.1.5 Example 5

$$\begin{aligned} y' &= \sqrt{1 - y^2} \\ y(0) &= 1 \end{aligned}$$

$f(x, y) = \sqrt{1 - y^2}$ is continuous in x everywhere. For y we want $1 - y^2 \geq 0$ or $y^2 \leq 1$. The point $y_0 = 1$ satisfies this. Now $f_y = \frac{-2y}{2\sqrt{1-y^2}}$. We want $1 - y^2 > 1$ or $y^2 < 1$. The point y_0 does not satisfy this. Hence theory says nothing about uniqueness. Solution can be unique or not. When the ode has form $y' = f(y)$ we always check if IC satisfies the ode. In this case $y(x) = 1$ does satisfy the ode. So this means $y(x) = 1$ is solution. We do not need to solve by integration. But if we did, we will obtain the following

$$\begin{aligned} \frac{dy}{\sqrt{1 - y^2}} &= dx \\ \arcsin(y) &= x + c \\ y &= \sin(x + c) \end{aligned}$$

At initial conditions the above gives $1 = \sin c$. Hence $c = \frac{\pi}{2}$. Therefore solution is $y = \sin\left(x + \frac{\pi}{2}\right) = \cos x$. So this is another solution that satisfies the ode. Solution is not unique.

1.1.1.6 Example 6

$$\begin{aligned}y' &= \sqrt{1-y^2}+x\\ y(0) &= 1\end{aligned}$$

$f(x,y)=\sqrt{1-y^2}+x$ is continuous in x everywhere. For y we want $1-y^2\geq 0$ or $y^2\leq 1$. The point $y_0=1$ satisfies this. Now $f_y=\frac{-2y}{2\sqrt{1-y^2}}$. We want $1-y^2>1$ or $y^2<1$. The point y_0 does not satisfy this. Hence theory does not apply.

In this case the ode has form $y'=f(x,y)$ and not $y'=f(y)$. So we can not just check if initial conditions satisfies the ode and use that as solution. If we did, we see that $y(x)=1$ does satisfy the ode at $x=0$ but this will be wrong solution. In this case we have to go ahead and solve the ode. In this case we will find that no general solution exists.

1.1.1.7 Example 7

$$\begin{aligned}y' &= \sqrt{1-y^2}\\ y(0) &= 2\end{aligned}$$

$f(x,y)=\sqrt{1-y^2}$ is continuous in x everywhere. For y we want $1-y^2\geq 0$ or $y^2\leq 1$. The point $y_0=2$ does not satisfy. Hence theorem does not apply. We just need any solution that satisfies the ode. Since the ode has form $y'=f(y)$ and not $y'=f(x,y)$ then we always try $y(x)=y_0$ to see if it satisfies the ode. Substituting $y=2$ into the ode gives

$$\begin{aligned}0 &= \sqrt{1-y^2}\\ &= \sqrt{1-4}\end{aligned}$$

Therefore this solution did not work. In this case we have to solve the ode by integration which gives

$$\begin{aligned}\frac{dy}{\sqrt{1-y^2}} &= dx\\ \arcsin(y) &= x+c\\ y &= \sin(x+c)\end{aligned}$$

At initial conditions the above gives $2=\sin c$. Or $c=\arcsin(2)$. Hence the solution is

$$y(x)=\sin(x+\arcsin(2))$$

1.1.1.8 Example 8

$$
\begin{aligned}
y' &= \frac{1}{y} \\
y(1) &= 0
\end{aligned}
$$

By Existence and uniqueness, we see $f(x,y)$ is not defined at $y_0 = 0$. Hence theorem does not apply. Since ode has form $y' = f(y)$ we now check if IC satisfies the ode itself. Plugging in $y = 0$ into the ode is not satisfied due to $\frac{1}{0}$. So we have to solve the ode in this case. integrating gives

$$
\begin{aligned}
\int y dy &= \int dx \\
\frac{1}{2} y^2 &= x + c
\end{aligned}
$$

At IC this gives

$$
\begin{aligned}
0 &= 1 + c \\
c &= -1
\end{aligned}
$$

Hence solution is

$$
\begin{aligned}
\frac{1}{2} y^2 &= x - 1 \\
y(x) &= \pm \sqrt{2\,(x-1)}
\end{aligned}
$$

We see solution is not unique.

1.1.2 Existence and uniqueness for linear first order ode in y

These are ode's in the form

$$
y' + p(x)\, y = q(x)
$$

The theorem says that if both $p(x)\,, q(x)$ are continuous at x_0 then solution exists and is unique. Notice that now we do not check on y_0 but only on x_0. We get both existence and uniqueness all in one test. If either p or q are not continuous, then no guarantee solution exist or be unique.

1.1.2.1 Example 1

$$\begin{aligned} y' &= \frac{y}{x} \\ y(0) &= 1 \end{aligned}$$

In standard form $y' - p(x)\,y = q(x)$. So $p = \frac{-1}{x}, q = 0$. Hence the domain of p is all x except $x = 0$. Domain of q is all x. Since the IC includes $x = 0$ then no guarantee solution exists or be unique. Theory does not say anything. We have to try to solve the ode to find out. Solving gives

$$y = cx$$

As solution. Applying I.C. gives

$$1 = 0$$

Not possible. Therefore no solution exist.

1.1.2.2 Example 2

$$\begin{aligned} y' &= \frac{y}{x} \\ y(0) &= 0 \end{aligned}$$

In standard form $y' - p(x)\,y = q(x)$. So $p = \frac{-1}{x}, q = 0$. Domain of p is $x \neq 0$. Domain of q is all x. Since IC includes $x = 0$ then theory says nothing about existence and uniqueness. We have to solve the ode to find out. Solving gives

$$y = cx$$

Applying I.C. gives

$$0 = 0$$

Which is true for any c. Hence solution exist which is $y = cx$ for any c. Hence solution is not unique. There are ∞ number of solutions.

1.1.2.3 Example 3

$$\begin{aligned} y' &= \frac{y}{x} \\ y(1) &= 0 \end{aligned}$$

In standard form $y' - p(x)\,y = q(x)$. So $p = \frac{-1}{x}, q = 0$. The domain of p is all x except $x = 0$. Domain of q is all x. Since IC does not include $x = 0$ then solution is guaranteed to exist and be unique in some region near $x = 1$. Solving gives

$$y = cx$$

As solution. Applying I.C. gives

$$0 = c$$

Hence the unique solution is

$$y = 0 \qquad x > 0$$

Solution exists and is unique. Solution can only be in the right hand plan which includes $x = 1$ and it can not cross $x = 0$. i.e. solution is $y = 0$ for all $x > 0$. If IC was $y(-1) = 0$ then the solution would have been $y = 0$ for all $x < 0$.

1.1.2.4 Example 4

$$y' = \frac{1}{2\sqrt{x}}$$
$$y(0) = 1$$

In standard form $y' - p(x)\,y = q(x)$. Hence $p = \frac{-1}{2\sqrt{x}}, q = 0$. Domain of p is $x > 0$ (to avoid complex numbers) and the domain for q is all x. Combining these gives $x > 0$. Since IC includes $x = 0$ then the theory does not apply. Solving the ode gives

$$y = \sqrt{x} + c$$

At (x_0, y_0) the above gives

$$1 = c$$

Hence solution is

$$y = \sqrt{x} + 1 \qquad x > 0$$

So here solution exists and is unique. Even though theory did not apply.

1.2 First order linear in derivative

$$F(x, y, y') = 0$$

These are first order ode's which are linear in y' but can be nonlinear in y.

1.2.1 Flow charts

1.2.1.1 First flow chart

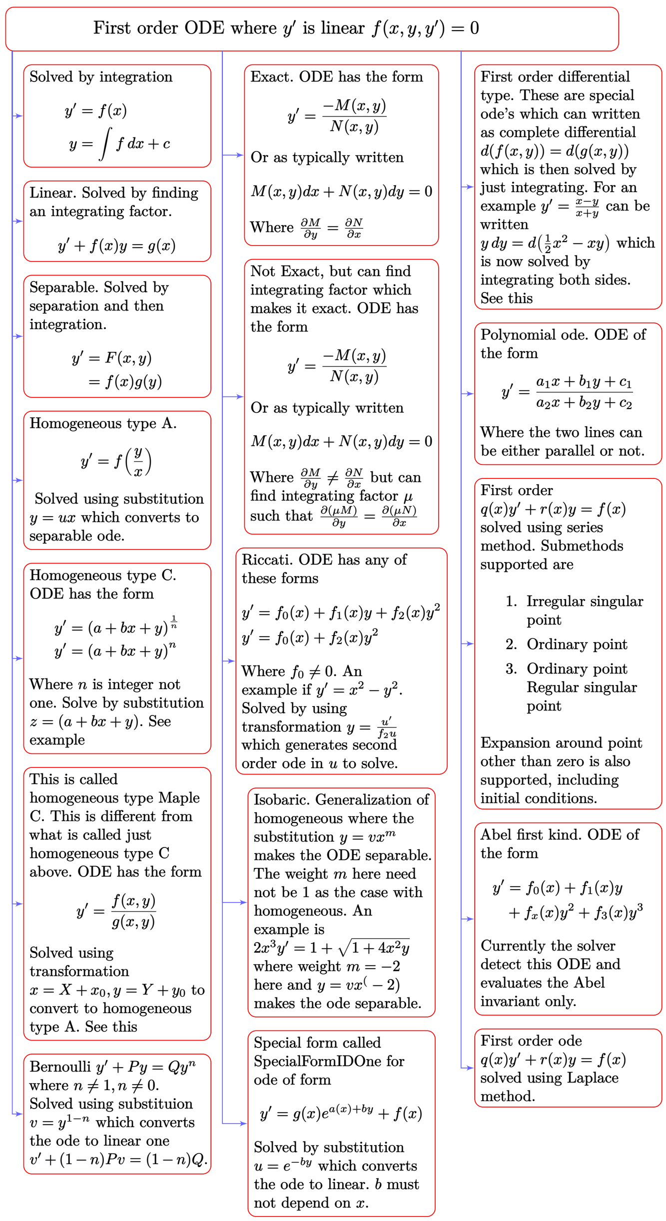

Figure 1.1: Flow chart for first order linear in y' solver

1.2.1.2 Second flow chart

This flow chart contains more details on the exact solver for first order ode.

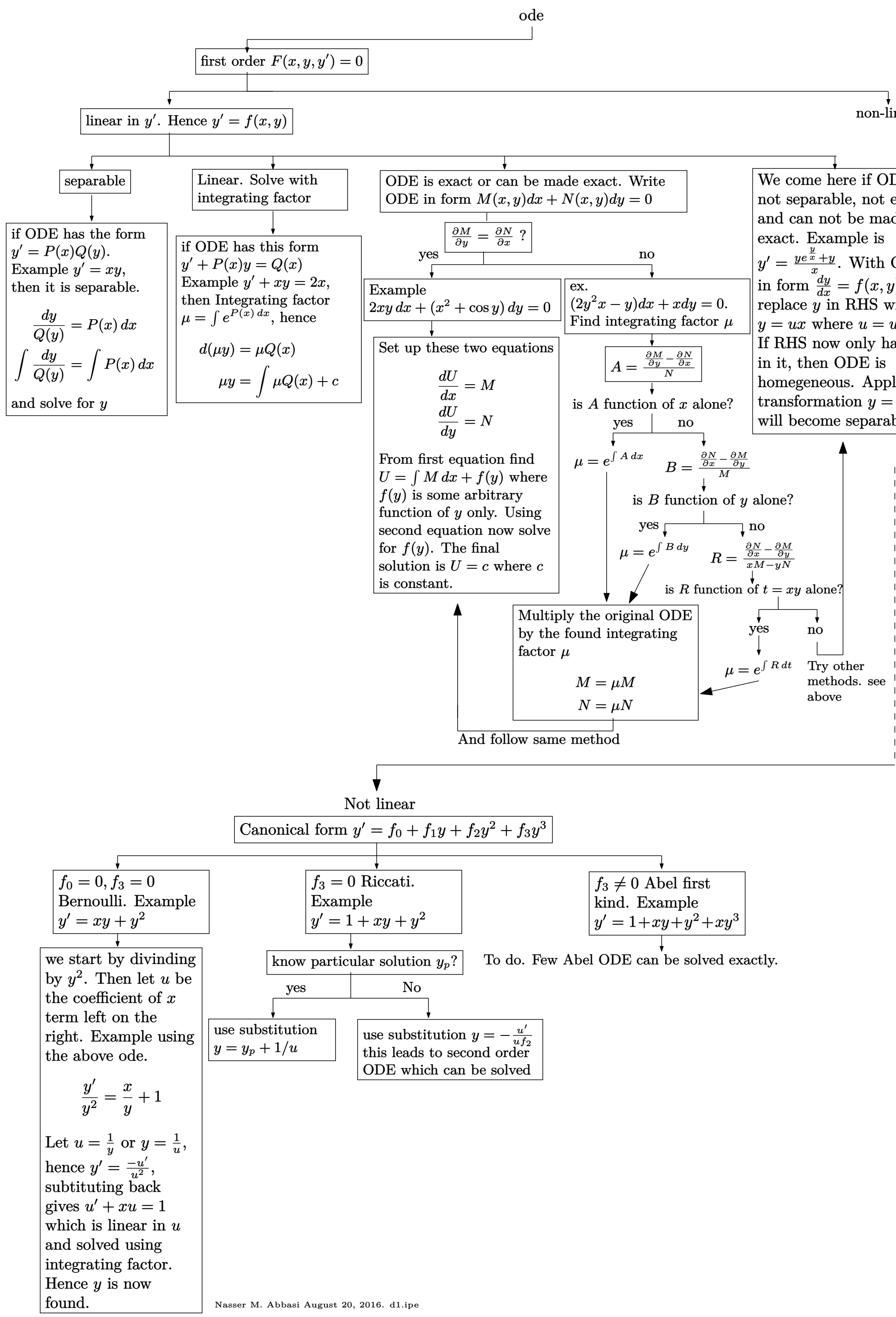

Figure 1.2: Additional flow chart for first order linear in y' and exact solver

1.2.2 Quadrature ode

$$
\begin{aligned}
y' &= f(x) \\
y' &= f(y)
\end{aligned}
$$

The following flow chart gives the algorithm for solving quadrature ode.

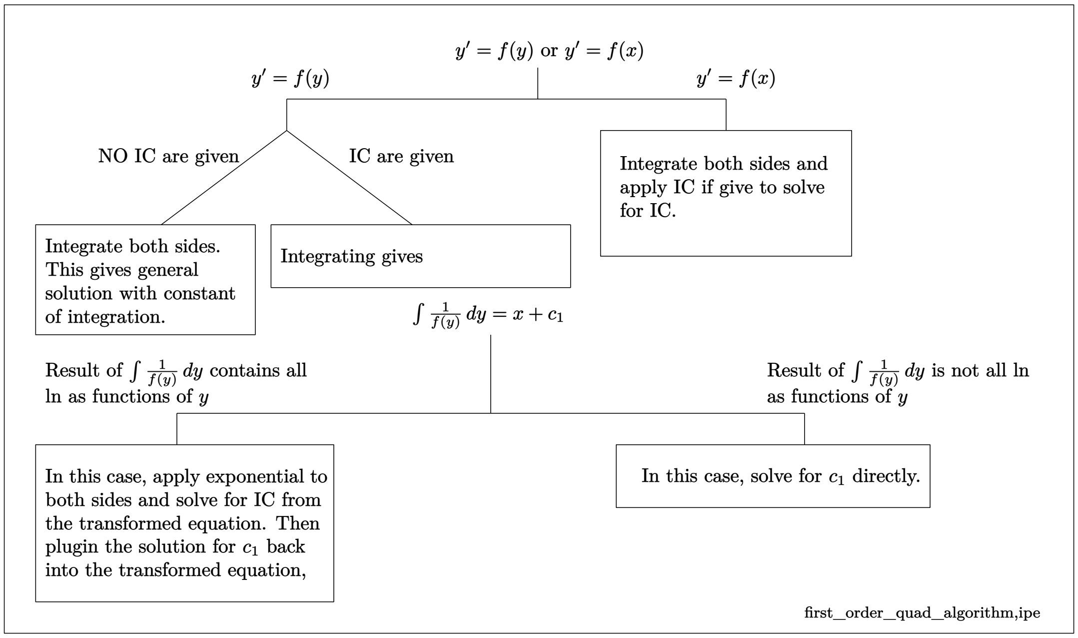

Figure 1.3: Flow chart for first order quadrature

ode internal name "quadrature"

Solved by direct integration. For the first form the solution is $y = \int f(x)\,dx + c$. And for the second form the solution is $\int \frac{dy}{f(y)} = \int dx$ or $\int \frac{dy}{f(y)} = x + c$. These two forms are special cases of separable first order ode $y' = f(x)\,g(y)$.

1.2.2.1 Example 1

$$
\begin{aligned}
y' &= y \\
y(0) &= 1
\end{aligned}
$$

Solution exists and unique. Integrating gives

$$
\begin{aligned}
\ln y &= x + c \\
y &= ce^x
\end{aligned}
$$

Applying IC gives

$$1 = c$$

Hence solution is

$$y = e^x$$

1.2.2.2 Example 2

$$\begin{aligned} y' &= y - 1 \\ y(0) &= 1 \end{aligned}$$

Solution exists and unique. Integrating gives

$$\begin{aligned} \ln(y - 1) &= x + c \\ y - 1 &= ce^x \end{aligned}$$

Applying IC gives

$$0 = c$$

Hence solution is

$$\begin{aligned} y - 1 &= 0 \\ y &= 1 \end{aligned}$$

1.2.2.3 Example 3

$$\begin{aligned} y' &= x \\ y(0) &= 1 \end{aligned}$$

Integrating gives

$$y = \frac{x^2}{2} + c$$

Applying IC gives

$$1 = c$$

Hence solution is

$$y(x) = \frac{x^2}{2} + 1$$

1.2.2.4 Example 4

$$
\begin{aligned}
y' &= \sin y + 1 \\
y(0) &= \pi
\end{aligned}
$$

This has unique solution. Integrating and solving for c results in the solution

$$
y = 2\arccos\left(\frac{-x}{\sqrt{4+4x+2x^2}}\right)
$$

1.2.2.5 Example 5

$$
\begin{aligned}
y' &= y(y-1)(y-3) \\
y(0) &= 4
\end{aligned}
$$

A solution exist an is unique. Integrating gives

$$
\begin{aligned}
\int \frac{dy}{y(y-1)(y-3)} &= \int dx \\
\frac{1}{3}\ln y + \frac{1}{6}\ln(y-3) - \frac{1}{2}\ln(y-1) &= x + c_1 \qquad (1)
\end{aligned}
$$

Applying initial conditions gives

$$
\begin{aligned}
\frac{1}{3}\ln 4 + \frac{1}{6}\ln(1) - \frac{1}{2}\ln(3) &= c_1 \\
\frac{1}{3}\ln 4 - \frac{1}{2}\ln(3) &= c_1
\end{aligned}
$$

Hence the solution from (1) is

$$
\frac{1}{3}\ln y + \frac{1}{6}\ln(y-3) - \frac{1}{2}\ln(y-1) = x + \frac{1}{3}\ln 4 - \frac{1}{2}\ln(3)
$$

Lets see what happens if we convert to exponential first. Applying exponential to both sides of (1) gives

$$
\begin{aligned}
\exp\left(\ln y^{\frac{1}{3}} + \ln(y-3)^{\frac{1}{6}} + \ln(y-1)^{\frac{-1}{2}}\right) &= c_2 e^x \\
y^{\frac{1}{3}}(y-3)^{\frac{1}{6}}\left(\frac{1}{\sqrt{y-1}}\right) &= c_2 e^x \\
\frac{y^{\frac{1}{3}}(y-3)^{\frac{1}{6}}}{\sqrt{y-1}} &= c_2 e^x \qquad (2)
\end{aligned}
$$

At IC

$$\frac{4^{\frac{1}{3}}(4-3)^{\frac{1}{6}}}{\sqrt{4-1}} = c_2$$
$$\frac{4^{\frac{1}{3}}}{\sqrt{3}} = c_2$$

Hence the solution from (2) is

$$\frac{y^{\frac{1}{3}}(y-3)^{\frac{1}{6}}}{\sqrt{y-1}} = \frac{4^{\frac{1}{3}}}{\sqrt{3}}e^{x}$$

And this is also correct. I prefer to convert to exponential when the solution has the form $f(y) = cg(x)$ where $f(y)$ is made up of all ln as functions of y. This makes finding constant of integration easier in all cases.

1.2.2.6 Example 6

$$y' = ay - by^2$$
$$y(0) = y_0$$

A solution exist an is unique. Integrating gives

$$\begin{aligned}
\int \frac{dy}{ay-by^2} &= \int dx \\
\frac{1}{a}\ln y - \frac{1}{a}\ln(by-a) &= x + c_1 \\
\ln y - \ln(by-a) &= ax + ac_1 \\
\frac{y}{by-a} &= e^{ax+ac_1} \\
\frac{y}{by-a} &= c_2e^{ax} \\
y &= c_2bye^{ax} - ac_2e^{ax} \\
y(1-c_2be^{ax}) &= -ac_2e^{ax} \\
y &= \frac{-ac_2e^{ax}}{1-c_2be^{ax}} \\
&= \frac{ac_2e^{ax}}{c_2be^{ax}-1} \\
&= \frac{ac_2}{c_2b-e^{-ax}} \\
&= \frac{a}{b-\frac{1}{c_2}e^{-ax}} \\
&= \frac{a}{b+c_3e^{-ax}}
\end{aligned}$$

Applying IC

$$\begin{aligned} y_0 &= \frac{a}{b+c_3} \\ (b+c_3)\,y_0 &= a \\ by_0 + c_3 y_0 &= a \\ c_3 &= \frac{a-by_0}{y_0} \end{aligned}$$

Hence the solution becomes

$$\begin{aligned} y &= \frac{a}{b+\left(\frac{a-by_0}{y_0}\right)e^{-ax}} \\ &= \frac{ay_0}{by_0+(a-by_0)\,e^{-ax}} \end{aligned}$$

1.2.3 Linear ode

$$y' + p(x)\,y = q(x)$$

ode internal name "linear"

Solved by finding integration factor $\mu = e^{\int p(x)dx}$. The ode then becomes $\frac{d}{dx}(\mu y) = \mu q$. Integrating gives $\mu y = \int \mu q dx + c$ or

$$y = \frac{1}{\mu}\left(\int \mu q dx + c\right)$$

1.2.3.1 Example 1

$$\begin{aligned} y' - \frac{1}{2\sqrt{x}}y &= x \\ y(0) &= 1 \end{aligned}$$

The above shows that $p(x) = \frac{1}{2\sqrt{x}}$ and $q(x) = x$. The domain of $p(x)$ is all the real line except $x = 0$ and domain of $q(x)$ is all the real line. Combining domains gives all the real line except $x = 0$. Since initial x_0 is $x = 0$ which is outside the domain, then uniqueness and existence theory do not apply. Solving gives

$$y = -2x^{\frac{3}{2}} - 12\sqrt{x} - 6x - 12 + c_1 e^{\sqrt{x}}$$

Applying IC

$$\begin{aligned} 1 &= -12 + c_1 \\ c_1 &= 13 \end{aligned}$$

Hence solution is

$$y = -2x^{\frac{3}{2}} - 12\sqrt{x} - 6x - 12 + 13e^{\sqrt{x}} \qquad x \neq 0$$

In this case, solution exists and unique.

1.2.3.2 Example 2

$$\begin{aligned} y' - \frac{y}{x} &= 0 \\ y(0) &= 1 \end{aligned}$$

The above shows that $p(x) = \frac{1}{1}$.The domain of $p(x)$ is all the real line except $x = 0$. Since initial x_0 is $x = 0$ which is outside the domain, then uniqueness and existence theory do not apply. We are not guaranteed solution exist or if it exist, is unique. Solving gives

$$y = c_1 x$$

Applying IC gives

$$1 = 0$$

Which is not possible. Hence no solution exist.

1.2.3.3 Example 3

$$y' + 2y\cot(2x) = 4x\csc(x)\sec^2(x)$$

Hence $p = 2\cot(2x)\,, q = 4x\csc(x)\sec(x)^2$. Therefore the integrating factor is

$$\begin{aligned} \mu &= e^{\int p(x)dx} \\ &= e^{\int 2\cot(2x)dx} \\ &= e^{-\frac{1}{2}\ln(1+\cot^2(2x))} \\ &= \frac{1}{\sqrt{1+\cot^2(2x)}} \end{aligned}$$

Then the ode becomes

$$\begin{aligned} \frac{d}{dx}(y\mu) &= \mu 4x\csc(x)\sec^2(x) \\ \frac{d}{dx}\left(y\frac{1}{\sqrt{1+\cot^2(2x)}}\right) &= \frac{1}{\sqrt{1+\cot^2(2x)}}4x\csc(x)\sec^2(x) \\ \frac{y}{\sqrt{1+\cot^2(2x)}} &= \int \frac{4x\csc(x)\sec^2(x)}{\sqrt{1+\cot^2(2x)}}dx + c_1 \\ y &= \sqrt{1+\cot^2(2x)}c_1 + \sqrt{1+\cot^2(2x)}\int \frac{4x\csc(x)\sec^2(x)}{\sqrt{1+\cot^2(2x)}}dx \end{aligned}$$

1.2.3.4 Example 4

$$
\begin{aligned}
y' + y\cot(x) &= \cos x \\
y(0) &= 0
\end{aligned}
$$

We see that $y' = \cos x - y\cot(x)$. Because $\cot(x)$ is $\frac{1}{\tan(x)}$ which is not defined at $x = 0$ then uniqueness and existence theory do not apply. Here we have $p = \cot(x), q = \cos(x)$. Therefore the integrating factor is

$$
\begin{aligned}
\mu &= e^{\int p(x)dx} \\
&= e^{\int \cot(x)dx} \\
&= e^{\ln(\sin x)} \\
&= \sin x
\end{aligned}
$$

Then the ode becomes

$$
\begin{aligned}
\frac{d}{dx}(y\mu) &= \mu\cos x \\
\frac{d}{dx}(y\sin x) &= \sin x\cos x \\
y\sin x &= \int \sin x\cos x\ dx + c_1 \\
y &= \frac{1}{\sin x}c_1 + \frac{1}{\sin x}\int \sin x\cos x\ dx \\
&= \frac{1}{\sin x}c_1 + \frac{1}{\sin x}\frac{\sin^2 x}{2} \\
&= \frac{1}{\sin x}c_1 + \frac{\sin x}{2} \\
y\sin x &= c_1 + \frac{1}{2}\sin x
\end{aligned}
$$

At $y(0) = 0$ the above results $c_1 = 0$. Hence the solution is

$$
y = \frac{\sin x}{2}
$$

Therefore no solution exists.

1.2.4 Separable ode

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)\,g(y)
\end{aligned}
$$

The following flow chart gives the algorithm for solving separable ode.

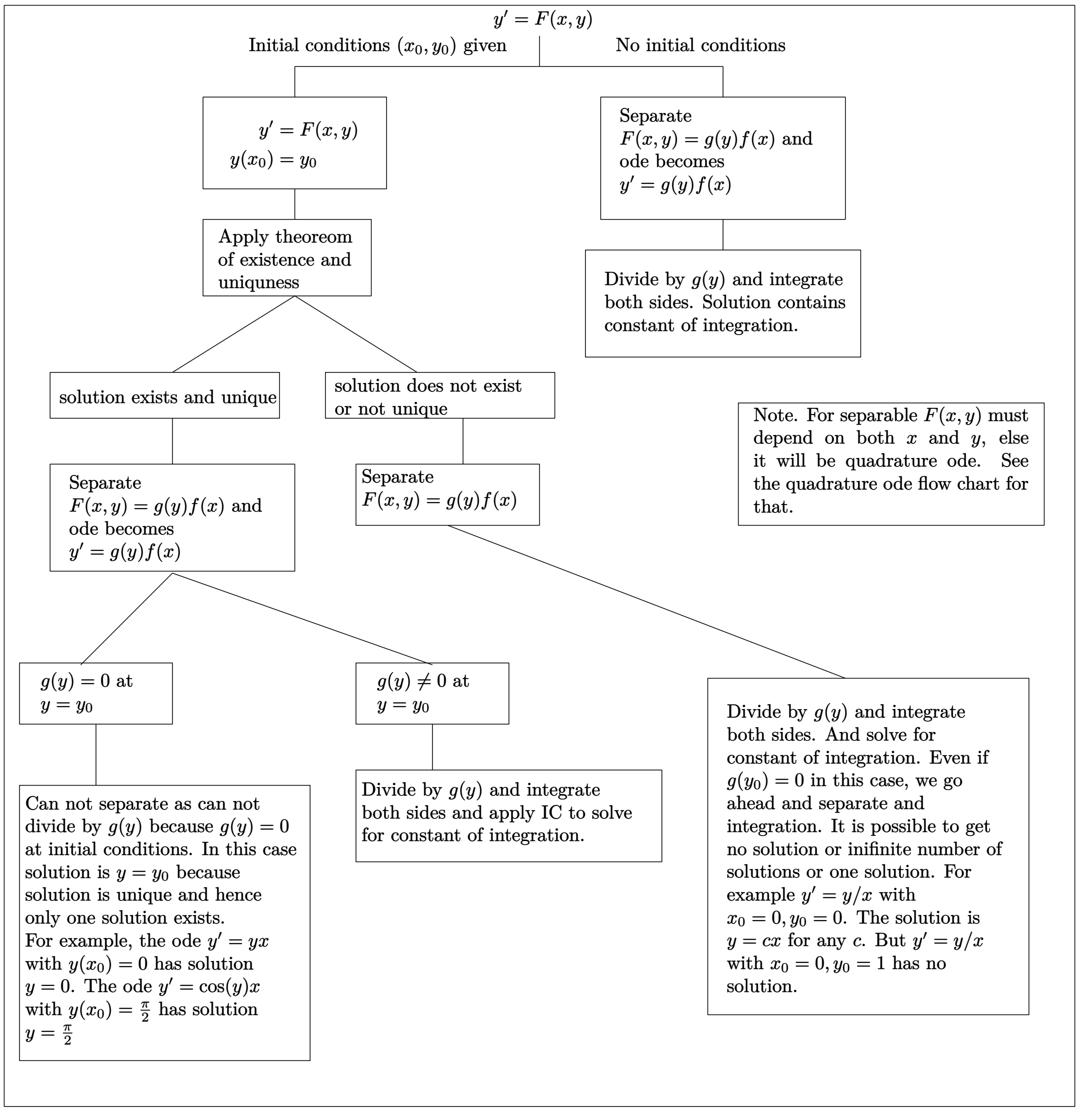

Figure 1.4: Flow chart for first order separable

ode internal name "separable"

Solved by separating and integrating. $\frac{dy}{dx}\frac{1}{g(y)} = f(x)$. Integrating gives $\int \frac{1}{g}dy = \int f dx$. If it is possible to do the integration of the LHS then explicit solution in y is obtained else the solution is implicit. The most difficult part is to determine that a given expression $F(x,y)$ is separable or not. i.e. given $y' = F(x,y)$ to find $f(x)$ and $g(y)$. Code in solver is over 600 lines long just to determine this due to many edge cases.

1.2.4.1 Example

Solve

$$y' = y^3 \sin x$$
$$y(0) = 0$$

From uniqueness and existence theory we see that solution to $y' = y^3 \sin x$ exist and is unique. This is because $f = y^3 \sin x$ is continuous everywhere (hence solution exist) and $f_y = 3y^2 \sin x$ is also continuous everywhere (hence uniqueness is guaranteed).

This is little more tricky than it looks. Notice that $y = 0$ at $x = 0$. This is special IC, because this means if we start by dividing both sides by y^3 to separate them as we normally do, this gives

$$\frac{dy}{y^3} = \sin x dx$$

But when we get to later on (after integration and adding constant of integration) to solve for c we will have problems. The reason is, we should not divide by y in first place, since $y = 0$ at initial conditions. In this special IC case, then at $x = 0$ the ode is

$$y' = 0$$

Hence $y = C_1$. But since the solution is guaranteed to be unique, then C_1 must be zero to give $y = 0$ as only one value of $y(x)$ can exist. Hence this is the solution. This way we do not even have to integrate or solve for constant of integration. If we were not given IC, then we do as normal and now can divide by y. Assuming $y \neq 0$ then the ode becomes

$$\frac{dy}{y^3} = \sin x dx$$

Integrating gives

$$-\frac{1}{2y^2} = -\cos x + c$$
$$\frac{1}{y^2} = 2\cos x - 2c$$
$$\frac{1}{y^2} = 2\cos x + c_1 \tag{1}$$

Hence

$$y^2 = \frac{1}{2\cos x + c_1}$$

Therefore

$$y = \pm \frac{1}{\sqrt{2\cos x + c_1}} \tag{2}$$

So we should always start, when IC are given, by checking uniqueness and existence and never divide by y if $y = 0$ at initial conditions. In all other cases, we can divide to separate. Lets do more examples on this to practice.

1.2.4.2 Example

Solve

$$\begin{aligned} y' &= y(x-1) \\ y(2) &= 0 \end{aligned}$$

$f = y(x-1)$ which is clearly continuous everywhere and so is f_y. Hence it is guaranteed that solution exist and unique. Since $y = 0$ at initial conditions, then we can't divide by y to separate. So we use the alternative method. At IC the ode itself becomes

$$y' = 0$$

Hence

$$y = c$$

Since y is constant, then $y = 0$ because it can only have one value due to uniqueness. Therefore the solution is

$$y = 0$$

Let now look at the general case to make things more clear.

1.2.4.3 Example

Solve

$$y' = f(y)\, g(x)$$

Such that $f(y)\, g(x)$ is continuos everywhere and $f_y g$ is also. Hence it is guaranteed that solution exist and unique. Let initial conditions be such that $f(y_0) = 0$. For example, if $f(y) = y$ and $y(0) = 0$. In this case, we can not separate using

$$\frac{dy}{f(y)} = g(x)$$

Since $f(y) = 0$ at I.C. So we use the short cut method. Substituting IC into the ode gives

$$
\begin{aligned}
y' &= 0 \\
y &= c
\end{aligned}
$$

But since the solution is unique, then $C_1 = 0$ since $y = 0$ is given and only one solution $y(x)$ can exist. Hence this is the solution.

$$
y = 0
$$

So the bottom line is this: Given a first order ode $y' = f(y)\,g(x)$ where the solution exist and unique and $f(y) = 0$ at IC, then the solution is always

$$
y = 0
$$

Lets look at another special case ode.

1.2.4.4 Example

Solve

$$
\begin{aligned}
y' &= \frac{y}{x} \\
y(0) &= 1
\end{aligned}
$$

We see that $f = \frac{y}{x}$ is not continuous at $x = 0$. Hence by uniqueness and existence theorem, there is no guarantee that solution exist. (Notice we do not say that no solution exist, as there might be one, but there is no guarantee that one exists using the theorem).

1.2.5 Homogeneous ode

$$
y' = F\left(\frac{y}{x}\right)
$$

ode internal name "homogeneous"

This is called Homogeneous type A in Maple. Solved by substituting $y = ux$ which converts it to separable ode. A homogeneous ode has the form $y' = f(x, y)$ where $tf(x, y) = f(tx, ty)$. In solving these types of problems, separable is called. It is best to return implicit solution from separable and not explicit. This makes the substitution $u = \frac{y}{x}$ easier. If explicit solution is needed, it can be done after this operation is done.

1.2.5.1 Example 1

$$
\begin{aligned}
xy' - y - 2\sqrt{yx} &= 0 \\
y' &= \frac{y}{x} + \frac{2}{x}\sqrt{yx}
\end{aligned}
$$

For real x

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x} + 2\sqrt{\frac{yx}{x^2}} \\
&= \frac{y}{x} + 2\sqrt{\frac{y}{x}}
\end{aligned}
$$

Let $u = \frac{y}{x}$, hence $\frac{dy}{dx} = x\frac{du}{dx} + u$ and the above ode becomes

$$
\begin{aligned}
x\frac{du}{dx} + u &= u + 2\sqrt{u} \\
x\frac{du}{dx} &= 2\sqrt{u} \\
\frac{du}{u^{\frac{1}{2}}} &= \frac{2}{x}dx
\end{aligned}
$$

Which is separable. If we do not obtain separable ode, then we have made mistake. Integrating gives

$$
\begin{aligned}
\int u^{\frac{-1}{2}} du &= \int \frac{2}{x} dx \\
2u^{\frac{1}{2}} &= 2\ln x + c_1 \\
u^{\frac{1}{2}} &= \ln x + c_2
\end{aligned}
$$

Replacing $u = \frac{y}{x}$ gives

$$
\sqrt{\frac{y}{x}} = \ln x + c_2
$$

1.2.5.2 Example 2

$$
\frac{dy}{dx} = \frac{2y^2 - xy}{3xy - 2x^2}
$$

Let $y = ux$ or $u = \frac{y}{x}$, hence $\frac{dy}{dx} = x\frac{du}{dx} + u$ and the above ode becomes

$$\begin{aligned}
x\frac{du}{dx} + u &= \frac{2u^2x^2 - x^2u}{3x^2u - 2x^2} \\
x\frac{du}{dx} + u &= \frac{2u^2 - u}{3u - 2} \\
x\frac{du}{dx} &= \frac{2u^2 - u}{3u - 2} - u \\
&= \frac{2u^2 - u}{3u - 2} - \frac{u(3u-2)}{3u - 2} \\
&= \frac{(2u^2 - u) - u(3u-2)}{3u - 2} \\
&= \frac{2u^2 - u - 3u^2 + 2u}{3u - 2} \\
&= \frac{-u^2 + u}{3u - 2} \\
&= \frac{u(1-u)}{3u - 2}
\end{aligned}$$

Hence

$$\frac{du}{dx} = \left(\frac{1}{x}\right)\left(\frac{u(1-u)}{3u-2}\right)$$

Which is separable. If we do not obtain separable ode, then we have made mistake. Integrating gives

$$\begin{aligned}
\int \frac{3u-2}{u(1-u)} du &= \int \frac{1}{x} dx \\
-2\ln u - \ln(u-1) &= \ln x + c_1
\end{aligned}$$

Replacing $u = \frac{y}{x}$ gives

$$\begin{aligned}
-2\ln\left(\frac{y}{x}\right) - \ln\left(\frac{y}{x} - 1\right) &= \ln x + c_1 \\
\ln\left(\frac{x^2}{y^2}\right) - \ln\left(\frac{y-x}{x}\right) &= \ln x + c_1 \\
\ln\left(\frac{x^2}{y^2}\right) + \ln\left(\frac{x}{y-x}\right) &= \ln x + c_1
\end{aligned}$$

Applying exponential to each side gives

$$\left(\frac{x^2}{y^2}\right)\left(\frac{x}{y-x}\right) = c_2 x \tag{1}$$

Lets say that we had also initial conditions $y(1) = -1$, then the above gives

$$\begin{aligned}\left(\frac{1}{-1-1}\right) &= c_2\\ -\frac{1}{2} &= c_2\end{aligned}$$

Therefore the solution (1) becomes

$$\left(\frac{x^2}{y^2}\right)\left(\frac{x}{y-x}\right) = -\frac{1}{2}x$$

1.2.5.3 Example 3

$$\begin{aligned}\frac{dy}{dx} &= \frac{2(2y-x)}{x+y}\\ y(0) &= 2\end{aligned}$$

Let $y = ux$ or $u = \frac{y}{x}$, hence $\frac{dy}{dx} = x\frac{du}{dx} + u$ and the above ode becomes

$$\begin{aligned}x\frac{du}{dx} + u &= \frac{2(2ux-x)}{x+ux}\\ x\frac{du}{dx} + u &= \frac{2(2u-1)}{1+u}\\ x\frac{du}{dx} &= \frac{2(2u-1)}{1+u} - u\\ &= \frac{2(2u-1)-u(1+u)}{1+u}\\ &= \frac{-u^2+3u-2}{1+u}\end{aligned}$$

This is separable

$$\frac{1+u}{-u^2+3u-2}du = \frac{1}{x}dx$$

Integrating

$$\begin{aligned}\int \frac{1+u}{-u^2+3u-2}du &= \int \frac{1}{x}dx\\ -3\ln(u-2) + 2\ln(u-1) &= \ln x + c\end{aligned}$$

Replacing $u = \frac{y}{x}$ gives

$$
\begin{aligned}
-3\ln\left(\frac{y}{x}-2\right)+2\ln\left(\frac{y}{x}-1\right) &= \ln x + c\\
-3\ln\left(\frac{y-2x}{x}\right)+2\ln\left(\frac{y-x}{x}\right) &= \ln x + c\\
\ln\left(\frac{x}{y-2x}\right)^3+\ln\left(\frac{y-x}{x}\right)^2 &= \ln x + c \qquad (1)
\end{aligned}
$$

Note on the power rule for log. $n\ln(m) = \ln(m^n)$ is valid for $m > 0$ and in real domain. So in this above we implicitly assumed this is true in order to write $-3\ln\left(\frac{y-2x}{x}\right)$ as $\ln\left(\frac{x}{y-2x}\right)^3$. Now, taking exponential of (1) gives

$$
\begin{aligned}
\left(\frac{x}{y-2x}\right)^3\left(\frac{y-x}{x}\right)^2 &= c_1 x\\
\frac{x^3}{(y-2x)^3}\frac{(y-x)^2}{x^2} &= c_1 x\\
\frac{x(y-x)^2}{(y-2x)^3} &= c_1 x\\
\frac{(y-x)^2}{(y-2x)^3} &= c_1 \qquad (2)
\end{aligned}
$$

At $y(0) = 2$ then

$$
\begin{aligned}
\frac{(2)^2}{(2)^3} &= c_1\\
\frac{1}{2} &= c_1
\end{aligned}
$$

Hence the solution from (2) becomes

$$
\frac{(y-x)^2}{(y-2x)^3} = \frac{1}{2}
$$

It is important in these kind of problems where left side has ln as function of $y(x)$ is to take exponential. Lets see what happens of we do not. Starting again from (1) and let us try to solve for IC from (1) as is

$$
\ln\left(\frac{x}{y-2x}\right)^3+\ln\left(\frac{y-x}{x}\right)^2 = \ln x + c
$$

At $y(0) = 2$ the above becomes

$$
\ln(0)^3+\ln\left(\frac{2}{0}\right)^2 = \ln 0 + c
$$

We see this will not work. These types of issues are easy to work around when solving by hand and looking at equations. But very hard to program since the code has to handle any form of expression.

1.2.5.4 Example 4

$$\begin{aligned}\frac{dy}{dx} &= 1+\frac{y}{2x}\\ y(0) &= 0\end{aligned}$$

The RHS is not defined at $x=0$, therefore existence and uniqueness theorem does not apply. Lets solve this as linear ode and not as homogeneous first to show that we obtain same solution. It is much easier to solve this as linear ode.

$$\frac{dy}{dx}-\frac{y}{2x}=1$$

Integrating factor is $I=e^{\int -\frac{1}{2x}dx}=e^{-\frac{1}{2}\ln x}=x^{-\frac{1}{2}}=\frac{1}{\sqrt{x}}$. Hence the above becomes

$$\frac{d}{dx}(yI)=I$$

Integrating

$$\begin{aligned}\frac{y}{\sqrt{x}} &= \int \frac{1}{\sqrt{x}}dx\\ &= 2\sqrt{x}+c\\ y &= 2x+c\sqrt{x}\end{aligned}$$

At $y(0)=0$

$$0=0+(0)\,c$$

Which is true for any c. Therefore there are infinite number of solutions. The solution is

$$y=2x+c\sqrt{x}$$

Now we solve as homogeneous ode. Let $y=ux$ or $u=\frac{y}{x}$, hence $\frac{dy}{dx}=x\frac{du}{dx}+u$ and the above ode becomes

$$\begin{aligned}x\frac{du}{dx}+u &= 1+\frac{ux}{2x}\\ x\frac{du}{dx}+u &= 1+\frac{u}{2}\\ x\frac{du}{dx} &= 1+\frac{u}{2}-u\\ x\frac{du}{dx} &= \frac{2-u}{2}\end{aligned}$$

This is separable

$$\frac{2}{2-u}du = \frac{1}{x}dx$$

Integrating

$$\begin{aligned}\int \frac{2}{2-u}du &= \int \frac{1}{x}dx\\ -2\ln(u-2) &= \ln x + c\\ &= \ln(c_1 x)\end{aligned}$$

Replacing $u = \frac{y}{x}$ gives

$$\begin{aligned}-2\ln\left(\frac{y}{x}-2\right) &= \ln(c_1 x)\\ -2\ln\left(\frac{y}{x}-2\right) - \ln(c_1 x) &= 0\\ \ln\left(\frac{x}{(y-2x)^2 c_1}\right) &= 0\end{aligned}$$

Taking exponential

$$\begin{aligned}\frac{x}{c_1 (y-2x)^2} &= 1\\ x &= c_1(y-2x)^2\end{aligned}$$

Apply IC $y(0) = 0$

$$0 = c_1(0)$$

Which is true for any c_1. Hence solution is

$$\begin{aligned}\frac{1}{c_1}\sqrt{x} &= y - 2x\\ y &= 2x + \frac{1}{c_1}\sqrt{x}\end{aligned}$$

Or

$$y = 2x + c_2\sqrt{x}$$

Which is same as earlier solution.

1.2.5.5 Example 5

$$\frac{dy}{dx} = \frac{y^2 - x^2 - 2xy}{y^2 - x^2 + 2xy}$$
$$y(1) = -1$$

At $x = 1, y = -1$ then $f(x,y) = \frac{y^2-x^2-2xy}{y^2-x^2+2xy}$ is defined. And f_y is also defined at $x = 1, y = -1$. Hence a unique solution exist.

Let $y = ux$ or $u = \frac{y}{x}$, hence $\frac{dy}{dx} = x\frac{du}{dx} + u$ and the above ode becomes

$$\begin{aligned}
x\frac{du}{dx} + u &= \frac{u^2x^2 - x^2 - 2ux^2}{u^2x^2 - x^2 + 2ux^2} \\
x\frac{du}{dx} + u &= \frac{u^2 - 1 - 2u}{u^2 - 1 + 2u} \\
x\frac{du}{dx} &= \frac{u^2 - 1 - 2u}{u^2 - 1 + 2u} - u \\
&= \frac{u^2 - 1 - 2u - u(u^2 - 1 + 2u)}{u^2 - 1 + 2u} \\
&= -\frac{u^3 + u^2 + u + 1}{u^2 - 1 + 2u}
\end{aligned}$$

This is separable.

$$\frac{du}{dx}\left(\frac{u^2 + 2u - 1}{u^3 + u^2 + u + 1}\right) = \frac{-1}{x}$$

Integrating gives

$$\begin{aligned}
\int \frac{u^2 + 2u - 1}{u^3 + u^2 + u + 1} du &= -\int \frac{1}{x} dx \\
-\ln(1 + u) + \ln\left(1 + u^2\right) &= -\ln x + c_1
\end{aligned}$$

Replacing $u = \frac{y}{x}$ gives

$$-\ln\left(1 + \frac{y}{x}\right) + \ln\left(1 + \frac{y^2}{x^2}\right) = -\ln x + c$$

Applying exponential to each side gives

$$
\begin{aligned}
\left(1+\frac{y}{x}\right)^{-1}\left(1+\frac{y^2}{x^2}\right) &= c_1\frac{1}{x}\\
\left(\frac{x}{x+y}\right)\left(\frac{x^2+y^2}{x^2}\right) &= c_1\frac{1}{x}\\
\left(\frac{x^2}{x+y}\right)\left(\frac{x^2+y^2}{x^2}\right) &= c_1\\
x^2+y^2 &= c_1(x+y)\\
\frac{1}{c_1}\left(x^2+y^2\right) &= x+y\\
c_2\left(x^2+y^2\right) &= x+y \qquad (1)
\end{aligned}
$$

At $y(1)=-1$ the above gives

$$2c_2 = 0$$

Hence

$$c_2 = 0$$

Therefore from (1)

$$
\begin{aligned}
x+y &= 0\\
y &= -x
\end{aligned}
$$

1.2.6 Homogeneous type C $y'=(a+bx+cy)^{\frac{n}{m}}$

<u>ode internal name</u> "homogeneousTypeC"

Ode has the form $y'=(a+bx+cy)^{\frac{n}{m}}$ where n,m integers. Solved by substituting $z=(a+bx+cy)$.

1.2.6.1 Introduction

This note is about solving a first order ode of the form $y'=(a+bx+cy)^{\frac{1}{n}}$ and $y'=(a+bx+cy)^{m}$ where $n,m\neq 1$ and are integers. This is of the form $y'=f(x,y)^{\frac{1}{n}}$ and $y'=f(x,y)^m$. Where $f(x,y)$ <u>must be linear</u> in both y and x. The reason it needs to be linear in x so that the transformed ode in z becomes separable.

One way to solve $y'=(a+bx+cy)^{\frac{1}{n}}$ is to raise both sides to n. For example for $n=2$ the ode becomes $(y')^2=(a+bx+cy)$ which can be solved as d'Alembert.

This is what Maple seems to do based on what the Maple advisor says about the type of this ode being d'Alembert.

But the problem with squaring both sides or raising both sides of ode to some power is that this will introduce extraneous solutions to the original ode. Hence it is will be better to avoid doing this if at all possible.

The following methods solve these odes without having to square or raise both sides to same power and eliminate the introduction of extraneous solutions.

It is important to note that $f(x,y)$ must be linear in x,y and not have product terms xy.

1.2.6.2 Solving $y' = (a+bx+cy)^{\frac{1}{n}}$

For n integer $\neq 1$ which can be negative or positive, the ode is

$$\frac{dy}{dx} = (a+bx+cy)^{\frac{1}{n}} \tag{1}$$

Let $z = a+bx+cy$ then

$$\begin{aligned}\frac{dz}{dx} &= b + c\frac{dy}{dx}\\ \frac{dy}{dx} &= \left(\frac{dz}{dx} - b\right)\frac{1}{c}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\left(\frac{dz}{dx} - b\right)\frac{1}{c} &= z^{\frac{1}{n}}\\ \frac{dz}{dx} &= cz^{\frac{1}{n}} + b\\ \int \frac{dz}{cz^{\frac{1}{n}} + b} &= \int dx \end{aligned} \tag{2}$$

If the left side is integrable, then the solution to (1) can be found. For n integer it is possible to find antiderivative. For example for $n=2$ then (2) becomes

$$\frac{2}{c}\sqrt{z} - \frac{2b\ln\left(b + c\sqrt{z}\right)}{c^2} = x + C_1$$

Replacing back $z = a+bx+cy$ the above becomes

$$\frac{2}{c}\sqrt{a+bx+cy} - \frac{2b\ln\left(b + c\sqrt{a+bx+cy}\right)}{c^2} = x + C_1 \tag{3}$$

Which is the implicit solution to (1).

To show that the above does not work if we had xy term, lets give an example. Let $y' = (a+xy)^{\frac{1}{2}}$, then following the above, let $z = a + xy$ and $\frac{dz}{dx} = y + xy'$ or $y' = \frac{\frac{dz}{dx} - y}{x}$. Hence $z^{\frac{1}{2}} = \frac{\frac{dz}{dx} - y}{x}$ or $xz^{\frac{1}{2}} + y = \frac{dz}{dx}$ and this is not separable. (it is Chini ode, where is very hard to solve).

for $n = 2$. Using $a = 1, b = 1, c = 1$ Eq. (1) becomes

$$\frac{dy}{dx} = (1 + x + y)^{\frac{1}{2}}$$

And (3) becomes

$$2\sqrt{1+x+y} - 2\ln\left(1 + \sqrt{1+x+y}\right) = x + C_1 \tag{4}$$

And for $n = 3$ Eq. (2) becomes

$$\frac{3\left(-2b + cz^{\frac{1}{3}}\right)}{2c^2} z^{\frac{1}{3}} + \frac{3b^2 \ln\left(b + cz^{\frac{1}{3}}\right)}{c^3} = x + C_1$$

Replacing back $z = a + bx + cy$ the above becomes

$$\frac{3\left(-2b + c(a+bx+cy)^{\frac{1}{3}}\right)}{2c^2} z^{\frac{1}{3}} + \frac{3b^2 \ln\left(b + c(a+bx+cy)^{\frac{1}{3}}\right)}{c^3} = x + C_1 \tag{5}$$

Which is the implicit solution to (1) for $n = 3$. Using $a = 1, b = 1, c = 1$ then (1) becomes

$$\frac{dy}{dx} = (1 + x + y)^{\frac{1}{3}}$$

And its solution (5) becomes

$$\frac{3}{2}\left(-2 + (1+x+y)^{\frac{1}{3}}\right)(1+x+y)^{\frac{1}{3}} + 3\ln\left(1 + (1+x+y)^{\frac{1}{3}}\right) = x + C_1$$

And so on for higher values of n. This also works negative values of n. For example, for $n = -2$ then (1) becomes

$$\frac{dy}{dx} = (a + bx + cy)^{\frac{-1}{2}}$$

And the integral equation (2) now becomes

$$\int \frac{dz}{cz^{\frac{-1}{n}} + b} = \int dx$$

Which for $n = 2$ gives

$$\frac{1}{b^3}\left(-2bc\sqrt{z} + b^2 z + 2c^2 \ln\left(c + b\sqrt{z}\right)\right) = x + C_1$$

Replacing back $z = a + bx + cy$ the above becomes

$$\frac{1}{b^3}\left(-2bc\sqrt{a+bx+cy} + b^2(a+bx+cy) + 2c^2\ln\left(c + b\sqrt{a+bx+cy}\right)\right) = x + C_1$$

For $a = 1, b = 1, c = 1$ the above becomes

$$\left(-2\sqrt{1+x+y} + (1+x+y) + 2\ln\left(1 + \sqrt{1+x+y}\right)\right) = x + C_1$$

And so on.

1.2.6.3 Solving $y' = (a + bx + cy)^m$

For m integer $\neq 1$ which can be negative or positive, the ode is

$$\frac{dy}{dx} = (a + bx + cy)^m \tag{1}$$

Let $z = a + bx + cy$ then

$$\begin{aligned}\frac{dz}{dx} &= b + c\frac{dy}{dx}\\ \frac{dy}{dx} &= \left(\frac{dz}{dx} - b\right)\frac{1}{c}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\left(\frac{dz}{dx} - b\right)\frac{1}{c} &= z^m\\ \frac{dz}{dx} &= cz^m + b\\ \int \frac{dz}{cz^m + b} &= \int dx \qquad (2)\end{aligned}$$

If the left side is integrable, then the solution to (1) can be found. For m integer it is possible to find antiderivative. For example for $n = 2$ then (2) becomes

$$\frac{1}{\sqrt{bc}}\arctan\left(\sqrt{\frac{c}{b}}z\right) = x + C_1$$

Replacing back $z = a + bx + cy$ the above becomes

$$\frac{1}{\sqrt{bc}}\arctan\left(\sqrt{\frac{c}{b}}(a + bx + cy)\right) = x + C_1 \tag{3}$$

Which is the implicit solution to (1).

for $m=2$. For an example, for $a=1,b=1,c=1$ Eq. (1) becomes

$$\frac{dy}{dx}=(1+x+y)^2$$

And (3) becomes

$$\begin{aligned}\arctan(1+x+y)&=x+C_1\\ 1+x+y&=\tan(x+C_1)\\ y&=\tan(x+C_1)-1-x \qquad (4)\end{aligned}$$

And for $m=3$ Eq. (2) becomes

$$\frac{-1}{6b^{\frac{2}{3}}c^{\frac{1}{3}}}\left(2\sqrt{3}\arctan\left(\frac{1-2\left(\frac{c}{b}\right)^{\frac{1}{3}}z}{\sqrt{3}}\right)-2\ln\left(b^{\frac{1}{3}}+c^{\frac{1}{3}}z\right)+\ln\left(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}z^2\right)\right)=x+C_1$$

Replacing back $z=a+bx+cy$ the above becomes

$$\frac{-1}{6b^{\frac{2}{3}}c^{\frac{1}{3}}}\left(2\sqrt{3}\arctan\left(\frac{1-2\left(\frac{c}{b}\right)^{\frac{1}{3}}(a+bx+cy)}{\sqrt{3}}\right)-2\ln\left(b^{\frac{1}{3}}+c^{\frac{1}{3}}(a+bx+cy)\right)+\ln\left(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}(a+\right.\right. \qquad (5)$$

Which is the implicit solution to (1) for $m=3$. Using $a=1,b=1,c=1$ then (1) becomes

$$\frac{dy}{dx}=(1+x+y)^3$$

And its solution (5) now simplifies to

$$\frac{-1}{6}\left(2\sqrt{3}\arctan\left(\frac{1-2(1+x+y)}{\sqrt{3}}\right)-2\ln(2+x+y)+\ln\left((1+x+y)^2\right)\right)=x+C_1$$

And so on for higher values of m, but solution get complicated very quickly. This method also works for negative m.

For example, for $m=-2$ then (1) becomes

$$\frac{dy}{dx}=(a+bx+cy)^{-2}$$

And the integral equation (2) now becomes

$$\int\frac{dz}{cz^{-2}+b}=\int dx$$

Which gives

$$\frac{z}{b}-\frac{\sqrt{c}\arctan\left(\sqrt{\frac{b}{c}}z\right)}{b^{\frac{3}{2}}}=x+C_1$$

Replacing back $z = a + bx + cy$ the above becomes

$$\frac{a+bx+cy}{b} - \frac{\sqrt{c}\arctan\left(\sqrt{\frac{b}{c}}(a+bx+cy)\right)}{b^{\frac{3}{2}}} = x + C_1$$

For $a = 1, b = 1, c = 1$ the above becomes

$$\begin{aligned}(1+x+y) - \arctan(1+x+y) &= x + C_1\\ \arctan(1+x+y) &= (1+x+y) - x - C_1\\ \arctan(1+x+y) &= 1 + y - C_1\\ \arctan(1+x+y) &= y + C_2\end{aligned}$$

And and so on for $= -3, -4, \cdots$ as all of these are integrable but become complicated very quickly and the computer is needed to find the antiderivatives in these cases.

1.2.6.4 Examples

1.2.6.4.1 Example 1 $y' = (1+5x+y)^{\frac{1}{2}}$

Let $z = 1 + 5x + y$, then $\frac{dz}{dx} = 5 + y'$. This simplifies to

$$\begin{aligned}y' &= z' - 5\\ \left(1+x^2+y\right)^{\frac{1}{2}} &= z' - 5\\ z^{\frac{1}{2}} &= z' - 5\\ \frac{dz}{dx} &= z^{\frac{1}{2}} + 5\end{aligned}$$

Which is separable. Hence

$$\begin{aligned}\frac{dz}{z^{\frac{1}{2}}+5} &= dx\\ 2\sqrt{z} - 5\ln\left(5+\sqrt{z}\right) + 5\ln\left(\sqrt{z}-5\right) - 5\ln(z-25) &= x + C_1\end{aligned}$$

Hence the implicit solution is

$$\begin{aligned}2\sqrt{1+5x+y} - 5\ln\left(5+\sqrt{1+5x+y}\right) + 5\ln\left(\sqrt{1+5x+y}-5\right) - 5\ln(1+5x+y-25) &= x + C_1\\ 2\sqrt{1+5x+y} - 5\ln\left(5+\sqrt{1+5x+y}\right) + 5\ln\left(\sqrt{1+5x+y}-5\right) - 5\ln(5x+y-24) &= x + C_1\end{aligned} \tag{1}$$

The above method is now compared to using d'Alembert for solving the ode, which results after squaring both sides of the given ode. Squaring the ode gives

$$
\begin{aligned}
(y')^2 &= (1+5x+y) \\
y &= (y')^2 - 1 - 5x \\
&= x(-5) + (p^2 - 1) \\
&= xf(p) + g(p) \qquad (2)
\end{aligned}
$$

Where $p = \frac{dy}{dx}$. This is d'Alembert of the form $y = xf(p) + g(p)$ where $f(p) = 5$ and $g(p) = p^2 - 1$. Taking derivative of (2) w.r.t. x gives

$$
\begin{aligned}
p &= f(p) + x\frac{df}{dp}\frac{dp}{dx} + \frac{dg}{dp}\frac{dp}{dx} \\
p - f(p) &= \left(x\frac{df}{dp} + \frac{dg}{dp}\right)\frac{dp}{dx} \qquad (3)
\end{aligned}
$$

Using $f(p) = 5$ and $g(p) = p^2 - 1$ the above becomes

$$
\begin{aligned}
p - 5 &= 2p\frac{dp}{dx} \\
\frac{dp}{dx} &= \frac{p-5}{2p}
\end{aligned}
$$

Which is separable. Solving for p gives

$$
p = 5\,\mathrm{LambertW}\left(\frac{C}{5}e^{\frac{x}{10}-1}\right) + 5
$$

Substituting this back into (2) gives

$$
y = -5x + \left(\left(5\,\mathrm{LambertW}\left(\frac{C}{5}e^{\frac{x}{10}-1}\right) + 5\right)^2 - 1\right) \qquad (4)
$$

This is an explicit general solution for the ode $y' = (1+5x+y)^{\frac{1}{2}}$. The singular solution is found when $\frac{dp}{dx} = 0$ in (3) which gives

$$
\begin{aligned}
p - 5 &= 0 \\
p &= 5
\end{aligned}
$$

Eq (2) now becomes

$$
\begin{aligned}
y &= -5x + (5^2 - 1) \\
&= 24 - 5x \qquad (5)
\end{aligned}
$$

However, and this is the problem with squaring the ode, it can be shown that both solution (4) and (5) do not verify the given $y' = (1 + 5x + y)^{\frac{1}{2}}$. What went wrong? They do verify the ode $y' = -(1 + 5x + y)^{\frac{1}{2}}$ (with minus sign). This example shows why one must be careful when squaring both sides of an ode and solving the squared version. Because the squared version of the ode, when also squaring it, results with same solutions. Therefore It is better to avoid the squaring operation and to try to find a method to solve the original ode in its original form.

1.2.6.5 References

1. will-squaring-both-sides-of-the-ode-change-its-type Thanks to this answer which gave the main hint on how to solve such ode. I expanded this idea for a more general cases and different exponents.
2. Wikipedia entry on D'Alembert's equation This show alternative method to solve the ode for $\frac{1}{2}$.
3. Wikipedia entry on Riccati equation
4. Wikipedia entry on Abel ode
5. paper: Exactness of Second Order Ordinary Differential Equations and Integrating Factors by R. AlAhmad, M. Al-Jararha and H. Almefleh

1.2.7 Homogeneous Maple type C

$$y' = \frac{f(x,y)}{g(x,y)}$$

ode internal name "homogeneousTypeMapleC"

This is different than the above homogeneous type C. This has the form $y' = \frac{f(x,y)}{g(x,y)}$ solved by transformation $x = X + x_0, y = Y + y_0$. If able to solve for y_0, x_0 then the ode becomes Homogeneous type A.

So what is *homogeneous ode of class C* ? It is an ode $y' = F(x,y)$ which is not *homogeneous ode of class A* but using the transformation $x = X + x_0, y = Y + y_0$ it can become one. This means if given an ode and it is not *homogeneous ode of class A* then if such transformation can be found to convert it to one, it is called *homogeneous ode of class C*. The transformed ode is then solved in $Y(X)$ as homogeneous ode and the solution is transformed back to $y(x)$ using $x = X + x_0, y = Y + y_0$. This however required finding (if possible) the x_0, y_0. This section illustrates this method with an example.

1.2.7.1 Example

$$y' = \frac{8y^2 + 12xy - 10y - 6x + 3}{y^2 + 6xy - 2y + 9x^2 - 6x + 1}$$

Using methods in earlier sections it can be shown that this is not isobaric for any degree including $m = 1$ (which means it is not even *homogeneous* ode of class A, which is special case of isobaric). Let

$$\begin{aligned} x &= X + x_0 \\ y &= Y + y_0 \end{aligned}$$

The above ode becomes

$$\begin{aligned} Y' &= \frac{8(Y+y_0)^2 + 12(X+x_0)(Y+y_0) - 10(Y+y_0) - 6(X+x_0) + 3}{(Y+y_0)^2 + 6(X+x_0)(Y+y_0) - 2(Y+y_0) + 9(X+x_0)^2 - 6(X+x_0) + 1} \qquad (1) \\ &= F(X,Y) \end{aligned}$$

The question now becomes to find x_0, y_0 such that the above ode is isobaric of degree 1. (*homogeneous* ode of class A). Earlier section showed that this becomes the condition that

$$m = -\frac{XF_X}{YF_Y} \qquad (2)$$

Where $m = 1$. Applying the above to (1) and setting $m = 1$ gives

$$\begin{aligned} 1 &= -\frac{X\frac{d}{dX}\left(\frac{8(Y+y_0)^2+12(X+x_0)(Y+y_0)-10(Y+y_0)-6(X+x_0)+3}{(Y+y_0)^2+6(X+x_0)(Y+y_0)-2(Y+y_0)+9(X+x_0)^2-6(X+x_0)+1}\right)}{Y\frac{d}{dY}\left(\frac{8(Y+y_0)^2+12(X+x_0)(Y+y_0)-10(Y+y_0)-6(X+x_0)+3}{(Y+y_0)^2+6(X+x_0)(Y+y_0)-2(Y+y_0)+9(X+x_0)^2-6(X+x_0)+1}\right)} \\ &= -\frac{X\left(\frac{-6(3X+3Y+3x_0+3y_0-2)(2Y+2y_0-1)}{(y_0-1+3x_0+Y+3X)^3}\right)}{Y\left(\frac{2(3X+3Y+3x_0+3y_0-2)(6X+6x_0-1)}{(y_0-1+3x_0+Y+3X)^3}\right)} \\ &= -\frac{X(-6(3X+3Y+3x_0+3y_0-2)(2Y+2y_0-1))}{Y(2(3X+3Y+3x_0+3y_0-2)(6X+6x_0-1))} \\ 1 &= 3\frac{X}{Y}\frac{2Y+2y_0-1}{6X+6x_0-1} \end{aligned}$$

The above is satisfied is $\frac{2Y+2y_0-1}{6X+6x_0-1} = \frac{1}{3}\frac{Y}{X}$. Which means $\frac{6Y+6y_0-3}{6X+6x_0-1} = \frac{Y}{X}$. This implies if $6y_0 - 3 = 0$ and $6x_0 - 1 = 0$ then the equation is satisfied. Therefore a solution is found which is

$$\begin{aligned} 6y_0 - 3 &= 0 \\ y_0 &= \frac{1}{2} \end{aligned}$$

And

$$
\begin{aligned}
6x_0 - 1 &= 0 \\
x_0 &= \frac{1}{6}
\end{aligned}
$$

Since transformation is found, then substituting the above two equations in (1) gives

$$
\begin{aligned}
Y' &= \frac{8\left(Y+\frac{1}{2}\right)^2 + 12\left(X+\frac{1}{6}\right)\left(Y+\frac{1}{2}\right) - 10\left(Y+\frac{1}{2}\right) - 6\left(X+\frac{1}{6}\right) + 3}{\left(Y+\frac{1}{2}\right)^2 + 6\left(X+\frac{1}{6}\right)\left(Y+\frac{1}{2}\right) - 2\left(Y+\frac{1}{2}\right) + 9\left(X+\frac{1}{6}\right)^2 - 6\left(X+\frac{1}{6}\right) + 1} \\
&= 4\frac{3XY + 2Y^2}{(3X+Y)^2} \\
&= G(X,Y)
\end{aligned}
$$

The above ode is now *homogeneous ode of class A*. We can verify this using method from above section as follows

$$
\begin{aligned}
m &= -\frac{XG_X}{YG_Y} \\
&= \frac{-X\frac{d}{dX}\left(4\frac{Y(3X+2Y)}{(3X+Y)^2}\right)}{Y\frac{d}{dY}\left(4\frac{Y(3X+2Y)}{(3X+Y)^2}\right)} \\
&= \frac{-X\left(-36\frac{Y}{(3X+Y)^3}(X+Y)\right)}{Y\left(36\frac{X}{(3X+Y)^3}(X+Y)\right)} \\
&= 1
\end{aligned}
$$

We see that this is indeed *homogeneous ode of class A*. Now this is solved easily using the substitution $Y = uX$. This results in

$$
-\ln\left(\frac{Y+X}{X}\right) + 3\ln\left(\frac{Y}{X}\right) - 3\ln\left(-\frac{3X-Y}{X}\right) - \ln X = c_1 \tag{3}
$$

But from earlier

$$
\begin{aligned}
X &= x - x_0 \\
&= x - \frac{1}{6} \\
Y &= y - y_0 \\
&= y - \frac{1}{2}
\end{aligned}
$$

Hence the solution (3) in $y(x)$ now becomes

$$
\begin{aligned}
-\ln\left(\frac{y-\frac{1}{2}+x-\frac{1}{6}}{x-\frac{1}{6}}\right)+3\ln\left(\frac{y-\frac{1}{2}}{x-\frac{1}{6}}\right)-3\ln\left(-\frac{3\left(x-\frac{1}{6}\right)-\left(y-\frac{1}{2}\right)}{x-\frac{1}{6}}\right)-\ln\left(x-\frac{1}{6}\right)&=c_2\\
-\ln\left(\frac{x+y-\frac{2}{3}}{x-\frac{1}{6}}\right)+3\ln\left(\frac{6y-3}{6x-1}\right)-3\ln\left(\frac{6y-18x}{6x-1}\right)-\ln\left(x-\frac{1}{6}\right)&=c_2\\
-\ln\left(\frac{6\left(x+y-\frac{2}{3}\right)}{6x-1}\right)+3\ln\left(\frac{6y-3}{6x-1}\right)-3\ln\left(6\frac{y-3x}{6x-1}\right)-\ln\left(x-\frac{1}{6}\right)&=c_2
\end{aligned}
$$

The above is the solution (implicit) to the original ode. The main difficulty with this method is in solving (if possible) equation (2) when $m=1$ which is

$$
1=-\frac{XF_X}{YF_Y}
$$

For x_0, y_0. In other words, to find explicit values for x_0, y_0 which makes the RHS above 1. If we can find such x_0, y_0 then the original ode can now be solved. If not, then this method will not work and we say the ode is not *homogeneous ode of class C*. Using the software Maple this can be found as follows

```
restart;
eq:=1=3*X/Y*(2*Y+2*y0-1)/(6*X+6*x0-1);
solve(identity(eq,X),[x0,y0])
```

Which gives

```
[[x0 = 1/6, y0 = 1/2]]
```

And Using Mathematica

```
eq = 1 == 3*X/Y*(2*Y + 2*y0 - 1)/(6*X + 6*x0 - 1);
SolveAlways[eq, {X, Y}]
```

Which gives

```
{{x0 -> 1/6, y0 -> 1/2}}
```

1.2.8 Homogeneous type D

ode internal name "homogeneousTypeD"

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned} \frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\ u' &= \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \end{aligned} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$.

1.2.8.1 Examples

1.2.8.1.1 Example

The first step is to see if we can write the above as

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Hence

$$y' = \frac{y}{x} - \frac{2}{x}e^{\frac{-y}{x}} \tag{2}$$

Comparing (2) to (1) shows that

$$\begin{aligned} n &= 1 \\ m &= 1 \\ g(x) &= -\frac{2}{x} \\ b &= -1 \\ f\left(b\frac{y}{x}\right) &= e^{-\frac{y}{x}} \end{aligned}$$

Hence the solution is

$$y = ux \tag{A}$$

Where u is the solution to

$$u' = \frac{1}{x} g(x) f(u) \tag{3}$$

Therefore $f(bu) = e^{-u}$ and (3) becomes

$$u' = -\frac{2}{x^2} e^{-u}$$

This is separable.

$$\begin{aligned}
e^u du &= -\frac{2}{x^2} dx \\
\int e^u du &= -2 \int \frac{1}{x^2} dx \\
e^u &= \frac{2}{x} + c_1 \\
u &= \ln\left(\frac{2}{x} + c_1\right)
\end{aligned}$$

Hence (A) becomes

$$y = x \ln\left(\frac{2}{x} + c_1\right)$$

1.2.8.1.2 Example $y'x - y - 2e^{x - \frac{y}{x}} = 0$

The first step is to see if we can write the above as

$$y' = \frac{y}{x} + g(x) f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Hence

$$\begin{aligned}
y'x - y - 2e^{x \frac{-y}{x}} &= 0 \\
y' &= \frac{y}{x} - \frac{2}{x} e^x e^{\frac{-y}{x}} \qquad (2)
\end{aligned}$$

Comparing (2) to (1) shows that

$$\begin{aligned}
n &= 1 \\
m &= 1 \\
g(x) &= -\frac{2}{x} e^x \\
b &= -1 \\
f\left(b\frac{y}{x}\right) &= e^{-\frac{y}{x}}
\end{aligned}$$

Hence the solution is

$$y = ux \tag{A}$$

Where u is the solution to

$$u' = \frac{1}{x} g(x) f(u) \tag{3}$$

Therefore $f(u) = e^{-u}$ and (3) becomes

$$u' = -\frac{2}{x^2} e^x e^{-u}$$

This is separable.

$$\begin{aligned}
e^u du &= -\frac{2}{x^2} e^x dx \\
\int e^u du &= -2 \int \frac{e^x}{x^2} dx \\
e^u &= -2\left(-\frac{e^x}{x} + \operatorname{Ei}(x)\right) + c_1
\end{aligned}$$

Where $\operatorname{Ei}(x)$ is the exponential integral $\operatorname{Ei}(x) = \int_{-x}^{\infty} \frac{e^{-t}}{t} dt$. Hence

$$u = \ln\left(c_1 - 2\left(-\frac{e^x}{x} + \operatorname{Ei}(x)\right)\right)$$

And (A) becomes

$$y = x \ln\left(c_1 - 2\left(-\frac{e^x}{x} + \operatorname{Ei}(x)\right)\right)$$

1.2.8.1.3 Example $y'x - y - 2\sin\left(3\frac{y}{x}\right) = 0$

The first step is to see if we can write the above as

$$y' = \frac{y}{x} + g(x) f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Hence

$$\begin{aligned}
y'x - y - 2\sin\left(3\frac{y}{x}\right) &= 0 \\
y' &= \frac{y}{x} - \frac{2}{x}\sin\left(3\frac{y}{x}\right) \qquad (2)
\end{aligned}$$

Comparing (2) to (1) shows that

$$\begin{aligned}
n &= 1 \\
m &= 1 \\
g(x) &= -\frac{2}{x} \\
b &= 3 \\
f\left(b\frac{y}{x}\right) &= \sin\left(3\frac{y}{x}\right)
\end{aligned}$$

Hence the solution is

$$y = ux \tag{A}$$

Where u is the solution to

$$u' = \frac{1}{x} g(x) f(u) \tag{3}$$

Therefore $f(u) = \sin(3u)$ and (3) becomes

$$u' = -\frac{2}{x^2}\sin(3u)$$

This is separable.

$$\begin{aligned}
\frac{1}{\sin(3u)} du &= -\frac{2}{x^2} dx \\
\int \frac{1}{\sin(3u)} du &= -2\int \frac{1}{x^2} dx \\
\frac{1}{3}\left(\ln\sin\left(\frac{3u}{2}\right) - \ln\cos\left(\frac{3u}{2}\right)\right) &= \frac{2}{x} + c_1 \\
\ln\sin\left(\frac{3u}{2}\right) - \ln\cos\left(\frac{3u}{2}\right) &= -\frac{6}{x} + c_2 \\
\ln\frac{\sin\left(\frac{3u}{2}\right)}{\cos\left(\frac{3u}{2}\right)} &= -\frac{6}{x} + c_2 \\
\ln\tan\left(\frac{3u}{2}\right) &= -\frac{6}{x} + c_2 \\
\tan\left(\frac{3u}{2}\right) &= c_3 e^{-\frac{6}{x}} \\
\frac{3u}{2} &= \arctan\left(c_3 e^{-\frac{6}{x}}\right) \\
u &= \frac{2}{3}\arctan\left(c_3 e^{-\frac{6}{x}}\right)
\end{aligned}$$

And (A) becomes

$$y = \frac{2}{3} x \arctan\left(c_3 e^{-\frac{6}{x}}\right)$$

1.2.8.1.4 Example $y' = \frac{y}{x} - \frac{2}{x}\sqrt{\sin\left(3\frac{y}{x}\right)}$

The first step is to see if we can write the above as

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Hence

$$y' = \frac{y}{x} - \frac{2}{x}\left(\sin\left(3\frac{y}{x}\right)\right)^{\frac{1}{2}} \tag{2}$$

Comparing (2) to (1) shows that

$$\begin{aligned} n &= 1 \\ m &= 2 \\ g(x) &= -\frac{2}{x} \\ b &= 3 \\ f\left(b\frac{y}{x}\right) &= \sin\left(3\frac{y}{x}\right) \end{aligned}$$

Hence the solution is

$$y = ux \tag{A}$$

Where u is the solution to

$$u' = \frac{1}{x} g(x)\, f(u)^{\frac{1}{2}} \tag{3}$$

Therefore $f(u) = \sin(3u)$ and (3) becomes

$$u' = -\frac{2}{x^2}\sin(3u)^{\frac{1}{2}}$$

This is separable.

$$\begin{aligned} \frac{1}{\sqrt{\sin(3u)}} du &= -\frac{2}{x^2} dx \\ \int \frac{1}{\sqrt{\sin(3u)}} du &= -2\int \frac{1}{x^2} dx \\ \int \frac{1}{\sqrt{\sin(3u)}} du &= \frac{2}{x} + c_1 \end{aligned}$$

Leaving the integral as is, since it is too complicated to solve, then using $y = ux$ where u is the solution of the above.

1.2.8.1.5 Example $y - 2x^3 \tan\left(\frac{y}{x}\right) - y'x = 0$

The first step is to see if we can write the above as

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Hence

$$\begin{aligned} y - 2x^3 \tan\left(\frac{y}{x}\right) - y'x = 0 \\ y'x &= y - 2x^3 \tan\left(\frac{y}{x}\right) \\ y' &= \frac{y}{x} - 2x^2 \tan\left(\frac{y}{x}\right) \end{aligned} \tag{2}$$

Comparing (2) to (1) shows that

$$\begin{aligned} n &= 1 \\ m &= 1 \\ g(x) &= -2x^2 \\ b &= 1 \\ f\left(b\frac{y}{x}\right) &= \tan\left(\frac{y}{x}\right) \end{aligned}$$

Hence the solution is

$$y = ux \tag{A}$$

Where u is the solution to

$$u' = \frac{1}{x} g(x)\, f(u) \tag{3}$$

Therefore $f(u) = \tan u$ and (3) becomes

$$u' = -2x \tan u$$

This is separable.

$$\begin{aligned} \frac{1}{\tan} du &= -2x dx \\ \int \frac{1}{\tan} du &= -2 \int x dx \\ \ln(\sin u) &= -x^2 + c_1 \\ \sin u &= c_2 e^{-x^2} \\ u &= \arcsin\left(c_2 e^{-x^2}\right) \end{aligned}$$

Hence (A) becomes

$$y = x \arcsin\left(c_2 e^{-x^2}\right)$$

1.2.8.1.6 Example $y' = \frac{y}{x} + x\sin\left(\frac{y}{x}\right)$

The first step is to see if we can write the above as

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Hence

$$y' = \frac{y}{x} + x\sin\left(\frac{y}{x}\right) \tag{2}$$

Comparing (2) to (1) shows that

$$\begin{aligned} n &= 1 \\ m &= 1 \\ g(x) &= x \\ b &= 1 \\ f\left(b\frac{y}{x}\right) &= \sin\left(\frac{y}{x}\right) \end{aligned}$$

Hence the solution is

$$y = ux \tag{A}$$

Where u is the solution to

$$u' = \frac{1}{x} g(x)\, f(u) \tag{3}$$

Therefore $f(u) = \sin u$ and (3) becomes

$$u' = \frac{1}{x}(x)\sin(u)$$

This is separable.

$$\begin{aligned} \frac{1}{\sin u} du &= dx \\ \int \frac{1}{\sin u} du &= \int dx \\ \ln\sin\frac{u}{2} - \ln\cos\frac{u}{2} &= x + c_1 \\ \ln\tan\frac{u}{2} &= x + c_1 \\ \tan\frac{u}{2} &= c_2 e^x \\ \frac{u}{2} &= \arctan(c_2 e^x) \\ u &= 2\arctan(c_2 e^x) \end{aligned}$$

Hence (A) becomes

$$y = 2x\arctan(c_2 e^x)$$

1.2.9 Homogeneous type D2

$$y' = f(x, y)$$

ode internal name "homogeneousTypeD2"

These are ode of any form, in which the change of variables results in either separable or quadrature ode. Hence given an ode $y' = f(x, y)$ the change of variables $y(x) = u(x)\,x$ is made and the resulting ode in $u(x)$ is examined. If it is separable or quadrature, then it is solved for u and hence the solution $y = ux$ is found.

1.2.9.1 Examples

1.2.9.1.1 Example $y' = -\frac{y(y^2+3x^2+2x)}{x^2+y^2}$

Applying change of variables $y = ux$ results in

$$u' = -\frac{u(u^2+3)}{u^2+1}\frac{x+1}{x}$$

Which is separable. Solving this for $u(x)$ by integration gives

$$\frac{1}{3}\ln\left(\left(u^2+3\right)u\right) + x + \ln(x) = c_1$$

Hence the solution in $y(x)$ is

$$\frac{1}{3}\ln\left(\left(\left(\frac{y}{x}\right)^2+3\right)\frac{y}{x}\right) + x + \ln(x) = c_1$$

1.2.10 isobaric ode

1.2.10.1 Introduction

ode internal name "isobaric"

This is a generalization of the above homogeneous ODE, where the substitution $y = v(x)\,x^m$ makes the ODE separable. The weight m needs to be found first.

These are examples showing how to solve isobaric ode's step by step method. The same method is also used to solve homogeneous odes, which is special case of isobaric.

The hardest part is to determine if the ode is isobaric or homogeneous and to find the degree of the isobaric.

An ode $y' = f(x, y)$ is called isobaric of degree m if

$$f(tx, t^m y) = t^{m-1} f(x, y)$$

It is called homogeneous ode if $m = 1$

$$f(tx, ty) = f(x, y)$$

So homogeneous ode is special case of isobaric ode when $m = 1$. Another common definition of a homogeneous ode is that when writing the ode as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{M(x, y)}{N(x, y)} \end{aligned}$$

Then M, N must both be homogeneous functions of same degree. Care is needed here, Homogeneous function is not the same as a homogeneous ode. A function $M(x, y)$ is homogeneous function of degree n if $M(tx, ty) = t^n M(x, y)$ where n here do not have to be zero.

Using this second definition of homogeneous ode of $\frac{M(x,y)}{N(x,y)}$, we can check if $M(x, y)$ and $N(x, y)$ are both homogeneous functions and also have same degree (whatever this degree happened to be). If this is the case, then we say the ode itself is homogeneous ode.

It is possible to have an ode $y' = \frac{M(x,y)}{N(x,y)}$ where M, N are both homogeneous functions but with *different* degrees. In this case the ode is *not* homogeneous ode even though both M, N are each homogeneous functions.

We can use similar way to view isobaric ode. By saying that an isobaric ode is one when it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{M(x, y)}{N(x, y)} \end{aligned}$$

Then given $M(tx, t^m y) = t^r M(x, y)$ is homogeneous function of degree r and $N(tx, t^m y) = t^{r-m+1} N(x, y)$ is homogeneous function of degree $r - m + 1$. In this case we say that the ode itself is isobaric of degree m, since

$$\begin{aligned} f(tx, t^m y) &= \frac{t^r M(x, y)}{t^{r-m+1} N(x, y)} \\ &= t^{m-1} \frac{M(x, y)}{N(x, y)} \\ &= t^{m-1} f(x, y) \end{aligned}$$

The above gives us another method to determine if an ode is homogeneous ode or isobaric ode. We start by writing the ode as $y' = \frac{M(x,y)}{N(x,y)}$. If M, N are both homogeneous functions of same degree, then the ode is homogeneous ode and we stop. If however M satisfies $M(tx, t^m y) = t^r M(x, y)$ and N satisfies $N(tx, t^m y) = t^{r-m+1} N(x, y)$ where r is positive integer, then we say the ode is isobaric of degree m.

Why is it important to know if an ode is homogeneous or isobaric? This is because if an ode is isobaric of degree m then the substitution $y = ux^m$ or $u = \frac{y}{x^m}$ converts to separable ode in u. If an ode is homogeneous then the substitution $y = ux$ or $u = \frac{y}{x}$ converts to separable ode in u.

This is why it is very useful to determine if an ode is isobaric or homogeneous ode. Because it allows us to use this substitution to convert it to separable. Separable ode's are easy to solve, since they involve only integration. Of course the integrals can be very difficult to solve, but this is another issue.

How to determine if an ode is homogeneous or isobaric in practice? To check if an ode is homogeneous, we start with the definition that ode $y' = f(x, y)$ is homogeneous ode if in

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{A}$$

then if $m = 1$ then the ode is homogenous. If not, then the ode is not homogenous and we check if it is isobaric by solving for m. How to find m?

This is done by taking derivative of both sides of equation (A) w.r.t. t and setting $t = 1$ after that. This results in

$$\begin{aligned} xf_x + myf_y &= (m-1) f \\ xf_x + myf_y &= mf - f \\ xf_x + f &= m(f - yf_y) \end{aligned}$$

Hence

$$m = \frac{f + xf_x}{f - yf_y}$$

Here is the important point. *If it is possible* to simplify the RHS above to an actual numerical value, then m is the degree of isobaric and the ode is indeed isobaric. If it is not possible to obtain a numerical m value, then the ode is not isobaric. The best way to learn how to do this is by examples. Note in the above f_x is partial derivative. Which means taking derivative of f w.r.t while keeping y *fixed*.

1.2.10.2 Examples

1.2.10.2.1 Example 1

$$\frac{dy}{dx} = \frac{-\left(y^2 + \frac{2}{x}\right)}{2yx} \tag{1}$$

Here $f(x,y) = \frac{-\left(y^2+\frac{2}{x}\right)}{2yx}$. We start by checking if it is isobaric or not. To find m such that $f(tx, t^m y) = t^{m-1} f(x,y)$ we do (as given in the introduction)

$$\begin{aligned} m &= \frac{f + x f_x}{f - y f_y} \\ &= \frac{\frac{-\left(y^2+\frac{2}{x}\right)}{2yx} + x\left(\frac{xy^2+4}{2x^3y}\right)}{\frac{-\left(y^2+\frac{2}{x}\right)}{2yx} - y\left(-\frac{xy^2-2}{2x^2y^2}\right)} \\ &= \frac{\frac{1}{x^2y}}{-\frac{2}{x^2y}} \\ &= -\frac{1}{2} \end{aligned} \tag{2}$$

Hence this is isobaric of index $m = -\frac{1}{2}$ because it has a numerical solution as a result.

To verify this result, here $M(x,y) = \left(-y^2 - \frac{2}{x}\right), N(x,y) = 2yx$. Let us start by checking for isobaric (since homogeneous is special case).

$$\begin{aligned} M(tx, t^m y) &= \left(-t^{2m} y^2 + \frac{2}{tx}\right) \\ &= \frac{1}{t}\left(-t^{2m+1} y^2 + \frac{2}{x}\right) \\ &= t^{-1}\left(-t^{2m+1} y^2 + \frac{2}{x}\right) \end{aligned}$$

The above is same as $\left(-y^2 - \frac{2}{x}\right)$ when $2m+1 = 0$ or $m = -\frac{1}{2}$. From the above we also see that $r = -1$. This is by comparing the last result above to $t^r M(x,y)$. Now that we found candidate m and r, then all what we have to do is check $N(tx, t^m y) = t^{r-m-1} N(x,y)$ or not. If it is, then we are done and the ode is isobaric of degree m

$$\begin{aligned} N(tx, t^m y) &= 2t^m ytx \\ &= 2t^{\frac{-1}{2}} ytx \\ &= t^{\frac{1}{2}}(2yx) \\ &= t^{\frac{1}{2}} N(x,y) \end{aligned}$$

Now we check if $\frac{1}{2} = r - m + 1$. Which it is. Since $r - m + 1 = -1 - \left(-\frac{1}{2}\right) + 1 = \frac{1}{2}$. Hence this ode is isobaric. From now on Eq (2) will be used to find m.

Hence the substitution $y = vx^m$ will make the ode separable. This is the whole point of isobaric ode's. The hardest part is to find m. Substituting $y = vx^{\frac{=1}{2}}$ in (1) results in

$$v\frac{dv}{dx} = -\frac{1}{x}$$

This is solved for v easily since separable, and then y is found from $y = vx^{\frac{=1}{2}}$.

1.2.10.2.2 Example 2

$$\frac{dy}{dx} = x\sqrt{x^4 + 4y} - x^3 \tag{1}$$

We start by checking if it is isobaric or not. Using

$$\begin{aligned}
m &= \frac{f + xf_x}{f - yf_y} \\
&= \frac{\left(x\sqrt{x^4+4y} - x^3\right) + x\left(\sqrt{x^4+4y} + \frac{2x^4}{\sqrt{x^4+4y}} - 3x^2\right)}{\left(x\sqrt{x^4+4y} - x^3\right) - x^3 - \frac{2xy}{\sqrt{x^4+4y}}} \\
&= \frac{4\frac{x}{\sqrt{x^4+4y}}\left(2y - x^2\sqrt{x^4+4y} + x^4\right)}{\frac{x}{\sqrt{x^4+4y}}\left(2y - 2x^2\sqrt{x^4+4y} + x^4\right)} \\
&= \frac{4\frac{x}{\sqrt{x^4+4y}}}{\frac{x}{\sqrt{x^4+4y}}} \\
&= 4
\end{aligned}$$

Therefore this is isobaric of order 4. Substituting $y = vx^m = vx^4$ in (1) results in

$$v' = \frac{-4v + \sqrt{1+4v} - 1}{x}$$

Which is separable. This is solved easily for $v(x)$ and then y is found from $y = vx^4$.

1.2.10.2.3 Example 3

$$\begin{aligned}
x\left(x - y^3\right)\frac{dy}{dx} &= \left(3x + y^3\right)y \\
\frac{dy}{dx} &= \frac{\left(3x + y^3\right)y}{x\left(x - y^3\right)} \qquad (1)
\end{aligned}$$

We start by checking if it is isobaric or not. Using

$$\begin{aligned}
m &= \frac{f + x f_x}{f - y f_y} \\
&= \frac{\frac{(3x+y^3)y}{x(x-y^3)} + x\left(\frac{3y}{x(-y^3+x)} - \frac{(y^3+3x)y}{x^2(-y^3+x)} - \frac{(y^3+3x)y}{x(-y^3+x)^2}\right)}{\frac{(3x+y^3)y}{x(x-y^3)} - y\left(\frac{3y^3}{x(-y^3+x)} + \frac{y^3+3x}{x(-y^3+x)} + \frac{3(y^3+3x)y^3}{x(-y^3+x)^2}\right)} \\
&= \frac{-4\frac{y^4}{(x-y^3)^2}}{-12\frac{y^4}{(x-y^3)^2}} \\
&= \frac{1}{3}
\end{aligned}$$

$m = \frac{1}{3}$ makes each term the same weight $\frac{4}{3}$. Hence the substitution $y = vx^{\frac{1}{3}}$ will make the ode separable. Substituting this in (1) results in

$$\frac{dv}{dx} = \frac{-4}{3x}\frac{v(v^3+2)}{(v^3-1)}$$

Which is separable. This is solved for v, and then y is found from $y = vx^{\frac{1}{3}}$.

1.2.10.2.4 Example 4

$$y' = \frac{y}{x}\ln(xy-1) \tag{1}$$

We start by checking if it is isobaric or not. Using

$$\begin{aligned}
m &= \frac{f + x f_x}{f - y f_y} \\
&= \frac{\frac{y}{x}\ln(xy-1) + x\left(\frac{-y\ln(xy-1)}{x^2} + \frac{y^2}{x(xy-1)}\right)}{\frac{y}{x}\ln(xy-1) - y\left(\frac{\ln(xy-1)}{x} + \frac{y}{xy-1}\right)} \\
&= \frac{\frac{y^2}{xy-1}}{-\frac{y^2}{xy-1}} \\
&= -1
\end{aligned}$$

Hence the substitution $y = \frac{v}{x}$ will make the ode separable. Substituting this in (1) results in

$$v' = \frac{v\ln(v)}{x}$$

Which is separable. This is solved for v, and then y is found from $y = \frac{v}{x}$.

1.2.10.2.5 Example 5

$$(y')^2 = y(y - 2y'x)^3 \tag{1}$$

One way to handle this is to first solve for y' and then apply the above method. This will result in $m = -1$.

1.2.10.2.6 Example 6

$$\begin{aligned}(x-y)\,y' - x - y &= 0\\ y' &= \frac{x+y}{x-y}\\ &= f(x,y)\end{aligned} \tag{1}$$

We start by checking if it homogenous or not. Using

$$\begin{aligned}m &= \frac{f + xf_x}{f - yf_y}\\ &= \frac{\frac{x+y}{x-y} + x\left(\frac{1}{x-y} - \frac{x+y}{(x-y)^2}\right)}{\frac{x+y}{x-y} - y\left(\left(\frac{1}{x-y} + \frac{x+y}{(x-y)^2}\right)\right)}\\ &= \frac{x\left(\frac{1}{x-y} - \frac{x+y}{(x-y)^2}\right)}{-y\left(\left(\frac{1}{x-y} + \frac{x+y}{(x-y)^2}\right)\right)}\\ &= 1\end{aligned}$$

Since $m = 1$ then this is homogeneous ode (special case of isobaric). Hence the substitution $v = \frac{y}{x}$ makes the ode (1) separable.

1.2.10.2.7 Example 7

$$\begin{aligned}y'x - y - 2\sqrt{xy} &= 0\\ y' &= \frac{y + 2\sqrt{xy}}{x}\end{aligned} \tag{1}$$

We start by checking if it homogenous or not. Using

$$\begin{aligned}m &= \frac{f + xf_x}{f - yf_y}\\ &= \frac{\frac{y+2\sqrt{xy}}{x} + x\left(\frac{y}{x\sqrt{xy}} - \frac{y+2\sqrt{xy}}{x^2}\right)}{\frac{y+2\sqrt{xy}}{x} - y\left(\frac{1+\frac{x}{\sqrt{xy}}}{x}\right)}\\ &= 1\end{aligned}$$

Since $m = 1$ then this is homogeneous ode (special case of isobaric). Hence the substitution $v = \frac{y}{x}$ makes the ode (1) separable.

1.2.10.2.8 Example 8

$$y' = \frac{-y(y^2 + 3x^2 + 2x)}{x^2 + y^2} \tag{1}$$

We start by checking if it homogenous or not. Using

$$\begin{aligned}
m &= \frac{f + x f_x}{f - y f_y} \\
&= \frac{\frac{-y(y^2+3x^2+2x)}{x^2+y^2} + x \frac{d}{dx}\left(\frac{-y(y^2+3x^2+2x)}{x^2+y^2}\right)}{\frac{-y(y^2+3x^2+2x)}{x^2+y^2} - y \frac{d}{dy}\left(\frac{-y(y^2+3x^2+2x)}{x^2+y^2}\right)} \\
&= \frac{\frac{-y(y^2+3x^2+2x)}{x^2+y^2} + x\left(-2\frac{y(-x^2+2xy^2+y^2)}{(x^2+y^2)^2}\right)}{\frac{-y(y^2+3x^2+2x)}{x^2+y^2} - y\left(-\frac{3x^4+2x^3-2xy^2+y^4}{(x^2+y^2)^2}\right)} \\
&= \frac{3x^4 + 8x^2y^2 + 4xy^2 + y^4}{4x^2y^2 + 4xy^2}
\end{aligned}$$

Since this does not simplify to numerical value, it is not homogenous ode. This turns out to be homogenous type D. See earlier note on this. There is a slight difference in definition between homogenous ode and homogenous type D. In Maple terms, homogenous ode is called homogenous ode type A. A homogenous type D is one in which the substitution $y = ux$ makes the ode separable or quadrature.

1.2.11 First order special form ID 1 $y' = g(x)\, e^{a(x)+by} + f(x)$

ode internal name "first order special form ID 1"

Special form which did not fit in any of the above ones. Solved by the substitution $u = e^{-by}$ which converts the ode to a linear first order ode in $u(x)$ which is solved, then y is found. b must not depend on for this to work.

1.2.11.1 Example

$$y' = 5e^{x^2+20y} + \sin x \tag{1}$$

Here $a(x) = x^2, b = 20, f(x) = \sin x, g(x) = 5$. Hence

$$u = e^{-by} = e^{-20y}$$

Therefore

$$\begin{aligned}\frac{du}{dx} &= -20y'e^{-20y}\\ &= -20y'u\end{aligned}$$

Or

$$y' = -\frac{u'}{20u} \tag{2}$$

Comparing (1,2) gives

$$\begin{aligned}-\frac{u'}{20u} &= 5e^{x^2+20y} + \sin x\\ &= 5e^{20y}e^{x^2} + \sin x\\ &= 5\frac{1}{u}e^{x^2} + \sin x\end{aligned}$$

Or

$$\begin{aligned}-u' &= 100e^{x^2} + 20u\sin x\\ u' &= -100e^{x^2} - 20u\sin x\\ u' + 20u\sin x &= -100e^{x^2} \qquad (3)\end{aligned}$$

This is linear first order ode. The integrating factor is

$$\begin{aligned}I &= e^{\int 20\sin x dx}\\ &= e^{-20\cos x}\end{aligned}$$

(3) becomes

$$\begin{aligned}
\frac{d}{dx}(uI) &= -I100e^{x^2} \\
ue^{-20\cos x} &= -100\int e^{x^2}e^{-20\cos x}dx + c \\
u &= -100e^{20\cos x}\int e^{x^2-20\cos x}dx + ce^{20\cos x} \\
&= e^{20\cos x}\left(-100\int e^{x^2-20\cos x}dx + c\right)
\end{aligned}$$

But $u = e^{-20y}$ therefore

$$\begin{aligned}
e^{-20y} &= e^{20\cos x}\left(-100\int e^{x^2-20\cos x}dx + c\right) \\
-20y &= \ln\left(e^{20\cos x}\left(-100\int e^{x^2-20\cos x}dx + c\right)\right) \\
y &= -\frac{1}{20}\ln\left(e^{20\cos x}\left(-100\int e^{x^2-20\cos x}dx + c\right)\right)
\end{aligned}$$

1.2.12 Polynomial ode $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

ode internal name "polynomial"

Special form for first order ode where the lines $a_1x+b_1y+c_1 = 0, a_2x+b_2y+c_2 = 0$ can be either parallel or not parallel. If the lines are not parallel then the transformation $X = x-x_0, Y = y-y_0$ transforms the ode to homogeneous ode. If the lines are parallel then the transformation $U(x) = a_1x + b_1y$ converts the ode to separable in $U(x)$. The not parallel case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case is when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.

1.2.12.1 Example lines are not parallel

$$y' = \frac{-6x+y-3}{2x-y-1}$$

Comparing to $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ shows that $a_1 = -6, b_1 = 1, a_2 = 2, b_2 = -1$. Hence $\frac{a_1}{b_1} = -6$, $\frac{a_2}{b_2} = -2$. This shows the lines are not parallel. Let

$$\begin{aligned}
X &= x - x_0 \\
Y &= y - y_0
\end{aligned}$$

The constant x_0, y_0 are found by solving

$$\begin{aligned}
a_1x_0 + b_1y_0 + c_1 &= 0 \\
a_2x_0 + b_2y_0 + c_2 &= 0
\end{aligned}$$

Or

$$
\begin{aligned}
-6x_0 + y_0 - 3 &= 0 \\
2x_0 - y_0 - 1 &= 0
\end{aligned}
$$

Solving for x_0, y_0 gives

$$
\begin{aligned}
x_0 &= -1 \\
y_0 &= -3
\end{aligned}
$$

Hence

$$
\begin{aligned}
X &= x + 1 \\
Y &= y + 3
\end{aligned}
$$

Using this transformation in $y' = \frac{-6x+y-3}{2x-y-1}$ results in the ode

$$
\frac{dY}{dX} = \frac{6X - Y}{-2X + Y}
$$

This is a homogeneous ode

$$
\frac{dY}{dX} = \frac{6 - \frac{Y}{X}}{-2 + \frac{Y}{Y}}
$$

Let $u = \frac{Y}{X}$. Now it is solved as was shown in the above sections. At the end, Y is replaced by $y - y_0$ to obtain the solution in $y(x)$.

1.2.12.2 Example lines are parallel

$$
y' = -\frac{x + y}{3x + 3y - 4}
$$

Comparing to $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ shows that $a_1 = -1, b_1 = -1, a_2 = 3, b_2 = 3$. Hence $\frac{a_1}{b_1} = 1, \frac{a_2}{b_2} = 1$. This shows the lines are parallel. Let

$$
\begin{aligned}
U(x) &= a_1 x + b_1 y \\
&= -x - y
\end{aligned}
$$

Hence $y' = -1 - U'(x)$. Hence the ode becomes

$$
\begin{aligned}
-1 - U' - \frac{U}{-3U - 4} &= 0 \\
U' &= -\frac{2U + 4}{3U + 4}
\end{aligned}
$$

This is separable. After solving for $U(x)$, then y is found from $U(x) = -x - y$

$$
y = -x - U
$$

1.2.13 Bernoulli ode $y' + Py = Qy^n$

ode internal name "bernoulli"

This has the form $y' + Py = Qy^n$ where $n \neq 1, n \neq 0$. Solved by dividing by y^n and then using the substitution $v = y^{1-n}$. This converts the ode to linear ode $v' + (1-n) Pv = (1-n) Q$ which is solved for v, then y is found.

1.2.14 Exact ode $M(x,y) + N(x,y)\, y' = 0$

ode internal name "exact"

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{1}$$

If the above ODE is exact, then there it can be written as a complete differential

$$\begin{aligned} M(x,y) + N(x,y)\frac{dy}{dx} &= d\phi(x,y) \\ &= \frac{\partial \phi}{\partial x}\frac{dx}{dx} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} \end{aligned} \tag{2}$$

Comparing (1,2) shows that

$$\frac{\partial \phi}{\partial x} = M \tag{3}$$

$$\frac{\partial \phi}{\partial y} = N \tag{4}$$

But since $\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$ then this implies

$$\begin{aligned} \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) \\ \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \end{aligned}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. Given the ode is exact, then integrating (3) gives

$$\phi = \int M dx + f(y) \tag{5}$$

Where $f(y)$ is arbitrary function to be found. Taking derivative of the above w.r.t. y gives

$$\frac{\partial \phi}{\partial y} = \frac{d}{dy}\int M dx + f'(y)$$

Comparing the above to (4) gives an equation to solve for f

$$\left(\frac{d}{dy}\int M dx\right) + f'(y) = N \tag{6}$$

Once $f(y)$ is found then from (5) and since ϕ is constant it becomes

$$c = \int M dx + f(y)$$

This is an implicit solution for $y(x)$.

1.2.14.1 Examples

1.2.14.1.1 Example1

$$\left(3x^2 + 2xy^2\right) + \left(2x^2y + 4y^3\right) y' = 0$$

Hence $M = \left(3x^2 + 2xy^2\right), N = \left(2x^2y + 4y^3\right)$. We see that $\frac{\partial M}{\partial y} = 4xy$ and $\frac{\partial N}{\partial x} = 4xy$, hence exact. Then (5) gives

$$\begin{aligned}
\phi &= \int M dx + f(y) \\
&= \int 3x^2 + 2xy^2 dx + f(y) \\
&= x^3 + x^2y^2 + f(y)
\end{aligned}$$

Hence (6) gives

$$\begin{aligned}
\frac{d}{dy}\left(x^3 + x^2y^2 + f(y)\right) &= N \\
2yx^2 + f'(y) &= 2x^2y + 4y^3 \\
f'(y) &= 4y^3
\end{aligned}$$

Therefore $f(y) = y^4 + c_1$. Therefore

$$\begin{aligned}\phi &= \int M dx + f(y)\\ &= x^3 + x^2y^2 + f(y)\\ &= x^3 + x^2y^2 + y^4 + c_1\end{aligned}$$

But $\phi = c$, since constant. Hence combining constants the above becomes

$$x^3 + x^2y^2 + y^4 = C$$

Which is implicit solution for $y(x)$.

1.2.14.1.2 Example2

$$\left(\ln\left(\frac{y+x}{x+3}\right) - \frac{y+x}{x+3}\right) dx + \ln\left(\frac{y+x}{x+3}\right) dy = 0$$

Hence $M = \left(\ln\left(\frac{y+x}{x+3}\right) - \frac{y+x}{x+3}\right)$, $N = \ln\left(\frac{y+x}{x+3}\right)$. We see that $\frac{\partial M}{\partial y} = \frac{3-y}{(y+x)(x+3)}$ and $\frac{\partial N}{\partial x} = \frac{3-y}{(y+x)(x+3)}$, hence the ode is exact. Eq (5) gives

$$\begin{aligned}\phi &= \int M dx + f(y)\\ &= \int \left(\ln\left(\frac{y+x}{x+3}\right) - \frac{y+x}{x+3}\right) dx + f(y)\\ &= (3-y)\ln\left(\frac{y-3}{x+3}\right) + (y+x)\ln\left(\frac{y+x}{x+3}\right) + (3-y)\ln(x+3) - x + f(y)\\ &= (3-y)\left(\ln\left(\frac{y-3}{x+3}\right) + \ln(x+3)\right) + (y+x)\ln\left(\frac{y+x}{x+3}\right) - x + f(y)\\ &= (3-y)\ln(y-3) + (y+x)\ln\left(\frac{y+x}{x+3}\right) - x + f(y)\end{aligned}$$

Hence (6) gives

$$\begin{aligned}\frac{d}{dy}(\phi) &= N\\ \frac{d}{dy}\left((3-y)\ln(y-3) + (y+x)\ln\left(\frac{y+x}{x+3}\right) - x + f(y)\right) &= \ln\left(\frac{y+x}{x+3}\right)\\ \ln\left(\frac{y+x}{x+3}\right) - \ln(y-3) + f'(y) &= \ln\left(\frac{y+x}{x+3}\right)\\ -\ln(y-3) + f'(y) &= 0\\ f'(y) &= \ln(y-3)\end{aligned}$$

Therefore

$$\begin{aligned}f(y) &= \int \ln(y-3)\,dy\\ &= \ln(y-3)(y-3)+3-y+c_1\end{aligned}$$

Hence from above

$$\begin{aligned}\phi &= (3-y)\ln(y-3)+(y+x)\ln\left(\frac{y+x}{x+3}\right)-x+f(y)\\ &= (3-y)\ln(y-3)+(y+x)\ln\left(\frac{y+x}{x+3}\right)-x+\ln(y-3)(y-3)+3-y+c_1\\ &= -(y-3)\ln(y-3)+(y+x)\ln\left(\frac{y+x}{x+3}\right)-x+\ln(y-3)(y-3)+3-y+c_1\\ &= (y+x)\ln\left(\frac{y+x}{x+3}\right)-x+3-y+c_1\\ &= (y+x)\ln\left(\frac{y+x}{x+3}\right)-x-y+c_2\end{aligned}$$

But $\phi = c$, since constant. Hence combining constants the above becomes

$$(y+x)\ln\left(\frac{y+x}{x+3}\right)-x-y=C$$

1.2.15 Not exact ode but can be made exact with integrating factor

ode internal name "exactWithIntegrationFactor"

This has the form $M(x,y)+N(x,y)\,y'=0$ where $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ where there exist integrating factor μ such that $\mu M(x,y)+\mu N(x,y)\,y'=0$ becomes exact. Three methods are implemented to find the integrating factor.

1.2.15.1 First integrating factor $\mu(x)$ that depends on x only

Let

$$\begin{aligned}\mu M(x,y)+\mu N(x,y)\frac{dy}{dx} &= d\phi(x,y) \qquad (1)\\ &= \frac{\partial\phi}{\partial x}\frac{dx}{dx}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}\\ &= \frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx} \qquad (2)\end{aligned}$$

Comparing (1),(2) then

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \mu M\\ \frac{\partial \phi}{\partial y} &= \mu N\end{aligned}$$

The compatibility condition is $\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$ then this implies

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)\\ \frac{\partial \mu M}{\partial y} &= \frac{\partial \mu N}{\partial x}\\ \mu_y M + \mu M_y &= \mu_x N + \mu N_x\\ \mu_x N &= \mu_y M + \mu M_y - \mu N_x\\ \mu_x N &= \mu_y M + \mu (M_y - N_x)\\ \mu_x &= \frac{\mu_y M}{N} + \frac{\mu}{N}(M_y - N_x)\end{aligned}$$

Assuming $\mu \equiv \mu(x)$ then $\mu_y = 0$ and the above simplifies to

$$\begin{aligned}\mu_x &= \frac{\mu}{N}(M_y - N_x)\\ \frac{d\mu}{dx}\frac{1}{\mu} &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\end{aligned}$$

Let $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = A$. If $A \equiv A(x)$ which depends only on x then we can solve the above.

$$\begin{aligned}\frac{d\mu}{dx}\frac{1}{\mu} &= A\\ \mu &= e^{\int A dx}\end{aligned}$$

Let $\overline{M} = \mu M, \overline{N} = \mu N$ then the ode

$$\overline{M}(x,y) + \overline{N}(x,y)\, y' = 0$$

is now exact.

1.2.15.2 Second integrating factor $\mu(y)$ that depends on y only

Let

$$\begin{aligned}\mu M(x,y)+\mu N(x,y)\frac{dy}{dx} &= d\phi(x,y) \qquad (1)\\ &= \frac{\partial\phi}{\partial x}\frac{dx}{dx}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}\\ &= \frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx} \qquad (2)\end{aligned}$$

Comparing (1),(2) then

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \mu M\\ \frac{\partial\phi}{\partial y} &= \mu N\end{aligned}$$

The compatibility condition is $\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\phi}{\partial x\partial y}$ then this implies

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)\\ \frac{\partial\mu M}{\partial y} &= \frac{\partial\mu N}{\partial x}\\ \mu_y M+\mu M_y &= \mu_x N+\mu N_x\\ \mu_y M &= \mu_x N+\mu N_x-\mu M_y\\ \mu_y M &= \mu_x N+\mu(N_x-M_y)\\ \mu_y &= \frac{\mu_x N}{M}+\frac{1}{M}\mu(N_x-M_y)\end{aligned}$$

Assuming $\mu\equiv\mu(y)$ then $\mu_x=0$ and the above simplifies to

$$\begin{aligned}\mu_y &= \frac{1}{M}\mu(N_x-M_y)\\ \frac{d\mu}{dy}\frac{1}{\mu} &= \frac{1}{M}(N_x-M_y)\end{aligned}$$

Let $\frac{1}{M}(N_x-M_y)=B$. If $B\equiv B(y)$ which depends only on y then we can solve the above.

$$\begin{aligned}\frac{d\mu}{dy}\frac{1}{\mu} &= B(y)\\ \mu &= e^{\int B dy}\end{aligned}$$

Let $\overline{M}=\mu M,\overline{N}=\mu N$ then the ode

$$\overline{M}(x,y)+\overline{N}(x,y)\,y'=0$$

is now exact.

1.2.15.3 Third integrating factor $\mu(xy)$

Using similar method If the above did not work, then we try

$$R = \frac{1}{xM - yN}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

If R is function of $t = xy$ only then the integrating factor is $\mu = e^{\int R dt}$ and let $\overline{M} = \mu M, \overline{N} = \mu N$ then the ode $\overline{M}(x,y) + \overline{N}(x,y)\, y' = 0$ is now exact.

1.2.16 Not exact first order ode where integrating factor is found by inspection

ode internal name "exactByInspection"

This has the form $M(x,y)+N(x,y)\,y' = 0$ where $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (i.e. the ode is not exact) and none of the above three known methods for finding integrating factor were successful. This solver uses trial and error using a number of built-in common integrating factor to see if any one of them makes the ode exact.

1.2.16.1 Example

$$\begin{aligned} y dx + x\left(x^2 y - 1\right) dy &= 0 \\ M(x,y) + N(x,y)\, y' &= 0 \end{aligned}$$

Where

$$\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 3x^2 y - 1 \end{aligned}$$

Hence not exact. Trying the above 3 methods shows it is not possible to find an integrating factor. But by inspection let $I = \frac{y}{x^3}$. Then the ode becomes

$$\begin{aligned} yI dx + Ix\left(x^2 y - 1\right) dy &= 0 \\ y\frac{y}{x^3} dx + \frac{y}{x^3} x\left(x^2 y - 1\right) dy &= 0 \\ \frac{y^2}{x^3} dx + \left(y^2 - \frac{y}{x^2}\right) dy &= 0 \\ M(x,y) + N(x,y)\, y' &= 0 \end{aligned}$$

Where

$$
\begin{aligned}
M &= \frac{y^2}{x^3} \\
N &= \left(y^2 - \frac{y}{x^2}\right)
\end{aligned}
$$

Now we see that the ode is exact by checking:

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{2y}{x^3} \\
\frac{\partial N}{\partial x} &= -\left(-2\frac{y}{x^3}\right) = \frac{2y}{x^3}
\end{aligned}
$$

Since ode is now exact, we need to find ϕ from

$$
\frac{\partial \phi}{\partial x} = M \tag{3}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{4}
$$

From (3)

$$
\frac{\partial \phi}{\partial x} = \frac{y^2}{x^3}
$$

Therefore

$$
\begin{aligned}
\phi &= \int M dx + f(y) \\
&= \int \frac{y^2}{x^3} dx + f(y) \\
&= y^2 \int x^{-3} dx + f(y) \\
&= y^2 \frac{x^{-2}}{-2} + f(y) \\
&= \frac{y^2}{-2x^2} + f(y)
\end{aligned} \tag{5}
$$

Where $f(y)$ is arbitrary function to be found. Taking derivative of the above w.r.t. y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{d}{dy}\left(-\frac{y^2}{2x^2} + f(y)\right) \\
&= -\frac{y}{x^2} + f'(y)
\end{aligned}
$$

Comparing the above to (4) shows that

$$
\begin{aligned}
N &= -\frac{y}{x^2} + f'(y) \\
y^2 - \frac{y}{x^2} &= -\frac{y}{x^2} + f'(y) \\
f'(y) &= y^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(y) &= \int y^2 dy \\
&= \frac{y^3}{3} + c
\end{aligned}
$$

Substituting this into (5) gives

$$
\begin{aligned}
\phi &= \frac{y^2}{-2x^2} + f(y) \\
&= \frac{y^2}{-2x^2} + \frac{y^3}{3} + c
\end{aligned}
$$

Since ϕ is also constant function then we can simplify the above to

$$
\begin{aligned}
\frac{y^2}{-2x^2} + \frac{y^3}{3} &= C \\
3y^2 - 2x^2y^3 &= 6x^2C \\
3y^2 - 2x^2y^3 &= x^2C_1
\end{aligned}
$$

1.2.17 Riccati ode $y' = f_0 + f_1 y + f_2 y^2$

1.2.17.1 Direct solution of Riccati

There is no general method to solve the general Riccati ode. These are special cases to try

1.2.17.1.1 Case 1

If f_0, f_1, f_2 are constants then this is separable ode and can easily be solved.

1.2.17.1.2 Case 2

If $f_1 = 0$ then we have the reduced Riccati

$$y' = f_0(x) + f_2(x)\,y^2$$

For the case of $f_0 = cx^n$ where c_1 is constant and $f_2 = c_2$ is also a constant, then the above becomes

$$y' = c_1 x^n + c_2 y^2$$

Now it depends on n. The case of $n = -2$ is $y' = \frac{c_1}{x^2} + c_2 y^2$ can be solved using the substitution $y = \frac{1}{u}$. Hence $y' = -\frac{u'}{u^2}$ and the ode becomes

$$\begin{aligned}
-\frac{u'}{u^2} &= \frac{c_1}{x^2} + c_2\frac{1}{u^2}\\
-u' &= c_1\frac{u^2}{x^2} + c_2\\
u' &= -c_1\frac{u^2}{x^2} - c_2
\end{aligned}$$

Which is first order Homogeneous ode type (see earlier section).

The case of $n = -4k(2k-1)$ where $k = 0, \pm 1, \pm 2, \cdots$ are all solvable by algebraic, exponential and logarithmic function. For all other values, Liouville proved no solution exist in terms of elementary functions. These n values come out to be $n = \{\cdots, -112, -60, -24, -4, 0, -12, -40, -84, \cdots\}$. For example for $n = -4$

$$y' = \frac{c_1}{x^4} + c_2 y^2$$

This is solved by converting to second order ode using $y = \frac{-u'}{c_2 u}$ which result in ode which can be solved as Bessel ode. Similarly for all other n values listed above. I need to look into this. When I tried $n = -3$ I also got solution in terms of Bessel functions. So what is the difference?

1.2.17.1.3 Case 3

Assume we can find a particular solution y_1 to the general Riccati ode $y' = f_0(x) + f_1(x)\,y + f_2(x)\,y^2$. Then let $y = y_1 + u$. The Riccati ode becomes a Bernoulli ode.

$$\begin{aligned}
(y_1 + u)' &= f_0 + f_1(y_1 + u) + f_2(y_1 + u)^2\\
y_1' + u' &= f_0 + f_1 y_1 + f_1 u + f_2\left(y_1^2 + u^2 + 2y_1 u\right)\\
y_1' + u' &= f_0 + f_1 y_1 + f_1 u + f_2 y_1^2 + f_2 u^2 + 2 f_2 y_1 u\\
y_1' + u' &= \overbrace{f_0 + f_1 y_1 + f_2 y_1^2} + f_1 u + f_2 u^2 + 2 f_2 y_1 u\\
u' &= f_1 u + f_2 u^2 + 2 f_2 y_1 u\\
&= u(f_1 + 2 f_2 y_1) + f_2 u^2
\end{aligned}$$

Which is Bernoulli ode. But this assumes we are able to find particular solution y_1 to the general Riccati ode. There is no method to do that. So this case will not be tried.

1.2.17.1.4 References used

1. `https://mathworld.wolfram.com/RiccatiDifferentialEquation.html`
2. `https://math24.net/riccati-equation.html`
3. `https://encyclopediaofmath.org/wiki/Riccati_equation`
4. `https://www.youtube.com/watch?v=iuHDmZ8VutM`
5. paper: Methods of Solution of the Riccati Differential Equation. By D. Robert Haaheim and F. Max Stein. 1969

1.2.17.2 Conversion of Riccati to second order ode

ode internal name "riccati"

Solved using transformation $y = \frac{-u'}{f_2 u}$ which generates second order ode in u. This is solved for u (if possible) then y is found.

1.2.18 Abel first kind ode $y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$

ode internal name "abelFirstKind"

Currently the program detect this ODE and evaluates the Abel invariant only. This ODE has the form

$$y'(x) = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 \tag{1}$$

Any of the following forms is called an Abel ode of first kind

$$\begin{aligned}
y' &= f_0 + f_1 y + f_2 y^2 + f_3 y^3 \\
y' &= f_1 y + f_2 y^2 + f_3 y^3 \\
y' &= f_2 y^2 + f_3 y^3 \\
y' &= f_0 + f_2 y^2 + f_3 y^3 \\
y' &= f_0 + f_3 y^3 \\
y' &= f_0 + f_1 y + f_3 y^3 \\
y' &= f_2 y^2 + f_3 y^3
\end{aligned}$$

The case for both $f_0(x)=0, f_2(x)=0$ is not allowed, else it becomes Bernoulli ode. Either $f_0=0$ or $f_2=0$ is allowed but not both at same time. The term $f_3(x)$ must be there in all cases. When $f_2=0$ then Abel invariant is given by

$$\Delta = -\frac{\left(-f_0' f_3 + f_0 f_3' + 3 f_0 f_3 f_1\right)^3}{27 f_3^4 f_0^5}$$

In the case when $f_2 \neq 0$, then f_2 is removed from the original ode using the change of dependent variable $y = u(x) - \frac{f_2}{3f_3}$. Now the new ode will not have f_2 in it, and the above invariant can now be applied to it.

There are two possibilities. Δ can be constant (does not depend on x) or not constant (i.e. function of x). The constant invariant is the easier case and can be solved. The non constant case is not fully solved and only few cases can be solved analytically.

1.2.18.1 Solution method

Find what is called the abel invariant and check if constant.

$$\Delta = -\frac{\left(-f_0' f_3 + f_0 f_3' + 3 f_0 f_3 f_1\right)^3}{27 f_3^4 f_0^5}$$

The substitution $y=\frac{1}{u}$ is now applied. Therefore $y' = -\frac{1}{u^2}u'$. Substituting this in (1) gives

$$\begin{aligned}
-\frac{1}{u^2}u' &= f_0(x) + f_1(x)\frac{1}{u} + f_2(x)\frac{1}{u^2} + f_3(x)\frac{1}{u^3}\\
-uu' &= u^3 f_0(x) + u^2 f_1(x) + u f_2(x) + f_3(x)\\
uu' &= -u^3 f_0(x) - u^2 f_1(x) - u f_2(x) - f_3(x) \qquad (2)
\end{aligned}$$

Using the substitution $u = \frac{1}{E}\left(y + \frac{f_2}{3f_3}\right)$ where $E = \exp\left(\int f_1 - \frac{f_2^2}{3f_3}dx\right)$ in the above gives

$$\frac{1}{E}\left(y + \frac{f_2}{3f_3}\right)u' = -u^3 f_0(x) - u^2 f_1(x) - u f_2(x) - f_3(x)$$

Hence

$$\begin{aligned}
u' &= \frac{1}{E^2}\frac{dE}{dx}\left(y+\frac{f_2}{3f_3}\right)+\frac{1}{E}\left(y'+\frac{1}{3}\frac{f_2'f_3-f_2f_3'}{f_3^2}\right)\\
&= \frac{1}{E^2}\frac{dE}{dx}\left(\frac{1}{u}+\frac{f_2}{3f_3}\right)+\frac{1}{E}\left(-\frac{1}{u^2}u'+\frac{1}{3}\frac{f_2'f_3-f_2f_3'}{f_3^2}\right)\\
u'+\frac{u'}{Eu^2} &= \frac{1}{E^2}\frac{dE}{dx}\left(\frac{1}{u}+\frac{f_2}{3f_3}\right)+\frac{1}{3E}\frac{f_2'f_3-f_2f_3'}{f_3^2}\\
u'\left(1+\frac{1}{Eu^2}\right) &= \frac{1}{E^2}\frac{dE}{dx}\left(\frac{1}{u}+\frac{f_2}{3f_3}\right)+\frac{1}{3E}\frac{f_2'f_3-f_2f_3'}{f_3^2}\\
u' &= \frac{Eu^2}{1+Eu^2}\left(\frac{1}{E^2}\frac{dE}{dx}\left(\frac{1}{u}+\frac{f_2}{3f_3}\right)+\frac{1}{3E}\frac{f_2'f_3-f_2f_3'}{f_3^2}\right)\\
u' &= \frac{u^2}{1+Eu^2}\left(\frac{1}{E}\frac{dE}{dx}\left(\frac{1}{u}+\frac{f_2}{3f_3}\right)+\frac{1}{3}\frac{f_2'f_3-f_2f_3'}{f_3^2}\right)
\end{aligned}$$

Substituting the above into (2) gives

$$u\frac{u^2}{1+Eu^2}\left(\frac{1}{E}\frac{dE}{dx}\left(\frac{1}{u}+\frac{f_2}{3f_3}\right)+\frac{1}{3}\frac{f_2'f_3-f_2f_3'}{f_3^2}\right)=-u^3f_0-u^2f_1-uf_2-f_3$$

Therefore

$$\begin{aligned}
E &= \exp\left(\int f_1(x)-\frac{f_2^2(x)}{3f_3\,(x)}dx\right)\\
\xi &= \int f_3(x)\,E^2dx\\
u &= \frac{1}{E}\left(y+\frac{f_2(x)}{3f_3\,(x)}\right)
\end{aligned}$$

The above are used to convert the first kind Abel ode to canonical form. (To finish).

1.2.18.2 About equivalence between two Abel ode's

Given one Abel ode $y'(x)=f_0(x)+f_1(x)y+f_2(x)y^2+f_3(x)y^3$, it is called equivalent to another Abel ode $u'(t)=g_0(t)+g_1(t)u+g_2(t)u^2+g_3(t)u^3$ if there is *transformation* which converts one to the other. This transformation is given by

$$\begin{aligned}
x &= F(t) \qquad (1)\\
y(x) &= P(t)\,u(t)+Q(t)
\end{aligned}$$

Where $F'\neq 0, P\neq 0$. If such transformation can be found, then if given the solution of one of these ode's, the solution to the other ode can directly be fond using this transformation. In this case, we also call these two ode as belonging to same Abel equivalence class. In other words, an Abel equivalence class is the set of all Abel ode's that can be transformed to each others using the same transformation given in (1).

There are many disjoint Abel equivalence classes, each class will have all the ode that can be transformed to each others using some specific transformation (1). Here is one example below taken from paper by A.D.Roch and E.S.Cheb-Terrab called "Abel ODEs: Equivalence and integrable classes".

Given one Abel ode

$$y'(x) = \frac{1}{2x+8}y^2 + \frac{x}{2x+8}y^3 \tag{2}$$

Which is known to have solution

$$c_1 + \frac{\sqrt{y^2x - 4y - 1}}{y} + 2\arctan\left(\frac{1+2y}{\sqrt{y^2x-4y-1}}\right) = 0 \tag{3}$$

And now we are given a second Abel ode

$$u'(t) = \frac{1}{t}u + \frac{f't - f}{2(f+3t)}u^2 + \frac{(f't-f)(t-f)}{2(f+3t)}u^3 \tag{4}$$

And asked to find its solution. If we can determine if (4) is equivalent to (2) then the solution of (4) can be obtained directly. It can be found that

$$\begin{aligned} F(t) &= \frac{f(t)}{t} - 1 \\ Q(t) &= 0 \\ P(t) &= t \end{aligned}$$

Where see that $F'(t) \neq 0$ and $P(t) \neq 0$. Hence (1) becomes

$$\begin{aligned} x &= \frac{f(t)}{t} - 1 \\ y(x) &= tu(t) \end{aligned} \tag{5}$$

Applying the transformation (5) on the solution (3) results in the solution of (4) as

$$A = \sqrt{\left(\frac{f}{t} - 1\right)t^2u^2 - 4tu - 1}$$

$$c_1 + \frac{A}{tu} + 2\arctan\left(\frac{1+2tu}{A}\right) = 0 \tag{6}$$

Equation (6) above is the implicit solution to (4) obtained from the solution to (2) by using equivalence transformation as the two ode's are found to be equivalent. Finding the transformation (5) requires more calculation and not trivial. See the above paper for more information.

1.2.19 differential type ode $y' = f(x,y)$

ode internal name "differentialType"

These are special case ode where the ode can be written as complete differential $d(f(y)) = d(g(x))$ which is then solved by just integrating.

1.2.19.1 Example 1

$$\begin{aligned}\frac{dy}{dx} &= \frac{x-y}{x+y}\\(x+y)\,dy &= (x-y)\,dx\\xdy+ydy &= (x-y)\,dx\\ydy &= -xdy+xdx-ydx \qquad (1)\end{aligned}$$

But RHS is complete differential because

$$-xdy+xdx-ydx = d\left(\frac{1}{2}x^2-xy\right)$$

Hence (1) becomes

$$ydy = d\left(\frac{1}{2}x^2-xy\right)$$

Integrating

$$\begin{aligned}\int ydy &= \int d\left(\frac{1}{2}x^2-xy\right)\\\frac{1}{2}y^2 &= \frac{1}{2}x^2-xy+c\\y^2 &= x^2-2xy+2c\end{aligned}$$

Which is an implicit solution. This method works if it is possible by the solver to detect that the ode can be written as complete differentials or not.

1.2.19.2 Example 2

$$\begin{aligned}\frac{dy}{dx} &= -\frac{y}{x}+x^2\\dy &= \left(\frac{-y+x^3}{x}\right)dx\\xdy &= -ydx+x^3dx\\0 &= -xdy-ydx+x^3dx \qquad (1)\end{aligned}$$

But RHS is complete differential because

$$-xdy - ydx + x^3dx = d\left(\frac{x^4}{4} - xy\right)$$

Hence (1) becomes

$$0 = d\left(\frac{x^4}{4} - xy\right)$$

Integrating gives

$$0 = \frac{x^4}{4} - xy + c$$

solving for y gives

$$y = \frac{x^3}{4} + \frac{c}{x}$$

1.2.20 Series method

1.2.20.1 Algorithm flow chart

The algorithms are summarized in the following flow chart.

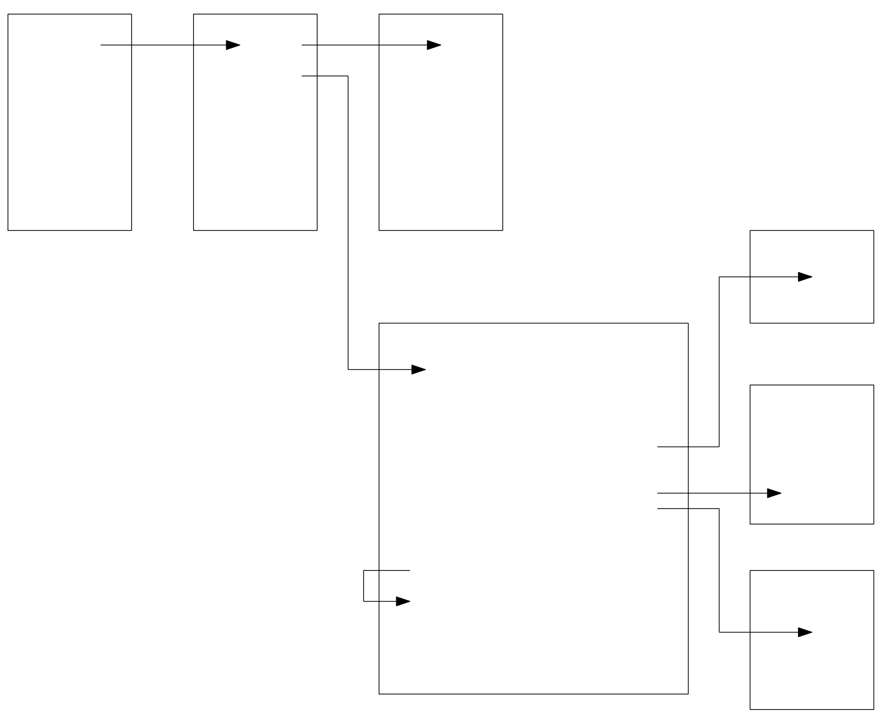

Figure 1.5: Flow chart for series solution for first order

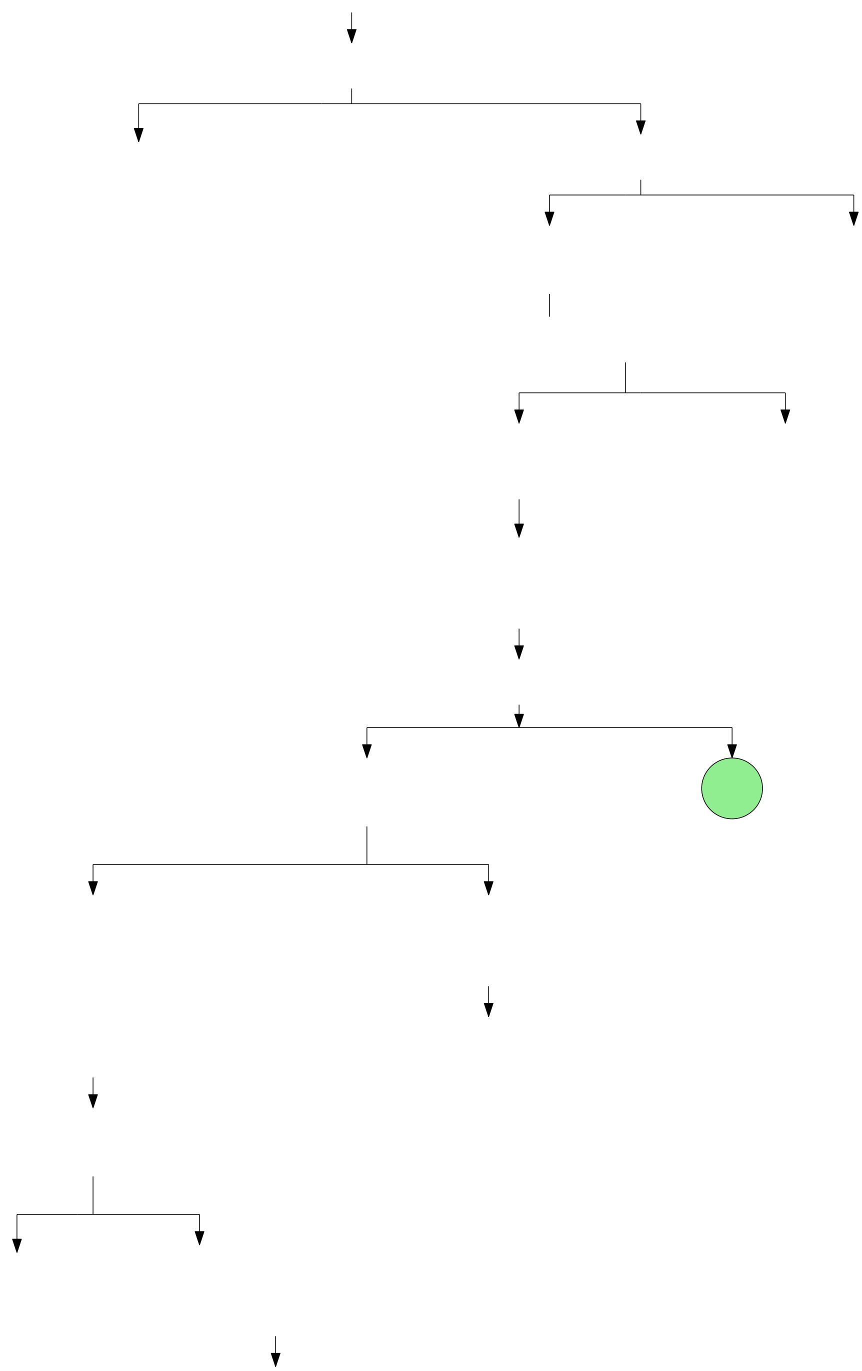

Figure 1.6: Algorithm for series solution for first orde

1.2.20.2 Algorithm pseudocode

function SOLVE__FIRST__ORDER__ODE__SERIES($y' = f(x,y)$)
    if $f(x,y)$ analytic at expansion point x_0 **then**
        Apply Taylor series defintion directly to find the series expansion. Let $y_0 = y(x_0)$ and

$$y = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x,y)\bigg|_{\substack{x=x_0\\y=y_0}}$$

        where

$$\begin{aligned} F_0 &= f(x,y) \\ F_n &= \frac{d}{dx} F_{n-1} \\ &= \frac{\partial F_{n-1}}{\partial x} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \end{aligned}$$

        return y as the solution
    else
        if $f(x,y)$ not linear in $y(x)$ **then**
            return - Not supported.
        else
            Write the ode as $y' + p(x)y = q(x)$
            if $\lim_{x\to x_0}(x - x_0)p(x)$ does not exist **then**
                return Irregular singular point. Not supported.
            else
                Regular singular point. Expand $p(x)$ in series if not already a polynomial.
                if unable to obtain series for $p(x)$ **then**
                    return Not supported.
                else
                    Use Frobenius series. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \end{aligned}$$

Figure 1.7: Algorithm for series solution for first orde

1.2.20.3 Ordinary point using standard power series method

<u>ode internal name</u> "first_order_ode_power_series_method_ordinary_point"

Expansion point is an ordinary point. Standard power series. The ode must be linear in y' and y at this time. See below for examples.

1.2.20.4 Ordinary point using Taylor series method

ode internal name "first_order_ode_taylor_series_method_ordinary_point"

Alternative method to solving the above example is given here which is to use the Taylor series method. This is derived as follows.

Let

$$y' = f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \qquad (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned} F_0 &= f(x,y) \qquad (4)\\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)F_0 \qquad (5) \end{aligned}$$

For example, for $n=1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f \end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \qquad (6)$$

See below for examples.

1.2.20.4.1 Example 1

$$y' + 2xy = x$$

Solved using power series

Expansion is around $x=0$. The (homogeneous) ode has the form $y' + p(x)\,y = 0$. We see that $p(x)$ is defined as is at $x=0$. Hence this is an ordinary point, also the RHS has series expansion at $x=0$. It is very important to check that the RHS has series expansion at $x = 0$. Otherwise this method will fail and we must use Frobenius even if $x = 0$ is ordinary point for the LHS of the ode. For example for the ode $y' + 2xy = \frac{1}{x}$ or $y' + 2xy = \sqrt{x}$ standard power series will fail. See examples below.

Using standard power series, let

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}
\end{aligned}
$$

The ode now becomes

$$
\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n &= x \\
\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1} &= x
\end{aligned}
$$

Reindex so that all powers on x are n gives

$$
\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n + \sum_{n=1}^{\infty} 2 a_{n-1} x^n = x
$$

For $n = 0$, the RHS is zero, since there is no matching term with x^0, therefore the above gives

$$
a_1 = 0
$$

For $n = 1$, the RHS is x^1 which gives

$$
\begin{aligned}
(n+1)\, a_{n+1} + 2a_{n-1} &= 1 \\
2a_2 + 2a_0 &= 1 \\
a_2 &= \frac{1 - 2a_0}{2}
\end{aligned}
$$

For $n \geq 2$ the RHS is zero and we have recurrence relation. Therefore we have

$$
(n+1)\, a_{n+1} + 2a_{n-1} = 0
$$

For $n = 2$

$$
\begin{aligned}
3a_3 + 2a_1 &= 0 \\
a_3 &= -\frac{2a_1}{3} = 0
\end{aligned}
$$

For $n = 3$

$$
\begin{aligned}
4a_4 + 2a_2 &= 0 \\
a_4 = -\frac{1}{2}a_2 = -\frac{1}{2}\left(\frac{1-2a_0}{2}\right) &= \frac{2a_0 - 1}{4}
\end{aligned}
$$

And so on. The solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= a_0 + \left(\frac{1-2a_0}{2}\right) x^2 + \left(\frac{2a_0-1}{4}\right) x^4 + \cdots \\
&= a_0\left(1 - x^2 + \frac{1}{2}x^4 + \cdots\right) + \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 + \cdots\right)
\end{aligned}
$$

Which can be written as

$$
y = y(0)\left(1 - x^2 + \frac{1}{2}x^4 + \cdots\right) + \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 + \cdots\right)
$$

Solved using Taylor series

$$
\begin{aligned}
y' + 2xy &= x \\
y' &= x - 2xy \\
&= f(x, y)
\end{aligned}
$$

For this method to work, $f(x, y)$ must be analytic at $x = x_0$, the expansion point. Let expansion point be $x = 0$. Let $y(0) = y_0$. Then

$$
y = y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x, y)\Big|_{x=0, y_0}
$$

Where $F_0 = f(x,y)$ and $F_n = \frac{\partial F_{n-1}}{\partial x} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0$. Hence

$$
\begin{aligned}
F_0 &= (x - 2xy)\\
F_1 &= \frac{d}{dx} F_0\\
&= \left(\frac{\partial F_0}{\partial x}\right) + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \left(\frac{\partial (x-2xy)}{\partial x}\right) + \left(\frac{\partial (x-2xy)}{\partial y}\right)(x-2xy)\\
&= (1-2y) - 2x(x-2xy)\\
&= 4x^2y - 2y - 2x^2 + 1\\
F_2 &= \frac{d^2}{dx^2} F_1\\
&= \left(\frac{\partial F_1}{\partial x}\right) + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \left(\frac{\partial}{\partial x}\left(4x^2y - 2y - 2x^2 + 1\right)\right) + \left(\frac{\partial}{\partial y} 4x^2y - 2y - 2x^2 + 1\right)(x-2xy)\\
&= (8xy - 4x) + \left(4x^2 - 2\right)(x - 2xy)\\
&= 12xy - 8x^3y - 6x + 4x^3\\
F_3 &= \frac{d^3}{dx^3} F_2\\
&= \left(\frac{\partial F_2}{\partial x}\right) + \left(\frac{\partial F_2}{\partial y}\right) F_0\\
&= \left(\frac{\partial}{\partial x}\left(12xy - 8x^3y - 6x + 4x^3\right)\right) + \left(\frac{\partial}{\partial y}\left(12xy - 8x^3y - 6x + 4x^3\right)\right)(x-2xy)\\
&= 12y - 24x^2y - 6 + 12x^2 + \left(12x - 8x^3\right)(x - 2xy)\\
&= 12y - 48x^2y + 16x^4y + 24x^2 - 8x^4 - 6
\end{aligned}
$$

And so on. Evaluating the above at $x = 0, y = y_0$ gives

$$
\begin{aligned}
F_0 &= 0\\
F_1 &= -2y_0 + 1\\
F_2 &= 0\\
F_3 &= 12y_0 - 6
\end{aligned}
$$

Hence

$$\begin{aligned}
y &= y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x,y)\big|_{x=0,y_0} \\
&= y_0 + xF_0 + \frac{x^2}{2}F_1 + \frac{x^3}{6}F_2 + \frac{x^4}{24}F_3 + \cdots \\
&= y_0 + 0 + \frac{x^2}{2}(-2y_0+1) + 0 + \frac{x^4}{24}(12y_0-6) + \cdots \\
&= y_0 - 2y_0\frac{x^2}{2} + \frac{x^2}{2} + \frac{1}{2}y_0x^4 - \frac{x^4}{4} + \\
&= y_0\left(1 - x^2 + \frac{1}{2}x^4\right) + \frac{x^2}{2} - \frac{x^4}{4} + \cdots
\end{aligned}$$

1.2.20.4.2 Example 2

Solved using Taylor series

Another example using Taylor series method.

$$\begin{aligned}
y' + 2xy &= 1 + x + x^2 \\
y' &= 1 + x + x^2 - 2xy \\
&= f(x,y)
\end{aligned}$$

Let expansion point be $x = 0$. Let $y(0) = y_0$. Then

$$y = y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x,y)\big|_{x=0,y_0}$$

Where $F_0 = f(x,y)$ and $F_n = \frac{\partial F_{n-1}}{\partial x} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0$. Hence

$$\begin{aligned}
F_0 &= 1 + x + x^2 - 2xy \\
F_1 &= \left(\frac{\partial F_0}{\partial x}\right) + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= 1 + 2x - 2y + (-2x)\left(1 + x + x^2 - 2xy\right) \\
&= 4x^2y - 2y - 2x^2 - 2x^3 + 1 \\
F_2 &= \left(\frac{\partial F_1}{\partial x}\right) + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \left(8xy - 4x - 6x^2\right) + \left(4x^2 - 2\right)(x - 2xy) \\
&= 12xy - 8x^3y - 6x - 6x^2 + 4x^3 \\
F_3 &= \left(\frac{\partial F_2}{\partial x}\right) + \left(\frac{\partial F_2}{\partial y}\right) F_0 \\
&= 12y - 24x^2y - 6 - 12x + 12x^2 + \left(12x - 8x^3\right)\left(1 + x + x^2 - 2xy\right) \\
&= 12y - 48x^2y + 16x^4y + 24x^2 + 4x^3 - 8x^4 - 8x^5 - 6
\end{aligned}$$

And so on. Evaluating the above at $x = 0, y = y_0$ gives

$$\begin{aligned}
F_0 &= 1 \\
F_1 &= -2y_0 + 1 \\
F_2 &= 0 \\
F_3 &= 12y_0 - 6
\end{aligned}$$

Hence

$$\begin{aligned}
y &= y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x,y)\big|_{x=0,y_0} \\
&= y_0 + F_0 x + F_1 \frac{x^2}{2} + F_2 \frac{x^3}{6} + F_3 \frac{x^4}{24} + \cdots \\
&= y_0 + x + (-2y_0 + 1)\frac{x^2}{2} + (12y_0 - 6)\frac{x^4}{24} + \cdots \\
&= y_0\left(1 - x^2 + \frac{1}{2}x^4 + \cdots\right) + \left(x + \frac{1}{2}x^2 - \frac{1}{4}x^4 + \cdots\right)
\end{aligned}$$

1.2.20.4.3 Example 3

Solved using Taylor series

$$
\begin{aligned}
y' + 2xy^2 &= 1 + x + x^2 \\
y' &= 1 + x + x^2 - 2xy^2 \\
&= f(x, y)
\end{aligned}
$$

Let expansion point be $x = 0$. Let $y(0) = y_0$. Then

$$
y = y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x,y)\big|_{x=0,y_0}
$$

Where $F_0 = f(x,y)$ and $F_n = \frac{\partial F_{n-1}}{\partial x} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0$. Hence

$$
\begin{aligned}
F_0 &= 1 + x + x^2 - 2xy^2 \\
F_1 &= \left(1 + 2x - 2y^2\right) + (-4xy)\left(1 + x + x^2 - 2xy^2\right) \\
&= -4x^3y + 8x^2y^3 - 4x^2y - 4xy + 2x - 2y^2 + 1 \\
F_2 &= \left(\frac{\partial F_1}{\partial x}\right) + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \left(-12x^2y + 16xy^3 - 8xy - 4y + 2\right) + \left(-4x^3 + 24x^2y^2 - 4x^2 - 4x - 4y\right)\left(1 + x + x^2 - 2xy^2\right) \\
&= -4x^5 + 32x^4y^2 - 8x^4 - 48x^3y^4 + 32x^3y^2 - 12x^3 + 32x^2y^2 - 16x^2y - 8x^2 + 24xy^3 - 12xy - 4x - 8
\end{aligned}
$$

$$
F_3 = \left(\frac{\partial F_2}{\partial x}\right) + \left(\frac{\partial F_2}{\partial y}\right) F_0
$$

And so on. Evaluating the above at $x = 0, y = y_0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= -2y_0^2 + 1 \\
F_2 &= -8y_0 + 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x,y)\big|_{x=0,y_0} \\
&= y_0 + F_0 x + F_1 \frac{x^2}{2} + F_2 \frac{x^3}{6} + F_3 \frac{x^4}{24} + \cdots \\
&= y_0 + x + \left(-2y_0^2 + 1\right)\frac{x^2}{2} + (-8y_0 + 2)\frac{x^3}{6} + \cdots \\
&= y_0\left(1 - \frac{4}{3}x^3 + \cdots\right) + y_0^2\left(-x^2 + \cdots\right) + \cdots + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right)
\end{aligned}
$$

1.2.20.4.4 Example 4

Solved using power series

$$y' + y = \sin x$$

Expansion is around $x = 0$. The (homogenous) ode has the form $y' + p(x)\,y = 0$. We see that $p(x)$ is defined as is at $x = 0$. Hence this is ordinary point, also the RHS has series expansion at $x = 0$.

Let $y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$. The ode becomes

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sin x$$

Indexing so all powers of x start at n gives

$$\sum_{n=0}^{\infty} (n+1)\,a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = \sin x$$

Expanding $\sin x$ in series gives

$$\sum_{n=0}^{\infty} (n+1)\,a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

For $n = 0$, there is no term on RHS with x^0, hence we obtain

$$\begin{aligned} a_1 + a_0 &= 0 \\ a_1 &= -a_0 \end{aligned}$$

For $n = 1$ there is one term x^1 on RHS, hence

$$\begin{aligned} 2a_2 + a_1 &= 1 \\ a_2 &= \frac{1 - a_1}{2} = \frac{1 + a_0}{2} \end{aligned}$$

For $n = 2$ there is no term on RHS with x^2 hence

$$\begin{aligned} 3a_3 + a_2 &= 0 \\ a_3 &= -\frac{a_2}{3} = -\frac{\frac{1+a_0}{2}}{3} = -\frac{1}{6}a_0 - \frac{1}{6} \end{aligned}$$

For $n = 3$ there is term $-\frac{1}{6}x^3$ on RHS, hence

$$\begin{aligned} 4a_4 + a_3 &= -\frac{1}{6} \\ a_4 &= \frac{-\frac{1}{6} - a_3}{4} = \frac{-\frac{1}{6} - \left(-\frac{1}{6}a_0 - \frac{1}{6}\right)}{4} = \frac{1}{24}a_0 \end{aligned}$$

And so on. The solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + \cdots \\
&= a_0 - a_0 x + \left(\frac{1+a_0}{2}\right) x^2 + \left(-\frac{1}{6}a_0 - \frac{1}{6}\right) x^3 + \left(\frac{1}{24}a_0\right) x^4 + \cdots \\
&= a_0\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \cdots\right) + \left(\frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots\right)
\end{aligned}
$$

1.2.20.5 Regular singular point using Frobenius series method.

ode internal name "first_order_ode_series_method_regular_singular_point"

expansion point is a regular singular point. Standard power series. The ode must be linear in y' and y at this time.

1.2.20.5.1 Example 1

$$
y' + 2xy = \sqrt{x}
$$

Expansion is around $x = 0$. The (homogenous) ode has the form $y' + p(x)\,y = 0$. We see that $p(x)$ is analytic at $x = 0$. However the RHS has no series expansion at $x = 0$ (not analytic there). Therefore we must use Frobenius series in this case. Let

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}
\end{aligned}
$$

The (homogenous) ode becomes

$$
\begin{aligned}
\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + 2x \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r+1} &= 0
\end{aligned}
$$

Reindex so all powers on x are the lowest gives

$$
\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r-1} = 0 \tag{1}
$$

For $n=0$, Eq(1) gives

$$ra_0x^{r-1}=0$$

Hence $r=0$ since $a_0\neq 0$. Therefore the balance equation is

$$mc_0x^{m-1}=\sqrt{x}$$

Where r is replaced my m and a_n is replaced by c_n. The above will used below to find y_p. For $n=1$, Eq(1) gives

$$\begin{aligned}(1+r)\,a_1x^r&=0\\ a_1&=0\end{aligned}$$

For $n\geq 2$ the recurrence relation is from (1)

$$\begin{aligned}(n+r)\,a_n+2a_{n-2}&=0\\ a_n&=-\frac{2a_{n-2}}{(n+r)}\end{aligned}\tag{2}$$

Or for $r=0$ the above simplifies to

$$a_n=-\frac{2}{n}a_{n-2}\tag{2A}$$

Eq (2A) is what is used to find all a_n for For $n\geq 2$. Hence for $n=2$ and remembering that $a_0=1$ gives

$$a_2=-1$$

For $n=3$

$$a_3=-\frac{2}{3}a_1=0$$

For $n=4$

$$a_4=-\frac{1}{2}a_2=\frac{1}{2}$$

For $n=5,7,\cdots$ and all odd n then $a_n=0$. For $n=6$

$$a_6=-\frac{1}{3}a_4=-\frac{1}{6}$$

And so on. Hence (using $a_0=1$)

$$\begin{aligned}y_h&=c_1\sum_{n=0}^{\infty}a_nx^{n+r}\\ &=c_1\sum_{n=0}^{\infty}a_nx^n\\ &=c_1\left(a_0+a_1x+a_2x^2+a_3x^3+\cdots\right)\\ &=c_1\left(1-x^2+\frac{1}{2}x^4-\frac{1}{6}x^6+\cdots\right)\end{aligned}$$

Now we need to find y_p using the balance equation. From above we found that

$$ra_0x^{r-1} = x^{\frac{1}{2}}$$

Renaming a to c and r as m so not to confuse terms used for y_h, the above becomes

$$mc_0x^{m-1} = x^{\frac{1}{2}}$$

Hence $m - 1 = \frac{1}{2}$ or $m = \frac{3}{2}$. Therefore $mc_0 = 1$ or $c_0 = \frac{2}{3}$. Now we can find the series for y_p using

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= x^{\frac{3}{2}} \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

To find c_m we use the same recurrence relation found for y_h but change r to m and a to c. From above we found

$$(n + r)\, a_n + 2a_{n-2} = 0$$

Hence it becomes

$$(n + m)\, c_n + 2c_{n-2} = 0$$

The above is valid for $n \geq 2$. For $n = 0$ we have found c_0 already. For c_1 using the above $ra_1 = 0$ hence it becomes $mc_1 = 0$ which implies

$$c_1 = 0$$

since $m \neq 0$. Now we are ready to find few c_n terms. The above recurrence relation becomes for $m = \frac{3}{2}$

$$\begin{aligned} \left(n + \frac{3}{2}\right) c_n + 2c_{n-2} &= 0 \\ c_n &= \frac{-2c_{n-2}}{\left(n + \frac{3}{2}\right)} \end{aligned}$$

Hence for $n = 2$

$$c_2 = \frac{-2c_0}{\left(2 + \frac{3}{2}\right)} = \frac{-2\left(\frac{2}{3}\right)}{\left(2 + \frac{3}{2}\right)} = -\frac{8}{21}$$

For $n = 3$

$$c_3 = \frac{-2c_1}{\left(3 + \frac{3}{2}\right)} = 0$$

For $n=4$

$$c_4 = \frac{-2c_2}{\left(4+\frac{3}{2}\right)} = \frac{-2\left(-\frac{8}{21}\right)}{\left(4+\frac{3}{2}\right)} = \frac{32}{231}$$

And so on. Hence

$$\begin{aligned} y_p &= x^{\frac{3}{2}} \sum_{n=0}^{\infty} c_n x^n \\ &= x^{\frac{3}{2}} \left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots\right) + x^{\frac{3}{2}} \left(\frac{2}{3} + -\frac{8}{21}x^2 + \frac{32}{231}x^4 - \cdots\right) \end{aligned}$$

1.2.20.5.2 Example 2

$$y' + 2xy = \frac{1}{x}$$

Expansion is around $x=0$. The (homogenous) ode has the form $y' + p(x)\,y = 0$. We see that $p(x)$ is defined as is at $x=0$. However the RHS has no series expansion at $x=0$. Therefore we must use Frobenius series. This is the same ode as example 1. So we go straight to find y_p as y_h is the same. Now we need to find y_p using the balance equation. From above we found that

$$r a_0 x^{r-1} = \frac{1}{x}$$

Renaming a to c and r as m so not to confuse terms used for y_h, the above becomes

$$m c_0 x^{m-1} = x^{-1}$$

Hence $m-1=-1$ or $m=0$. Therefore $mc_0 = 1$. But since $m=0$ then no solution for c_0. Hence it is not possible to find series solution. This is an example where the balance equation fails and so we have to use asymptotic expansion to find solution, which is not supported now.

1.2.20.5.3 Example 3

$$y' = \frac{1}{x}$$

Expansion is around $x = 0$. The (homogenous) ode has the form $y' + p(x)\,y = 0$. We see that $p(x) = 0$ is analytic at $x = 0$. However the RHS has no series expansion at $x = 0$ (not analytic there). Therefore we must use Frobenius series in this case. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \end{aligned}$$

The (homogenous) ode becomes

$$\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} = 0 \tag{1}$$

For $n = 0$

$$r a_0 x^{r-1} = 0$$

Hence $r = 0$ since $a_0 \neq 0$. Therefore the ode satisfies

$$y' = r a_0 x^{r-1}$$

Eq (1) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} n a_n x^{n-1} &= 0 \\ n a_n x^{n-1} &= 0 \end{aligned} \tag{2}$$

Therefore for all $n \geq 1$ we have $a_n = 0$. Hence

$$y_h = a_0$$

Now we need to find y_p using the balance equation. From above we found that

$$r a_0 x^{r-1} = \frac{1}{x}$$

Changing r to m and a_0 to c_0 so not to confuse notation gives

$$m c_0 x^{m-1} = x^{-1}$$

Hence $m - 1 = -1$ or $m = 0$. Therefore there is no solution for c_0. Unable to find y_p therefore no series solution exists. Asymptotic methods are needed to solve this. Mathematica AsymptoticDSolveValue gives the solution as $y(x) = c + \ln x$.

1.2.20.5.4 Example 4

$$y' = \frac{1}{x^2}$$

Expansion is around $x = 0$. The (homogenous) ode has the form $y' + p(x)\,y = 0$. We see that $p(x) = 0$ is analytic at $x = 0$. However the RHS has no series expansion at $x = 0$ (not analytic there). Therefore we must use Frobenius series in this case. Let

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$
$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

The (homogenous) ode becomes

$$\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} = 0 \tag{1}$$

For $n = 0$

$$r a_0 x^{r-1} = 0$$

Hence $r = 0$ since $a_0 \neq 0$. Therefore the balance equation is

$$r a_0 x^{r-1} = \frac{1}{x^2}$$

Or by changing r to m and a_0 to c_0 so not to confuse notation with y_h gives

$$m c_0 x^{m-1} = x^{-2} \tag{2}$$

Eq (1) becomes, where $r = 0$ now

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = 0$$
$$n a_n x^{n-1} = 0 \tag{2}$$

$n = 0$ is not used since that was used to find r. Therefore we start from $n = 1$. For all $n \geq 1$ we see from (2) that $a_n = 0$. Hence

$$y_h = c_1(a_0 + O(x))$$

Letting $a_0 = 1$ the above becomes

$$y_h = c_1(1 + O(x))$$

Now we need to find y_p using the balance equation. From (2) above we found that

$$mc_0x^{m-1} = x^{-2}$$

To balance, we need $m-1=-2$ or $m=-1$ and $mc_0 = 1$ or $c_0=-1$. Therefore

$$y_p = x^m \sum_{n=0}^{\infty} c_0 x^n$$

Where $c_0=-1$ and all c_n for $n \geq 1$ are found using the recurrence relation from finding y_h. But from above we found that all $a_n = 0$ for $n \geq 1$. Hence $c_n = 0$ also for $n \geq 1$. Therefore

$$\begin{aligned} y_p &= x^m c_0 \\ &= \frac{-1}{x} + O(x^2) \end{aligned}$$

Hence the solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1(1+O(x^2)) + \left(\frac{-1}{x} + O(x^2)\right) \end{aligned}$$

If we to ignore the big O, the above becomes

$$y = c_1 - \frac{1}{x}$$

To verify, we see that $y' = \frac{1}{x^2}$.

1.2.20.5.5 Example 5

$$y' + \frac{y}{x} = 0$$

Expansion is around $x=0$. The (homogenous) ode has the form $y'+p(x)\,y=0$. We see that $p(x)=\frac{1}{x}$ is not analytic at $x=0$ but $\lim_{x\to 0} xp(x) = 0$ is analytic. Therefore we must use Frobenius series in this case. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \qquad \text{(A)} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \end{aligned}$$

The ode becomes

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\frac{1}{x}\sum_{n=0}^{\infty}a_n x^{n+r}&=0\\
\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r-1}&=0\\
\sum_{n=0}^{\infty}\left((n+r)\,a_n+a_n\right)x^{n+r-1}&=0\\
\sum_{n=0}^{\infty}(n+r+1)\,a_n x^{n+r-1}&=0 \qquad (1)
\end{aligned}$$

For $n=0$

$$(r+1)\,a_0=0$$

Hence $r=-1$ since $a_0\neq 0$. Eq (1) becomes, where $r=-1$ now

$$\begin{aligned}
\sum_{n=0}^{\infty}n a_n x^{n}&=0\\
n a_n x^{n-1}&=0 \qquad (2)
\end{aligned}$$

$n=0$ is not used since that was used to find r. Therefore we start from $n=1$. For $n=1$ the above gives $a_1=0$ and same for all $n\geq 1$. Hence from Eq (A), since $y=\sum_{n=0}^{\infty}a_n x^{n+r}$ then (note: When there is only one $\sum$ term left in (1) as in this case, then this means there is no recurrence relation and all $a_n=0$ for $n>0$).

$$\begin{aligned}
y&=c_1\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)\\
&=c_1\left(\sum_{n=0}^{\infty}a_n x^{n-1}\right)\\
&=c_1\left(a_0 x^{-1}+0+0+\cdots+O(x)\right)
\end{aligned}$$

Letting $a_0=1$ the above becomes

$$y=c_1\left(x^{-1}+O(x)\right)$$

1.2.20.6 irregular singular point

ode internal name "first order ode series method. Irregular singular point"

expansion point is an irregular singular point. Not supported.

1.2.21 Laplace method

ode internal name "first_order_laplace"

These are ode's solved using Laplace method. Currently only linear constant coefficient is supported.

1.2.21.1 Examples

1.2.21.1.1 Example 1

$$y' - 2y = 6e^{5t}$$

With initial conditions $y(0) = 3$. Taking the Laplace transform gives

$$\begin{aligned}
\mathcal{L}(y) &= Y(s) \\
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}\left(6e^{5t}\right) &= \frac{6}{s-5}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
sY(s) - y(0) - 2Y(s) &= \frac{6}{s-5} \\
Y(s)\,(s-2) - y(0) &= \frac{6}{s-5} \\
Y(s)\,(s-2) &= \frac{6}{s-5} + y(0) \\
Y(s)\,(s-2) &= \frac{6}{s-5} + 3 \\
Y(s)\,(s-2) &= \frac{6+3(s-5)}{s-5} \\
Y(s)\,(s-2) &= \frac{3s-9}{s-5} \\
Y(s) &= \frac{3s-9}{(s-5)\,(s-2)} \\
&= \frac{2}{s-5} + \frac{1}{s-2}
\end{aligned}$$

Applying inverse Laplace transform and using $\mathcal{L}^{-1}\left(\frac{2}{s-5}\right) = 2e^{5t}, \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$ then the above gives

$$y(t) = 2e^{5t} + e^{2t}$$

1.2.21.1.2 Example 2

$$y' - ty = 0$$

With initial conditions $y(0) = 0$. For this we will use relation $\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s)$. Hence taking the Laplace transform gives

$$\begin{aligned}\mathcal{L}(ty) &= -\frac{d}{ds}\mathcal{L}(y) \\ &= -\frac{d}{ds}Y(s) \\ \mathcal{L}(y') &= sY(s) - y(0)\end{aligned}$$

The ode becomes

$$\begin{aligned}sY(s) - y(0) + \frac{d}{ds}Y(s) &= 0 \\ sY(s) + \frac{d}{ds}Y(s) &= 0\end{aligned}$$

This is linear ode in $Y(s)$. The integrating factor is $e^{\int s ds} = e^{\frac{s^2}{2}}$. Hence the above becomes

$$\frac{d}{ds}\left(Ye^{\frac{s^2}{2}}\right) = 0$$

Integrating gives

$$\begin{aligned}Ye^{\frac{s^2}{2}} &= c_1 \\ Y &= c_1 e^{\frac{-s^2}{2}}\end{aligned}$$

Applying inverse Laplace transform and using

$$y(t) = c_1 \mathcal{L}^{-1}\left(e^{\frac{-s^2}{2}}\right)$$

Using Laplace transform on time varying coefficient ode is not good idea. I need to look more into this. There is no $\mathcal{L}^{-1}\left(e^{\frac{-s^2}{2}}\right)$. Solving this in time domain is much easier of course.

1.2.21.1.3 Example 3

$$
\begin{aligned}
y' - 6y &= 0 \\
y(-1) &= 4
\end{aligned}
$$

Taking the Laplace transform gives

$$
\begin{aligned}
\mathcal{L}(y) &= Y(s) \\
\mathcal{L}(y') &= sY(s) - y(0)
\end{aligned}
$$

The ode becomes

$$
sY(s) - y(0) - 6Y = 0
$$

Since IC is not at zero, we let $y(0) = c_1$ and solving for Y gives

$$
\begin{aligned}
Y(s-6) - c_1 &= 0 \\
Y &= \frac{c_1}{s-6}
\end{aligned}
$$

Taking inverse Laplace transform gives

$$
y(t) = c_1 e^{6t}
$$

At $t = -1$, from IC, we obtain

$$
\begin{aligned}
4 &= c_1 e^{-6} \\
c_1 &= 4e^{6}
\end{aligned}
$$

Hence solution is

$$
\begin{aligned}
y(t) &= 4e^{6} e^{6t} \\
&= 4e^{6t+6}
\end{aligned}
$$

1.2.22 Lie symmetry method for solving first order ODE

1.2.22.1 Terminology used and high level introduction

1. x, y are the natural coordinates used in the input ode $\frac{dy}{dx} = \omega(x, y)$.

2. $\bar{x}, \bar{y}$ are called the Lie group (local) transformation coordinates. The ode remains invariant (same shape) when written in $\bar{x}, \bar{y}$. The coordinates R, S (some books use lower case r, s) are called the canonical coordinates in which the input ode becomes a quadrature and therefore easily solved by just integration.

3. ξ, η are called the Lie infinitesimals. $\xi(x, y), \eta(x, y)$ can be calculated knowing $\bar{x}, \bar{y}$. Also $\bar{x}, \bar{y}$ can be calculated given ξ, η. It is ξ, η which are the most important quantities that need to be determined in order to find the canonical coordinates R, S. These quantities are called the tangent vectors. These specify how the orbit moves. The orbit is the path the point (x, y) point travels on as it move toward $\bar{x}, \bar{y}$. The tangent vectors ξ, η are calculated at $\epsilon = 0$. The point $\bar{x} = x + \xi\epsilon$ and the point $\bar{y} = y + \eta\epsilon$.

4. The ultimate goal is write $\frac{dy}{dx} = \omega(x, y)$ in R, S coordinates where it is solved by integration only as it will have the form $\frac{dS}{dR} = F(R)$. The right hand side should always be a function of R only in canonical coordinates.

5. $\bar{x}, \bar{y}$ can be calculated knowing the canonical coordinates R, S.

6. The ideal transformation has the form $(\bar{x}, \bar{y}) \to (x, y + \epsilon)$ because with this transformation the ode becomes quadrature in the transformed coordinates. But because not all ode's have this transformation available, the ode is transformed to canonical coordinates (R, S) where the transformation $(\bar{R}, \bar{S}) \to (R, S + \epsilon)$ can be used.

7. The main goal of Lie symmetry method is to determine S, R. To be able to do this, the quantities ξ, η must be determined first.

8. The remarkable thing about this method, is that regardless of how complicated the original ode $\frac{dy}{dx} = \omega(x, y)$ is, if the similarity condition PDE can be solved for ξ, η, then R, S are found and the ode becomes quadrature $\frac{dS}{dR} = F(R)$. The ode is then solved in canonical coordinates and the solution transformed back to x, y.

9. The quantity ϵ is called the Lie parameter. This is a real quantity which as it goes to zero, gives the identity transformation. In other words, when $\epsilon = 0$ then $(x, y) = (\bar{x}, \bar{y})$.

10. But there is no free lunch, even in Mathematics. The problem comes down to

finding ξ, η. This requires solving a PDE. This is done using ansatz and trial and error. This reason possibly explains why the Lie symmetry method have not become standard in textbooks for solving ODE's as the algebra and computation needed to find ξ, η from the PDE becomes very complex to do by hand.

11. Total derivative operator: Given $f(x,y)$ then $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$ where it is assumed that $y(x)$ depends on x. Total derivative operator will be used extensively in all the derivatiations below, so good to practice this. It is written as $D_x = \partial_x + \partial_y y'$ for first order ode, and as $D_x = \partial_x + \partial_y y' + \partial_{y'} y''$ for second order ode and as $D_x = \partial_x + \partial_y y' + \partial_{y'} y'' + \partial_{y''} y'''$ for third order ode and so on.

12. The notation f_x means partial derivative. Hence $\frac{\partial f}{\partial x}$ is written as f_x. Total derivative will always be written as $\frac{df}{dx}$. It is important to distinguish between these two as the algebra will get messy with Lie symmetry. Sometimes we write f' to mean $\frac{df}{dx}$ but it is better to avoid f' and just write $\frac{df}{dx}$ when f is function of more than one variable.

13. Given first ode $\frac{dy}{dx} = \omega(x,y)$, where $\bar{y} \equiv \bar{y}(x,y)$ and $\bar{x} \equiv \bar{x}(x,y)$ then then$\frac{d\bar{y}}{d\bar{x}}$ is given by the following (using the total derivative operator)

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{D_x \bar{y}}{D_x \bar{x}}\\ &= \frac{\bar{y}_x + \bar{y}_y y'}{\bar{x}_x + \bar{x}_y y'}\\ &= \frac{\bar{y}_x + \bar{y}_y \omega}{\bar{x}_x + \bar{x}_y \omega}\end{aligned}$$

14. Given second order ode $\frac{d^2y}{dx^2} = \omega(x,y,y')$ where $\bar{y} \equiv \bar{y}(x,y,y')$ and $\bar{x} \equiv \bar{x}(x,y,y')$ then $\frac{d^2\bar{y}}{d\bar{x}^2}$ is given by

$$\begin{aligned}\frac{d^2\bar{y}}{d\bar{x}^2} &= \frac{D_x \frac{d\bar{y}}{d\bar{x}}}{D_x \bar{x}}\\ &= \frac{\bar{y}'_x + \bar{y}'_y y' + \bar{y}'_{y'} y''}{\bar{x}'_x + \bar{x}'_y y'}\end{aligned}$$

To simplify notation we used $\bar{y}'$ for $\frac{d\bar{y}}{d\bar{x}}$ above. The above simplifies to

$$\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{\bar{y}'_x + \bar{y}'_y y' + \bar{y}'_{y'} \omega}{\bar{x}'_x + \bar{x}'_y y'}$$

Keeping in mind that $(\circ)_x$ or $(\circ)_y$ mean partial derivative.

15. Given third order ode $\frac{d^3y}{dx^3} = \omega(x,y,y',y'')$ where $\bar{y} \equiv \bar{y}(x,y,y',y'')$ and $\bar{x} \equiv \bar{x}(x,y,y',y')$ then $\frac{d^3\bar{y}}{d\bar{x}^3}$ is given by

$$\begin{aligned}
\frac{d^3\bar{y}}{d\bar{x}^3} &= \frac{D_x \frac{d^2\bar{y}}{d\bar{x}^2}}{D_x \bar{x}} \\
&= \frac{\bar{y}''_x + \bar{y}''_y y' + \bar{y}''_{y'} y'' + \bar{y}''_{y''} y'''}{\bar{x}'_x + \bar{x}'_y y'} \\
&= \frac{\bar{y}''_x + \bar{y}''_y y' + \bar{y}''_{y'} y'' + \bar{y}''_{y''} \omega}{\bar{x}'_x + \bar{x}'_y y'}
\end{aligned}$$

To simplify notation we used $\bar{y}''$ for $\frac{d^2\bar{y}}{d\bar{x}^2}$ above. And so on for higher order ode's.

1.2.22.2 Introduction

Given any first order ODE

$$\frac{dy}{dx} = \omega(x,y) \tag{A}$$

The first goal is to find a one parameter invariant Lie group transformation that keeps the ode invariant. The Lie parameter the transformation depends on is called ϵ. This means finding transformation of (x,y) to new coordinates $(\bar{x},\bar{y})$ that keeps the ode the same form when written using $\bar{x},\bar{y}$.

This view looks at the transformation on the ode itself. Another view is to look at the family of the solution curves of the ode instead. Looking at solution curves transformation is geometrical in nature and can lead to more insight.

What does the transformation mean when looking at solution curves instead of the ODE itself? It is the mapping of a point (x,y) on one solution curve to another point $(\bar{x},\bar{y})$ on another solution curve. If the mapping sends point (x,y) to another point $(\bar{x},\bar{y})$ on the same solution curve, then it is called a trivial mapping or trivial transformation.

As an example, given the ode $y' = 0$, this has solutions $y = c_1$. For any constant c_1 there is a solution curve. There are infinite number of solution curves. All solution curves are horizontal lines. The mapping $(x,y) \to (x+\epsilon, y)$ is trivial transformation as it moves the point (x,y) to another point $(\bar{x},\bar{y})$ on the *same* solution curve.

The transformation $(x,y) \to (x, y+\epsilon)$ however is non trivial as it moves the point (x,y) to point $(\bar{x},\bar{y})$ on another solution curve. Here $\bar{x} = x$ and $\bar{y} = y + \epsilon$. This can also be written $(x,y) \to (x, e^{\epsilon} y)$ which is the preferred way.

The transformation $(x,y) \to (x+\epsilon, y+\epsilon)$ is non trivial for this ode. The simplest non trivial transformation that map all points on one solution curve to another solution curve is selected. In canonical coordinates the transformation used has the form $(R,S) \to (R,S+\epsilon)$.

Another example is $y' = y$. This has solution curves given by $y = ce^x$. This is a plot showing two such curves for different c values.

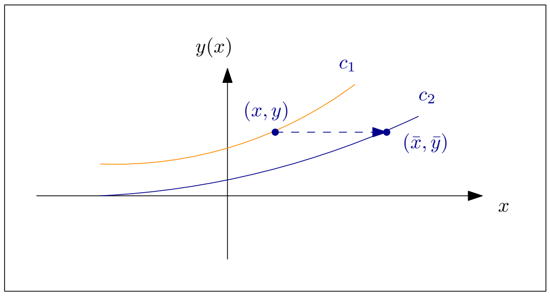

Figure 1.8: Point transformation example for $y' = y$

The above shows that a non trivial transformation is given by $\bar{x} = x+\epsilon, \bar{y} = y$. This can be found analytically by solving the symmetry condition as will be illustrated below using examples. For this case, the tangent vectors are $\xi = \frac{\partial \bar{x}}{\partial \epsilon}\big|_{\epsilon=0} = 1$ and $\eta = \frac{\partial \bar{y}}{\partial \epsilon}\big|_{\epsilon=0} = 0$. In Maple this is found using

```
ode:=diff(y(x),x)=y(x);
DEtools:-symgen(ode)
[_xi = 1, _eta = 0]
```

But the following transformation $\bar{x} = x, \bar{y} = y+\epsilon$ does not work

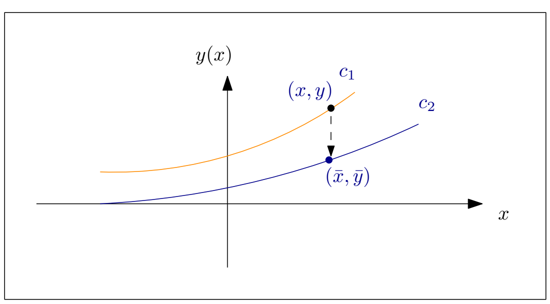

Figure 1.9: Possible Point transformation for $y' = y$

This is because it does not leave the ode invariant because $\frac{d\bar{y}}{d\bar{x}} = \bar{y}$ becomes $\frac{\bar{y}_x+\bar{y}_y y'}{\bar{x}_x+\bar{x}_y y'} = \bar{y}$, where now $\bar{y}_x = 0, \bar{y}_y = 1, \bar{x}_x = 1, \bar{x}_y = 0, \bar{y} = y+\epsilon$, and hence $\frac{\bar{y}_x+\bar{y}_y y'}{\bar{x}_x+\bar{x}_y y'} = \bar{y}$ simplifies to

$y' = y + \epsilon$ which is not the same ode. This shows that $\bar{x} = x, \bar{y} = y + \epsilon$ is not valid Lie point symmetry.

However $\bar{x} = x + \epsilon, \bar{y} = y$ leaves the ODE invariant. In this case $\bar{y}_x = 0, \bar{y}_y = 1, \bar{x}_x = 1, \bar{x}_y = 0, \bar{y} = y$ and hence $\frac{\bar{y}_x + \bar{y}_y y'}{\bar{x}_x + \bar{x}_y y'} = \bar{y}$ becomes $y' = y$ which is the same ode.

The transformation must keep the ode invariant as this is the main definition of symmetry transformation.

In the above, the path the point (x, y) travels over as it moves to $(\bar{x}, \bar{y})$ as ϵ changes is called the *orbit*. Each point (x, y) travels on its orbit during transformation.

In all such transformations, there is a parameter ϵ that the transformation depends on. This is why this is called the Lie one parameter symmetry transformation group. There are infinite number of such transformations.

Lie symmetry is called *point symmetry*, because of the above. It transforms points from an ODE solution curves to points on another solution curves for the same ODE. The identity transformation is when $\epsilon = 0$, since then the point is transformed to itself.

An example using an ODE. The Clairaut ode of the form $y = xf(p) + g(p)$ where $p \equiv y'$.

$$x(y')^2 - yy' + m = 0 \tag{1}$$

$$y = x\frac{(y')^2}{m} + y\frac{y'}{m}$$

Where $f(p) = \frac{(y')^2}{m}$ and $g(p) = \frac{y'}{m}$. Using the dilation transformation Lie group

$$\bar{x} \equiv \bar{x}(x, y; \epsilon) = e^{2\epsilon}x \tag{2}$$

$$\bar{y} \equiv \bar{y}(x, y; \epsilon) = e^{\epsilon}y \tag{3}$$

Eq. (1) is now expressed in the new coordinates $\bar{x}, \bar{y}$. If this results in same same ode form but written in $\bar{x}, \bar{y}$ then the transformation is invariant. But how to find $\frac{d\bar{y}}{d\bar{x}}$? This is done as follows

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} \\ &= \frac{\bar{y}_x + \bar{y}_y \frac{dy}{dx}}{\bar{x}_x + \bar{x}_y \frac{dy}{dx}}\end{aligned}$$

In this example $\bar{y}_x = 0, \bar{y}_y = e^{\epsilon}, \bar{x}_x = e^{2\epsilon}, \bar{x}_y = 0$. The above now becomes

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{e^{\epsilon}\frac{dy}{dx}}{e^{2\epsilon}} \\ &= e^{-\epsilon}\frac{dy}{dx}\end{aligned}$$

Writing (1) in terms of $\bar{x}, \bar{y}$ now gives

$$\bar{x}\left(\frac{d\bar{y}}{d\bar{x}}\right)^2 - \bar{y}\frac{d\bar{y}}{d\bar{x}} + m = 0 \tag{4}$$

$$\left(e^{2\epsilon}x\right)\left(e^{-\epsilon}\frac{dy}{dx}\right)^2 - \left(e^{\epsilon}y\right)e^{-\epsilon}\frac{dy}{dx} + m = 0$$

$$x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + m = 0 \tag{5}$$

Which gives the same ode. The above method starts by replacing the given ode by $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$ and finds if the result gives back the original ode in $x, y, \frac{dy}{dx}$. This is simpler than having to transform the original ode to $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$. This transformation can be verified in Maple as follows

```
ode:=x*diff(y(x),x)^2-y(x)*diff(y(x),x)+m=0;
the_tr:={x=X*exp(-2*s),y(x)=Y(X)*exp(-s)};
newode:=PDEtools:-dchange(the_tr,ode,{Y(X),X},'known'={y(x)},'uknown'={Y(X)});
diff(Y(X), X)^2*X - Y(X)*diff(Y(X), X) + m = 0
```

Comparing (4) to (5) shows that the ode form did not change, only the letters changed from x to $\bar{x}$ and y to $\bar{y}$. The resulting ode must never have the parameter ϵ show or remain in it.

The above shows how to verify that a transformation is invariant or not. In Lie group transformation there is only one parameter ϵ and the transformation is obtained by evaluating the group as ϵ goes to zero.

But how does this help in solving the ode? If the ode in x, y is hard to solve, then the ode written with $\bar{x}, \bar{y}$ will also be hard to solve since it is the same. But Eq. (4) is not what is used to solve the ode, but the above is just to verify the transformation is *invariant*. Similarity transformation is used to determine tangent vectors ξ, η only. Then the ode in canonical coordinates is used instead. In the canonical coordinates (R, S) the ode becomes quadrature and solved by integration. The transformation found above is only one step toward finding (R, S) and it is these canonical coordinates that are the goal and not $\bar{x}, \bar{y}$.

1.2.22.3 Outline of the steps in solving a differential equation using Lie symmetry method

These are the steps in solving an ODE using Lie symmetry method.

1. Given an ode $y' = \omega(x,y)$ to solve in natural coordinates.
2. Now the tangent vector $\xi(x,y)\,,\eta(x,y)$ are found. There are two options.
   (a) If Lie group coordinates $(\bar{x},\bar{y})$ are given, then it is easy to determine $\xi(x,y)\,,\eta(x,y)$ using
   $$\xi(x,y) = \left.\frac{\partial \bar{x}}{\partial \epsilon}\right|_{\epsilon=0}$$
   $$\eta(x,y) = \left.\frac{\partial \bar{y}}{\partial \epsilon}\right|_{\epsilon=0}$$
   Lie group coordinates $(\bar{x},\bar{y})$ must also satisfy
   $$\bar{x}_x\bar{y}_y - \bar{x}_y\bar{y}_x \neq 0$$
   (b) In practice Lie group coordinates $(\bar{x},\bar{y})$ are not given and are not known. In this case $\xi(x,y)\,,\eta(x,y)$ are found by solving the similarity condition which results in a PDE (derivation is given below). The PDE is
   $$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0$$
3. ξ,η are now used to determine the canonical coordinates (R,S). In the canonical coordinates, only S translation is needed to make the ode quadrature. The transformation is $(R,S) \to (R,S+\varepsilon)$. This transforms the original ode $y' = \omega(x,y)$ to $\frac{dS}{dR} = F(R)$ which is then solved by only integration. This is the main advantage of moving to canonical coordinates (R,S).
4. The ODE is solved in (R,S) space where $R \equiv R(x,y)\,,S \equiv S(x,y)$. The transformation from (x,y) to (R,S) is found by solving two set of PDEs using the characteristic method. After finding $R(x,y)\,,S(x,y)$ the ode will then be given by $\frac{dS}{dR} = \frac{S_x+S_y\frac{dy}{dx}}{R_x+R_y\frac{dy}{dx}}$ which will be quadrature. If this ode does not come out as $\frac{dS}{dR} = F(R)$ then something went wrong in the process. This ode is now solved for $S(R)$. It is the symmetry of the form $(R,S) \to (R,S+\varepsilon)$ which is of the most interest in the Lie method. This is called a translation transformation along the y axis (or the S axis). This is because this transformation leads to an ode which is solved by just integration.

5. Transform the solution from $S(R)$ to $y(x)$.

6. An alternative to steps (3) to (5) is to use ξ, η to determine an integrating factor $\mu(x,y)$ which is given by $\mu(x,y) = \frac{1}{\eta - \xi\omega}$ then the general solution to $y' = \omega(x,y)$ can be written directly as $\int \mu(x,y)\,(dy - \omega dx) = c_1$ or $\int \frac{dy - \omega dx}{\eta - \xi\omega} = c_1$ but this requires finding a function $F(x,y)$ whose *differential* is $dF = \frac{dy - \omega dx}{\eta - \xi\omega}$ and now the solution becomes $\int dF = c_1$ or $F = c_1$. If we can integrate this using $\int \mu dy - \int \mu\omega dx = c_1$ then this is the solution to the ode. It is implicit in $y(x)$. Currently my program does not implement Lie symmetry to find an integrating factor due to difficulty of finding dF that satisfies $dF = \frac{dy - \omega dx}{\eta - \xi\omega}$ or in carrying out the integration in all general cases but I hope to add this soon as a backup algorithm if the main one fails.

7. An important property, at least for first order ode's (I do not know now if this carries to higher order) is that given $\xi = f(x,y)\,, \eta = g(x,y)$, then we can always shift and use $\xi \equiv 0, \eta = g - \omega f$ where $y' = \omega(x,y)$. This means we can always base everything on $\xi \equiv 0$ after this shift is done to η. This can simplify some parts of the computation. Ofcourse if ξ was found to be zero initially, i.e. just after solving the linearized similarity PDE, then there is nothing more to do.

The *most difficult* step in all of the above is 2(b) which requires finding $\xi(x,y)\,, \eta(x,y)$. In practice Lie group $\bar{x}, \bar{y}$ transformation is not given. Lie infinitesimal $\xi(x,y)\,, \eta(x,y)$ have to be found directly from the linearized symmetry condition PDE using ansatz and by trial and error. The following diagram illustrates the above steps.

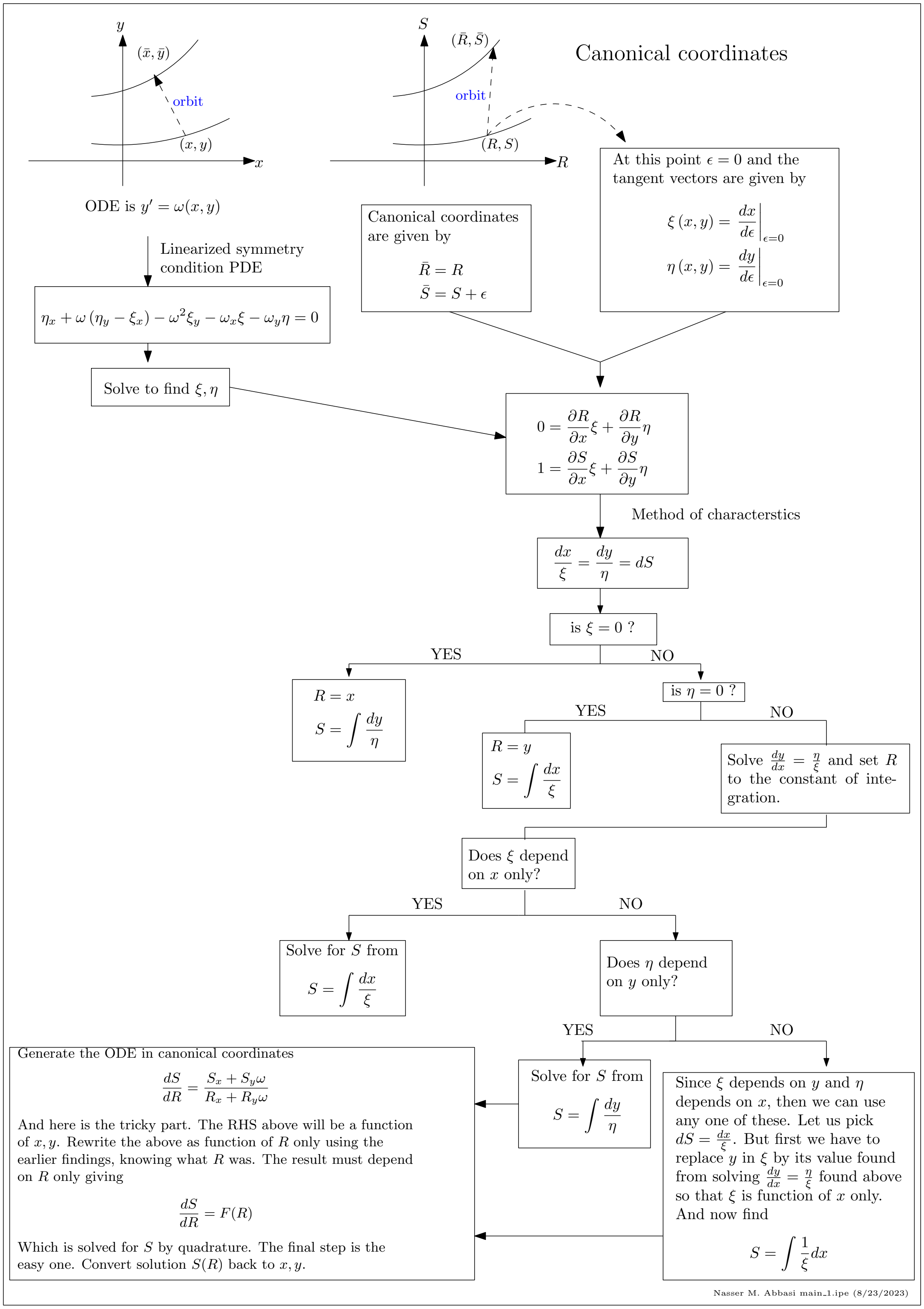

Figure 1.10: General steps to solve ode using Lie symmetry method

The following diagram illustrates the above steps when we carry the shifting step in order to force $\xi = 0$. We see that It simplifies the algorithm as now we can just assume $\xi = 0$ and we do not have to check for different cases as before.

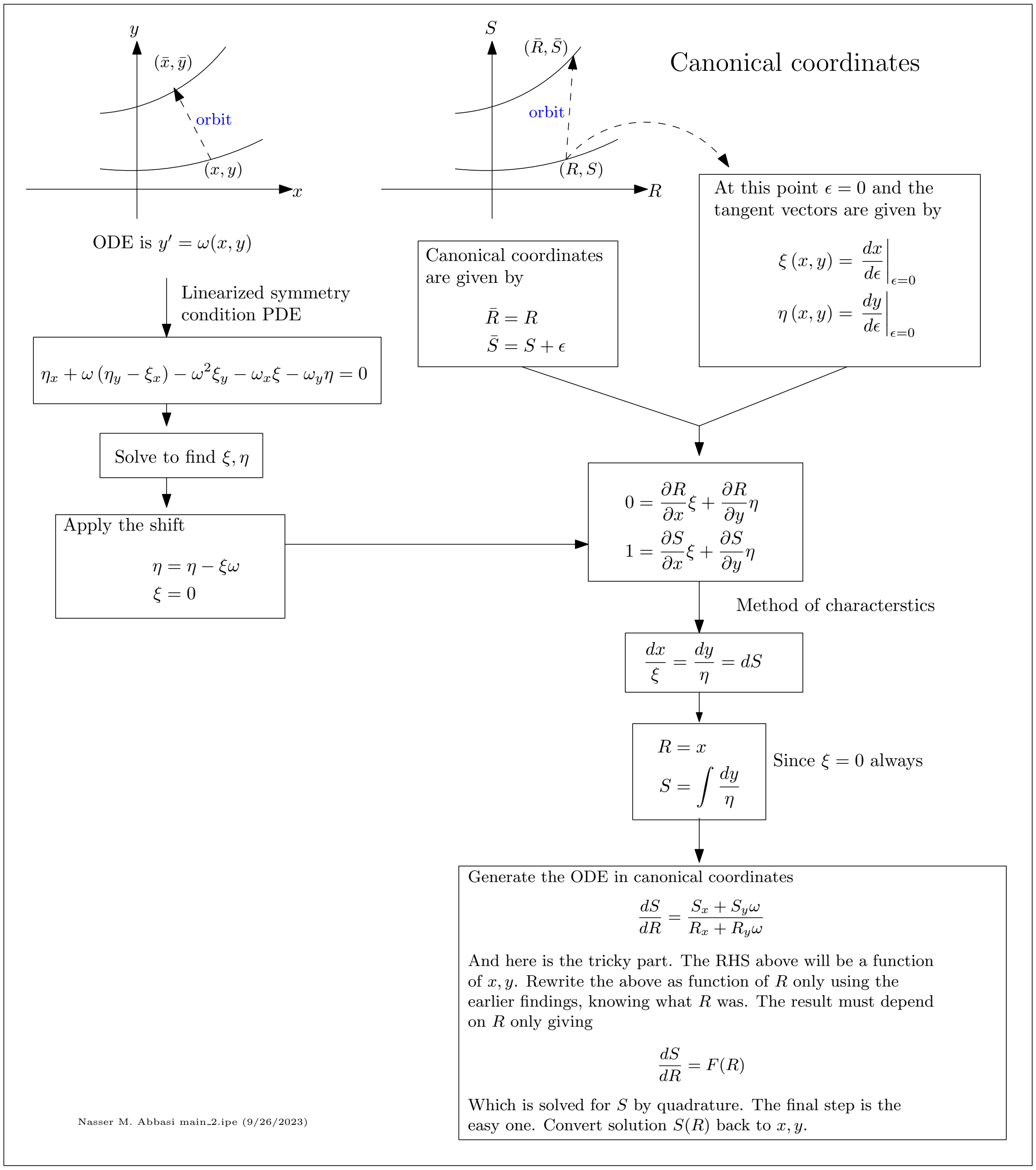

Figure 1.11: General steps to solve ode using Lie symmetry method. Shifting method

1.2.22.4 Finding $\xi(x,y)\,,\eta(x,y)$ knowing the first order ode type. Table lookup method.

There is a short cut to obtaining $\xi(x,y)\,,\eta(x,y)$ if the first order ode type is known or can be determined. (of course, if we know the ode type, then a direct method for solving the ode can be used, since the type is known and there is no need to use Lie symmetry), but still Lie symmetry can be useful in this case, and also it allows us to find the integrating factor quickly, which provides one more method to solve the ode. An example of a first order ode which does not have known type is

$$\left(x\cos y - e^{-\sin y}\right) y' + 1 = 0$$

The above can be solved using Lie symmetry but with functional form of anstaz $\xi = f(x)\,g(y)\,,\eta = 0$. which gives $\xi = e^{-\sin y}, \eta = 0$.

I am in the process of building table for ready to use infinitesimal based on the first ode type. The following small list is the current ones determined. For some first order ode such as linear $y' = f(x)\,y(x) + g(x)$ or separable $y' = f(x)\,g(y)$ the infinitesimals can be written directly (but again, for these simple ode's Lie method is not really needed but it provides good illustration on how to use it. Lie method is meant to be used for ode's which have no known type or difficult to solve otherwise). For an ode type not given in this list, an anstaz have to be used to solve the similarity PDE.

ode type	form	ξ	η	notes
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$	Notice that $g(x)$ does not affect the result
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0	This works for any g function that depends on y only
quadrature ode	$y' = f(x)$	0	1	of course for quadrature we do not need Lie symmetry as ode is already quadrature
quadrature ode	$y' = g(y)$	1	0	For example $y' = \frac{x+y}{-x+y}$ or $y' = \frac{y+2\sqrt{yx}}{x}$
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y	

homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$	Also $\xi=0,\eta=c(bx+cy+a)^{\frac{n}{m}}$ are possible. For example, for $y'=(1+2x+3y)^{\frac{1}{2}}$ then use the first option as simpler which is $\xi=1,\eta=-\frac{2}{3}$. Notice that $\xi=1,\eta=-\frac{b}{c}$ does not depend on a and not on n,m. Hence these odes $y'=(1+x+y)^{\frac{1}{3}}$,$y'=(10+x+y)^{\frac{1}{3}}$ and $y'=(10+x+y)^{\frac{2}{3}}$ all have the same infinitesimals $\xi=1,\eta=-\frac{b}{c}=-1$
homogeneous class D	$y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy	example $y'=\frac{y}{x}+\frac{1}{x}e^{-\frac{y}{x}}$. Where here $g(x)=\frac{1}{x},F\left(\frac{y}{x}\right)=e^{-\frac{y}{x}}$.
First order special form ID 1	$y'=g(x)\,e^{h(x)+by}+f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$	For an example, for the ode $y'=5e^{x^2+20y}+\sin x$, here $g(x)=5,h(x)=x^2,b=20,f(x)=\sin x$, hence $\xi=\frac{e^{-\int 20\sin dx-x^2}}{5},\eta=\frac{\sin xe^{-\int 20\sin xdx-x^2}}{5}$ or $\xi=\frac{1}{5}\sin x\left(e^{20\cos(x)-x^2}\right),\eta=\frac{\sin(x)}{5}\left(e^{20\cos(x)-x^2}\right)$. In this form, b must be constant.
polynomial type ode	$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$	For example for $y'=\frac{x+y+3}{2x+y}$ then $a_1=1,b_1=1,c_1=3,a_2=2,b_2=1,c_2=0$. Hence $\xi=x-3,\eta=y+6$.

Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$y^n e^{\int (1-n) f(x)\, dx}$	n is integer $n \neq 1, n \neq 0$. For example, for $y' = -\sin(x)\, y + x^2 y^2$ then $f(x) = -\sin x, g(x) = x^2, n = 2$ and $\xi = 0, \eta = e^{\int \sin x dx} y^2$ or $\xi = 0, \eta = e^{-\cos x} y^2$. Notice that $g(x)$ does not show up in the infinitesimals Another example is $y' = 2\frac{y}{x} + \frac{y^3}{x^2}$ where here $f(x) = \frac{2}{x}$. Hence $\xi = 0, \eta = e^{-\int (3-1)\frac{2}{x} dx} y^3$ or $\xi = 0, \eta = \eta = \frac{y^3}{x^4}$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$	For example, for $y' = xy + \sin(x)\, y^2$ then $f_1 = x, f_2 = \sin x$ and hence $\xi = 0, \eta = e^{-\int x dx}$ or $\xi = 0, \eta = e^{\frac{1}{2}x^2}$. Notice that $f_2(x)$ does not show up in the infinitesimals. I could not find infinitesimals for the full Riccati ode $y' = f_0(x) + f_1(x)\, y + f_2(x)\, y^2$. Notice that f_1, f_2 can not be both constants, else this becomes separable
Abel first kind	$y' = f_0(x) + f_1(x)\, y + f_2(x)\, y^2 + f_3(x)\, y^3$			No infinitesimals found

Currently the above are the ones I am able to determine for known first order ode's. If I find more, will add them. The table lookup is much faster to use than having to solve the similarity PDE each time using anstaz in order to find ξ, η.

1.2.22.5 Finding $\xi(x,y)\,,\eta(x,y)$ from linearized symmetry condition

Given any first order ODE

$$\frac{dy}{dx} = \omega(x,y) \tag{A}$$

$\xi(x,y)\,,\eta(x,y)$ are called the infinitesimals of the transformation. Maple has function called symgen in the DEtools package to determine these using 16 different algorithms. Starting with the Lie point transformation group

$$\begin{aligned}\bar{x} &\equiv \bar{x}(x,y;\epsilon)\\ \bar{y} &\equiv \bar{y}(x,y;\epsilon)\end{aligned}$$

Expanding using Taylor series near $\epsilon = 0$ gives

$$\begin{aligned}\bar{x} &= x + \left.\frac{\partial \bar{x}}{\partial \epsilon}\right|_{\epsilon=0} \epsilon + O(\epsilon^2)\\ &= x + \epsilon\xi(x,y) + O(\epsilon^2)\\ \bar{y} &= y + \left.\frac{\partial \bar{y}}{\partial \epsilon}\right|_{\epsilon=0} \epsilon + O(\epsilon^2)\\ &= y + \epsilon\eta(x,y) + O(\epsilon^2)\end{aligned}$$

Ignoring higher order terms gives

$$\bar{x}(x,y) = x + \epsilon\xi(x,y) \tag{1}$$

$$\bar{y}(x,y) = y + \epsilon\eta(x,y) \tag{2}$$

In the above ϵ is the one parameter in the Lie symmetry group. The symmetry condition for (A) is that

$$\frac{d\bar{y}}{d\bar{x}} = \omega(\bar{x},\bar{y})$$

Whenever

$$\frac{dy}{dx} = \omega(x,y)$$

Symmetry of an ODE means the ODE in (x,y) remain the same form (but using new variables $(\bar{x},\bar{y})$) after applying the (non-trivial) transformation (1,2).

Nontrivial transformation means $\epsilon \neq 0$. The first goal is to find the functions $\xi(x,y)\,,\eta(x,y)$ which satisfy the symmetry condition above.

The symmetry condition is written as

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} = \omega(\bar{x},\bar{y}) \tag{3}$$

Where $\frac{d\bar{y}}{dx}$ is the total derivative with respect to the x variable. Similarly for $\frac{d\bar{x}}{dx}$. But

$$\begin{aligned}\frac{d\bar{y}}{dx} &= \bar{y}_x + \bar{y}_y \frac{dy}{dx} \\ &= \bar{y}_x + \bar{y}_y \omega(x,y)\end{aligned} \tag{4}$$

And

$$\begin{aligned}\frac{d\bar{x}}{dx} &= \bar{x}_x + \bar{x}_y \frac{dy}{dx} \\ &= \bar{x}_x + \bar{x}_y \omega(x,y)\end{aligned} \tag{5}$$

Substituting (4,5) into (3) gives the *symmetry condition* as

$$\frac{\bar{y}_x + \omega(x,y)\,\bar{y}_y}{\bar{x}_x + \omega\,(x,y)\,\bar{x}_y} = \omega(\bar{x},\bar{y}) \tag{6}$$

But

$$\bar{x}_x = 1 + \epsilon \xi_x \tag{7}$$

And similarly

$$\bar{x}_y = \epsilon \xi_y \tag{8}$$

And

$$\bar{y}_x = \epsilon \eta_x \tag{9}$$

And

$$\bar{y}_y = 1 + \epsilon \eta_y \tag{10}$$

Substituting (7,8,9,10) back into the symmetry condition (6) gives

$$\begin{aligned}\frac{\epsilon \eta_x + \omega(1+\epsilon \eta_y)}{(1+\epsilon \xi_x) + \omega \epsilon \xi_y} &= \omega(x+\epsilon\xi, y+\epsilon\eta) \\ \frac{\epsilon \eta_x + \omega + \omega s \eta_y}{1+\epsilon \xi_x + \omega \epsilon \xi_y} &= \omega(x+\epsilon\xi, y+\epsilon\eta) \\ \frac{\omega + s(\eta_x + \omega \eta_y)}{1+\epsilon\,(\xi_x + \omega \xi_y)} &= \omega(x+\epsilon\xi, y+\epsilon\eta)\end{aligned} \tag{11}$$

The above is used to determine $\xi(x,y)\,,\eta(x,y)$. The above PDE is too complicated to use as is. It is linearized, and the linearized version is used to solve for ξ,η near small ϵ.

Eq. (11) is linearized by expanding the LHS and the RHS using Taylor series around $\epsilon = 0$. Starting with the LHS first, let $\frac{\omega+\epsilon(\eta_x+\omega\eta_y)}{1+\epsilon(\xi_x+\omega\xi_y)} = \Delta_{LHS}$. Expanding this using Taylor series around $\epsilon = 0$ gives

$$\Delta_{LHS} = \Delta_{\epsilon=0} + \epsilon \frac{d}{d\epsilon}\,(\Delta)_{\epsilon=0} + h.o.t. \tag{11A}$$

But $\Delta_{\epsilon=0} = \omega$ and

$$\begin{aligned}\frac{d}{d\epsilon}\left(\Delta_{LHS}\right) &= \frac{\frac{d}{d\epsilon}\left[\omega+\epsilon(\eta_x+\omega\eta_y)\right]\left(1+\epsilon(\xi_x+\omega\xi_y)\right)-\left(\omega+\epsilon(\eta_x+\omega\eta_y)\right)\frac{d}{d\epsilon}\left[1+\epsilon(\xi_x+\omega\xi_y)\right]}{\left(1+\epsilon\left(\xi_x+\omega\xi_y\right)\right)^2}\\ &= \frac{\left(\eta_x+\omega\eta_y\right)\left(1+\epsilon(\xi_x+\omega\xi_y)\right)-\left(\omega+\epsilon(\eta_x+\omega\eta_y)\right)\left(\xi_x+\omega\xi_y\right)}{\left(1+\epsilon\left(\xi_x+\omega\xi_y\right)\right)^2}\end{aligned}$$

At $\epsilon = 0$ the above reduces to

$$\begin{aligned}\frac{d}{d\epsilon}\left(\Delta_{LHS}\right)_{\epsilon=0} &= \left(\eta_x+\omega\eta_y\right)-\omega(\xi_x+\omega\xi_y)\\ &= \eta_x+\omega\eta_y-\omega\xi_x-\omega^2\xi_y\\ &= \eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y \end{aligned} \tag{12}$$

Therefore the LHS of Eq. (11A) becomes

$$\Delta_{LHS} = \omega+\epsilon\left(\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y\right) \tag{11B}$$

Now the RHS of Eq. (11) is linearized. Let $\omega(x+s\xi, y+s\eta) = \Delta_{RHS}$. Expansion around $\epsilon = 0$ gives

$$\Delta_{RHS} = \Delta_{\epsilon=0}+\epsilon\left(\frac{d}{d\epsilon}\Delta\right)_{\epsilon=0}+h.o.t.$$

But $\Delta_{\epsilon=0} = \omega(x,y)$ and

$$\frac{d}{d\epsilon}\Delta_{RHS} = \omega_x\xi+\omega_y\eta$$

Hence the linearized RHS of (11) becomes

$$\Delta_{RHS} = \omega(x,y)+\epsilon(\omega_x\xi+\omega_y\eta) \tag{13}$$

Substituting (11B,13) back into (11), gives the linearized version of (11) as

$$\begin{aligned}\Delta_{LHS} &= \Delta_{RHS}\\ \omega+\epsilon\left(\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y\right) &= \omega+\epsilon(\omega_x\xi+\omega_y\eta)\\ \epsilon\left(\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y\right) &= \epsilon(\omega_x\xi+\omega_y\eta)\\ \eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y &= \omega_x\xi+\omega_y\eta\end{aligned}$$

Hence

$$\boxed{\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0} \tag{14}$$

The above equation (14) is what is used to determine ξ, η. It is the linearized symmetry condition. There is an additional constraint not mentioned above which is

$$\bar{x}_x\bar{y}_y \neq \bar{x}_y\bar{y}_x$$

The restricted form of (14) is

$$\chi_x + \chi_y \omega - \chi \omega_y = 0$$

An important property is the following. Given any

$$\xi = A, \eta = B$$

Then we can always write the above as

$$\xi = 0, \eta = B - \omega A$$

So that $\xi = 0$ can always be used if needed to simplify some things.

After finding ξ, η from (14), the question now becomes is how to use them to solve the original ODE?

1.2.22.6 Moving to canonical coordinates R, S

The next step is to determine what is called the canonical coordinates (R, S). In these canonical coordinates the ODE becomes a quadrature and solved by integration. Once solved, the solution is transformed back to (x, y). The canonical coordinates (R, S) are found as follows. Selecting the transformation to be

$$\bar{R} = R \tag{15}$$

$$\bar{S} = S + \epsilon \tag{16}$$

Eq. (15) becomes

$$\left.\frac{\partial \bar{R}}{\partial \epsilon}\right|_{\epsilon=0} = \left.\left(\frac{\partial \bar{R}}{\partial x}\frac{dx}{d\epsilon}\right)\right|_{\epsilon=0} + \left.\left(\frac{\partial \bar{R}}{\partial y}\frac{dy}{d\epsilon}\right)\right|_{\epsilon=0}$$

But $\left.\frac{\partial \bar{R}}{\partial x}\right|_{\epsilon=0} = \frac{\partial R}{\partial x}$ and $\left.\frac{dx}{d\epsilon}\right|_{\epsilon=0} = \xi(x, y)$ and similarly $\left.\frac{\partial \bar{R}}{\partial y}\right|_{\epsilon=0} = \frac{\partial R}{\partial y}$ and $\left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = \eta(x, y)$. The above becomes

$$\left.\frac{\partial \bar{R}}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial R}{\partial x}\xi + \frac{\partial R}{\partial y}\eta$$

But $\left.\frac{\partial \bar{R}}{\partial \epsilon}\right|_{\epsilon=0} = 0$ since $\bar{R} = R$. The above reduces to

$$0 = \frac{\partial R}{\partial x}\xi + \frac{\partial R}{\partial y}\eta$$

This PDE have solution using symmetry method given by

$$\frac{dR}{dt} = 0 \tag{15A}$$

$$\frac{dx}{dt} = \xi \tag{15B}$$

$$\frac{dy}{dt} = \eta \tag{15C}$$

The same procedure is applied to Eq. (16) which gives

$$\left.\frac{\partial \bar{S}}{\partial \epsilon}\right|_{\epsilon=0} = \left.\left(\frac{\partial \bar{S}}{\partial x}\frac{dx}{d\epsilon}\right)\right|_{\epsilon=0} + \left.\left(\frac{\partial \bar{S}}{\partial y}\frac{dy}{d\epsilon}\right)\right|_{\epsilon=0}$$

But $\left.\frac{\partial \bar{S}}{\partial x}\right|_{\epsilon=0} = \frac{\partial S}{\partial x}$ and $\left.\frac{dx}{d\epsilon}\right|_{\epsilon=0} = \xi(x,y)$ and similarly $\left.\frac{\partial \bar{S}}{\partial y}\right|_{\epsilon=0} = \frac{\partial S}{\partial y}$ and $\left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = \eta(x,y)$. The above becomes

$$\left.\frac{\partial \bar{S}}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial R}{\partial x}\xi + \frac{\partial R}{\partial y}\eta$$

But $\left.\frac{\partial \bar{S}}{\partial \epsilon}\right|_{\epsilon=0} = 1$ since $\bar{S} = S + \epsilon$. The above reduces to

$$1 = \frac{\partial S}{\partial x}\xi + \frac{\partial S}{\partial y}\eta$$

This PDE have solution using symmetry method given by

$$\frac{dS}{dt} = 1 \tag{16A}$$
$$\frac{dx}{dt} = \xi \tag{16B}$$
$$\frac{dy}{dt} = \eta \tag{16C}$$

Equations (15A,B,C) are used to solve for $R(x,y)$ and equations (16A,B,C) are used to solve for $S(x,y)$. Starting with R. In the case when $\xi = 0$ the equations become

$$\frac{dR}{dt} = 0$$
$$\frac{dx}{dt} = 0$$
$$\frac{dy}{dt} = \eta$$

First equation above gives $R = c_1$. Second equation gives $x = c_2$. Letting $c_1 = c_2$ then

$$R = x$$

If $\xi \neq 0$ then combining Eqs. (15B,15C) gives

$$\frac{dy}{dx} = \frac{\eta}{\xi}$$
$$R = c_1$$

The ODE $\frac{dy}{dx}=\frac{\eta}{\xi}$ is solved first and the constant of integration is replaced by R. Hence R is now found. $S(x,y)$ is found similarly using Eqs. (16A,B,C). If $\xi=0$ then

$$\begin{aligned}\frac{dS}{dt}&=1\\ \frac{dx}{dt}&=0\\ \frac{dy}{dt}&=\eta\end{aligned}$$

The first and third equations give

$$\begin{aligned}\frac{dS}{dy}&=\frac{1}{\eta}\\ S&=\int\frac{1}{\eta}dy\end{aligned}$$

If $\xi\neq 0$ then using the second and third equation gives

$$\begin{aligned}\frac{dS}{dx}&=\frac{1}{\xi}\\ S&=\int\frac{1}{\xi}dx\end{aligned}$$

Now that R,S are found and the problem is solved. The ode in (R,S) space is set up using

$$\frac{dS}{dR}=\frac{S_x+S_y\frac{dy}{dx}}{R_x+R_y\frac{dy}{dx}} \tag{16}$$

Where $\frac{dy}{dx}=\omega(x,y)$ which is given. The solution $S(R)$ is next converted back to $y(x)$.

Examples below illustrate how this done on a number of ODE's. Eq. (16) is solved by quadrature. This is the whole point of Lie symmetry method, is that the original ode is solved in canonical coordinates where it is much easier to solve and the solution is transformed back to natural coordinates.

The only way to understand this method well, is to workout some problems. To learn more about the theory of Lie transformation itself and why it works, there are many links in my links page on the subject.

1.2.22.7 Definitions and various notes

1. infinitesimal generator operator. $\mathbf{\Gamma} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$. Any first order ode has such generator. For instance, for the ode $y' = \omega(x,y)$ then $\mathbf{\Gamma}\omega = \xi\frac{\partial \omega}{\partial x} + \eta\frac{\partial \omega}{\partial y}$. The ode $y' = \omega(x,y) = \frac{y}{x} + x$ has solution $y = x^2 + xc_1$, therefore the solution family is $\phi(x,y) = \frac{y-x^2}{x} = c$. Using $\xi = 0, \eta = x$ then $\mathbf{\Gamma}\phi = x\frac{\partial\left(\frac{y-x^2}{x}\right)}{\partial y} = 1$. This is another example: using $\xi = x, \eta = 2y$, hence $\mathbf{\Gamma}\phi = x\frac{\partial\left(\frac{y-x^2}{x}\right)}{\partial x} + 2y\frac{\partial\left(\frac{y-x^2}{x}\right)}{\partial y} = x\left(-\frac{y}{x^2} - 1\right) + 2y\left(\frac{1}{x}\right) = -\frac{y}{x} - 1 + 2\frac{y}{x} = \frac{y}{x} - 1 \neq 1$. I must be not applying the symmetry generator correct as the result supposed to be 1. Need to visit this again. See book Bluman and Anco, page 109. Maybe some of the assumptions for using this generator are not satisfied for this ode.

2. $\omega(x,y)$ is invariant iff $\mathbf{\Gamma}\omega = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} = 0$.

3. The linearized PDE from the symmetry condition is $\omega\xi_x + \omega^2\xi_y + \omega_x\xi = \omega_y\eta + \eta_x + \omega\eta_y$. This is used to determine tangent vector $(\xi(x,y), \eta(x,y))$ which is one of the core parts of the algorithm to solve the ode using symmetry methods. There are infinite number of solutions and only one is needed.

4. Symmetries and first integrals are the two most important structures of differential equations. First integral is quantity that depends on x, y and when integrated over any solution curve is constant.

5. Lie symmetry allows one to reduce the order of an ode by one. So if we have third order ode and we know the symmetry for it, we can change the ode to second order ode. Then if apply the symmetry for this second order ode, its order is reduced to one now.

6. If ξ, η are known then the canonical coordinates R, S can now be found as functions of x, y. We just ξ, η to find R, S. Once R, S are known then $\frac{dS}{dR} = f(R)$ can be formulated. This ode is solved for S by quadrature. Final solution is found by replacing R, S back by x, y. I have functions and a solver now written and complete to do all of this but just for first order ode's only. I need to start on second order ode's after that. The main and most difficult step is in finding ξ, η. Currently I only use multivariable polynomial ansatz up to second order for ξ and multivariable polynomial ansatz up to third order for η and then try all possible combinations. This is not very efficient. But works for now. I need to add better and more efficient methods to finding ξ, η but need to do more research on this.

7. When using polynomial ansatz to find ξ, η do not mix x, y in both ansatz. For

example if we use $\xi = p(x)$ then can use $\eta = q(x)$ or $\eta = q(x,y)$ polynomial ansatz to find η. But do not try $\xi = p(x,y)$ ansatz with $\eta = q(x,y)$ ansatz. In other words, if one ansatz polynomial is multivariable, then the other should be single variable. Otherwise results will be complicated and this defeats the whole ides of using Lie symmetry as the ode generated will be as complicated or more than the original ode we are trying to solve. I found this the hard way. I was generating all permutations of ξ, η ansatz's but with both as multivariable polynomials. This did not work well.

8. Symmetries on the ode itself, is same as talking about symmetries on solution curves. i.e. given an ode $y' = \omega(x,y)$ with solution $y = f(x)$, then when we look for symmetry on the ode which leaves the ode looking the same but using the new variables $\bar{x}, \bar{y}$. This is the same as when we look for symmetry which maps any point (x,y) on solution curve $y = f(x)$ to another solution curve. In other words, the symmetry will map all solution curves of $y' = \omega(x,y)$ to the same solution curves. i.e. a specific solution curve $y = f(x, c_1)$ will be mapped to $y = f(x, c_2)$. All solution curves of $y' = \omega(x,y)$ will be mapped to the same of solution curves. But each curve maps to another curve within the same set. If the same curve maps to itself, then this is called invariant curve.

9. An orbit is the name given to the path the transformation moves the point (x,y) from one solution curve to another point on another solution curve due to the symmetry transformation.

10. A solution curve of $y' = \omega(x,y)$ that maps to itself under the symmetry transformation is called an invariant curve.

11. Not every first order ode has symmetry. At least according to Maple. For example $y' + y^3 + xy^2 = 0$ which is Abel ode type, it found no symmetries using way=all. May be with special hint it can find symmetry?

12. After trying polynomials ansatz, I find it is limited. Since it will only find symmetries that has polynomials form. A more powerful ansatz is the functional form. But these are much harder to work with but they are more general at same time and can find symmetries that can't be found with just polynomials. So I have to learn how to use functional ansatz's. Currently I only use Polynomials.

13. ξ, η are called Lie infinitesimal and $\bar{x}, \bar{y}$ are called the Lie group.

14. If we given the ξ, η then we can find Lie group $(\bar{x}, \bar{y})$. See example below.

15. If we are given Lie group $(\bar{x}, \bar{y})$ then we can find the infinitesimal using $\xi(x,y) =$

$\frac{\partial}{\partial\epsilon}\bar{x}\Big|_{\epsilon=0}$ and $\eta(x,y) = \frac{\partial}{\partial\epsilon}\bar{y}\Big|_{\epsilon=0}$.

16. First order ode have infinite number of symmetries. Talking about symmetry of an ode is the same as talking about symmetry between solution curves of the ode itself. i.e. symmetry then becomes finding mapping that maps each solution curve to another one in the same family of solutions of the ode.

17. ξ,η can also be used to find the integrating factor for the first order ode. This is given by $\mu(x,y) = \frac{1}{\eta-\xi\omega}$ where the ode is $y'(x) = \omega(x,y)$. This gives an alternative approach to solve the ode. I still need to add examples using $\mu(x,y)$.

18. For first order ode, to find Lie infinitesmilas, we have to solve first order PDE in 2 variables. For second order ode, to find Lie infinitesmilas, we have to solve second order PDE in 3 variables. For third order ode, to find Lie infinitesmilas, we have to solve third order PDE in 4 variables and so on. Hence in general, for n^{th} order ode, we have to solve n^{th} order PDE in $n+1$ variables to find the required Lie infinitesmilas. For first order, these variables are ξ,η and the PDE is $\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0$. Currently my program only handles first order odes. Once I am more familar with Lie method for second order ode, will update these notes. See at the end a section on just second order ode that I started working on.

1.2.22.8 Closer look at orbits and tangent vectors

This section takes a closer look at orbits and tangent vectors ξ,η which are the core of Lie symmetry method. By definition

$$\begin{aligned}\xi(x,y) &= \frac{d\bar{x}}{d\epsilon}\bigg|_{\epsilon=0} \\ \eta(x,y) &= \frac{d\bar{y}}{d\epsilon}\bigg|_{\epsilon=0}\end{aligned} \tag{1}$$

Hence $\xi(x,y)$ shows how $\bar{x}$ changes as function of (x,y). And $\eta(x,y)$ shows how $\bar{y}$ changes as function of (x,y). This is because

$$\begin{aligned}\bar{x} &= x + \xi\epsilon \\ \bar{y} &= y + \eta\epsilon\end{aligned} \tag{2}$$

Comparing (2) to equation of motion where $\bar{x}$ represents final position and x is initial position, then ξ is the speed and ϵ is the time. When time is zero, initial and final position is the same. As time increases final position changes depending on the speed as time (here represented as ϵ) increases. So it helps to think of ξ,η as the rate at which

$\bar{x}, \bar{y}$ change location depending on the value ϵ. ξ, η are calculated when ϵ is very small in the limit as it reaches zero.

As ϵ increases the point (x, y) moves closer to the final destination point $(\bar{x}, \bar{y})$. So these quantities ξ, η specify the orbit shape. The orbit is the path taken by point transformation from (x, y) to $(\bar{x}, \bar{y})$ and depends on ϵ such that the ode remain invariant in $\bar{x}, \bar{y}$ and points on solution curves are mapped to points on other solution curves.

Different ξ, η give different orbits between two solution curves. The following example shows this. Given the ode

$$y' = \frac{x - y}{x + y}$$

This is Abel type ode. Also Homogeneous class A.

It has two solutions. One solution is given by Mathematica as $y = -x - \sqrt{c_1 + 2x^2}$. A small program was now written that plots the orbit for 4 solutions ξ, η found for the similarity conditions. The similarity solution were found by Maple's symgen command.

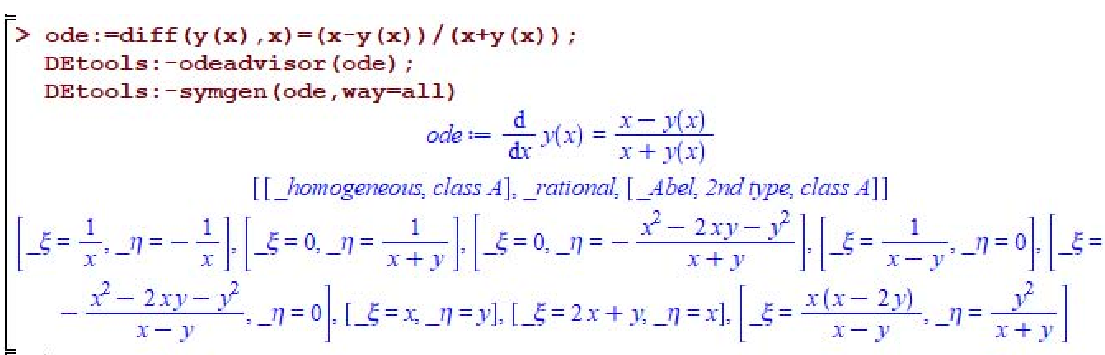

Figure 1.12: Command used to find ξ, η

The program starts from the same (x, y) point from one solution curve and determines $(\bar{x}, \bar{y})$ location on anther solution curve using each pair of ξ, η found. The same solution curves are used in order to compare the orbits. The following plot was generated showing the result

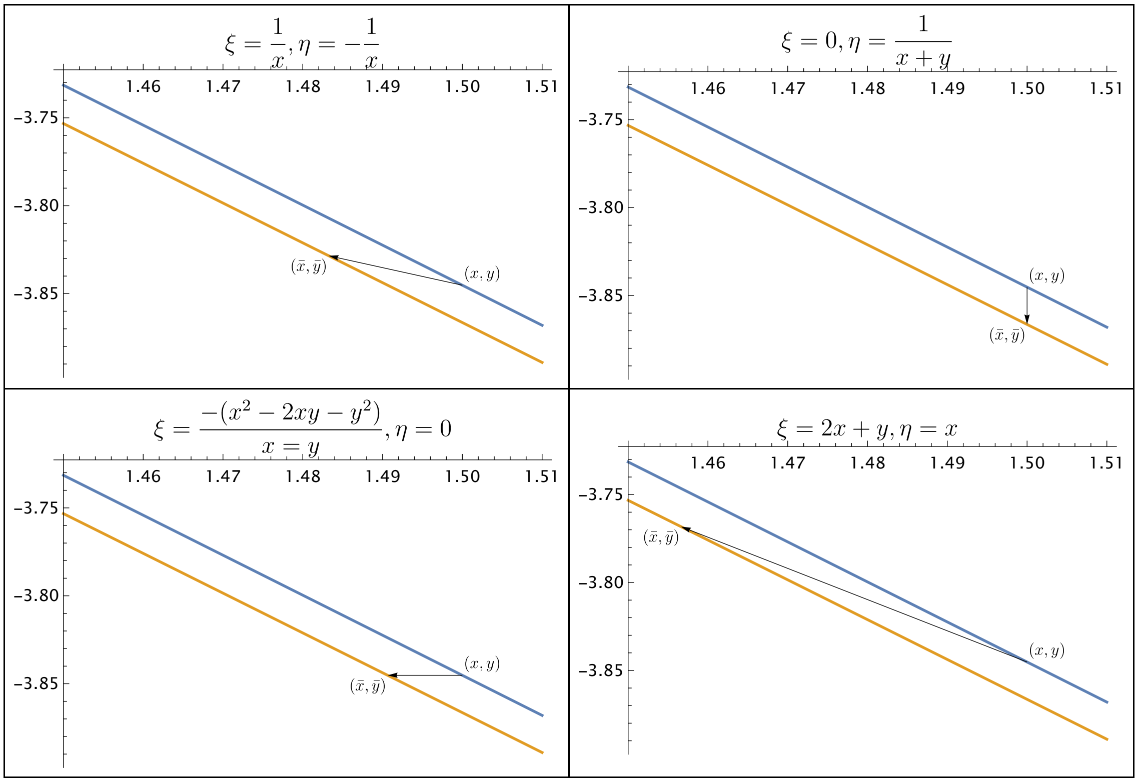

Figure 1.13: Different orbits using different ξ, η

The source code used to generate the above plot is

```
<<MaTeX`
ode=y'[x]==(x-y[x])/(x+y[x]);
ysol=DSolve[ode,y[x],x]
ysol=-x-Sqrt[C[1]+2 x^2];

x1 = 1.5;
y1 = ysol /. {C[1] -> 1, x -> x1};

ysol2=ysol/.C[1]->1.1

getSolutions[inf_List, titles_List, x_Symbol, ysol1_, ysol2_, x1_,
  y1_, from_, to_] :=
 Module[{xbar, ybar, eps, eq, soleps, p, data, n, xi, eta, texStyle},
  data = Table[0, {n, Length@inf}];
  texStyle = {FontFamily -> "Latin Modern Roman", FontSize -> 12};
```

```
  Do[
   xi = First[inf[[n]]];
   eta = Last[inf[[n]]];
   xbar = x1 + eps*xi ;
   ybar = y1 + eps*eta;
   eq = ybar == ysol2 /. x -> xbar;
   soleps = SolveValues[eq, eps];
   soleps = First@SortBy[soleps, Abs];
   ybar = ybar /. eps -> soleps;
   xbar = xbar /. eps -> soleps;
   p = Plot[{ysol1, ysol2}, {x, from, to},
     PlotLabel -> MaTeX[titles[[n]], Magnification -> 1.5],
     BaseStyle -> texStyle,
     Epilog -> {{Arrowheads[.02], Arrow[{{x1, y1}, {xbar, ybar}}]},
       Text[MaTeX["\\left( x,y \\right)"], {x1, y1}, {-1, -1}],
       Text[
        MaTeX["\\left( \\bar{x},\\bar{y}\\right)"], {xbar, ybar}, {1,
         1}]},
     ImageSize -> 400];
   data[[n]] = p
   ,
   {n, 1, Length@inf}
   ];

  data

  ];

inf = {{1/x1, -1/x1},
   {0, 1/(x1 + y1)},
   {-(x1^2 - 2*x1*y1 - y1^2)/(x1 - y1), 0},
   {2*x1 + y1, x1}
   };
titles = {"\\xi=\\frac{1}{x},\\eta=-\\frac{1}{x}",
   "\\xi=0,\\eta=\\frac{1}{x+y}",
   "\\xi=\\frac{-(x^2-2 x y-y^2)}{x=y},\\eta=0", "\\xi=2 x+y,\\eta=x"};
data = getSolutions[inf, titles, x, ysol /. C[1] -> 1, ysol2, x1, y1,
   1.45, 1.51];
p = Grid[Partition[data, 2], Frame -> All, Spacings -> {1, 1}]
```

1.2.22.9 Selection of ansatz to try

The following are selection of ansatz to try for solving the linearized PDE above generated from the symmetry condition in order to solve for $\xi(x,y), \eta(x,y)$. These use the functional form. As a general rule, the simpler that ansatz that works, the better it is. Functional form of ansatz is better than explicit polynomials but much harder to use and implement. Maple's symgen has 16 different algorithms include HINT option to support functional forms. The following are possible cases to use.

1. $\xi = 0, \eta = f(x)$
2. $\xi = 0, \eta = f(y)$
3. $\xi = f(x), \eta = 0$
4. $\xi = f(y), \eta = 0$
5. $\xi = f(x), \eta = xg(y)$. An example: applied to $y' = \frac{x+\cos(e^y+(1+x)e^{-x})}{e^{y+x}}$ should give $\xi = e^x, \eta = xe^{-y}$ which leads to solution $y = \ln\left(2\arctan\left(\frac{e^{-(c_1+e^{-x})}-1}{e^{-(c_1+e^{-x})}+1}\right) - (1+x)e^{-x}\right)$.
6. $\xi = f(x), \eta = g(y)$
7. $\xi = 0, \eta = f(x)g(y)$. For example, applied to $y' = \frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}$ should give $f(x) = \sqrt{1+x}, g(y) = \sqrt{1+y}$.
8. $\xi = f(x)g(y), \eta = 0$

1.2.22.10 Examples

1.2.22.11 Example 1 on how to find Lie group $(\bar{x}, \bar{y})$ given Lie infinitesimal ξ, η

Given $\xi = 1, \eta = 2x$ find Lie group $\bar{x}, \bar{y}$. Since

$$\xi(x,y) = \left.\frac{\partial \bar{x}}{\partial \epsilon}\right|_{\epsilon=0}$$

Then

$$\begin{aligned}\frac{d\bar{x}}{d\epsilon} &= \xi(\bar{x}, \bar{y})\\ &= 1 \end{aligned} \tag{1}$$

Similarly, since

$$\eta(x,y) = \left.\frac{\partial \bar{y}}{\partial \epsilon}\right|_{\epsilon=0}$$

Then

$$\begin{aligned}\frac{d\bar{y}}{d\epsilon} &= \eta(\bar{x},\bar{y}) \\ &= 2\bar{y} \end{aligned} \tag{2}$$

Where in both odes (1,2) we have the condition that at $\epsilon = 0$ then $\bar{x} = x, \bar{y} = y$. Starting with (1), solving it gives

$$\bar{x} = \epsilon + c_1(x,y)$$

Where $c_1(x,y)$ is arbitrary function which acts like constant of integration since $\bar{x}(x,y)$ is function of two variables. At $\epsilon = 0$ then $c_1(x,y) = x$. Hence the above is

$$\bar{x} = \epsilon + x \tag{3}$$

And from (2), solving give

$$\bar{y} = 2\bar{x}\epsilon + c_2(x,y)$$

But at $\epsilon = 0$, $\bar{y} = y, \bar{x} = x$ then the above gives $c_2 = y$. Hence the above becomes

$$\bar{y} = 2\bar{x}\epsilon + y$$

But $\bar{x} = \epsilon + x$ from (3), hence the above becomes

$$\begin{aligned}\bar{y} &= 2(\epsilon + x)\,\epsilon + y \\ &= 2\epsilon^2 + 2\epsilon x + y\end{aligned}$$

Therefore Lie group is

$$\begin{aligned}\bar{x} &= \epsilon + x \\ \bar{y} &= 2\epsilon^2 + 2\epsilon x + y\end{aligned}$$

1.2.22.12 Example how to find Lie group $(\bar{x},\bar{y})$ given canonical coordinates R,S

Given $R = x, S = \frac{y}{x}$ find Lie group $\bar{x},\bar{y}$. Solving for x,y from R,S gives

$$\begin{aligned}x &= R \\ y &= SR\end{aligned}$$

Hence

$$\begin{aligned}\bar{x} &= \bar{R} \\ \bar{y} &= \bar{S}\bar{R}\end{aligned}$$

But $\bar{S} = S + \epsilon$ by definition of canonical coordinates and $\bar{R} = R$ by definition of canonical coordinates. Hence the above becomes

$$\begin{aligned}\bar{x} &= R\\ \bar{y} &= (S+\epsilon)\,R\end{aligned}$$

Using the values given for R, S in terms of x, y the above becomes

$$\begin{aligned}\bar{x} &= x\\ \bar{y} &= \left(\frac{y}{x}+\epsilon\right)x\\ &= y+\epsilon x\end{aligned}$$

1.2.22.13 Example $y' = \frac{y}{x} + x$

This is linear first order which can be easily solved using integrating factor. But this is just to illustrate Lie symmetry method.

$$\begin{aligned}y' &= \frac{y}{x}+x \qquad (1)\\ y' &= \omega(x,y)\end{aligned}$$

The first step is to find ξ and η. Using lookup method, since this is linear ode of form $y' = f(x)\,y + g(x)$ then

$$\begin{aligned}\xi &= 0\\ \eta &= e^{\int f dx} = e^{\int \frac{1}{x} dx} = x\end{aligned}$$

The end of this problem shows also how to find these from the symmetry conditions. Therefore we write

$$\begin{aligned}\bar{x} &= x+\xi\epsilon\\ &= x\\ \bar{y} &= y+\eta\epsilon\\ &= y+\eta x \qquad (2)\end{aligned}$$

The integrating factor is therefore

$$\begin{aligned}\mu(x,y) &= \frac{1}{\eta-\xi\omega}\\ &= \frac{1}{x}\end{aligned}$$

Before solving this, let us first verify that transformation (2) is invariant which means it leaves the ode in same form but using $\bar{x}, \bar{y}$. We do the same as in the above introduction.

$$
\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} \\
&= \frac{\bar{y}_x + \bar{y}_y \frac{dy}{dx}}{\bar{x}_x + \bar{x}_y \frac{dy}{dx}}
\end{aligned}
$$

But $\bar{y}_x = s, \bar{y}_y = 1, \bar{x}_x = 1, \bar{x}_y = 0$ and the above becomes

$$
\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= \frac{\epsilon + \frac{dy}{dx}}{1} \\
&= \epsilon + \frac{dy}{dx}
\end{aligned}
$$

Substituting $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$ in the original ode gives

$$
\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= \frac{\bar{y}}{\bar{x}} + \bar{x} \\
\epsilon + \frac{dy}{dx} &= \frac{y + \epsilon x}{x} + x \\
\epsilon + \frac{dy}{dx} &= \frac{y}{x} + \epsilon + x \\
\frac{dy}{dx} &= \frac{y}{x} + x
\end{aligned}
$$

Which is the original ODE. Therefore (2) are indeed an invariant Lie group transformation as it leaves the ODE unchanged. The next step is to determine what is called the canonical coordinates R, S. Where R is the independent variable and S is the dependent variable. So we are looking for $S(R)$ function. This is done by using the standard characteristic equation by writing

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\
\frac{dx}{0} &= \frac{dy}{x} = dS \qquad (1)
\end{aligned}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Which is a first order PDE. This is solved for S, which gives (1) using the method of characteristic to solve first order PDE which is standard method. In the special case when $\xi = 0$ and

$\eta \neq 0$ these give

$$
\begin{aligned}
R &= x \\
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x} dy \\
&= \frac{y}{x} + c
\end{aligned}
$$

We are free to set $c = 0$, hence $S = \frac{y}{x}$. Therefore the transformation to canonical coordinates is

$$(x, y) \to (R, S) = \left(x, \frac{y}{x}\right)$$

The derivative in (R, S) is found same as with $\frac{d\bar{y}}{d\bar{x}}$ giving

$$\frac{dS}{dR} = \frac{S_x + S_y \frac{dy}{dx}}{R_x + R_y \frac{dy}{dx}}$$

But $S_x = -\frac{y}{x^2}, S_y = \frac{1}{x}, R_x = 1, R_y = 0$ and the above becomes

$$
\begin{aligned}
\frac{dS}{dR} &= \frac{-\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx}}{1} \\
&= -\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx}
\end{aligned}
$$

But $\frac{dy}{dx} = \frac{y}{x} + x$ hence the above becomes

$$
\begin{aligned}
\frac{dS}{dR} &= -\frac{y}{x^2} + \frac{1}{x}\left(\frac{y}{x} + x\right) \\
&= 1
\end{aligned}
$$

Solving this gives

$$S = R + c_1$$

But $S = \frac{y}{x}, R = x$. Therefore the above becomes

$$
\begin{aligned}
\frac{y}{x} &= x + c_1 \\
y &= x^2 + c_1 x
\end{aligned}
$$

Which is the solution to the original ode. Of course this was just an example showing how to use Lie symmetry method. The original ode is linear and can be easily solved using an integrating factor

$$
\begin{aligned}
y' - \frac{y}{x} &= x \\
I = e^{-\int \frac{1}{x} dx} &= e^{-\ln x} = \frac{1}{x}
\end{aligned}
$$

Multiplying the ode by I gives

$$
\begin{aligned}
\frac{d}{dx}(yI) &= Ix \\
\frac{y}{x} &= \int \frac{x}{x} dx \\
&= x + c_1
\end{aligned}
$$

Hence

$$
y = x^2 + xc_1
$$

Which is same solution. But Lie symmetry method works the same way for any given ode. And this is where it powers are. It can solve much more complicated odes than this using the same procedure. The main difficulty is in finding the infinitesimals for the group, which are ξ, η that leaves the ode invariant.

Finding Lie symmetries for this example

$$
\begin{aligned}
y' &= \frac{y}{x} + x \\
&= \omega(x,y)
\end{aligned}
$$

The condition of symmetry is a the linearized PDE given above in equation (14) as

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{14}
$$

We first find the determining equation before solving for ξ, η. Since $\omega = \frac{y}{x} + x$ then $\omega_y = \frac{1}{x}, \omega_x = -\frac{y}{x^2} + 1$. Hence the above becomes

$$
\begin{aligned}
\eta_x + \left(\frac{y}{x} + x\right)(\eta_y - \xi_x) - \left(\frac{y}{x} + x\right)^2 \xi_y - \left(-\frac{y}{x^2} + 1\right)\xi - \frac{1}{x}\eta = 0 \\
\eta_x + \left(\frac{y}{x} + x\right)(\eta_y - \xi_x) - \left(\frac{y^2}{x^2} + x^2 + 2y\right)\xi_y - \left(-\frac{y}{x^2} + 1\right)\xi - \frac{1}{x}\eta = 0 \\
\eta_x + \left(\frac{y}{x} + x\right)\eta_y - \xi_x\left(\frac{y}{x} + x\right) - \left(\frac{y^2}{x^2} + x^2 + 2y\right)\xi_y - \left(-\frac{y}{x^2} + 1\right)\xi - \frac{1}{x}\eta = 0
\end{aligned}
$$

Multiplying by x^2 to normalize gives

$$
x^2\eta_x + \left(yx + x^3\right)\eta_y - \xi_x\left(yx + x^3\right) - \left(y^2 + x^4 + 2yx^2\right)\xi_y - \left(-y + x^2\right)\xi - x\eta = 0 \tag{A}
$$

Equation (A) is called the determining equation. Using different ansatz can result in more solutions.

Trying ansatz

$$
\begin{aligned}
\xi &= 0 \\
\eta &= b_0 x
\end{aligned}
$$

Plugging these into (A) and comparing coefficients to solve for the unknown gives

$$
\begin{aligned}
x^2(b_0) - x\eta &= 0 \\
b_0 x^2 - x(b_0 x) &= 0 \\
b_0 x^2 - b_0 x^2 &= 0 \\
b_0(0) &= 0
\end{aligned}
$$

So any b_0 will work. Let $b_0 = 1$. Hence

$$
\begin{aligned}
\xi &= 0 \\
\eta &= x
\end{aligned}
$$

Now Trying ansatz as

$$
\begin{aligned}
\xi &= a_0 + a_1 x \\
\eta &= b_0 + b_1 y
\end{aligned}
$$

Then $\xi_x = a_1, \xi_y = 0, \eta_x = 0, \eta_y = b_1$ and the determining equation (A) becomes

$$
\begin{aligned}
(b_0 + b_1 y)\, x + (a_0 + a_1 x)\left(x^2 - y\right) + b_1\left(-yx - x^3\right) + a_1\left(yx + x^3\right) &= 0 \\
(b_0 + b_1 y)\, x + (a_0 + a_1 x)\left(x^2 - y\right) + (b_1 - a_1)\left(-yx - x^3\right) &= 0 \\
x b_0 - y a_0 + x^2 a_0 + x^3(2a_1 - b_1) &= 0
\end{aligned}
$$

Setting each coefficient to zero gives

$$
\begin{aligned}
b_0 &= 0 \\
a_0 &= 0 \\
a_0 &= 0 \\
2a_1 - b_1 &= 0
\end{aligned}
$$

Hence the solution is $a_0 = 0, b_0 = 0, a_1 = \frac{b_1}{2}$. Using $b_1 = 2$ gives $a_1 = 1$ and therefore

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

And Trying ansatz as

$$\xi = a_0 + a_1 x + a_2 y$$
$$\eta = b_0 + b_1 y + b_2 x$$

Hence $\xi_x = a_1, \xi_y = a_2, \eta_x = b_2, \eta_y = b_1$ and the determining equation (A) becomes

$$(b_0 + b_1 y + b_2 x)\, x + (a_0 + a_1 x + a_2 y)\left(x^2 - y\right) + b_1\left(-yx - x^3\right) + a_2\left(y^2 + x^4 + 2yx^2\right) + b_2\left(-x^2\right) + a_1\left(yx$$
$$x^4(-a_2) + x^3(-2a_1) + x^2 y(-3a_2) + x^3(b_1) + x^2(-a_0) + y(a_0) -$$

Setting each coefficient to zero gives

$$b_0 = 0$$
$$a_0 = 0$$
$$a_1 = 0$$
$$b_1 = 0$$
$$a_2 = 0$$
$$b_2 = 0$$

This shows there is no solution for this ansatz. There are more solutions depending on what ansatz we used. We just need one to obtain the final solution. In Maple, these solutions can be found as follows

```
ode:=diff(y(x),x)= y(x)/x+x;
DEtools:-symgen(ode,y(x),way=all)
[_xi = 0, _eta = x],
[_xi = 0, _eta = x],
[_xi = 0, _eta = x^2 - y],
[_xi = x, _eta = 2*y],
[_xi = 1, _eta = y/x],
[_xi = x^2 + y, _eta = 4*y*x],
[_xi = x^2 - 3*y, _eta = -4*y^2/x]
```

Trying ansatz using functional form. Let $\xi = 0, \eta = f(x)$ then $\xi_x = 0, \xi_y = 0, \eta_x = f'(x), \eta_y = 0$ and the determining equation (A) becomes

$$x^2\eta_x + \left(yx + x^3\right)\eta_y - \xi_x\left(yx + x^3\right) - \left(y^2 + x^4 + 2yx^2\right)\xi_y - \left(-y + x^2\right)\xi - x\eta = 0$$
$$x^2 f'(x) - x f(x) = 0$$
$$x f'(x) - f(x) = 0$$

This is easily solved to give $f = cx$. Hence $\xi = 0, \eta = x$ by choosing $c = 1$. We see that this choice of ansatz was the easiest in this case, as the ode generated was linear. Let us try another and see what happens.

Trying ansatz as $\xi = 0, \eta = f(y)$ then $\xi_x = 0, \xi_y = 0, \eta_x = 0, \eta_y = f'(y)$ and the determining equation (A) becomes

$$\begin{aligned} \left(yx + x^3\right) f'(y) - xf(y) &= 0 \\ \left(y + x^2\right) f'(y) - f(y) &= 0 \end{aligned}$$

This is separable and its solution is $f = c_1(x^2 + y)$. Hence $\xi = 0, \eta = (x^2 + y)$ by using $c_1 = 1$. But this is not function of y only. So this choice did not work. Trying $[\xi = f(x), \eta = 0], [\xi = f(y), \eta = 0]$ shows these also do not work.

ξ, η can be checked for validity by substituting them in the PDE. Maple's *symtest* command does this. These functional ansatz's lead to an ode which have to be solved.

1.2.22.14 Example $y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$

Solve

$$\begin{aligned} y' &= xy^2 - \frac{2y}{x} - \frac{1}{x^3} \qquad (1) \\ y' &= \omega(x, y) \end{aligned}$$

For $x \neq 0$. Given dilation transformation

$$\begin{aligned} \bar{x} &= e^{\epsilon} x \qquad (2) \\ \bar{y} &= e^{-2\epsilon} y \end{aligned}$$

Hence

$$\begin{aligned} \xi(x, y) &= \left.\frac{d\bar{x}}{d\epsilon}\right|_{\epsilon=0} = x \\ \eta(x, y) &= \left.\frac{d\bar{y}}{d\epsilon}\right|_{\epsilon=0} = -2y \end{aligned}$$

(At end shows how to obtain these). The integrating factor is therefore

$$\begin{aligned} \mu(x, y) &= \frac{1}{\eta - \xi\omega} \\ &= \frac{1}{-2y - x\left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)} \\ &= -\frac{x^2}{x^4y^2 - 1} \end{aligned}$$

Now

$$\begin{aligned}\bar{x} &= x + \xi\epsilon = x + \epsilon x \\ \bar{y} &= y + \eta\epsilon = y - 2y\epsilon\end{aligned} \tag{3}$$

This transformation $\bar{x} = e^{\epsilon}x, \bar{y} = e^{-2\epsilon}y$ is now verified that it keeps the ode invariant.

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}_x + \bar{y}_y \frac{dy}{dx}}{\bar{x}_x + \bar{x}_y \frac{dy}{dx}} = \frac{e^{-2\epsilon}\frac{dy}{dx}}{e^{\epsilon}} = e^{-3\epsilon}\frac{dy}{dx}$$

Substituting $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$ in the original ode gives

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \bar{x}\bar{y}^2 - \frac{2\bar{y}}{\bar{x}} - \frac{1}{\bar{x}^3} \\ e^{-3\epsilon}\frac{dy}{dx} &= (e^{\epsilon}x)\left(e^{-2\epsilon}y\right)^2 - \frac{2(e^{-2\epsilon}y)}{(e^{\epsilon}x)} - \frac{1}{(e^{\epsilon}x)^3} \\ e^{-3\epsilon}\frac{dy}{dx} &= e^{-3\epsilon}xy^2 - \frac{2e^{-3\epsilon}y}{x} - \frac{e^{-3\epsilon}}{x^3} \\ \frac{dy}{dx} &= xy^2 - \frac{2y}{x} - \frac{1}{x^3}\end{aligned}$$

Which is the original ode. Hence the transformation (2) is invariant. It is important to use (2) and not (3) when doing the verification.

The next step is to determine what is called the canonical coordinates R, S. Where R is the independent variable and S is the dependent variable. So we are looking for $S(R)$ function. This is done by using the standard characteristic equation by writing

$$\begin{aligned}\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\ \frac{dx}{x} &= \frac{dy}{-2y} = dS\end{aligned} \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Which is a first order PDE. This is solved for S, which gives (1) using the method of characteristic to solve first order PDE which is standard method. Starting with the first pair of ODE gives

$$\frac{dy}{dx} = -\frac{2y}{x}$$

Integrating gives $yx^2 = c$ where c is constant of integration. In this method R is always c. Hence

$$R = yx^2$$

$S(x, y)$ is now found from the first equation in (1) and the last equation which gives

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
S &= \int \frac{dx}{x} \\
S &= \ln x
\end{aligned}
$$

Now that $R(x,y)\,, S(x,y)$ are found, the ODE $\frac{dS}{dR} = \Omega(R)$ is setup. The ODE comes out to be function of R only, so it is quadrature. This is the main idea of this method. By solving for R we go back to x, y and solve for $y(x)$. How to find $\frac{dS}{dR}$? There is an equation to determine this given by

$$
\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{dS}{dx} + \omega(x,y)\,\frac{dS}{dy}}{\frac{dR}{dx} + \omega\,(x,y)\,\frac{dR}{dy}} \\
&= \frac{S_x + \omega(x,y)\,S_y}{R_x + \omega\,(x,y)\,R_y}
\end{aligned}
$$

Everything on the RHS is known. But

$$
\begin{aligned}
S_x &= \frac{1}{x} \\
S_y &= 0 \\
R_x &= 2yx \\
R_y &= x^2
\end{aligned}
$$

Substituting gives

$$
\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{1}{x} + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)(0)}{2xy + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)x^2} \\
&= \frac{\frac{1}{x}}{2xy + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)x^2} \\
&= \frac{1}{x^4y^2 - 1}
\end{aligned}
$$

But $R = yx^2$, hence the above becomes

$$
\frac{dS}{dR} = \frac{1}{R^2 - 1}
$$

This is just quadrature. Integrating gives

$$
S = -\operatorname{arctanh}(R) + c_1
$$

This solution is converted back to x, y. Since $S = \ln x, R = yx^2$, the above becomes

$$\ln|x| = -\operatorname{arctanh}\left(yx^2\right) + c_1$$

Or

$$\begin{aligned}
-\ln|x| + c_1 &= \operatorname{arctanh}\left(yx^2\right) \\
yx^2 &= \tanh\left(-\ln|x| + c_1\right) \\
y &= \frac{\tanh\left(-\ln|x| + c_1\right)}{x^2}
\end{aligned}$$

Which is the solution to the original ODE.

The above shows the basic steps in this method. Let us solve more ODE's to practice this method more.

Finding Lie symmetries for this example

The condition of symmetry is given above in equation (14) as

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{14}$$

We now need to solve the above for ξ, η given a specific $\omega(x, y)$ for the ODE at hand. This PDE can not be solved as is for ξ, η without an ansatz. One common ansatz is to use $\xi = \alpha(x)$ and $\eta = \beta(x)\, y + \gamma(x)$ and plugging these into the above and then compare coefficients to solve for $\alpha(x), \beta(x), \gamma(x)$.

Another ansatz is to use a polynomials for ξ, η. And this is what we will start with.

Using polynomial as ansatz

We start with order 1 polynomials. Hence

$$\xi = a_0 + a_1 x \tag{1}$$
$$\eta = b_0 + b_1 y \tag{2}$$

If this does not generate solution, we will try higher order polynomials. Eq (14) becomes

$$\begin{aligned}
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta &= 0 \\
0 + \omega(b_1 - a_1) - \omega^2(0) - \omega_x(a_0 + a_1 x) - \omega_y(b_0 + b_1 y) &= 0
\end{aligned}$$

But in this ODE $\omega = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$, hence $\omega_x = y^2 + \frac{2y}{x^2} + \frac{3}{x^4}$ and $\omega_y = 2yx - \frac{2}{x}$. The above becomes

$$\left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)(b_1 - a_1) - \left(y^2 + \frac{2y}{x^2} + \frac{3}{x^4}\right)(a_0 +$$

$$xy^2 b_1 - \frac{2y}{x} b_1 - \frac{1}{x^3} b_1 - xy^2 a_1 + \frac{2y}{x} a_1 + \frac{1}{x^3} a_1 - y^2 a_0 - \frac{2y}{x^2} a_0 - \frac{3}{x^4} a_0 - xy^2 a_1 - a_1 \frac{2y}{x} - a_1 \frac{3}{x^3} -$$

$$xy^2(b_1 - a_1 - a_1 - 2b_1) + \frac{y}{x}(-2b_1 + 2a_1 - 2a_1 + 2b_1) + \frac{1}{x^3}(-b_1 + a_1 - 3a_1) + y^2(-a_0) + \frac{y}{x^2}(-2a_0) + \frac{1}{x^4}$$

Each coefficient to each monomial must be zero. Hence

$$\begin{aligned}-2a_1 - b_1 &= 0\\ -b_1 - 2a_1 &= 0\\ -2a_1 - 2b_1 &= 0\\ a_0 &= 0\\ b_0 &= 0\end{aligned}$$

These are overdetermined equations. Solving gives $a_1 = -\frac{1}{2}b_1$ and $a_0 = b_0 = 0$. Choosing $b_1 = -2$ gives $a_1 = 1$. Hence

$$\begin{aligned}\xi &= a_0 + a_1 x = x\\ \eta &= b_0 + b_1 y = -2y\end{aligned}$$

Which is what we wanted to show for this ODE. These are the values we used earlier to solve the ODE using symmetry method.

Using functions as ansatz

Now ξ, η are found using $\xi = \alpha(x)$ and $\eta = \beta(x)\,y + \gamma(x)$ as ansatz. Eq. (14) is

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{14}$$

But

$$\eta_x = \beta'(x)\,y + \gamma'(x)$$

And

$$\eta_y = \beta(x)$$

And

$$\begin{aligned}\xi_y &= 0\\ \xi_x &= \alpha'(x)\end{aligned}$$

Substituting the above into EQ. (14) gives

$$\beta'(x)\,y + \gamma'(x) + \omega(\beta(x) - \alpha'(x)) - \omega_x \alpha(x) - \omega_y(\beta(x)\,y + \gamma(x)) = 0$$

But in this ODE $\omega = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$, hence $\omega_x = y^2 + \frac{2y}{x^2} + \frac{3}{x^4}$ and $\omega_y = 2yx - \frac{2}{x}$. The above becomes

$$\beta' y + \gamma' + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)(\beta - \alpha') - \left(y^2 + \frac{2y}{x^2} + \frac{3}{x^4}\right)\alpha - \left(2yx - \frac{2}{x}\right)(\beta y + \gamma) = 0$$

Or

$$\gamma' + y\beta' + \frac{2}{x}\gamma - \frac{1}{x^3}\beta - \frac{3}{x^4}\alpha - y^2\alpha + \frac{1}{x^3}\alpha' - 2xy\gamma - \frac{2}{x^2}y\alpha - xy^2\beta + \frac{2}{x}y\alpha' - xy^2\alpha' = 0$$

Collecting on y gives

$$y^0\left(\gamma' + \frac{2}{x}\gamma - \frac{1}{x^3}\beta - \frac{3}{x^4}\alpha + \frac{1}{x^3}\alpha'\right) + y\left(\beta' - 2xy\gamma - \frac{2}{x^2}\alpha + \frac{2}{x}\alpha'\right) + y^2(-\alpha - x\beta - x\alpha') = 0$$

Each term above is zero. This gives the following equations

$$\begin{aligned}
\gamma'(x) + \frac{2}{x}\gamma(x) - \frac{1}{x^3}\beta(x) - \frac{3}{x^4}\alpha(x) + \frac{1}{x^3}\alpha'(x) &= 0 \\
\beta'(x) - 2xy\gamma(x) - \frac{2}{x^2}\alpha(x) + \frac{2}{x}\alpha'(x) &= 0 \\
-\alpha(x) - x\beta(x) - x\alpha'(x) &= 0
\end{aligned}$$

Solving these coupled ODE on the computer gives

$$\begin{aligned}
\alpha(x) &= \frac{1}{x}\left(c_3x^4 + c_1x^2 + c_2\right) \\
\beta(x) &= -4c_3x^2 - 2c_1 \\
\gamma(x) &= -2c_3 - 2\frac{c_2}{x^4}
\end{aligned}$$

Where the c_1, c_2, c_3 above are constant of integration. Let $c_2 = c_3 = 0$. Hence

$$\begin{aligned}
\alpha(x) &= \frac{1}{x}\left(c_3x^4 + c_1x^2\right) \\
\beta(x) &= -4c_3x^2 - 2c_1 \\
\gamma(x) &= 0
\end{aligned}$$

Let $c_3 = 0$. Hence

$$\begin{aligned}
\alpha(x) &= \frac{1}{x}c_1x^2 \\
\beta(x) &= -2c_1 \\
\gamma(x) &= 0
\end{aligned}$$

Let $c_1 = 1$, hence

$$\begin{aligned}
\alpha(x) &= x \\
\beta(x) &= -2 \\
\gamma(x) &= 0
\end{aligned}$$

Therefore, since $\xi = \alpha(x)$ and $\eta = \beta(x)\,y + \gamma(x)$ then $\xi = x, \eta = -2y$ which is the same as the earlier method. After working using this ansatz, I find using the polynomial ansatz better. First of all, I had to set constants above to values in order to obtain the same result as earlier. Setting these constants other values will give different result. For example, the following are another set of possible solutions obtained from Maple for this ODE

$$
\begin{aligned}
&\left\{\alpha(x) = \frac{1}{x}, \beta(x) = 0, \gamma(x) = -\frac{2}{x^4}\right\} \\
&\left\{\alpha(x) = -\frac{x}{2}, \beta(x) = 1, \gamma(x) = 0\right\} \\
&\left\{\alpha(x) = -\frac{x^3}{4}, \beta(x) = x^2, \gamma(x) = \frac{1}{2}\right\}
\end{aligned}
$$

Which gives

$$
\begin{aligned}
&\left\{\xi = \frac{1}{x}, \eta = -\frac{2}{x^4}\right\} \\
&\left\{\xi = -\frac{x}{2}, \eta = y\right\} \\
&\left\{\xi = \frac{-x^3}{4}, \eta = x^2 y + \frac{1}{2}\right\}
\end{aligned}
$$

1.2.22.15 Example $y' = \frac{y+1}{x} + \frac{y^2}{x^3}$

Solve

$$
\begin{aligned}
y' &= \frac{y+1}{x} + \frac{y^2}{x^3} \\
y' &= \omega(x, y)
\end{aligned}
$$

This can be written as

$$
\begin{aligned}
y' &= \frac{y}{x} + \frac{1}{x} + \frac{y^2}{x^3} \\
&= \frac{y}{x} + \frac{x^2 + y^2}{x^3} \\
&= \frac{y}{x} + \frac{1}{x}\left(\frac{x^2 + y^2}{x^2}\right) \\
&= \frac{y}{x} + \frac{1}{x}\left(1 + \left(\frac{y}{x}\right)^2\right)
\end{aligned}
$$

Hence this has the form $y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$ where $g(x) = \frac{1}{x}$ and $F = \left(1 + \left(\frac{y}{x}\right)^2\right)$. Therefore this is homogeneous class D. Lookup table gives

$$\begin{aligned}\xi &= x^2\\ \eta &= xy\end{aligned}$$

Another way to find ξ, η is by solving the symmetry condition PDE and this is shown at the end of this problem. Hence

$$\begin{aligned}\bar{x} &= x + \xi\epsilon\\ &= x + x^2\epsilon\\ \bar{y} &= y + \eta\epsilon\\ &= y + xy\epsilon\end{aligned} \tag{2}$$

The integrating factor is therefore

$$\begin{aligned}\mu(x,y) &= \frac{1}{\eta - \xi\omega}\\ &= \frac{1}{xy - x^2\left(\frac{y+1}{x} + \frac{y^2}{x^3}\right)}\\ &= -\frac{x}{x^2 + y^2}\end{aligned}$$

The ode is now verified that it remains invariant under (2) transformation.

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}}\\ &= \frac{\bar{y}_x + \bar{y}_y \frac{dy}{dx}}{\bar{x}_x + \bar{x}_y \frac{dy}{dx}}\end{aligned}$$

But from (2) $\bar{y}_x = y\epsilon, \bar{y}_y = 1 + x\epsilon, \bar{x}_x = 1 + 2x\epsilon, \bar{x}_y = 0$ and the above becomes

$$\frac{d\bar{y}}{d\bar{x}} = \frac{1 + (1 + x\epsilon)\frac{dy}{dx}}{1 + 2x\epsilon}$$

Substituting $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$ in the original ode gives

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{\bar{y}+1}{\bar{x}} + \frac{\bar{y}^2}{\bar{x}^3}\\ \frac{1 + (1 + x\epsilon)\frac{dy}{dx}}{1 + 2x\epsilon} &= \frac{(y + xy\epsilon) + 1}{x + x^2\epsilon} + \frac{(y + xy\epsilon)^2}{(x + x^2\epsilon)^3}\end{aligned}$$

Which as $\lim_{\epsilon\to 0}$ gives

$$\frac{dy}{dx} = \frac{y+1}{x} + \frac{y^2}{x^3}$$

The same original ode showing the transformation is valid symmetry.

```
Y:=y/(1-s*x):
X:=x/(1-s*x):
eq:=(diff(Y,x)+diff(Y,y)*Z)/(diff(X,x)+diff(X,y)*Z)=simplify((Y+1)/X+Y^2/X^3):
solve(simplify(eq),Z)
y/x + 1/x + y^2/x^3
```

Hence the transformation in (2) is invariant.

The next step is to determine what is called the canonical coordinates R, S. Where R is the independent variable and S is the dependent variable. So we are looking for $S(R)$ function. This is done by using the standard characteristic equation by writing

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS$$

$$\frac{dx}{x^2} = \frac{dy}{xy} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Which is a first order PDE. We need to solve this for S, which gives (1) using method of characteristic to solve first order PDE which is standard method. Starting with the first pair of ODE in (1) gives

$$\frac{dy}{dx} = \frac{xy}{x^2} = \frac{y}{x}$$

Integrating gives $\frac{y}{x} = c$ where c is constant of integration. In this method R is always c. Hence

$$R(x,y) = \frac{y}{x}$$

Now we find $S(x,y)$ from the first equation in (1) and the last equation

$$dS = \frac{dx}{\xi}$$

$$S = \int \frac{dx}{x^2}$$

$$S = \frac{-1}{x}$$

Now that we found R and S, we determine the ODE $\frac{dS}{dR} = \Omega(R)$. The ODE comes out to be function of R only, so it is quadrature. This is the whole idea of this method.

By solving for R we go back to x, y and solve for $y(x)$. How to find $\frac{dS}{dR}$? There is an equation to determine this given by

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)\, S_y}{R_x + \omega\,(x,y)\, R_y}$$

We know everything on the RHS. Substituting gives

$$\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{1}{x^2} + \left(\frac{y+1}{x} + \frac{y^2}{x^3}\right)(0)}{-\frac{y}{x^2} + \left(\frac{y+1}{x} + \frac{y^2}{x^3}\right)\frac{1}{x}} \\
&= \frac{\frac{1}{x^2}}{-\frac{y}{x^2} + \left(\frac{y+1}{x} + \frac{y^2}{x^3}\right)\frac{1}{x}} \\
&= \frac{x^2}{x^2 + y^2} \\
&= \frac{1}{1 + \left(\frac{y}{x}\right)^2}
\end{aligned}$$

But $R = \frac{y}{x}$, hence the above becomes

$$\frac{dS}{dR} = \frac{1}{1 + R^2}$$

This is just quadrature. Integrating gives

$$S = \arctan\,(R) + c_1$$

Now we go back to x, y. Since $S = -\frac{1}{x}, R = \frac{y}{x}$, then the above becomes

$$\begin{aligned}
-\frac{1}{x} &= \arctan\left(\frac{y}{x}\right) + c_1 \\
\frac{-1}{x} + c_2 &= \arctan\left(\frac{y}{x}\right) \\
\frac{y}{x} &= \tan\left(\frac{-1}{x} + c_2\right) \\
y(x) &= x\tan\left(\frac{-1}{x} + c_2\right)
\end{aligned}$$

And the above is the solution to original ODE.

Finding Lie symmetries for this example

The symmetry condition was derived earlier as

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{14}$$

Let ansatz be

$$\xi = c_1 x + c_2 y + c_3$$
$$\eta = c_4 x + c_5 y + c_6$$

Eq 14 becomes

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0$$
$$c_4 + \omega(c_5 - c_1) - \omega^2 c_2 - \omega_x(c_1 x + c_2 y + c_3) - \omega_y(c_4 x + c_5 y + c_6) = 0$$

But in this ODE $\omega = \frac{y+1}{x} + \frac{y^2}{x^3}$, hence $\omega_x = -\frac{y+1}{x^2} - 3\frac{y^2}{x^4}$ and $\omega_y = \frac{1}{x} + \frac{2y}{x^3}$. The above becomes

$$c_4 + \left(\frac{y+1}{x} + \frac{y^2}{x^3}\right)(c_5 - c_1) - \left(\frac{y+1}{x} + \frac{y^2}{x^3}\right)^2 c_2 - \left(-\frac{y+1}{x^2} - 3\frac{y^2}{x^4}\right)(c_1 x + c_2 y + c_3) - \left(\frac{1}{x} + \frac{2y}{x^3}\right)($$
$$\frac{1}{x^2}c_3 - \frac{1}{x^2}c_2 + \frac{1}{x}c_5 - \frac{1}{x}c_6 + \frac{2}{x^3}y^2 c_1 - \frac{2}{x^4}y^2 c_2 + \frac{3}{x^4}y^2 c_3 + \frac{1}{x^4}y^3 c_2 - \frac{1}{x^3}y^2 c_5 - \frac{1}{x^6}y^4 c_2 - \frac{1}{x^2}y c_2 + \frac{1}{x^2}y c_3 -$$
$$x^4 c_3 - x^4 c_2 + x^5 c_5 - x^5 c_6 + 2x^3 y^2 c_1 - 2x^2 y^2 c_2 + 3x^2 y^2 c_3 + x^2 y^3 c_2 - x^3 y^2 c_5 - y^4 c_2 - x^4 y c_2 + x^4 y c_3 - 2$$
$$x^4(c_3 - c_2) + x^5(c_5 - c_6) + x^3 y^2(2c_1 - c_5) + x^2 y^2(-2c_2 + 3c_3) + x^2 y^3(c_2) + y^4(-c_2) + x^4 y(-c_2 + c_3 - 2c$$

Each coefficient to each monomial must be zero. Hence

$$c_3 - c_2 = 0$$
$$c_5 - c_6 = 0$$
$$2c_1 - c_5 = 0$$
$$-2c_2 + 3c_3 = 0$$
$$c_2 = 0$$
$$-c_2 + c_3 - 2c_4 = 0$$
$$-2c_6 = 0$$

Which simplifies to (since $c_2 = 0, c_6 = 0$)

$$c_3 = 0$$
$$c_5 = 0$$
$$c_1 - c_5 = 0$$
$$3c_3 = 0$$
$$c_3 - 2c_4 = 0$$

Which simplifies to (since $c_3 = 0, c_5 = 0$)

$$c_5 = 0$$
$$c_1 - c_5 = 0$$
$$c_4 = 0$$

Hence $c_5 = 0, c_1 = 0, c_4 = 0$. We see that all $c_i = 0$, therefore there is no solution using this ansatz.

Trying ansatz

$$\xi = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2$$
$$\eta = b_0 + b_1 x + b_2 y + b_3 xy + b_4 y^2$$

Eq 9 becomes

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0$$

Substituting the ansatz and simplifying gives

$$-x^2y^3a_2+y^4a_2+x^4(-a_0+a_2)+x^2y^2(-3a_0+2a_2)+xy^4a_3+2x^3yb_0+x^4y(-a_0+a_2+2b_1)+x^5(a_3+b_0-b_2)+x^3y^2$$

Each coefficient to each monomial must be zero. Hence

$$a_2 = 0$$
$$-a_0 + a_2 = 0$$
$$-3a_0 + 2a_2 = 0$$
$$a_3 = 0$$
$$b_0 = 0$$
$$-a_0 + a_2 + 2b_1 = 0$$
$$a_3 + b_0 - b_2 = 0$$
$$-2a_1 + 2a_3 + b_2 = 0$$
$$a_4 - b_3 = 0$$
$$2a_3 - 2b_4 = 0$$
$$a_3 - b_4 = 0$$

Since $a_2 = a_3 = b_0 = 0$ the above simplifies to

$$-a_0 = 0$$
$$-3a_0 = 0$$
$$-a_0 + 2b_1 = 0$$
$$-b_2 = 0$$
$$-2a_1 + b_2 = 0$$
$$a_4 - b_3 = 0$$
$$-2b_4 = 0$$
$$-b_4 = 0$$

Since $a_0 = b_2 = a_4 = b_4 = 0$, The above now simplifies to

$$a_4 - b_3 = 0$$

Therefore, if we let $a_4 = 1$ then $b_3 = 1$ and the solution is

$$\begin{aligned}\xi &= a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 \\ &= x^2 \\ \eta &= b_0 + b_1 x + b_2 y + b_3 xy + b_5 y^2 \\ &= xy\end{aligned}$$

Which is what we used above to solve the ode.

1.2.22.16 Example $y' = \frac{y-4xy^2-16x^3}{y^3+4x^2y+x}$

Solve

$$\begin{aligned}y' &= \frac{y - 4xy^2 - 16x^3}{y^3 + 4x^2y + x} \\ y' &= \omega(x, y)\end{aligned}$$

The first step is to find ξ and η. This is shown at the end of this problem below.

$$\begin{aligned}\xi &= -y \\ \eta &= 4x\end{aligned}$$

The integrating factor is therefore

$$\begin{aligned}\mu(x, y) &= \frac{1}{\eta - \xi\omega} \\ &= \frac{1}{4x + y\left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)} \\ &= \frac{x^2y + x + y^3}{4x^2 + y^2}\end{aligned}$$

The next step is to determine what is called the canonical coordinates R, S. Where R is the independent variable and S is the dependent variable. This is done by using the standard characteristic equation by writing

$$\begin{aligned}\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\ \frac{dx}{-y} &= \frac{dy}{4x} = dS \qquad (1)\end{aligned}$$

The first pair of ode's in (1) gives

$$\frac{dy}{dx} = -\frac{4x}{y}$$

Solving gives

$$y = \sqrt{-4x^2 + c}$$

Where c is constant of integration (For $y > 0$ only). In this method R is always c. Hence

$$\begin{aligned} y^2 &= -4x^2 + c \\ R &= y^2 + 4x^2 \end{aligned} \tag{2}$$

The first equation in (1) and the last equation gives

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ S &= -\int \frac{dx}{y} \end{aligned}$$

But $y = \sqrt{-4x^2 + c}$. The above becomes

$$\begin{aligned} S &= -\int \frac{dx}{\sqrt{-4x^2 + c}} \\ &= -\frac{1}{2}\arctan\left(\frac{2x}{\sqrt{-4x^2 + c}}\right) \\ &= -\frac{1}{2}\arctan\left(\frac{2x}{y}\right) \end{aligned}$$

For $y > 0$. Now that we found R and S, we determine the ODE $\frac{dS}{dR} = \Omega(R)$. The ODE comes out to be function of R only, so it is quadrature. This is the whole idea of this method. By solving for R we go back to x, y and solve for $y(x)$. How to find $\frac{dS}{dR}$? There is an equation to determine this given by

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)\, S_y}{R_x + \omega\,(x, y)\, R_y}$$

We know everything on the RHS. Substituting gives

$$\begin{aligned} \frac{dS}{dR} &= \frac{\frac{d}{dx}\left(-\frac{1}{2}\arctan\left(\frac{2x}{y}\right)\right) + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)\frac{d}{dy}\left(-\frac{1}{2}\arctan\left(\frac{2x}{y}\right)\right)}{\frac{d}{dx}\sqrt{y^2 + 4x^2} + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)\frac{d}{dy}\sqrt{y^2 + 4x^2}} \\ &= \frac{\frac{-1}{y\left(\frac{4x^2}{y^2}+1\right)} + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)\frac{x}{y^2\left(\frac{4x^2}{y^2}+1\right)}}{\frac{4x}{\sqrt{y^2+4x^2}} + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)\frac{y}{\sqrt{y^2+4x^2}}} \\ &= -\sqrt{4x^2 + y^2} \\ &= -R \end{aligned}$$

Hence

$$\frac{dS}{dR} = -R$$

This is just quadrature. Integrating gives

$$S = -\frac{R^2}{2} + c$$

Now we go back to x, y. Since $S = -\frac{1}{2}\arctan\left(\frac{2x}{y}\right), R = \sqrt{y^2+4x^2}$, then the above becomes

$$-\frac{1}{2}\arctan\left(\frac{2x}{y}\right) = -\left(\frac{y^2+4x^2}{2}\right) + c$$
$$\frac{y^2}{2} - \frac{1}{2}\arctan\left(\frac{2x}{y}\right) + 2x^2 - c = 0 \qquad y > 0$$

And the above is the solution to original ODE.

Finding Lie symmetries for this example

The symmetry condition was derived earlier as

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{14}$$

Let ansatz be

$$\xi = c_1x + c_2y + c_3$$
$$\eta = c_4x + c_5y + c_6$$

Eq 14 becomes

$$c_4 + \omega(c_5 - c_1) - \omega^2c_2 - \omega_x(c_1x + c_2y + c_3) - \omega_y(c_4x + c_5y + c_6) = 0$$

But in this ODE $\omega = \frac{y-4xy^2-16x^3}{y^3+4x^2y+x}$, hence $\omega_x = \frac{-4y^5-32x^2y^3-8xy^2+(-64x^4-1)y-32x^3}{(4x^2y+y^3+x)^2}$ and $\omega_y = \frac{64x^5+32x^3y^2+4xy^4-8x^2y-2y^3+x}{(4x^2y+y^3+x)^2}$. Above becomes

$$c_4 + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)(c_5-c_1) - \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)^2 c_2 - \left(\frac{-4y^5-32x^2y^3-8xy^2+(-64x^4-1)\,y-}{(4x^2y+y^3+x)^2}\right.$$

Which expands to

$$\frac{8c_1xy^2}{4x^2y+y^3+x}+\frac{4c_5xy^2}{4x^2y+y^3+x}-\frac{256c_2x^4y^2}{(4x^2y+y^3+x)^2}-\frac{48c_2x^2y^4}{(4x^2y+y^3+x)^2}+\frac{16c_2x^3y}{(4x^2y+y^3+x)^2}+\frac{12c_2xy^3}{(4x^2y+y^3+}$$
$$+\frac{48x^2c_2y}{4x^2y+y^3+x}-\frac{128x^5yc_1}{(4x^2y+y^3+x)^2}-\frac{128x^4yc_3}{(4x^2y+y^3+x)^2}-\frac{32x^3y^3c_1}{(4x^2y+y^3+x)^2}-\frac{32x^2y^3c_3}{(4x^2y+y^3+x)^2}$$
$$+\frac{4x^2y^2c_1}{(4x^2y+y^3+x)^2}+\frac{4xy^2c_3}{(4x^2y+y^3+x)^2}+\frac{yc_1x}{(4x^2y+y^3+x)^2}+\frac{8x^2yc_4}{4x^2y+y^3+x}+\frac{8xyc_6}{4x^2y+y^3+x}-$$
$$\frac{64x^5c_5y}{(4x^2y+y^3+x)^2}-\frac{64x^4y^2c_4}{(4x^2y+y^3+x)^2}-\frac{64x^3y^3c_5}{(4x^2y+y^3+x)^2}-\frac{64x^3y^2c_6}{(4x^2y+y^3+x)^2}-\frac{12x^2y^4c_4}{(4x^2y+y^3+x)^2}-\frac{16c_5x^}{4x^2y+y^3}$$
$$-\frac{256c_2x^6}{(4x^2y+y^3+x)^2}+\frac{64c_1x^3}{4x^2y+y^3+x}-\frac{c_1y}{4x^2y+y^3+x}+\frac{48x^2c_3}{4x^2y+y^3+x}+\frac{4y^3c_2}{4x^2y+y^3+x}+\frac{4y^2c_3}{4x^2y+y^3+x}$$
$$-\frac{16x^4c_1}{(4x^2y+y^3+x)^2}-\frac{16x^3c_3}{(4x^2y+y^3+x)^2}+\frac{yc_3}{(4x^2y+y^3+x)^2}-\frac{c_4x}{4x^2y+y^3+x}-\frac{64x^6c_4}{(4x^2y+y^3+x)^2}-\frac{64x^5c}{(4x^2y+y^3}$$
$$+\frac{3y^4c_5}{(4x^2y+y^3+x)^2}+\frac{3y^3c_6}{(4x^2y+y^3+x)^2}-\frac{c_6}{4x^2y+y^3+x}-\frac{12xy^5c_5}{(4x^2y+y^3+x)^2}-\frac{12xy^4c_6}{(4x^2y+y^3+x)^2}+\frac{4x^3yc}{(4x^2y+y^3}$$
$$+\frac{4x^2y^2c_5}{(4x^2y+y^3+x)^2}+\frac{4x^2yc_6}{(4x^2y+y^3+x)^2}+\frac{3y^3c_4x}{(4x^2y+y^3+x)^2}+c_4=0$$

Multiplying each term by $(4x^2y+y^3+x)^2$ and expanding gives the multivariable polynomial

$$128x^5yc_1+64x^3y^3c_1+8c_1xy^5-256c_2x^6-64c_2x^4y^2+16c_2x^2y^4+4c_2y^6-64x^6c_4-16x^4y^2c_4+4x^2y^4c_4+c_4y^6$$
$$-128x^5c_5y-64x^3y^3c_5-8xy^5c_5+64x^4yc_3+32x^2y^3c_3+4c_3y^5-64x^5c_6-32x^3y^2c_6-4xy^4c_6+48x^4c_1+$$
$$8x^2y^2c_1-c_1y^4+64c_2x^3y+16c_2xy^3+16x^3yc_4+4y^3c_4x-16c_5x^4+8x^2y^2c_5+3y^4c_5+32x^3c_3+8xy^2c_3+8x^2yc_6+2y$$

Each monomial coefficient must be zero. This gives the following equations to solve for c_i

equation
$-256c_2 - 64c_4 = 0$
$128c_1 - 128c_5 = 0$
$-64c_6 = 0$
$-64c_2 - 16c_4 = 0$
$64c_3 = 0$
$48c_1 - 16c_5 = 0$
$64c_1 - 64c_5 = 0$
$-32c_6 = 0$
$64c_2 + 16c_4 = 0$
$32c_3 = 0$
$16c_2 + 4c_4 = 0$
$32c_3 = 0$
$8c_1 + 8c_5 = 0$
$8c_6 = 0$
$8c_1 - 8c_5 = 0$
$-4c_6 = 0$
$16c_2 + 4c_4 = 0$
$8c_3 = 0$
$-c_6 = 0$
$4c_2 + c_4 = 0$
$4c_3 = 0$
$-c_1 + 3c_5 = 0$
$2c_6 = 0$
$c_3 = 0$

Hence we see that $c_6 = 0, c_3 = 0$. The above reduces to

equation
$-256c_2 - 64c_4 = 0$
$128c_1 - 128c_5 = 0$
$-64c_2 - 16c_4 = 0$
$48c_1 - 16c_5 = 0$
$64c_1 - 64c_5 = 0$
$64c_2 + 16c_4 = 0$
$16c_2 + 4c_4 = 0$
$8c_1 + 8c_5 = 0$
$8c_1 - 8c_5 = 0$
$16c_2 + 4c_4 = 0$
$4c_2 + c_4 = 0$
$-c_1 + 3c_5 = 0$

Hence $Ac = b$ gives

$$\begin{pmatrix} 0 & -256 & -64 & 0 \\ 128 & 0 & 0 & -128 \\ 0 & -64 & -16 & 0 \\ 48 & 0 & 0 & -16 \\ 64 & 0 & 0 & -64 \\ 0 & 64 & 16 & 0 \\ 0 & 16 & 4 & 0 \\ 8 & 0 & 0 & -8 \\ 0 & 16 & 4 & 0 \\ 0 & 4 & 1 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank of A is 3 and the number of columns is 4. Hence non-trivial solution exist. Solving the above gives $c_4 = -4$ and $c_2 = 1$ and all other coefficients are zero. this means that , since

$$\begin{aligned} \xi &= c_1 x + c_2 y + c_3 \\ \eta &= c_4 x + c_5 y + c_6 \end{aligned}$$

Then

$$\begin{aligned} \xi &= y \\ \eta &= -4x \end{aligned}$$

Which is what we wanted to show for this ODE.

1.2.22.17 Example $y' = \frac{-y^2}{e^x - y}$

Solve

$$
\begin{aligned}
y' &= \frac{-y^2}{e^x - y} \\
y' &= \omega(x, y)
\end{aligned}
$$

The symmetry condition results in the PDE

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0
$$

End of the problem shows how this is solved for ξ, η which results in

$$
\begin{aligned}
\xi(x, y) &= 1 \\
\eta(x, y) &= y
\end{aligned}
$$

The integrating factor is therefore

$$
\begin{aligned}
\mu(x, y) &= \frac{1}{\eta - \xi \omega} \\
&= \frac{1}{y - \left(\frac{-y^2}{e^x - y}\right)} \\
&= \frac{1 - y e^{-x}}{y}
\end{aligned}
$$

The next step is to determine what is called the canonical coordinates R, S. Where R is the independent variable and S is the dependent variable. So we are looking for $S(R)$ function. This is done by using the standard characteristic equation by writing

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\
\frac{dx}{1} &= \frac{dy}{y} = dS \qquad (1)
\end{aligned}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Which is a first order PDE. This is solved for S, which gives (1) using the method of characteristic to solve first order PDE which is standard method. Starting with the first pair of ODE gives

$$
\frac{dy}{dx} = y
$$

Integrating gives $\ln |y| = x + c$ or $y = c e^x$ where c is constant of integration. In this method R is always c. Hence

$$
R(x, y) = y e^{-x}
$$

$S(x,y)$ is now found from the first equation in (1) and the last equation which gives

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ dS &= \frac{dx}{1} \\ dS &= dx \\ S &= x \end{aligned}$$

Hence

$$\begin{aligned} R &= ye^{-x} \\ S &= x \end{aligned}$$

Now that $R(x,y)\,, S(x,y)$ are found, the ODE $\frac{dS}{dR} = \Omega(R)$ is setup. The ODE comes out to be function of R only, so it is quadrature. This is the main idea of this method. By solving for R we go back to x, y and solve for $y(x)$. How to find $\frac{dS}{dR}$? There is an equation to determine this given by

$$\begin{aligned} \frac{dS}{dR} &= \frac{\frac{dS}{dx} + \omega(x,y)\,\frac{dS}{dy}}{\frac{dR}{dx} + \omega\,(x,y)\,\frac{dR}{dy}} \\ &= \frac{S_x + \omega(x,y)\,S_y}{R_x + \omega\,(x,y)\,R_y} \end{aligned}$$

Everything on the RHS is known. $S_x = 1, R_x = -ye^{-x}, S_y = 0, R_y = e^{-x}$. Substituting gives

$$\begin{aligned} \frac{dS}{dR} &= \frac{1}{-ye^{-x} + \frac{-y^2}{e^x - y}e^{-x}} \\ &= \frac{ye^{-x} - 1}{ye^{-x}} \end{aligned}$$

But $R = ye^{-x}$, hence the above becomes

$$\frac{dS}{dR} = \frac{R-1}{R}$$

This is just quadrature. Integrating gives

$$\begin{aligned} S &= \int \frac{R-1}{R} dR \\ &= R - \ln R + c_1 \end{aligned}$$

This solution is converted back to x, y. Since $S = x, R = ye^{-x}$, the above becomes

$$x = ye^{-x} - \ln\left(ye^{-x}\right) + c_1$$

Which is the solution to the original ODE.

Finding Lie symmetries for this example

The condition of symmetry is given above in equation (14) as

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{14}$$

Try

$$\begin{aligned}\xi &= c_1x + c_2y + c_3\\ \eta &= c_4x + c_5y + c_6\end{aligned}$$

Hence $\xi_x = c_1, \xi_y = c_2, \eta_x = c_4, \eta_y = c_5$ and (14) becomes

$$\begin{aligned}\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta &= 0\\ c_4 + \omega(c_5 - c_1) - \omega^2 c_2 - \omega_x(c_1x + c_2y + c_3) - \omega_y(c_4x + c_5y + c_6) &= 0\end{aligned}$$

But $\omega = \frac{-y^2}{e^x - y}, \omega_x = \frac{y^2e^x}{(e^x-y)^2}, \omega_y = \left(-\frac{2y}{e^x-y} - \frac{y^2}{(e^x-y)^2}\right)$ and the above becomes

$$c_4 + \frac{-y^2}{e^x - y}(c_5 - c_1) - \left(\frac{-y^2}{e^x - y}\right)^2 c_2 - \frac{y^2e^x}{(e^x - y)^2}(c_1x + c_2y + c_3) - \left(-\frac{2y}{e^x - y} - \frac{y^2}{(e^x - y)^2}\right)(c_4x + c_5y + c_6)$$

Need to do this again. I should get $c_3 = 1, c_5 = 1$ and everything else zero.

$$\begin{aligned}\xi &= 1\\ \eta &= y\end{aligned}$$

1.2.22.18 Example $y' = \frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}$

Solve

$$\begin{aligned}y' &= \frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}\\ y' &= \omega(x,y)\end{aligned}$$

The symmetry condition results in the pde

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{1}$$

Let Ansatz be

$$\begin{aligned}\xi &= 0\\ \eta &= f(x)\, g(y)\end{aligned}$$

Hence (1) becomes

$$g(y)\frac{df}{dx} + \omega f(x)\frac{dg}{dy} - \omega_y f(x)\, g(y) = 0$$

But $\omega_x = \frac{d}{dx}\left(\frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}\right) = -\frac{(y+1)}{(x+1)^2}$ and $\omega_y = \frac{x+1+2\sqrt{1+y}}{\sqrt{1+y}(2+2x)}$. Hence the above becomes

$$g(y)\frac{df}{dx} + \left(\frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}\right) f(x)\frac{dg}{dy} - \frac{x+1+2\sqrt{1+y}}{\sqrt{1+y}\,(2+2x)}(f(x)\, g(y)) = 0 \tag{2}$$

The numerator of the normal form of the above is

$$2\frac{df}{dx}g\sqrt{1+y}x+2y\sqrt{1+y}f\frac{dg}{dy}+2f\frac{dg}{dy}xy+2\frac{df}{dx}g\sqrt{1+y}-2fg\sqrt{1+y}+2f\frac{dg}{dy}\sqrt{1+y}-fgx+2f\frac{dg}{dy}x+2fy\frac{d}{d} \tag{3}$$

We can now either collect on y or x and try. Let us start with collecting on all terms with y. This gives

$$g\sqrt{1+y}\left(2x\frac{df}{dx}+2\frac{df}{dx}-2f\right)+y\sqrt{1+y}\frac{dg}{dy}(2f)+\frac{dg}{dy}\sqrt{1+y}(2f)+g(xf-f)+y\frac{dg}{dy}(2xf+2f)+\frac{dg}{dy}(2xf \tag{3A}$$

The coefficients of all terms with $g(y)$ or y in them are from the above are the following, which each must be zero

$$\begin{aligned}2f &= 0\\ xf - f &= 0\\ 2xf + 2f &= 0\\ 2x\frac{df}{dx} + 2\frac{df}{dx} - 2f &= 0\end{aligned}$$

Now we set each to zero and see if this produces $f(x)$ which can be used. We have 4 choices to try above. Starting from the most simple one. The first one above gives $2f = 0$ or $f = 0$. But this is not function of x. We try the next one $xf - f = 0$. This gives $f = 0$ or $x = 1$. Hence this does not give f as function of x. Next we try $2xf + 2f$. This also does not give f as function of x. The last one is $2x\frac{df}{dx} + 2\frac{df}{dx} - 2f = 0$ or $\frac{df}{dx} = \frac{2f}{2x+2}$. Solving this gives $f = c_1(x+1)$. This is successful since f is function of x. Hence

$$\begin{aligned}f(x) &= c_1(x+1)\\ \frac{df}{dx} &= c_1\end{aligned}$$

Now we need to determine $g(y)$. Substituting the above into (3) gives

$$2c_1 g(y)\sqrt{1+y}x+2\sqrt{1+y}c_1(x+1)\frac{dg}{dy}y+2c_1(x+1)\frac{dg}{dy}xy+2c_1 g\sqrt{1+y}-2c_1(x+1)g\sqrt{1+y}+2c_1(x+1)\frac{dg}{dy}\sqrt{}$$

Which simplifies to

$$2c_1\sqrt{1+y}\frac{dg}{dy}yx+2c_1\frac{dg}{dy}x^2y-c_1gx^2+2c_1\frac{dg}{dy}\sqrt{1+y}x+2\sqrt{1+y}c_1\frac{dg}{dy}y+2c_1\frac{dg}{dy}x^2+4c_1\frac{dg}{dy}xy-2c_1xg+2c_1\frac{dg}{dy} \tag{4}$$

Now factoring on all terms with x, and these are $\{x, x^2\}$ gives

$$-c_1x^2\left(-2\frac{dg}{dy}y+g-2\frac{dg}{dy}\right)-c_1x\left(-2\sqrt{1+y}\frac{dg}{dy}y-2\sqrt{1+y}\frac{dg}{dy}-2\frac{dg}{dy}y+g-2\frac{dg}{dy}\right)+T=0 \tag{4A}$$

Where T are terms that depends on y only. Each factor of x, x^2 must be zero. Hence the first above implies

$$\begin{aligned}-2\frac{dg}{dy}y+g-2\frac{dg}{dy}&=0\\ g'(y)&=\frac{g}{2(1+y)}\end{aligned}$$

Solving gives

$$g=c_2\sqrt{1+y} \tag{5}$$

Substituting (5) into (4) gives

$$c_1(1+x)\,c_2(1+y)=0$$

Which is not zero. Hence this term does not work. Now we try the second term in (4A) which means

$$\begin{aligned}-2\sqrt{1+y}\frac{dg}{dy}y-2\sqrt{1+y}\frac{dg}{dy}-2\frac{dg}{dy}y+g-2\frac{dg}{dy}&=0\\ \frac{dg}{dy}&=\frac{-g}{-2\sqrt{1+y}y-2\sqrt{1+y}-2y-2}\end{aligned}$$

Solving gives

$$g(y)=c_2\frac{\sqrt{1+y}}{1+\sqrt{1+y}}$$

Again, substituting the above back in (4) gives

$$c_1(1+x)\,c_2\frac{(1+y)\,x}{\left(1+\sqrt{1+y}\right)^2}=0$$

Which is not zero. Therefore starting with $f(x) = c_1(x+1)$ has failed to produce a valid $g(y)$ to satisfy the pde. This means we need to start all over again. Going back to (3) and now collecting on all terms with x instead. Here is (3) again

$$2\frac{df}{dx}g\sqrt{1+y}x+2y\sqrt{1+y}f\frac{dg}{dy}+2f\frac{dg}{dy}xy+2\frac{df}{dx}g\sqrt{1+y}-2fg\sqrt{1+y}+2f\frac{dg}{dy}\sqrt{1+y}-fgx+2f\frac{dg}{dy}x+2fy\frac{d}{d} \tag{3}$$

Collecting on all terms that depend on x gives

$$x\frac{df}{dx}\left(2g\sqrt{1+y}\right)+f\left(2y\sqrt{1+y}\frac{dg}{dy}-2g\sqrt{1+y}+2\frac{dg}{dy}\sqrt{1+y}+2y\frac{dg}{dy}+2\frac{dg}{dy}-g\right)+xf\left(2\frac{dg}{dy}y-g+2\frac{d}{d}\right. \tag{3B}$$

Each term must be zero, hence this gives these trials

$$\begin{aligned}2g\sqrt{1+y}&=0\\ 2\frac{dg}{dy}y-g+2\frac{dg}{dy}y&=0\\ 2y\sqrt{1+y}\frac{dg}{dy}-2g\sqrt{1+y}+2\frac{dg}{dy}\sqrt{1+y}+2y\frac{dg}{dy}+2\frac{dg}{dy}-g&=0\end{aligned}$$

Starting with the first one above $2g\sqrt{1+y}=0$ which gives $g=0$ which does not match the ansatz. Now we try the second one above, which gives

$$\frac{dg}{dy}=\frac{g}{2+2y}$$

Solving gives

$$g=c_1\sqrt{1+y} \tag{6}$$

Which meets the requirements of the ansatz. Now we need to use the above to generate $f(x)$. We do not need to try the third one above unless this fails. Substituting (6) into (3) gives

$$\begin{aligned}c_2\left(2\frac{df}{dx}xy+2\frac{df}{dx}x+2\frac{df}{dx}y-fy+2\frac{df}{dx}-f\right)&=0\\ 2\frac{df}{dx}xy+2\frac{df}{dx}x+2\frac{df}{dx}y-fy+2\frac{df}{dx}-f&=0 \qquad (7)\end{aligned}$$

Collecting on y gives

$$c_1(1+y)\left(2\frac{df}{dx}x+2\frac{df}{dx}-f\right)=0$$

Hence $2\frac{df}{dx}x+2\frac{df}{dx}-f$ must be zero. This gives as solution

$$\begin{aligned}f(x)&=c_2\sqrt{1+x}\\ \frac{df}{dx}&=c_2\frac{1}{2\sqrt{1+x}}\end{aligned}$$

Substituting the above into (7) to verify gives

$$2\left(c_2\frac{1}{2\sqrt{1+x}}\right)xy+2\left(c_2\frac{1}{2\sqrt{1+x}}\right)x+2\left(c_2\frac{1}{2\sqrt{1+x}}\right)y-\left(c_2\sqrt{1+x}\right)y+2\left(c_2\frac{1}{2\sqrt{1+x}}\right)-c_2\sqrt{1+}$$
$$c_2\frac{1}{\sqrt{1+x}}xy+c_2\frac{1}{\sqrt{1+x}}x+c_2\frac{1}{\sqrt{1+x}}y-c_2\sqrt{1+x}y+c_2\frac{1}{\sqrt{1+x}}-c_2\sqrt{1+}$$
$$c_2\left(\frac{1}{\sqrt{1+x}}xy+\frac{1}{\sqrt{1+x}}x+\frac{1}{\sqrt{1+x}}y-\sqrt{1+x}y+\frac{1}{\sqrt{1+x}}-\sqrt{1+x}\right.$$

Verified, Hence we have found $f(x), g(y)$. Therefore

$$\begin{aligned}\xi &= 0\\ \eta &= f(x)\,g(y)\\ &= \sqrt{1+x}\sqrt{1+y}\end{aligned}$$

Where we set $c_1 = c_2 = 1$. The integrating factor is therefore

$$\begin{aligned}\mu(x,y) &= \frac{1}{\eta-\xi\omega}\\ &= \frac{1}{\sqrt{1+x}\sqrt{1+y}}\end{aligned}$$

The next step is to determine the canonical coordinates R, S. Where R is the independent variable and S is the dependent variable. This is done by using the standard characteristic equation by writing

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS$$

For the special case $\xi = 0$ we have $R = x$. $S(x,y)$ is now found from the last two pair of equations which gives

$$\begin{aligned}dS &= \frac{dy}{\eta}\\ dS &= \frac{dy}{\sqrt{1+x}\sqrt{1+y}}\\ S &= 2\frac{\sqrt{1+y}}{\sqrt{1+x}}\end{aligned}$$

Hence (constant of integration is set to zero)

$$\begin{aligned}R &= x\\ S &= 2\frac{\sqrt{1+y}}{\sqrt{1+x}}\end{aligned} \tag{2}$$

Now that $R(x,y), S(x,y)$ are found, the ODE $\frac{dS}{dR} = \Omega(R)$ is setup. The ODE comes out to be function of R only, so it is quadrature. This is the main idea of this method. By solving for R we go back to x, y and solve for $y(x)$. How to find $\frac{dS}{dR}$? There is an equation to determine this given by

$$
\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{dS}{dx} + \omega(x,y)\,\frac{dS}{dy}}{\frac{dR}{dx} + \omega\,(x,y)\,\frac{dR}{dy}} \\
&= \frac{S_x + \omega(x,y)\,S_y}{R_x + \omega\,(x,y)\,R_y}
\end{aligned}
$$

Everything on the RHS is known. $S_x = -\frac{\sqrt{1+y}}{(1+x)^{\frac{3}{2}}}, R_x = 1, S_y = \frac{1}{\sqrt{1+x}\sqrt{1+y}}, R_y = 0$. Substituting into the above gives

$$
\begin{aligned}
\frac{dS}{dR} &= -\frac{\sqrt{1+y}}{(1+x)^{\frac{3}{2}}} + \omega(x,y)\,\frac{1}{\sqrt{1+x}\sqrt{1+y}} \\
&= -\frac{\sqrt{1+y}}{(1+x)^{\frac{3}{2}}} + \left(\frac{x\sqrt{1+y} + \sqrt{1+y} + 1 + y}{1+x}\right)\frac{1}{\sqrt{1+x}\sqrt{1+y}} \\
&= \frac{1}{\sqrt{x+1}} \\
&= \frac{1}{\sqrt{R+1}}
\end{aligned}
$$

Hence

$$
\frac{dS}{dR} = \frac{1}{\sqrt{R+1}}
$$

This is quadrature. Solving gives

$$
S = 2\sqrt{R+1} + c_1
$$

Convecting back to x, y gives

$$
2\frac{\sqrt{1+y}}{\sqrt{1+x}} = 2\sqrt{x+1} + c_1
$$

1.2.22.19 Example $y' = \frac{-y}{2x-ye^y}$

Solve

$$
\begin{aligned}
y' &= \frac{-y}{2x - ye^y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The symmetry condition results in the pde

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{1}$$

Let anstaz be

$$\begin{aligned}\xi &= g(y)\\ \eta &= 0\end{aligned}$$

Substituting this into (1) gives

$$-\omega^2 \frac{dg}{dy} - \omega_x g = 0$$

But $\omega^2 = \frac{y^2}{(2x-ye^y)^2}, \omega_x = \frac{d}{dx}\left(\frac{-y}{2x-ye^y}\right) = \frac{2y}{(2x-ye^y)^2}$. The above becomes

$$\begin{aligned}-\frac{y^2}{(2x-ye^y)^2}\frac{dg}{dy} - \frac{2y}{(2x-ye^y)^2}g &= 0\\ -y^2\frac{dg}{dy} - 2yg &= 0\\ \frac{dg}{dy} + \frac{2}{y}g &= 0\end{aligned}$$

This is linear ode. The solution is

$$g = \frac{c_1}{y^2}$$

Hence

$$\begin{aligned}\xi &= \frac{1}{y^2}\\ \eta &= 0\end{aligned}$$

But taking $c_1 = 1$. The integrating factor is therefore

$$\begin{aligned}\mu(x,y) &= \frac{1}{\eta - \xi\omega}\\ &= \frac{1}{-\frac{1}{y^2}\left(\frac{-y}{2x-ye^y}\right)}\\ &= y(2x - ye^y)\end{aligned}$$

The next step is to determine the canonical coordinates R, S. Where R is the independent variable and S is the dependent variable. This is done by using the standard characteristic equation by writing

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS$$

Since $\eta = 0$, then in this special case $R = c_1 = y$. To find S we use $dS = \frac{dx}{\xi}$ or $dS = y^2 dx$. Hence $S = c_1^2 x + c_2 = c_1^2 x$ by taking $c_2 = 0$. Therefore $S = y^2 x$ since $c_1 = y$.

$$\begin{aligned} R &= y \\ S &= y^2 x \end{aligned} \tag{2}$$

Now that $R(x,y), S(x,y)$ are found, the ODE $\frac{dS}{dR} = \Omega(R)$ is setup. The ODE comes out to be function of R only, so it is quadrature. This is the main idea of this method. By solving for R we go back to x, y and solve for $y(x)$. How to find $\frac{dS}{dR}$? There is an equation to determine this given by

$$\begin{aligned} \frac{dS}{dR} &= \frac{\frac{dS}{dx} + \omega(x,y)\frac{dS}{dy}}{\frac{dR}{dx} + \omega(x,y)\frac{dR}{dy}} \\ &= \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \end{aligned}$$

Everything on the RHS is known. $S_x = y^2, R_x = 0, S_y = 2yx, R_y = 1$. Substituting into the above gives

$$\begin{aligned} \frac{dS}{dR} &= \frac{y^2 + \omega(x,y)\, 2yx}{\omega(x,y)} \\ &= \frac{y^2 + \left(\frac{-y}{2x - ye^y}\right) 2yx}{\left(\frac{-y}{2x - ye^y}\right)} \\ &= y^2 e^y \end{aligned}$$

Now we need to express the RHS in terms of R, S. From (2) we see that $y = R$, hence the above becomes

$$\frac{dS}{dR} = R^2 e^R$$

This is quadrature. Solving gives

$$S = \left(R^2 - 2R + 2\right) e^R + c_1$$

Convecting back to x, y gives

$$y^2 x = \left(y^2 - 2y + 2\right) e^y + c_1$$

1.2.22.20 Example $y' = \frac{-1-2yx}{x^2+2y}$

Solve

$$
\begin{aligned}
y' &= \frac{-1-2yx}{x^2+2y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The symmetry condition results in the pde

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{1}
$$

Let anstaz be

$$
\begin{aligned}
\xi &= 0 \\
\eta &= f(x)\, g(y)
\end{aligned}
$$

Substituting this into (1) gives

$$
g \frac{df}{dx} + \omega f \frac{dg}{dy} - \omega_y f g = 0
$$

But $\omega = \frac{-1-2yx}{x^2+2y}, \omega_y = \frac{d}{dy}\left(\frac{-1-2yx}{x^2+2y}\right) = \frac{2-2x^3}{(x^2+2y)^2}$. The above becomes

$$
g \frac{df}{dx} + \left(\frac{-1-2yx}{x^2+2y}\right) f \frac{dg}{dy} - \left(\frac{2-2x^3}{(x^2+2y)^2}\right) f g = 0
$$

The numerator of the normal form is

$$
\begin{aligned}
g \frac{df}{dx} (x^2+2y)^2 + (x^2+2y)(-1-2yx) f \frac{dg}{dy} - (2-2x^3) f g = 0 \\
g \frac{df}{dx} (x^4 + 4x^2 y + 4y^2) + (-2x^3 y - x^2 - 4x y^2 - 2y) f \frac{dg}{dy} - (2-2x^3) f g = 0 \quad (2)
\end{aligned}
$$

To solve this for $f(x), g(y)$ we start by collecting on either x or y. Let us start by collecting on y. This gives

$$
\left[4\frac{df}{dx}\right](g y^2) + \left[4\frac{df}{dx} x^2\right](y g) + \left[\frac{df}{dx} x^4 - (-2x^3+2) f\right] g + \left[(-2x^3 - 4x - 2) f\right]\left(\frac{dg}{dy}\right) - \left[x^2 f\right] \frac{dg}{dy} = 0 \tag{3}
$$

The other option was to collect on x terms. This would give

$$
\left[-2y\frac{dg}{dy} + 2g\right](x^3 f) - \left[x^2 f\right]\left(\frac{dg}{dy}\right) - [4x f]\left(y \frac{dg}{dy}\right) + \left[-2\frac{dg}{dy} y - 2g\right](f) + [g]\left(x^4 \frac{df}{dx}\right) + [y g]\left(4\frac{df}{dx} x^2\right) + [y \tag{4}
$$

We start from (3), and if this yields no solutions for $f(x), g(y)$ then we come back and try (4). In either form, the terms inside the $[\cdot]$ must all be zero to satisfy the ode. From (3) this gives

$$
\begin{aligned}
4\frac{df}{dx} &= 0\\
4\frac{df}{dx}x^2 &= 0\\
\frac{df}{dx}x^4 - \left(-2x^3+2\right)f &= 0\\
\left(-2x^3-4x-2\right)f &= 0\\
x^2 f &= 0
\end{aligned}
$$

If one of these results in $f(x)$ which is function of x. Then we try it to solve for $g(y)$. If the solutions end up verifying the pde, then we are done. From the above, we start with the first one. This gives $f = c_1$. Which is not function of x. The second give same result. The this option which is $\frac{df}{dx}x^4 - \left(-2x^3+2\right)f = 0$ gives

$$
f(x) = c_1 \frac{e^{-\frac{2}{3x^3}}}{x^2}
$$

Which is function of x. We now use this to find $g(y)$. It turns out this does not work. The whole anstaz will fail. So need to try different anstaz.

1.2.22.21 Example $y' = 3\sqrt{yx}$

Solve

$$
\begin{aligned}
y' &= 3\sqrt{yx}\\
y' &= \omega(x,y)
\end{aligned}
$$

The symmetry condition results in the pde

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{1}
$$

Trying polynomial anstaz

$$
\begin{aligned}
\xi &= a_0 + a_1 x\\
\eta &= b_0 + b_1 y
\end{aligned}
$$

And substituting these into (1) and simplifying gives

$$
\left(-9a_1 + 3b_1\right)yx - 3xb_0 - 3ya_0 = 0
$$

Setting all coefficients to zero gives

$$
\begin{aligned}
-9a_1 + 3b_1 &= 0 \\
b_0 &= 0 \\
a_0 &= 0
\end{aligned}
$$

Hence $a_1 = \frac{1}{3}b_1$. Letting $b_1 = 1$ then $a_1 = \frac{1}{3}$ and the infinitesimals are

$$
\begin{aligned}
\xi &= \frac{1}{3}x \\
\eta &= y
\end{aligned}
$$

The integrating factor is therefore

$$
\begin{aligned}
\mu(x,y) &= \frac{1}{\eta - \xi\omega} \\
&= \frac{1}{y - \frac{1}{3}x\left(3\sqrt{yx}\right)} \\
&= -\frac{y + x\sqrt{xy}}{x^3y - y^2}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. This is done by using the standard characteristic equation by writing

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS
$$

The first pair of equations gives

$$
\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{3y}{x}
$$

Solving gives

$$
y = c_1x^3
$$

Hence

$$
R = c_1 = \frac{y}{x^3} \tag{2}
$$

And S is found from

$$
dS = \frac{dx}{\xi} = 3\frac{dx}{x}
$$

Integrating gives

$$
\begin{aligned}
S &= 3\ln x + c_1 \\
&= 3\ln x
\end{aligned}
$$

By choosing $c_1 = 0$. Now that $R(x,y)\,,S(x,y)$ are found, the ODE $\frac{dS}{dR} = F(R)$ is determined. This is determined from

$$\begin{aligned}\frac{dS}{dR} &= \frac{\frac{dS}{dx} + \omega(x,y)\,\frac{dS}{dy}}{\frac{dR}{dx} + \omega\,(x,y)\,\frac{dR}{dy}}\\ &= \frac{S_x + \omega(x,y)\,S_y}{R_x + \omega\,(x,y)\,R_y}\end{aligned}$$

But $S_x = \frac{3}{x}, R_x = -3\frac{y}{x^4}, S_y = 0, R_y = \frac{1}{x^3}$. Substituting these into the above gives

$$\begin{aligned}\frac{dS}{dR} &= \frac{\frac{3}{x}}{-3\frac{y}{x^4} + \omega\,(x,y)\,\frac{1}{x^3}}\\ &= \frac{3x^3}{-3y + x\omega\,(x,y)}\end{aligned}$$

But $\omega(x,y) = 3\sqrt{yx}$. The above becomes

$$\begin{aligned}\frac{dS}{dR} &= \frac{3x^3}{-3y + 3x\sqrt{yx}}\\ &= \frac{x^3}{x\sqrt{yx} - y}\\ &= \frac{-1}{\sqrt{\frac{y}{x^3}} - \frac{y}{x^3}}\end{aligned} \tag{3}$$

But $R = \frac{y}{x^3}$ and the above becomes

$$\frac{dS}{dR} = \frac{-1}{R - \sqrt{R}}$$

Which is a quadrature. Solving gives

$$\begin{aligned}\int dS &= \int \frac{-1}{R - \sqrt{R}} dR\\ S &= -2\ln\left(\sqrt{R} - 1\right) + c_1\end{aligned}$$

Converting back to x, y gives

$$\begin{aligned}3\ln x &= -2\ln\left(\sqrt{\frac{y}{x^3}} - 1\right) + c_1\\ \ln x^3 + \ln\left(\sqrt{\frac{y}{x^3}} - 1\right)^2 &= c_1\\ \ln\left(x^3\left(\sqrt{\frac{y}{x^3}} - 1\right)^2\right) &= c_1\\ x^3\left(\sqrt{\frac{y}{x^3}} - 1\right)^2 &= c_2\end{aligned}$$

Or

$$y_1(x) = 2x\left(x^2 + x\sqrt{xc_1}\right) - x^3 + c_1$$
$$y_2(x) = -2x\left(-x^2 + x\sqrt{xc_1}\right) - x^3 + c_1$$

1.2.22.22 Example $y' = 4(yx)^{\frac{1}{3}}$

Solve

$$y' = 4(yx)^{\frac{1}{3}}$$
$$y' = \omega(x,y)$$

The symmetry condition results in the pde

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{1}$$

Trying polynomial anstaz

$$\xi = a_0 + a_1 x$$
$$\eta = b_0 + b_1 y$$

And substituting these into (1) and simplifying gives

$$(-16a_1 + 8b_1)\, yx - 4xb_0 - 4ya_0 = 0$$

Setting all coefficients to zero gives

$$-16a_1 + 8b_1 = 0$$
$$b_0 = 0$$
$$a_0 = 0$$

Hence $a_1 = \frac{1}{2}b_1$. Letting $b_1 = 1$ then $a_1 = \frac{1}{2}$ and the infinitesimals are

$$\xi = \frac{1}{2}x$$
$$\eta = y$$

The integrating factor is therefore

$$\begin{aligned}\mu(x,y) &= \frac{1}{\eta - \xi\omega}\\ &= \frac{1}{y - \frac{1}{2}x\left(4\,(yx)^{\frac{1}{3}}\right)}\\ &= \frac{1}{y - 2x\,(xy)^{\frac{1}{3}}}\end{aligned}$$

The next step is to determine the canonical coordinates R, S. This is done by using the standard characteristic equation by writing

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS$$

The first pair of equations gives

$$\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{2y}{x}$$

Solving gives

$$y = c_1 x^2$$

Hence

$$R = c_1 = \frac{y}{x^2} \tag{2}$$

And S is found from

$$dS = \frac{dx}{\xi} = 2\frac{dx}{x}$$

Integrating gives

$$\begin{aligned} S &= 2\ln x + c_1 \\ &= 2\ln x \end{aligned}$$

By choosing $c_1 = 0$. Now the ODE $\frac{dS}{dR} = F(R)$ is found from

$$\begin{aligned} \frac{dS}{dR} &= \frac{\frac{dS}{dx} + \omega(x,y)\,\frac{dS}{dy}}{\frac{dR}{dx} + \omega\,(x,y)\,\frac{dR}{dy}} \\ &= \frac{S_x + \omega(x,y)\,S_y}{R_x + \omega\,(x,y)\,R_y} \end{aligned}$$

But $S_x = \frac{2}{x}, R_x = -2\frac{y}{x^3}, S_y = 0, R_y = \frac{2}{x^2}$. Substituting these into the above and simplifying gives

$$\begin{aligned} \frac{dS}{dR} &= \frac{x^2}{2x\,(yx)^{\frac{1}{3}} - y} \\ &= \frac{1}{2\frac{1}{x}\,(yx)^{\frac{1}{3}} - \frac{y}{x^2}} \\ &= \frac{1}{2y^{\frac{1}{3}}x^{-\frac{2}{3}} - \frac{y}{x^2}} \\ &= \frac{1}{2\left(\frac{y}{x^2}\right)^{\frac{1}{3}} - \frac{y}{x^2}} \\ &= \frac{1}{2\,(R)^{\frac{1}{3}} - R} \end{aligned}$$

Hence

$$\frac{dS}{dR} = \frac{1}{2R^{\frac{1}{3}} - R}$$

Which is a quadrature. Solving gives

$$\int dS = \int \frac{1}{2R^{\frac{1}{3}} - R} dR$$
$$S = -\frac{3}{2}\ln\left(-2 + R^{\frac{2}{3}}\right) + c_1$$

Converting back to x, y gives

$$2\ln x = -\frac{3}{2}\ln\left(-2 + \left(\frac{y}{x^2}\right)^{\frac{2}{3}}\right) + c_1$$

The above can be simplified more if needed to solve for $y(x)$ explicitly.

1.2.22.23 Example $y' = 2y + 3e^{2x}$

Solve

$$y' = 2y + 3e^{2x}$$
$$y' = \omega(x, y)$$

From the lookup table, since this is linear ode $y' = f(x)\,y + g(x)$ then

$$\xi = 0$$
$$\eta = e^{\int f dx}$$
$$= e^{\int 2dx}$$
$$= e^{2x}.$$

If we were to use the integrating factor method, then

$$\mu(x, y) = \frac{1}{\eta - \xi\omega}$$
$$= \frac{1}{e^{2x}}$$
$$= e^{-2x}$$

Then the general solution is

$$\begin{aligned}
\int \mu(x,y)\,(dy-\omega dx) &= c_1\\
\int e^{-2x}\left(dy-\left(2y+3e^{2x}\right)dx\right) &= c_1\\
\int e^{-2x}dy-\left(2ye^{-2x}+3\right)dx &= c_1\\
\int e^{-2x}dy-2ye^{-2x}dx &= \int 3dx+c_1\\
\int d\left(e^{-2x}y\right) &= \int 3dx+c_1
\end{aligned}$$

Hence

$$\begin{aligned}
e^{-2x}y &= 3x+c_1\\
y &= e^{2x}(3x+c_1)
\end{aligned}$$

But if we were to use the basic Lie symmetry method, then the next step is to determine the canonical coordinates R, S. This is done by using the standard characteristic equation by writing

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS$$

Since $\xi=0$ then this is the special case where $R=x$. And S is found from

$$dS=\frac{dy}{\eta}=e^{-2x}dy$$

Integrating gives

$$\begin{aligned}
S &= e^{-2x}y+c_1\\
&= e^{-2x}y
\end{aligned}$$

By choosing $c_1=0$. Now the ODE $\frac{dS}{dR}=F(R)$ is found from

$$\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{dS}{dx}+\omega(x,y)\frac{dS}{dy}}{\frac{dR}{dx}+\omega(x,y)\frac{dR}{dy}}\\
&= \frac{S_x+\omega(x,y)\,S_y}{R_x+\omega(x,y)\,R_y}
\end{aligned}$$

But $S_x=-2e^{-2x}y, R_x=1, S_y=e^{-2x}, R_y=0$. Substituting these into the above and simplifying gives

$$\begin{aligned}
\frac{dS}{dR} &= -2e^{-2x}y+\left(2y+3e^{2x}\right)e^{-2x}\\
&= -2e^{-2x}y+2ye^{-2x}+3\\
&= 3
\end{aligned}$$

Which is a quadrature. Solving gives

$$\begin{aligned}\int dS &= \int 3dR\\ S &= 3R + c_1\end{aligned}$$

Converting back to x, y gives

$$\begin{aligned}e^{-2x}y &= 3x + c_1\\ y &= (3x + c_1)\, e^{2x}\end{aligned}$$

Of course, this ode is first order linear and can be solved much easier using integrating factor method. But this is just to illustrate the Lie symmetry method.

1.2.22.24 Example $y' = \frac{1}{3}\frac{2y+y^3-x^2}{x}$

Solve

$$\begin{aligned}y' &= \frac{1}{3}\frac{2y + y^3 - x^2}{x}\\ y' &= \omega(x, y)\end{aligned}$$

Using Maple the infinitesimals are

$$\begin{aligned}\xi &= \frac{3}{2x^{\frac{1}{3}}}\\ \eta &= \frac{y}{x^{\frac{4}{3}}}\end{aligned}$$

(Will need to show how to obtain these). Lets solve this using the integration factor method first. The integrating factor is given by

$$\begin{aligned}\mu(x, y) &= \frac{1}{\eta - \xi\omega}\\ &= \frac{1}{\frac{y}{x^{\frac{4}{3}}} - \frac{3}{2x^{\frac{1}{3}}}\left(\frac{1}{3}\frac{2y+y^3-x^2}{x}\right)}\\ &= 2\frac{x^{\frac{4}{3}}}{x^2 - y^3}\end{aligned}$$

Then the general solution is

$$
\begin{aligned}
\int \mu(x,y)\,(dy-\omega dx) &= c_1\\
\int 2\frac{x^{\frac{4}{3}}}{x^2-y^3}\left(dy-\left(\frac{1}{3}\frac{2y+y^3-x^2}{x}\right)dx\right) &= c_1\\
\int \left(2\frac{x^{\frac{4}{3}}}{x^2-y^3}dy-\left(2\frac{x^{\frac{4}{3}}}{x^2-y^3}\right)\left(\frac{1}{3}\frac{2y+y^3-x^2}{x}\right)dx\right) &= c_1\\
\int \left(2\frac{x^{\frac{4}{3}}}{x^2-y^3}dy-\left(\frac{2}{3}\frac{x^{\frac{1}{3}}}{x^2-y^3}\right)\left(2y+y^3-x^2\right)dx\right) &= c_1
\end{aligned}
$$

Hence we need to find $F(x,y)$ s.t. $dF = \left(2\frac{x^{\frac{4}{3}}}{x^2-y^3}dy-\left(\frac{2}{3}\frac{x^{\frac{1}{3}}}{x^2-y^3}\right)\left(2y+y^3-x^2\right)dx\right)$ which will make the solution $F=c$. Therefore

$$
\begin{aligned}
dF &= \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy\\
&= 2\frac{x^{\frac{4}{3}}}{x^2-y^3}dy-\left(\frac{2}{3}\frac{x^{\frac{1}{3}}}{x^2-y^3}\right)\left(2y+y^3-x^2\right)dx
\end{aligned}
$$

Hence

$$
\frac{\partial F}{\partial x} = -\frac{2}{3}\frac{x^{\frac{1}{3}}\left(2y+y^3-x^2\right)}{x^2-y^3} \tag{1}
$$

$$
\frac{\partial F}{\partial y} = 2\frac{x^{\frac{4}{3}}}{x^2-y^3} \tag{2}
$$

Integrating (1) gives

$$
\begin{aligned}
F &= \left(\int -\frac{2}{3}\frac{x^{\frac{1}{3}}\left(2y+y^3-x^2\right)}{x^2-y^3}dx\right)+g(y)\\
&= \frac{1}{2}x^{\frac{4}{3}}+\frac{1}{3}\ln\left(x^{\frac{4}{3}}+x^{\frac{2}{3}}y+y^2\right)-\frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\frac{\left(2x^{\frac{2}{3}}+y\right)\sqrt{3}}{y}\right)-\frac{2}{3}\ln\left(x^{\frac{2}{3}}-y\right)+g(y)
\end{aligned} \tag{3}
$$

Where $g(y)$ acts as the integration constant but F depends on x,y it becomes an arbitrary function. Taking derivative of the above w.r.t. y gives

$$
\frac{\partial F}{\partial y} = 2\frac{x^{\frac{4}{3}}}{x^2-y^3}+g'(y) \tag{4}
$$

Equating (4,2) gives

$$\begin{aligned}2\frac{x^{\frac{4}{3}}}{x^2-y^3} &= 2\frac{x^{\frac{4}{3}}}{x^2-y^3}+g'(y)\\ 0 &= g'(y)\\ g(y) &= c_1\end{aligned}$$

Hence (3) becomes

$$F = \frac{1}{2}x^{\frac{4}{3}}+\frac{1}{3}\ln\left(x^{\frac{4}{3}}+x^{\frac{2}{3}}y+y^2\right)-\frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\frac{\left(2x^{\frac{2}{3}}+y\right)\sqrt{3}}{y}\right)-\frac{2}{3}\ln\left(x^{\frac{2}{3}}-y\right)+c_1$$

Therefore the solution is

$$F = c$$

$$\frac{1}{2}x^{\frac{4}{3}}+\frac{1}{3}\ln\left(x^{\frac{4}{3}}+x^{\frac{2}{3}}y+y^2\right)-\frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\frac{\left(2x^{\frac{2}{3}}+y\right)\sqrt{3}}{y}\right)-\frac{2}{3}\ln\left(x^{\frac{2}{3}}-y\right)=c_2$$

Where constants c_1, c were combined into c_2. Now this ode will be solved using direct symmetry by converting to canonical coordinates. This is done by using the standard characteristic equation by writing

$$\begin{aligned}\frac{dx}{\xi} &= \frac{dy}{\eta} = dS\\ \frac{dx}{\frac{3}{2x^{\frac{1}{3}}}} &= \frac{dy}{\frac{y}{x^{\frac{4}{3}}}} = dS\end{aligned}$$

First pair of ode's give

$$\frac{dy}{dx} = \frac{\frac{y}{x^{\frac{4}{3}}}}{\frac{3}{2x^{\frac{1}{3}}}} = \frac{2}{3x}y$$

Hence

$$y = c_1 x^{\frac{2}{3}}$$

Therefore

$$R = yx^{-\frac{2}{3}}$$

And

$$dS = \frac{dx}{\xi} = \frac{2}{3}x^{\frac{1}{3}}dx$$

Integrating gives

$$\begin{aligned}
S &= \int \frac{2}{3} x^{\frac{1}{3}} dx \\
&= \frac{1}{2} x^{\frac{4}{3}} + c_1 \\
&= \frac{1}{2} x^{\frac{4}{3}}
\end{aligned}$$

By choosing $c_1 = 0$. Now the ODE $\frac{dS}{dR} = F(R)$ is found from

$$\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{dS}{dx} + \omega(x,y)\frac{dS}{dy}}{\frac{dR}{dx} + \omega(x,y)\frac{dR}{dy}} \\
&= \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y}
\end{aligned}$$

But $S_x = \frac{2}{3}x^{\frac{1}{3}}, R_x = -\frac{2}{3}yx^{-\frac{5}{3}}, S_y = 0, R_y = x^{-\frac{2}{3}}$. Substituting these into the above and simplifying gives

$$\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{2}{3}x^{\frac{1}{3}}}{-\frac{2}{3}yx^{-\frac{5}{3}} + \omega(x,y)\, x^{-\frac{2}{3}}} \\
&= \frac{\frac{2}{3}x^{\frac{1}{3}}}{-\frac{2}{3}yx^{-\frac{5}{3}} + \left(\frac{1}{3}\frac{2y+y^3-x^2}{x}\right) x^{-\frac{2}{3}}} \\
&= -2\frac{x^2}{x^2 - y^3}
\end{aligned}$$

But $R = yx^{-\frac{2}{3}}$ or $y = Rx^{\frac{2}{3}}$. The above becomes

$$\begin{aligned}
\frac{dS}{dR} &= -2\frac{x^2}{x^2 - R^3x^2} \\
&= \frac{-2}{1 - R^3}
\end{aligned}$$

Which is a quadrature. Solving gives

$$\begin{aligned}
\int dS &= \int \frac{-2}{1 - R^3} dR \\
S &= -\frac{1}{3}\ln\left(R^2 + x + 1\right) - \frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}(1 + 2R)\sqrt{3}\right) + \frac{2}{3}\ln(R - 1) + c_1
\end{aligned}$$

Converting back to x, y gives

$$\begin{aligned}
\frac{1}{2}x^{\frac{4}{3}} &= -\frac{1}{3}\ln\left(\left(yx^{-\frac{2}{3}}\right)^2 + x + 1\right) - \frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\left(1 + 2\left(yx^{-\frac{2}{3}}\right)\right)\sqrt{3}\right) + \frac{2}{3}\ln\left(\left(yx^{-\frac{2}{3}}\right) - 1\right) + c_1 \\
\frac{1}{2}x^{\frac{4}{3}} &= -\frac{1}{3}\ln\left(y^2x^{-\frac{4}{3}} + x + 1\right) - \frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\left(1 + 2yx^{-\frac{2}{3}}\right)\sqrt{3}\right) + \frac{2}{3}\ln\left(yx^{-\frac{2}{3}} - 1\right) + c_1
\end{aligned}$$

1.2.22.25 Example $y' = 3 - 2\frac{y}{x}$

This is homogeneous ODE of Class A of form $y' = F\left(\frac{y}{x}\right)$, hence from the lookup table

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

The first step is to verify that $\bar{x} = \epsilon x, \bar{y} = \epsilon y$ leaves the ode invariant.

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}_x + \bar{y}_y y'}{\bar{x}_x + \bar{x}_y y'} = \frac{\epsilon y'}{\epsilon} = y'$$

Hence the ode becomes

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= 3 - 2\frac{\bar{y}}{\bar{x}}\\ y' &= 3 - 2\frac{\epsilon y}{\epsilon x}\\ &= 3 - 2\frac{y}{x}\end{aligned}$$

Verified. Now the ode is solved. The tangent curves are computed directly from the Lie group symmetry given above

$$\begin{aligned}\xi &= \left.\frac{\partial \bar{x}}{\partial \epsilon}\right|_{\epsilon=0} = x\\ \eta &= \left.\frac{\partial \bar{y}}{\partial \epsilon}\right|_{\epsilon=0} = y\end{aligned}$$

The canonical coordinates (R, S) are now found. Using

$$\begin{aligned}\frac{dx}{\xi} &= \frac{dy}{\eta} = dS\\ \frac{dx}{x} &= \frac{dy}{y} = dS \qquad (1)\end{aligned}$$

The first pair gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x}\\ \ln y &= \ln x + c_1\\ y &= cx\end{aligned}$$

Hence

$$\begin{aligned}R &= c\\ &= \frac{y}{x}\end{aligned}$$

Now we find S from the last pair of equations

$$
\begin{aligned}
\frac{dy}{y} &= dS \\
S &= \ln y
\end{aligned}
$$

What is left is to find $\frac{dS}{dR}$. This is given by

$$
\frac{dS}{dR} = G(R)
$$

To find $G(R)$, we use $dS = S_x dx + S_y dy = \frac{1}{y} dy$ and $dR = R_x dx + R_y dy = -\frac{y}{x^2} dx + \frac{1}{x} dy$. Hence

$$
\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{1}{y} dy}{-\frac{y}{x^2} dx + \frac{1}{x} dy} \\
&= \frac{\frac{dy}{dx}}{-\frac{y^2}{x^2} + \frac{y}{x} \frac{dy}{dx}} \\
&= \frac{\frac{dy}{dx}}{-R^2 + R \frac{dy}{dx}}
\end{aligned}
$$

But $\frac{dy}{dx} = 3 - 2\frac{y}{x} = 3 - 2R$, hence

$$
\begin{aligned}
\frac{dS}{dR} &= \frac{3 - 2R}{-R^2 + R\,(3 - 2R)} \\
&= \frac{3 - 2R}{3\,(R - R^2)}
\end{aligned}
$$

Which is a quadrature. In Lie method, for first order ode, we always obtain $\frac{dS}{dR} = G(R)$. Integrating the above gives

$$
\begin{aligned}
\int dS &= \int \frac{3 - 2R}{3\,(R - R^2)} dR \\
S &= \ln R - \frac{1}{3} \ln\,(R - 1) + c_1
\end{aligned}
$$

Final step is to replace R,S back with x,y which gives

$$
\begin{aligned}
\ln y &= \ln \frac{y}{x} - \frac{1}{3}\ln\left(\frac{y}{x}-1\right) + c_1 \\
y &= c_1 \frac{\frac{y}{x}}{\left(\frac{y}{x}-1\right)^{\frac{1}{3}}} \\
\left(\frac{y}{x}-1\right)^{\frac{1}{3}} &= c_1 \frac{1}{x} \\
\frac{y}{x} - 1 &= c_2 \frac{1}{x^3} \\
y &= \left(c_2 \frac{1}{x^3} + 1\right) x
\end{aligned}
$$

1.2.22.26 Example $y' = \frac{-3+\frac{y}{x}}{-1-\frac{y}{x}}$

This is homogeneous ODE of Class A of form $y' = F\left(\frac{y}{x}\right)$, hence from the lookup table

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Canonical coordinates (R,S) are found similar to the above which gives

$$
\begin{aligned}
R &= \frac{y}{x} \\
S &= \ln y
\end{aligned}
$$

What is left is to find $\frac{dS}{dR}$. This is given by

$$
\frac{dS}{dR} = G(R)
$$

Which is the same as above

$$
\frac{dS}{dR} = \frac{\frac{dy}{dx}}{-R^2 + R\frac{dy}{dx}}
$$

But in this problem, the only difference is that $\frac{dy}{dx} = \frac{-3+\frac{y}{x}}{-1-\frac{y}{x}} = \frac{-3+R}{-1-R}$, hence

$$
\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{-3+R}{-1-R}}{-R^2 + R\left(\frac{-3+R}{-1-R}\right)} \\
&= \frac{1}{R}\frac{R-3}{R^2+2R-3}
\end{aligned}
$$

Which is a quadrature. In Lie method, for first order ode, we always obtain $\frac{dS}{dR} = G(R)$. Integrating the above gives

$$
\begin{aligned}
\int dS &= \int \frac{1}{R}\left(\frac{R-3}{R^2+2R-3}\right) dR \\
S &= \ln(R) - \frac{1}{2}\ln(R+3) - \frac{1}{2}\ln(R-1) + c_1
\end{aligned}
$$

Final step is to replace R, S back with x, y which gives

$$
\ln y = \ln\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{y}{x}+3\right) - \frac{1}{2}\ln\left(\frac{y}{x}-1\right) + c_1
$$

This can be solved for y if an explicit solution is needed.

1.2.22.27 Example $y' = \frac{1+3\left(\frac{y}{x}\right)^2}{2\frac{y}{x}}$

This is homogeneous ODE of Class A of form $y' = F\left(\frac{y}{x}\right)$, hence from the lookup table

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

The canonical ode is

$$
\frac{dS}{dR} = \frac{\frac{dy}{dx}}{-R^2 + R\frac{dy}{dx}}
$$

The above is the same ode in canonical coordinates for any ode of the form $y' = F\left(\frac{y}{x}\right)$. We just need to express y' as function of R. In this case the above becomes

$$
\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{1+3R^2}{2R}}{-R^2 + R\left(\frac{1+3R^2}{2R}\right)} \\
&= \frac{3R^2+1}{R^3+R}
\end{aligned}
$$

Integrating gives

$$
S = \ln\left(R\left(R^2+1\right)\right) + c_1
$$

Final step is to replace R, S back with x, y which gives

$$
\begin{aligned}
\ln y &= \ln\left(\frac{y}{x}\left(\left(\frac{y}{x}\right)^2+1\right)\right) + c_1 \\
y &= c_2\frac{y}{x}\left(\left(\frac{y}{x}\right)^2+1\right) \\
1 &= \frac{c_2}{x}\left(\left(\frac{y}{x}\right)^2+1\right) \\
\frac{y^2}{x^2} &= c_3 x - 1 \\
y^2 &= c_3 x^3 - x^2
\end{aligned}
$$

Hence

$$\begin{aligned} y &= \pm\sqrt{c_3x^3 - x^2} \\ &= \pm x\sqrt{c_3x - 1} \end{aligned}$$

Finding ξ,η from symmetry condition for the above ode This shows how to find ξ,η directly also. The condition of symmetry is given above in equation (14) as

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{14}$$

Try Ansatz

$$\begin{aligned} \xi &= c_0 + c_1x \\ \eta &= c_2 + c_3y \end{aligned}$$

And given

$$\begin{aligned} \omega &= \frac{1}{2}\frac{x^2+3y^2}{xy} \\ \omega^2 &= \frac{1}{4}\frac{(x^2+3y^2)^2}{x^2y^2} \\ \omega_x &= \frac{1}{2}\frac{x^2-3y^2}{yx^2} \\ \omega_y &= \frac{1}{2}\frac{3y^2-x^2}{xy^2} \end{aligned}$$

Hence (14) becomes

$$\eta_x + \frac{1}{2}\frac{x^2+3y^2}{xy}\eta_y - \frac{1}{2}\frac{x^2-3y^2}{yx^2}\xi - \frac{1}{2}\frac{3y^2-x^2}{xy^2}\eta = 0$$

Therefore the above becomes

$$\frac{1}{2}\frac{x^2+3y^2}{xy}c_3 - \frac{1}{2}\frac{x^2-3y^2}{yx^2}(c_0+c_1x) - \frac{1}{2}\frac{3y^2-x^2}{xy^2}(c_2+c_3y) = 0$$

Using the computer the above simplifies to

$$\frac{x}{y}(c_3-c_1) + \frac{1}{2}c_2\frac{x}{y^2} - \frac{1}{y}\left(\frac{1}{2}c_0\right) - \frac{1}{x}\frac{3}{2}c_2 + \frac{3}{2}c_0\frac{y}{x^2} = 0$$

Hence

$$
\begin{aligned}
c_3 - c_1 &= 0 \\
\frac{1}{2}c_2 &= 0 \\
-\frac{1}{2}c_0 &= 0 \\
-\frac{3}{2}c_2 &= 0 \\
\frac{3}{2}c_0 &= 0
\end{aligned}
$$

Solving gives $c_0 = 0, c_2 = 0$ and $c_3 = c_1$. Hence the solution is

$$
\begin{aligned}
\xi &= c_1 x \\
\eta &= c_3 y
\end{aligned}
$$

Let $c_1 = 1$, therefore $c_3 = 1$ and we obtain

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Which is the result we used in solving the above problem. Notice that any scaler will also work. Hence

$$
\begin{aligned}
\xi &= 5x \\
\eta &= 5y
\end{aligned}
$$

And

$$
\begin{aligned}
\xi &= 10x \\
\eta &= 10y
\end{aligned}
$$

This will also give same solution.

1.2.22.28 Example $y' = \frac{y}{x} + \frac{1}{x}F\left(\frac{y}{x}\right)$

This is homogeneous class D $y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$. Hence from lookup table

$$
\begin{aligned}
\xi &= x^2 \\
\eta &= xy
\end{aligned}
$$

Now we just need to find canonical coordinates (R, S) since ξ, η are known. Using

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\
\frac{dx}{x^2} &= \frac{dy}{xy} = dS
\end{aligned} \tag{1}
$$

The first pair gives

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x} \\
\ln y &= \ln x + c_1 \\
y &= cx
\end{aligned}
$$

Hence

$$
\begin{aligned}
R &= c \\
&= \frac{y}{x}
\end{aligned}
$$

Now we find S from the last pair of equations (we could also use the first and last equations in (1)).

$$
\begin{aligned}
\frac{dy}{xy} &= dS \\
S &= \frac{1}{x} \ln y
\end{aligned}
$$

What is left is to find $\frac{dS}{dR}$. This is given by

$$
\begin{aligned}
\frac{dS}{dR} &= G(R) \\
&= \frac{S_x + S_y y'}{R_x + R_y y'}
\end{aligned}
$$

To find $G(R)$, we use $S_x = \frac{-1}{x^2} \ln y, S_y = \frac{1}{xy}$ and $R_x = -\frac{y}{x^2}, R_y = \frac{1}{x}$. Hence

$$
\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{-1}{x^2} \ln y + \frac{1}{xy} y'}{-\frac{y}{x^2} + \frac{1}{x} y'} \\
&= \frac{-\ln y - \frac{x}{y} y'}{y + xy'} \\
&= \frac{-\ln y - \frac{1}{R} y'}{y + xy'}
\end{aligned}
$$

But $y' = \frac{y}{x} + \frac{1}{x}F\left(\frac{y}{x}\right) = R + \frac{1}{x}F(R)$. The above becomes

$$\begin{aligned}
\frac{dS}{dR} &= \frac{-\ln y - \frac{1}{R}\left(R + \frac{1}{x}F(R)\right)}{y + x\left(R + \frac{1}{x}F(R)\right)} \\
&= \frac{-\ln y - 1 - \frac{1}{xR}F(R)}{y + xR + F(R)} \\
&= \frac{-\ln y - 1 - \frac{1}{x\frac{y}{x}}F(R)}{y + x\frac{y}{x} + F(R)} \\
&= \frac{-\ln y - 1 - \frac{1}{y}F(R)}{2y + F(R)}
\end{aligned}$$

Something is wrong. $\frac{dS}{dR}$ should only be a function of R. Need to find out why. Let me try the other pair of equations from (1) to solve for S and see what happens.

$$\begin{aligned}
\frac{dx}{x^2} &= dS \\
S &= -\frac{1}{x}
\end{aligned}$$

What is left is to find $\frac{dS}{dR}$. This is given by

$$\begin{aligned}
\frac{dS}{dR} &= G(R) \\
&= \frac{S_x + S_y y'}{R_x + R_y y'}
\end{aligned}$$

To find $G(R)$, we use $S_x = \frac{1}{x^2}, S_y = 0$ and $R_x = -\frac{y}{x^2}, R_y = \frac{1}{x}$. Hence

$$\begin{aligned}
\frac{dS}{dR} &= \frac{\frac{1}{x^2}}{-\frac{y}{x^2} + \frac{1}{x}y'} \\
&= \frac{1}{-y + xy'}
\end{aligned}$$

But $y' = \frac{y}{x} + \frac{1}{x}F\left(\frac{y}{x}\right) = R + \frac{1}{x}F(R)$. The above becomes

$$\begin{aligned}
\frac{dS}{dR} &= \frac{1}{-y + x\left(R + \frac{1}{x}F(R)\right)} \\
&= \frac{1}{-y + xR + F(R)} \\
&= \frac{1}{-y + x\frac{y}{x} + F(R)} \\
&= \frac{1}{F(R)}
\end{aligned}$$

This worked. But why the first choice did not work? OK, let me continue now. Integrating the above gives

$$S=\int \frac{1}{F\left(R\right)}dR+c$$

But $S=-\frac{1}{x}$, hence

$$\begin{aligned}-\frac{1}{x}&=\int^{\frac{y}{x}} \frac{1}{F\left(r\right)}dr+c\\ 0&=\int^{\frac{y}{x}} \frac{1}{F\left(r\right)}dr+c+\frac{1}{x}\end{aligned}$$

This example shows that when solving for S from

$$\frac{dx}{x^2}=\frac{dy}{xy}=dS$$

There are two choice. One is $dS=\frac{dy}{xy}$ and the other $dS=\frac{dx}{x^2}$. Using the first choice did not work here (unless I made a mistake, but do not see it)., Only the second choice worked because we must end up with $\frac{dS}{dR}=G(R)$ where RHS is function of R only. I need to look more into this. In theory, any choice should have worked.

1.2.22.29 Example $y'=\frac{y}{x}+\frac{1}{x}e^{-\frac{y}{x}}$

This is homogeneous class D $y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$. Hence from lookup table

$$\begin{aligned}\xi&=x^2\\ \eta&=xy\end{aligned}$$

From above we found the solution to be

$$S=\int \frac{1}{F\left(R\right)}dR+c$$

In this case $F(R)=e^{-R}$. Hence

$$\begin{aligned}S&=\int e^{R}dR+c\\ S&=e^{R}+c\end{aligned}$$

Now we just need to find canonical coordinates (R,S) since ξ,η are known. From above

$$\begin{aligned}R&=\frac{y}{x}\\ S&=-\frac{1}{x}\end{aligned}$$

Hence the solution becomes

$$
\begin{aligned}
-\frac{1}{x} &= e^{\frac{y}{x}} + c \\
e^{\frac{y}{x}} &= c_2 - \frac{1}{x} \\
\frac{y}{x} &= \ln\left(c_2 - \frac{1}{x}\right) \\
y &= x \ln\left(c_2 - \frac{1}{x}\right)
\end{aligned}
$$

The nice thing about this method is that once we solve for one pattern of an ode, then the same solution in canonical coordinates is used, the only change need is to plug-in in the RHS of the original ode in the solution and integrate.

1.2.22.30 Example $y' = \frac{1-y^2+x^2}{1+y^2-x^2}$

$$
\begin{aligned}
y' &= \frac{1-y^2+x^2}{1+y^2-x^2} \\
&= \omega(x,y)
\end{aligned}
$$

Using anstaz's it is found that

$$
\begin{aligned}
\xi &= x - y \\
\eta &= y - x
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\
\frac{dx}{x-y} &= \frac{dy}{y-x} = dS \qquad (1)
\end{aligned}
$$

The first two give

$$
\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{y-x}{x-y} = -1
$$

Hence

$$
y = -x + c_1 \qquad (2)
$$

Therefore

$$
\begin{aligned}
R &= c_1 \\
&= y + x
\end{aligned}
$$

To find S, since both ξ,η depend on both x,y, then $\frac{dy}{\eta}=dS$ or $\frac{dx}{\xi}=dS$ can be used. Lets try both to show same answer results.

$$\frac{dy}{\eta}=dS$$
$$dS=\frac{dy}{y-x}$$

But from (2), $x=c_1-y$. The above becomes

$$\begin{aligned} dS &= \frac{dy}{y-(c_1-y)} \\ &= \frac{dy}{2y-c_1} \end{aligned}$$

Hence

$$S=\frac{1}{2}\ln(2y-c_1)$$

But $c_1=y+x$. So the above becomes

$$\begin{aligned} S &= \frac{1}{2}\ln(2y-(y+x)) \\ &= \frac{1}{2}\ln(y-x) \end{aligned} \tag{3}$$

Let us now try the other ode

$$\frac{dx}{\xi}=dS$$
$$dS=\frac{dx}{x-y}$$

But from (2) $y=-x+c_1$. The above becomes

$$\begin{aligned} dS &= \frac{dx}{x-(-x+c_1)} \\ &= \frac{dx}{2x-c_1} \end{aligned}$$

Therefore

$$S=\frac{1}{2}\ln(2x-c_1)$$

But $c_1=y+x$. Therefore

$$\begin{aligned} S &= \frac{1}{2}\ln(2x-(y+x)) \\ &= \frac{1}{2}\ln(x-y) \end{aligned} \tag{4}$$

The constant of integration is set to zero when finding S. What is left is to find $\frac{dS}{dR}$. This is given by

$$\frac{dS}{dR}=\frac{S_x+S_y\omega}{R_x+R_y\omega} \tag{5}$$

But, and using (4) for S we have

$$\begin{aligned} R_x &= 1\\ R_y &= 1\\ S_x &= \frac{-1}{y-x}\\ S_y &= \frac{1}{y-x} \end{aligned}$$

Hence (2) becomes

$$\begin{aligned} \frac{dS}{dR} &= \frac{\frac{-1}{y-x}+\frac{1}{y-x}\omega}{1+\omega}\\ &= \frac{-\frac{\omega-1}{x-y}}{1+\omega}\\ &= \frac{1-\omega}{(1+\omega)(x-y)}\\ &= \frac{1-\left(\frac{1-y^2+x^2}{1+y^2-x^2}\right)}{\left(1+\left(\frac{1-y^2+x^2}{1+y^2-x^2}\right)\right)(x-y)}\\ &= -x-y\\ &= -(x+y)\\ &= -R \end{aligned}$$

Hence

$$\begin{aligned} \frac{dS}{dR} &= -R\\ S &= -\frac{R^2}{2} \end{aligned}$$

Converting back to x, y gives

$$\ln(y-x)=-\frac{(y+x)^2}{2}$$

1.2.22.31 Example $y' = -\frac{1}{4}xe^{-2y} + \frac{1}{4}\sqrt{\left(e^{-2y}\right)^2 x^2 + 4e^{-2y}}$

$$\begin{aligned} y' &= -\frac{1}{4}xe^{-2y} + \frac{1}{4}\sqrt{\left(e^{-2y}\right)^2 x^2 + 4e^{-2y}} \\ &= \omega(x, y) \end{aligned}$$

Using anstaz's it is found that

$$\begin{aligned} \xi &= x \\ \eta &= 1 \end{aligned}$$

Hence

$$\begin{aligned} \frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\ \frac{dx}{x} &= dy = dS \end{aligned} \tag{1}$$

The first two give

$$\frac{dy}{dx} = \frac{1}{x}$$

Hence

$$y = \ln x + c_1$$

Therefore

$$\begin{aligned} R &= c_1 \\ &= y - \ln x \end{aligned}$$

And S is found from either $\frac{dy}{\eta} = dS$ or $\frac{dx}{\xi} = dS$. Since $\eta = 1$, it is simpler to use $\frac{dy}{\eta} = dS$ instead.

$$\begin{aligned} \frac{dy}{\eta} &= dS \\ dy &= dS \\ S &= y \end{aligned}$$

Where constant of integration is set to zero. What is left is to find $\frac{dS}{dR}$. This is given by

$$\frac{dS}{dR} = \frac{S_x + S_y\omega}{R_x + R_y\omega} \tag{2}$$

But

$$\begin{aligned} R_x &= -\frac{1}{x} \\ R_y &= 1 \\ S_x &= 0 \\ S_y &= 1 \end{aligned}$$

Hence (2) becomes

$$\begin{aligned} \frac{dS}{dR} &= \frac{\omega}{-\frac{1}{x} + \omega} = \frac{1}{-\frac{1}{x\omega} + 1} \\ &= \frac{1}{1 - \frac{1}{x\left(-\frac{1}{4}xe^{-2y} + \frac{1}{4}\sqrt{(e^{-2y})^2 x^2 + 4e^{-2y}}\right)}} \end{aligned}$$

But $y = R + \ln x$. The above becomes

$$\begin{aligned} \frac{dS}{dR} &= \frac{1}{1 - \frac{1}{x\left(-\frac{1}{4}xe^{-2(R+\ln x)} + \frac{1}{4}\sqrt{\left(e^{-2(R+\ln x)}\right)^2 x^2 + 4e^{-2R+\ln x}}\right)}} \\ &= \frac{1}{1 - \frac{1}{x\left(-\frac{1}{4}\frac{xe^{-2R}}{x^2} + \frac{1}{4}\frac{1}{x}\sqrt{e^{-4R} + 4e^{-2R}}\right)}} \\ &= \frac{1}{1 - \frac{1}{\left(-\frac{1}{4}e^{-2R} + \frac{1}{4}\sqrt{e^{-4R} + 4e^{-2R}}\right)}} \end{aligned}$$

Integrating gives

$$S = \frac{\sqrt{\frac{1+4e^{2R}}{e^{4R}}}\, e^{2R} \operatorname{arctanh}\left(\frac{1}{\sqrt{1+4e^{2R}}}\right)}{\sqrt{1 + 4e^{2R}}}$$

Converting back to x, y gives

$$y = \frac{\sqrt{\frac{1+4e^{2(y-\ln x)}}{e^{4(y-\ln x)}}}\, e^{2(y-\ln x)} \operatorname{arctanh}\left(\frac{1}{\sqrt{1+4e^{2(y-\ln x)}}}\right)}{\sqrt{1 + 4e^{2(y-\ln x)}}}$$

1.2.22.32 Example $y' = \frac{y - xf(x^2+ay^2)}{x+ayf(x^2+ay^2)}$

$$\begin{aligned} y' &= \frac{y - xf(x^2 + ay^2)}{x + ayf\left(x^2 + ay^2\right)} \\ &= \omega(x, y) \end{aligned}$$

Using anstaz's it is found that

$$\begin{aligned}\xi &= -ay\\ \eta &= x\end{aligned}$$

Hence

$$\begin{aligned}\frac{dx}{\xi} &= \frac{dy}{\eta} = dS\\ \frac{dx}{-ay} &= \frac{dy}{x} = dS \qquad (1)\end{aligned}$$

The first two give

$$\frac{dy}{dx} = \frac{x}{-ay}$$

This is separable. Solving gives (taking one root)

$$y = \frac{\sqrt{a\,(ac_1 - x^2)}}{a}$$

Solving for c_1 gives

$$c_1 = \frac{x^2 + ay^2}{a}$$

Hence

$$R = \frac{x^2 + ay^2}{a}$$

S is found from either $\frac{dy}{\eta} = dS$ or $\frac{dx}{\xi} = dS$. Using $\frac{dx}{-ay} = dS$ then

$$\frac{dx}{-ay} = dS$$

But $y = \frac{\sqrt{a(ac_1 - x^2)}}{a}$. Hence

$$\begin{aligned}\frac{dx}{-a\frac{\sqrt{a(ac_1-x^2)}}{a}} &= dS\\ \frac{dx}{-\sqrt{a\,(ac_1 - x^2)}} &= dS\\ -\frac{1}{\sqrt{a}}\arctan\left(\frac{\sqrt{a}x}{\sqrt{c_1 a^2 - x^2 a}}\right) &= S\\ -\frac{1}{\sqrt{a}}\arctan\left(\frac{\sqrt{a}x}{ay}\right) &= S\end{aligned}$$

Where constant of integration is set to zero. What is left is to find $\frac{dS}{dR}$. This is given by

$$\frac{dS}{dR} = \frac{S_x + S_y\omega}{R_x + R_y\omega} \tag{2}$$

But

$$\begin{aligned} R_x &= \frac{2x}{a} \\ R_y &= 2y \\ S_x &= -\frac{y}{x^2y^2 + a} \\ S_y &= -\frac{x}{a\left(1 + \frac{x^2y^2}{a}\right)} \end{aligned}$$

Hence (2) becomes

$$\frac{dS}{dR} = \frac{-\frac{y}{x^2y^2+a} + \left(-\frac{x}{a\left(1+\frac{x^2y^2}{a}\right)}\right)\omega}{\frac{2x}{a} + 2y\omega}$$

But $R = \frac{x^2+ay^2}{a}$. The above becomes

$$\frac{dS}{dR} = \frac{-\frac{y}{aR} + \left(-\frac{x}{a\left(1+\frac{x^2y^2}{a}\right)}\right)\omega}{\frac{2x}{a} + 2y\omega}$$

To finish. Another hard part of this Lie method is to convert back $\frac{dS}{dR} = \frac{S_x+S_y\omega}{R_x+R_y\omega}$ so that the RHS is only a function of R. Need to find a robust way to do this. This is now a weak point in my program as I have few ode's that it can't do it

1.2.22.33 Alternative form for the similarity condition PDE

This section shows how to obtain eq. (8) in paper "Computer Algebra Solving of First Order ODEs Using Symmetry Methods" 1996 by Durate, Terrab, Mota. Which is an alternative equation to solve instead of the main Lie condition for symmetry we were looking at above.

Starting with the main linearized symmetry pde

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{14}$$

Assuming anstaz

$$\eta = \xi\omega + \chi \tag{A}$$

Hence

$$\eta_x = \xi_x\omega + \xi\omega_x + \chi_x$$
$$\eta_y = \xi_y\omega + \xi\omega_y + \chi_y$$

Then (14) becomes

$$\begin{aligned}(\xi_x\omega + \xi\omega_x + \chi_x) + \omega((\xi_y\omega + \xi\omega_y + \chi_y) - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y(\xi\omega + \chi) &= 0\\ \xi_x\omega + \xi\omega_x + \chi_x + \xi_y\omega^2 + \xi\omega_y\omega + \chi_y\omega - \omega\xi_x - \omega^2\xi_y - \omega_x\xi - \xi\omega\omega_y - \omega_y\chi &= 0\\ \xi_x\omega + \chi_x + \xi_y\omega^2 + \xi\omega_y\omega + \chi_y\omega - \omega\xi_x - \omega^2\xi_y - \xi\omega\omega_y - \omega_y\chi &= 0\\ \chi_x + \xi_y\omega^2 + \xi\omega_y\omega + \chi_y\omega - \omega^2\xi_y - \xi\omega\omega_y - \omega_y\chi &= 0\\ \chi_x + \xi\omega_y\omega + \chi_y\omega - \xi\omega\omega_y - \omega_y\chi &= 0\end{aligned}$$

Or

$$\chi_x + \chi_y\omega - \omega_y\chi = 0 \tag{1}$$

And hence (1) is now solved for $\chi(x, y)$. If we are able to find χ then we can use the anstaz $\eta = \xi\omega + \chi$. This leaves only one unknown ξ. The paper does not explain how to solve for this, ξ, which I assume is by using (14) again. The paper only said

> The knowledge of χ, in turn, allows one to set ξ and η as desired using (A)

Which is not too clear how in practice this is done. I need to work an example showing this. The paper says that (1) is solved for $\chi(x, y)$ by using bivariate polynomial anstaz. The degree can be set by a user, or Maple internally determines this.

1.3 First order nonlinear in derivative $F(x, y, p) = 0$

1.3.1 Introduction and algorithm flow charts

This gives an overview on solving first order ode where y' enters the ode as nonlinear. Examples are $x(y')^2 + yy' + x = 0$ or $2y'x - y + \ln y' = 0$ and so on. Four general cases exist and these are summarized in the flow chart at the end of this section. Two of these cases are called the Clairaut ode and the d'Alembert ode. Following the flow chart, a number of examples are solved.

Given the ode $F(x, y, y') = 0$, we start by writing $y' = p$ which results in

$$F(x, y, p) = 0$$

This is the top level algorithm

---

function SOLVE_FIRST_ORDER_ODE_NONLINEAR_P$(F(x, y, p))$

Where $p = y'$ and the ode is non-linear in p. An example is $x(y')^2 - yy' = -1$ and $y = x\left(y' + a\sqrt{1 + (y')^2}\right)$

if degree of p an integer in $F(x, y, p)$ **then**

As an example $p^2x + yp + y = 0$ and it is possible to find the roots (i.e. solve for p) then let the roots be p_i and each generated ode is solved as a first order ode which is now linear in each in y_i'. So we need to solve $y_i' = f(x, y)$ for each root.

else if we can solve for x from $F(x, y, p)$ **then**

This is currently not implemented.

Let $x = \phi(y, p)$ then differentiating w.r.t. y gives

$$\begin{aligned} \frac{dx}{dy} &= \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial p}\frac{dp}{dy} \\ \frac{1}{p} &= \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial p}\frac{dp}{dy} \end{aligned} \tag{1}$$

Solving (1) for p from the above and substituting the result in $x = \phi(y, p)$ gives the solution.

else

CALL clairaut_dAlembert_solver$(F(x, y, p))$

end if

end function

Algorithm below is Clairaut dAlembert solver algorithm

---

function CLAIRAUT_DALEMBERT_SOLVER$(F(x, y, p))$

Solve for y and write the ode as (where $p = y'$)

$$y = xf(p) + g(p) \tag{1}$$

where $f(p) \neq 0$

if $f(p) = p$ **then** ▷ Example $y = xp + g(p)$

if $g(p) = 0$ **then** ▷ Example $y = xp$

return as this is neither Clairaut nor d'Alembert.

else if $g(p)$ is linear in p **then** ▷ Example $y = xp + p$

return as this is neither Clairaut nor d'Alembert.

else ▷ Example $y = xp + p^2$ or $y = xp + \sin(p)$

This is a Clairaut ode. Taking the derivative of (1) w.r.t. x gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}
$$

where g' is the derivative of $g(p)$ w.r.t. p. The general solution is

$$
\frac{dp}{dx} = 0 \qquad p = c_1
$$

where c_1 is constant. Substituting $p = c_1$ the in (1) gives the general solution y_g The singular solution y_s is now found from solving the ode $(x + g'(p)) = 0$ for p and substituting the solution p_i back in (1).

return y_g, y_s

end if

else

CALL dalembert_solver($F(x, y, p)$)

end if

end function

Algorithm below is just the dAlembert solver algorithm

---

function DALEMBERT_SOLVER($F(x, y, p)$)

Write the ode as (where $p = y'$)

$$
y = xf(p) + g(p) \tag{1}
$$

where $f(p) \neq 0$. Note that We get here when $f(p) \neq p$ else it is Clairaut.

if $g(p) = 0$ **then** ▷ Example $y = xf(p)$

$f(p)$ must be nonlinear in p but can not be the special case $p^{\frac{1}{n}}$ where n integer because then it is separable.

if $f(p) = p^{\frac{1}{n}}$ and $n \in \mathbb{Z}$ **then** ▷ Ex. $y = x(y')^{\frac{1}{2}}$

return as this is not dAlmbert ode.
end if
else
In this case any form of $f(p)$ is OK even $f(p) = p^{\frac{1}{n}}$ with n integer except ofcourse $f(p) = p$ since this would have made it Clairaut and not dAlembert. Example is $y = xf(p) + p$ is dAlembert.
if $g(p)$ is constant and does not depend on p **then** ▷ Ex. $y = xf(p) + 1$
return as this is not dAlmbert ode.
else
if $g(p) = f(p)$ **then**
if $g(p), f(p)$ have the form $p^{\frac{1}{n}}$ with n integer **then** ▷ Ex. $y = xp^{\frac{1}{2}} + p^{\frac{1}{2}}$
return as this is not dAlmbert ode.
else ▷ Ex. $y = xp^{\frac{2}{3}} + p^{\frac{2}{3}}$ or $y = xp^2 + p^2$
This is dAlmbert ode.
end if
end if
end if
end if

When we get here then (1) is dAlmbert ode. Note that all the above cases $f(p), g(p)$ can not be function of x in any case. Now we solve (1) using dAlmbert algorithm. Taking derivative of (1) w.r.t. x gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xf + g) \\
p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right)
\end{aligned}
$$

where f' means $\frac{df}{dp}$ and g' means $\frac{dg}{dp}$. The above becomes

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

The singular solution is given when $\frac{dp}{dx} = 0$ above. Hence

$$
p - f = 0
$$

Solving the above for p and substituting the result back in (1) gives the singular solution y_s. The general solution y_g is found by solving the ode in (2) for p and substituting the result in (1). there are two cases to consider.

if ode (2) is separable or linear in p as is **then**

Solve (2) for p directly and substitute the solution in (1). This gives the general solution y_g.

else

Inverting (2) first gives

$$\frac{dx}{dp} = \frac{xf' + g'}{p - f}$$

Which makes it linear ode in x. This is solved for $x(p)$ as function of p. Let

$$x = h(p) + c_1 \tag{3}$$

be the solution. Now two possible cases exist

if able to isolate p from (3) **then**

Substitute p in (1). This gives the general solution y_g.

else

Solve for p from (1) and substitute the result in (3). This gives an implicit solution for y_g instead of explicit one.

end if

end if

end function

1.3.2 Algorithm diagram

The following is the flow chart.

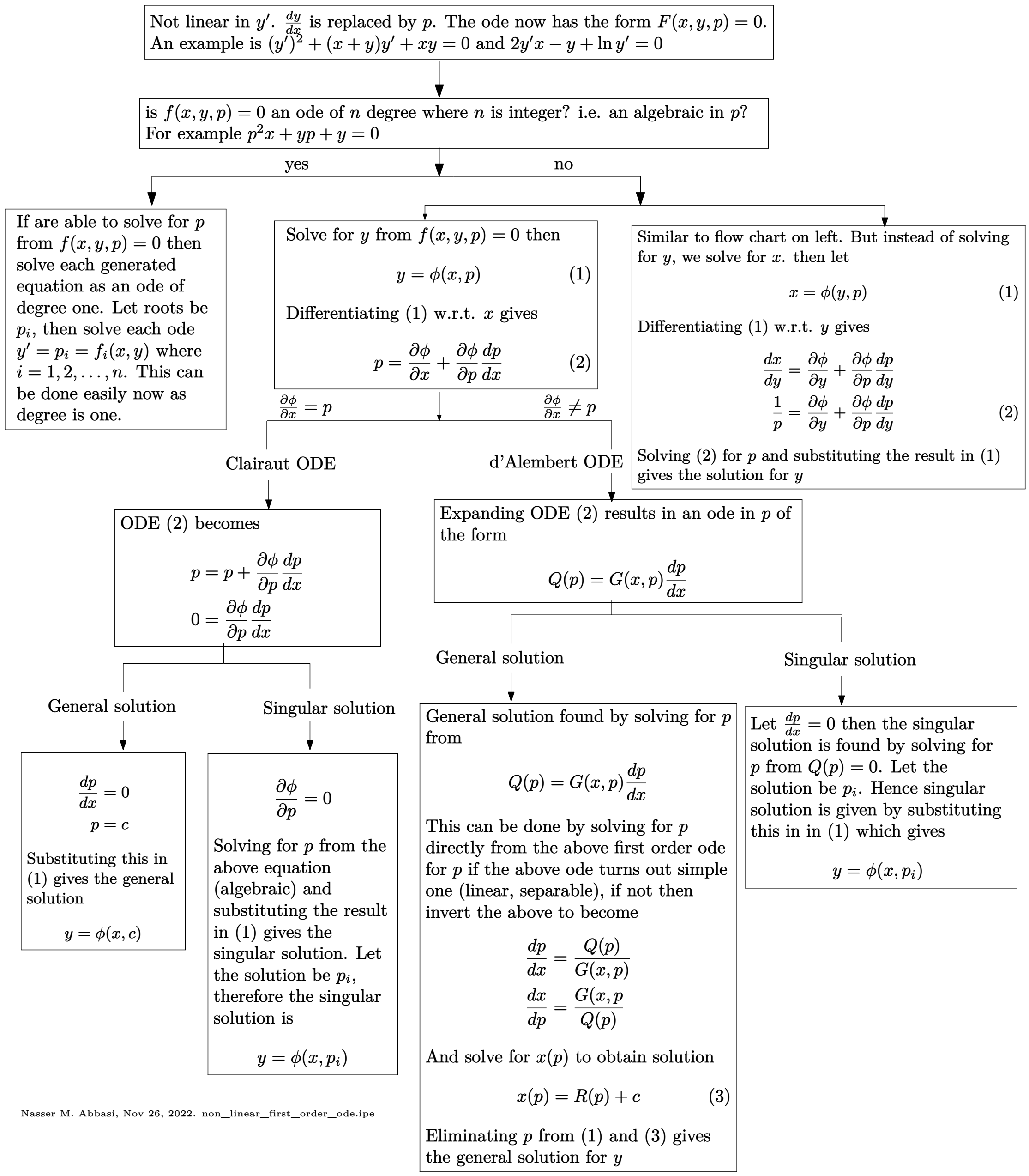

Figure 1.14: Algorithm for solving first order ode with nonlinear y'

1.3.3 Solved examples

#	original ode	$y=xf(p)+g(p)$	$f(p)$	$g(p)$	type
1	$x(y')^2-yy'=-1$	$y=xp+\frac{1}{p}$	p	$\frac{1}{p}$	Clairaut
2	$y=xy'-(y')^2$	$y=xp-p^2$	p	$-p^2$	Clairaut
3	$y=xy'-\frac{1}{4}(y')^2$	$y=xp-\frac{1}{4}p^2$	p	$-\frac{1}{4}p^2$	Clairaut
4	$y=x(y')^2$	$y=xp^2$	p^2	0	d'Alembe
5	$y=x+(y')^2$	$y=x+p^2$	1	p^2	d'Alembe
6	$(y')^2-1-x-y=0$	$y=-x+(p^2-1)$	-1	(p^2-1)	d'Alembe
7	$yy'-(y')^2=x$	$y=\frac{1}{p}x+p$	$\frac{1}{p}$	p	d'Alembe
8	$y=x(y')^2+(y')^2$	$y=xp^2+p^2$	p^2	p^2	d'Alembe
9	$y=\frac{x}{a}y'+\frac{b}{ay'}$	$y=\frac{x}{a}p+\frac{b}{a}p^{-1}$	$\frac{p}{a}$	$\frac{b}{a}p^{-1}$	d'Alembe
10	$y=x\left(y'+a\sqrt{1+(y')^2}\right)$	$y=x\left(p+a\sqrt{1+p^2}\right)$	$p+a\sqrt{1+p^2}$	0	d'Alembe
11	$y=x+(y')^2\left(1-\frac{2}{3}y'\right)$	$y=x+p^2\left(1-\frac{2}{3}p\right)$	1	$p^2\left(1-\frac{2}{3}p\right)$	d'Alembe
12	$y=2x-\frac{1}{2}\ln\left(\frac{(y')^2}{y'-1}\right)$	$y=2x-\frac{1}{2}\ln\left(\frac{p^2}{p-1}\right)$	2	$-\frac{1}{2}\ln\left(\frac{p^2}{p-1}\right)$	d'Alembe
13	$(y')^2-x(y')^2+y(1+y')-xy'=0$	$y=\frac{xp+xp^2-p^2}{p+1}=xp-\frac{p^2}{p+1}$	p	$-\frac{p^2}{p+1}$	Clairaut
14	$x(y')^2+(x-y)\,y'+1-y=0$	$y=xp+\frac{1}{1+p}$	p	$\frac{1}{1+p}$	Clairaut
15	$xyy'=y^2+x\sqrt{4x^2+y^2}$	$y=\text{RootOf}\left(h(p)\right)x$	$\text{RootOf}\left(h(p)\right)$	0	d'Alembe
16	$\ln(\cos y')+y'\tan y'=y$	$y=\ln(\cos y')+y'\tan y'$	0	$\ln(\cos p)+p\tan p$	d'Alembe

1.3.3.1 Example 1

$x(y')^2-yy'=-1$, is put in normal form (by replacing y' with p) and solving for y gives

$$
\begin{aligned}
y &= xp+\frac{1}{p} \qquad (1)\\
&= xf(p)+g(p)
\end{aligned}
$$

Where $f(p)=p$ and $g(p)=\frac{1}{p}$. Since $f(p)=p$ then this is Clairaut ode. Taking derivative of the above w.r.t. x gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xp+g(p))\\
p &= p+(x+g'(p))\frac{dp}{dx}\\
0 &= (x+g'(p))\frac{dp}{dx}
\end{aligned}
$$

The general solution is given by

$$\begin{aligned}\frac{dp}{dx} &= 0\\ p &= c_1\end{aligned}$$

Substituting this in (1) gives the general solution

$$y = c_1 x + \frac{1}{c_1}$$

The term $(x + g'(p)) = 0$ is used to find singular solutions.

$$\begin{aligned}x + g'(p) &= x + \frac{d}{dp}\frac{1}{p}\\ &= x - \frac{1}{p^2}\end{aligned}$$

Hence $x - \frac{1}{p^2} = 0$ or $p = \pm\frac{1}{\sqrt{x}}$. Substituting these back in (1) gives

$$\begin{aligned}y_1(x) &= xp + \frac{1}{p}\\ &= x\frac{1}{\sqrt{x}} + \sqrt{x}\\ &= 2\sqrt{x} \qquad (3)\\ y_2(x) &= -x\sqrt{\frac{1}{x}} - \sqrt{x}\\ &= -2\sqrt{x} \qquad (4)\end{aligned}$$

Eq. (2) is the general solution and (3,4) are the singular solutions.

1.3.3.2 Example 2

$y = xy' - (y')^2$ is put in normal form (by replacing y' with p) and solving for y gives

$$\begin{aligned}y &= xp - p^2 \qquad (1)\\ &= xf(p) + g(p)\end{aligned}$$

Where $f(p) = p$ and $g(p) = -p^2$. Taking derivative of the above w.r.t. x gives

$$\begin{aligned}p &= \frac{d}{dx}(xp + g(p))\\ p &= p + (x + g'(p))\frac{dp}{dx}\\ 0 &= (x + g'(p))\frac{dp}{dx}\end{aligned}$$

The general solution is given by

$$\begin{aligned}\frac{dp}{dx} &= 0\\ p &= c_1\end{aligned}$$

Substituting this in (1) gives the general solution

$$y = c_1 x - c_1^2$$

The term $(x + g'(p)) = 0$ is used to find singular solutions.

$$\begin{aligned}x + g'(p) &= x + \frac{d}{dp}(-p^2)\\ &= x + 2p\end{aligned}$$

Hence $x + 2p = 0$ or $p = \frac{x}{2}$. Substituting this back in (1) gives

$$\begin{aligned}y(x) &= \frac{x^2}{2} - \frac{x^2}{4}\\ &= \frac{x^2}{4} \qquad (3)\end{aligned}$$

Eq. (2) is the general solution and (3) is the singular solution.

1.3.3.3 Example 3

$y = xy' - \frac{1}{4}(y')^2$ is put in normal form (by replacing y' with p) and solving for y gives

$$\begin{aligned}y &= xp - \frac{1}{4}p^2 \qquad (1)\\ &= xf(p) + g(p)\end{aligned}$$

Where $f(p) = p$ and $g(p) = -\frac{1}{4}p^2$. Taking derivative of the above w.r.t. x gives

$$\begin{aligned}p &= \frac{d}{dx}(xp + g(p))\\ p &= p + (x + g'(p))\frac{dp}{dx}\\ 0 &= (x + g'(p))\frac{dp}{dx}\end{aligned}$$

The general solution is given by

$$\begin{aligned}\frac{dp}{dx} &= 0\\ p &= c_1\end{aligned}$$

Substituting this in (1) gives the general solution

$$y = c_1 x - \frac{1}{4}c_1^2$$

The term $(x + g'(p)) = 0$ is used to find singular solutions.

$$\begin{aligned} x + g'(p) &= x + \frac{d}{dp}\left(-\frac{1}{4}p^2\right) \\ &= x - \frac{1}{2}p \end{aligned}$$

Hence $x - \frac{1}{2}p = 0$ or $p = 2x$. Substituting this back in (1) gives

$$\begin{aligned} y(x) &= 2x^2 - x^2 \\ &= x^2 \end{aligned} \tag{3}$$

Eq. (2) is the general solution and (3) is the singular solution.

1.3.3.4 Example 4

$y = x(y')^2$ is put in normal form (by replacing y' with p) and solving for y gives

$$\begin{aligned} y &= xp^2 \qquad (1) \\ &= xf(p) \end{aligned}$$

This is the case when $f(p) = p^2$ and $g(p) = 0$. Writing $f \equiv f(p)$ and $g \equiv g(p)$ to make notation simpler but remembering that f is function of $p(x)$ which in turn is function of x. Same for $g(p)$.

$$y = xf$$

Taking derivative of the above w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xf) \\ p &= f + xf'\frac{dp}{dx} \\ p - f &= xf'\frac{dp}{dx} \end{aligned}$$

Since $f = p^2$ then the above becomes

$$p - p^2 = 2xp\frac{dp}{dx} \tag{2}$$

The singular solution is given when $\frac{dp}{dx} = 0$ or $p - p^2 = 0$. This gives $p = 0$ or $p = 1$. Substituting these values of p in (1) gives singular solutions

$$y_{s1} = 0 \tag{3}$$
$$y_{s2} = x \tag{4}$$

General solution is found when $\frac{dp}{dx} \neq 0$. Eq(2) is a first order ode in p. Now we could either solve ode (2) directly as it is for $p(x)$, or do an inversion and solve for $x(p)$. We always try the first option first. Since (2) is separable as is, no need to do an inversion. Eq (2) is separable. The solution is

$$\begin{aligned} p_1 &= 0 \\ p_2 &= 1 + \frac{c_1}{\sqrt{x}} \end{aligned}$$

For each p, there is a general solution. Substituting each of the above in (1) gives

$$\begin{aligned} y_1(x) &= 0 \\ y_2(x) &= x\left(1 + \frac{c_1}{\sqrt{x}}\right)^2 \end{aligned}$$

Hence the final solutions are

$$\begin{aligned} y &= x \qquad \text{(singular)} \\ y &= 0 \\ y &= x\left(1 + \frac{c_1}{\sqrt{x}}\right)^2 \end{aligned}$$

But $y = x$ can be obtained from the general solution when $c_1 = 0$. Hence it is removed. Therefore the final solutions are

$$y = 0 \tag{6}$$
$$y = x\left(1 + \frac{c_1}{\sqrt{x}}\right)^2 \tag{7}$$

What will happen if we had done an inversion to $x(p)$? Let us find out. ode(5) now becomes

$$\begin{aligned} \frac{p - p^2}{p}\frac{dx}{dp} &= 2x \\ \frac{dx}{2x} &= \frac{p}{p - p^2}dp \end{aligned}$$

This is also separable in x. Solving this for x gives

$$x = \frac{c_1}{(p-1)^2}$$

Solving for p from the above gives

$$\begin{aligned} p_1 &= \frac{x+\sqrt{xc_1}}{x} \\ p_2 &= \frac{x-\sqrt{xc_1}}{x} \end{aligned}$$

Substituting each of the above in (1) gives

$$\begin{aligned} y_1 &= x\left(\frac{x+\sqrt{xc_1}}{x}\right)^2 \\ &= \frac{\left(x+\sqrt{xc_1}\right)^2}{x} \\ y_2 &= x\left(\frac{x-\sqrt{xc_1}}{x}\right)^2 \\ &= \frac{\left(x-\sqrt{xc_1}\right)^2}{x} \end{aligned}$$

Now we see that singular solution $y = x$ can be obtained from the above general solutions from $c_1 = 0$. But $y = 0$ can not. Hence the final solutions are

$$y = 0 \qquad \text{(singular)} \tag{8}$$

$$y = \frac{\left(x+\sqrt{xc_1}\right)^2}{x} \tag{9}$$

$$y = \frac{\left(x-\sqrt{xc_1}\right)^2}{x} \tag{10}$$

All solutions (6,7,8,9,10) are correct and verified. Maple gives the solutions given in (8,9,10) and not those in (6,7).

1.3.3.5 Example 5

$y = x + (y')^2$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned} y &= x + p^2 \qquad (1) \\ &= xf + g \end{aligned}$$

Hence $f(p) = 1, g(p) = p^2$. Taking derivative w.r.t. x gives

$$\begin{aligned} p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Using $f = 1, g = p^2$ the above simplifies to

$$p - 1 = 2p\frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in (2) which results in $p - f = 0$ or $p - 1 = 0$. Hence $p = 1$. Substituting these values of p in (1) gives singular solution as

$$y = x + 1 \tag{3}$$

General solution is found when $\frac{dp}{dx} \neq 0$. Eq (2A) is a first order ode in p. Now we could either solve ode (2) directly as it is for $p(x)$, or do an inversion and solve for $x(p)$. We always try the first option first. Since (2) is separable as is, no need to do an inversion. Solving (2) for p gives

$$p = \operatorname{LambertW}\left(c_1 e^{\frac{x}{2}-1}\right) + 1$$

Substituting this in (1) gives the general solution

$$y(x) = x + \left(\operatorname{LambertW}\left(c_1 e^{\frac{x}{2}-1}\right) + 1\right)^2 \tag{4}$$

Note however that when $c_1 = 0$ then the general solution becomes $y(x) = x + 1$. Hence (3) is a particular solution and not a singular solution. (4) is the only solution.

1.3.3.6 Example 6

$(y')^2 - 1 - x - y = 0$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned} y &= -x + (p^2 - 1) \qquad (1) \\ &= xf + g \end{aligned}$$

Hence $f = -1, g(p) = (p^2 - 1)$. Taking derivative w.r.t. x gives

$$\begin{aligned} p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Using $f = -1, g = (p^2 - 1)$ the above simplifies to

$$p + 1 = 2p\frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p = -1$. Substituting this in (1) gives singular solution as

$$y(x) = -x \tag{3}$$

The general solution is found by finding p from (2A). No need here to do the inversion as (2) is separable already. Solving (2) gives

$$\begin{aligned} p &= -\operatorname{LambertW}\left(-e^{-\frac{x}{2}-1+\frac{c_2}{2}}\right) - 1 \\ &= -\operatorname{LambertW}\left(-c_1 e^{-\frac{x}{2}-1}\right) - 1 \end{aligned}$$

Substituting the above in (1) gives the general solution

$$\begin{aligned} y(x) &= -x + \left(p^2 - 1\right) \\ y(x) &= -x + \left(-\operatorname{LambertW}\left(-c_1 e^{-\frac{x}{2}-1}\right) - 1\right)^2 - 1 \end{aligned} \tag{4}$$

Note however that when $c_1 = 0$ then the general solution becomes $y(x) = -x$. Hence (3) is a particular solution and not a singular solution. Solution (4) is therefore the only solution.

1.3.3.7 Example 7

$yy' - (y')^2 = x$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned} y &= \frac{x + p^2}{p} \\ &= \frac{1}{p}x + p \\ &= xf + g \end{aligned} \tag{1}$$

Hence $f = \frac{1}{p}, g(p) = p$. Taking derivative w.r.t. x gives

$$\begin{aligned} p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned}$$

Using $f = \frac{1}{p}, g = p$ the above simplifies to

$$p - \frac{1}{p} = \left(-\frac{x}{p^2} + 1\right)\frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in (2) which results in $Q(p) = 0$ or $p - 1 = 0$ or $p = 1$. Substituting these values in (1) gives the solutions

$$y_1(x) = x + 1 \tag{3}$$

The general solution is found by finding p from (2A). Since (2A) is not linear and not separable in p, then inversion is needed. Writing (2) as

$$\begin{aligned}\frac{dx}{dp} &= \frac{1 - \frac{x}{p^2}}{p - \frac{1}{p}}\\ &= \frac{1}{p - p^3}(x - p^2)\end{aligned}$$

Hence

$$\frac{dx}{dp} + \frac{x}{p\,(p^2 - 1)} = \frac{p^2}{p\,(p^2 - 1)}$$

This is now linear ODE in $x(p)$. The solution is

$$\begin{aligned}x &= \frac{p\sqrt{(p-1)\,(1+p)}\ln\left(p + \sqrt{p^2-1}\right)}{(1+p)\,(p-1)} + c_1\frac{p}{\sqrt{(1+p)\,(p-1)}}\\ &= \frac{p\sqrt{p^2-1}\ln\left(p + \sqrt{p^2-1}\right)}{p^2-1} + c_1\frac{p}{\sqrt{p^2-1}}\end{aligned} \tag{4}$$

Now we need to eliminate p from (1,4). From (1) since $y = \frac{1}{p}x + p$ then solving for p gives

$$\begin{aligned}p_1 &= \frac{y}{2} + \frac{1}{2}\sqrt{y^2 - 4x}\\ p_2 &= \frac{y}{2} - \frac{1}{2}\sqrt{y^2 - 4x}\end{aligned}$$

Substituting each p_i in (4) gives the general solution (implicit) in $y(x)$. First solution is

$$x = \frac{\left(\frac{y}{2} + \frac{1}{2}\sqrt{y^2-4x}\right)\sqrt{\left(\frac{y}{2} + \frac{1}{2}\sqrt{y^2-4x}\right)^2 - 1}\ln\left(\frac{y}{2} + \frac{1}{2}\sqrt{y^2-4x} + \sqrt{\left(\frac{y}{2} + \frac{1}{2}\sqrt{y^2-4x}\right)^2 - 1}\right)}{\left(\frac{y}{2} + \frac{1}{2}\sqrt{y^2-4x}\right)^2 - 1} + c_1\frac{\quad}{\sqrt{\left(\frac{y}{2}\right.}}$$

And second solution is

$$x = \frac{\left(\frac{y}{2} - \frac{1}{2}\sqrt{y^2-4x}\right)\sqrt{\left(\frac{y}{2} - \frac{1}{2}\sqrt{y^2-4x}\right)^2 - 1}\ln\left(\frac{y}{2} - \frac{1}{2}\sqrt{y^2-4x} + \sqrt{\left(\frac{y}{2} - \frac{1}{2}\sqrt{y^2-4x}\right)^2 - 1}\right)}{\left(\frac{y}{2} - \frac{1}{2}\sqrt{y^2-4x}\right)^2 - 1} + c_1\frac{\quad}{\sqrt{\left(\frac{y}{2}\right.}}$$

1.3.3.8 Example 8

$y = x(y')^2 + (y')^2$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned} y &= xp^2 + p^2 \\ &= xf + g \end{aligned} \tag{1}$$

where $f = p^2, g = p^2$. Taking derivative and simplifying gives

$$\begin{aligned} p &= \left(f + xf' \frac{dp}{dx}\right) + \left(g' \frac{dp}{dx}\right) \\ p &= f + (xf' + g') \frac{dp}{dx} \\ p - f &= (xf' + g') \frac{dp}{dx} \end{aligned}$$

Using values for f, g the above simplifies to

$$p - p^2 = (2xp + 2p) \frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p = 0$ or $p = 1$. Substituting these values in (1) gives the singular solutions

$$y_1(x) = 0 \tag{3}$$

$$y_2(x) = x + 1 \tag{4}$$

The general solution is found by finding p from (2A). Since (2A) is not linear in p, then inversion is needed. Writing (A2) as

$$\frac{p(1-p)}{2p(x+1)} = \frac{dp}{dx}$$

Inverting gives

$$\begin{aligned} \frac{dx}{dp} &= \frac{2(x+1)}{(1-p)} \\ \frac{dx}{dp} - x\frac{2}{(1-p)} &= \frac{2}{(1-p)} \end{aligned}$$

This is now linear $x(p)$. The solution is

$$x = \frac{C^2}{(p-1)^2} - 1$$

Solving for p gives

$$\begin{aligned}\frac{C^2}{(p-1)^2} &= x+1\\ (p-1)^2 &= \frac{C^2}{x+1}\\ (p-1) &= \pm\frac{C}{\sqrt{x+1}}\\ p &= 1\pm\frac{C}{\sqrt{x+1}}\end{aligned}$$

Substituting the above in (1) gives the general solutions

$$y = (x+1)\,p^2$$

Therefore

$$\begin{aligned}y(x) &= (x+1)\left(1+\frac{C}{\sqrt{x+1}}\right)^2\\ y(x) &= (x+1)\left(1-\frac{C}{\sqrt{x+1}}\right)^2\end{aligned}$$

The solution $y_1(x) = 0$ found earlier can not be obtained from the above general solution hence it is singular solution. But $y_2(x) = x+1$ can be obtained from the general solution when $C = 0$. Hence there are only three solutions, they are

$$\begin{aligned}y_1(x) &= 0\\ y_2(x) &= (x+1)\left(1+\frac{C}{\sqrt{x+1}}\right)^2\\ y_3(x) &= (x+1)\left(1-\frac{C}{\sqrt{x+1}}\right)^2\end{aligned}$$

1.3.3.9 Example 9

$y = \frac{x}{a}y' + \frac{b}{ay'}$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned}y &= \frac{x}{a}p + \frac{b}{a}p^{-1} \qquad (1)\\ &= xf + g\end{aligned}$$

Where $f = \frac{p}{a}, g = \frac{b}{a}p^{-1}$. Taking derivative w.r.t. x gives

$$
\begin{aligned}
p &= \left(f + xf' \frac{dp}{dx}\right) + \left(g' \frac{dp}{dx}\right) \\
p &= f + (xf' + g') \frac{dp}{dx} \\
p - f &= (xf' + g') \frac{dp}{dx}
\end{aligned}
$$

Using values for f, g the above simplifies to

$$
p - \frac{p}{a} = \left(\frac{x}{a} - \frac{b}{a}p^{-2}\right) \frac{dp}{dx} \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p = 0$. Substituting this in (1) does not generate any solutions due to division by zero. Hence no singular solution exist.

The general solution is found by finding p from (2A). Since (2A) is not linear in p, then inversion is needed. Writing (2A) as

$$
\frac{p\left(1 - \frac{1}{a}\right)}{\frac{x}{a} - \frac{b}{a}p^{-2}} = \frac{dp}{dx}
$$

Since this is nonlinear, then inversion is needed

$$
\begin{aligned}
\frac{dx}{dp} &= \frac{\frac{x}{a} - \frac{b}{a}p^{-2}}{p\left(1 - \frac{1}{a}\right)} \\
\frac{dx}{dp} - x\frac{1}{p(a-1)} &= -\frac{b}{a}\frac{1}{p^3\left(1 - \frac{1}{a}\right)}
\end{aligned}
$$

This is now linear ode in $x(p)$. The solution is

$$
x = \frac{b}{(2a-1)p^2} + C_1 p^{\frac{1}{a-1}} \tag{3}
$$

There are now two choices to take. The first is by solving for p from the above in terms of x and then substituting the result in (1) to obtain explicit solution for $y(x)$, and the second choice is by solving for p algebraically from (1) and substituting the result in (3). The second choice is easier in this case but gives an implicit solution. Solving for p from (1) gives

$$
\begin{aligned}
p_1 &= \frac{ay + \sqrt{a^2y^2 - 4xb}}{2x} \\
p_1 &= \frac{ay - \sqrt{a^2y^2 - 4xb}}{2x}
\end{aligned}
$$

Substituting each one of these solutions back in (3) gives two implicit solutions

$$x = \frac{b}{(2a-1)\left(\frac{ay+\sqrt{a^2y^2-4xb}}{2x}\right)^2} + C_1\left(\frac{ay+\sqrt{a^2y^2-4xb}}{2x}\right)^{\frac{1}{a-1}}$$

$$x = \frac{b}{(2a-1)\left(\frac{ay-\sqrt{a^2y^2-4xb}}{2x}\right)^2} + C_1\left(\frac{ay-\sqrt{a^2y^2-4xb}}{2x}\right)^{\frac{1}{a-1}}$$

1.3.3.10 Example 10

$y = xy' + ax\sqrt{1+(y')^2}$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned} y &= x\left(p + a\sqrt{1+p^2}\right) \qquad (1) \\ &= xf \end{aligned}$$

where $f = p + a\sqrt{1+p^2}, g = 0$. Taking derivative and simplifying gives

$$\begin{aligned} p &= \left(f + xf'\frac{dp}{dx}\right) \\ p - f &= xf'\frac{dp}{dx} \end{aligned}$$

Using values for f, g the above simplifies to

$$-a\sqrt{1+p^2} = x\left(1 + \frac{ap}{\sqrt{1+p^2}}\right)\frac{dp}{dx} \qquad (2A)$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $-a\sqrt{1+p^2} = 0$. This gives no real solution for p. Hence no singular solution exists.

The general solution is when $\frac{dp}{dx} \neq 0$ in (2A). Since (2A) is nonlinear, inversion is needed.

$$\begin{aligned} \frac{-a\sqrt{1+p^2}}{x + \frac{1}{2}x\frac{2ap}{\sqrt{1+p^2}}} &= \frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{x\left(1 + \frac{1}{2}\frac{2ap}{\sqrt{1+p^2}}\right)}{-a\sqrt{1+p^2}} \\ \frac{dx}{x} &= \frac{1 + \frac{1}{2}\frac{2ap}{\sqrt{1+p^2}}}{-a\sqrt{1+p^2}}dp \\ \frac{dx}{x} &= \frac{\sqrt{1+p^2} + \frac{1}{2}2ap}{-a(1+p^2)}dp \\ \frac{dx}{x} &= \left(-\frac{1}{a\sqrt{1+p^2}} - \frac{p}{(1+p^2)}\right)dp \end{aligned}$$

Integrating gives

$$\ln x(p) = -\frac{1}{2}\ln\left(p^2+1\right) - \frac{1}{a}\operatorname{arcsinh}(p)$$

Therefore

$$x = c_1 \frac{-e^{-\frac{1}{a}(\operatorname{arcsinh}(p))}}{\sqrt{p^2+1}} \tag{3}$$

There are now two choices to take. The first is by solving for p from the above in terms of x and substituting the result in (1) to obtain explicit solution for $y(x)$, and the second choice is by solving for p algebraically from (1) and substituting the result in (3). The second choice is easier in this case but gives an implicit solution. Solving for p from (1) gives

$$\begin{aligned}
p_1 &= -\frac{1}{x}\frac{ay + \sqrt{-a^2x^2 + x^2 + y^2}a - y}{a^2-1}\\
p_2 &= \frac{1}{x}\frac{-ay + \sqrt{-a^2x^2 + x^2 + y^2}a - y}{a^2-1}
\end{aligned}$$

Substituting each one of these solutions back in (3) gives two implicit solutions

$$\begin{aligned}
x &= c_1 \frac{-e^{-\frac{1}{a}\left(\operatorname{arcsinh}\left(-\frac{1}{x}\frac{ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)\right)}}{\sqrt{\left(-\frac{1}{x}\frac{ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)^2+1}}\\
x &= c_1 \frac{-e^{-\frac{1}{a}\left(\operatorname{arcsinh}\left(\frac{1}{x}\frac{-ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)\right)}}{\sqrt{\left(\frac{1}{x}\frac{-ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)^2+1}}
\end{aligned}$$

1.3.3.11 Example 11

$$\begin{aligned}
y &= x + (y')^2\left(1 - \frac{2}{3}y'\right)\\
&= x + p^2\left(1 - \frac{2}{3}p\right)
\end{aligned}$$

Where $f = 1, g = p^2\left(1 - \frac{2}{3}p\right)$. Taking derivative w.r.t. x gives

$$\begin{aligned}
p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right)\\
p &= f + (xf' + g')\frac{dp}{dx}\\
p - f &= (xf' + g')\frac{dp}{dx}
\end{aligned}$$

Using values for f, g the above simplifies to

$$p - 1 = \left(2p - 2p^2\right) \frac{dp}{dx} \tag{2A}$$

The singular solution is when $\frac{dp}{dx} = 0$ which results in $p = 1$. Substituting this in (1) gives

$$\begin{aligned} y &= x - \left(1 - \frac{2}{3}\right) \\ &= x + \frac{1}{3} \end{aligned}$$

The general solution is when $\frac{dp}{dx} \neq 0$. Then (2A) is now separable. Solving for p gives

$$\begin{aligned} p &= -\sqrt{c_1 - x} \\ p &= \sqrt{c_1 - x} \end{aligned}$$

Substituting each one of the above solutions of p in (1) gives

$$\begin{aligned} y_1 &= x + \left(p^2 - \frac{2}{3} p^3\right) \\ &= x + \left(\left(-\sqrt{c_1 - x}\right)^2 - \frac{2}{3}\left(-\sqrt{c_1 - x}\right)^3\right) \\ &= x + \left(c_1 - x + \frac{2}{3}(c_1 - x)^{\frac{3}{2}}\right) \\ &= c_1 + \frac{2}{3}(c_1 - x)^{\frac{3}{2}} \end{aligned}$$

And

$$\begin{aligned} y_2 &= x + \left(p^2 - \frac{2}{3} p^3\right) \\ &= x + \left(\left(\sqrt{c_1 - x}\right)^2 - \frac{2}{3}\left(\sqrt{c_1 - x}\right)^3\right) \\ &= x + \left(c_1 - x - \frac{2}{3}(c_1 - x)^{\frac{3}{2}}\right) \\ &= c_1 - \frac{2}{3}(c_1 - x)^{\frac{3}{2}} \end{aligned}$$

Therefore the solutions are

$$\begin{aligned} y &= x + \frac{1}{3} \\ y &= c_1 + \frac{2}{3}(c_1 - x)^{\frac{3}{2}} \\ y &= c_1 - \frac{2}{3}(c_1 - x)^{\frac{3}{2}} \end{aligned}$$

1.3.3.12 Example 12

$$\begin{aligned}
(y')^2 &= e^{4x-2y}(y'-1) \\
\ln (y')^2 &= (4x-2y) + \ln (y'-1) \\
4x-2y &= \ln (y')^2 - \ln (y'-1) \\
4x-2y &= \ln \frac{(y')^2}{y'-1} \\
2y &= 4x - \ln \frac{(y')^2}{y'-1} \\
y &= 2x - \frac{1}{2} \ln \left(\frac{(y')^2}{y'-1} \right) \\
&= 2x - \frac{1}{2} \ln \left(\frac{p^2}{p-1} \right) \\
&= xf + g
\end{aligned}$$

Where $f = 2, g = -\frac{1}{2} \ln \left(\frac{p^2}{p-1} \right)$. Taking derivative w.r.t. x gives

$$\begin{aligned}
p &= \left(f + xf' \frac{dp}{dx} \right) + \left(g' \frac{dp}{dx} \right) \\
p &= f + (xf' + g') \frac{dp}{dx} \\
p - f &= (xf' + g') \frac{dp}{dx}
\end{aligned}$$

Using values for f, g the above simplifies to

$$p - 2 = \left(\frac{2-p}{2p^2 - 2p} \right) \frac{dp}{dx} \tag{2A}$$

The singular solution is when $\frac{dp}{dx} = 0$ which gives $p = 2$. From (1) this gives

$$y = 2x - \frac{1}{2} \ln 4$$

The general solution is when $\frac{dp}{dx} \neq 0$. Then (2) becomes

$$\begin{aligned}
\frac{dp}{dx} &= (p-2) \left(\frac{2p^2 - 2p}{2-p} \right) \\
&= 2p(1-p)
\end{aligned}$$

is now separable. Solving for p gives

$$p = \frac{1}{1 + ce^{-2x}}$$

Substituting the above solutions of p in (1) gives

$$\begin{aligned} y &= 2x - \frac{1}{2}\ln\left(\frac{\left(\frac{1}{1+ce^{-2x}}\right)^2}{\frac{1}{1+ce^{-2x}} - 1}\right) \\ &= 2x - \frac{1}{2}\ln\left(\frac{-e^{4x}}{c\,(c + e^{2x})}\right) \end{aligned}$$

1.3.3.13 Example 13

$$\begin{aligned} y &= \frac{xy' + x(y')^2 - (y')^2}{y' + 1} \\ &= \frac{xp + xp^2 - p^2}{p + 1} \\ &= xp - \frac{p^2}{p + 1} \\ &= xf + g \end{aligned} \tag{1}$$

Where $f = p$ and $g = -\frac{p^2}{p+1}$. Since $f(p) = p$ then this is Clairaut ode. Taking derivative of the above w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g(p)) \\ p &= p + (x + g'(p))\frac{dp}{dx} \\ 0 &= (x + g'(p))\frac{dp}{dx} \end{aligned}$$

The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution

$$y = xc_1 - \frac{c_1^2}{c_1 + 1}$$

The term $(x + g'(p)) = 0$ is used to find singular solutions.

$$\begin{aligned} x + g'(p) &= x + \frac{d}{dp}\frac{1}{p} \\ &= x - \frac{1}{p^2} \end{aligned}$$

Hence $x - \frac{1}{p^2} = 0$ or $p = \pm\frac{1}{\sqrt{x}}$. Substituting these back in (1) gives

$$\begin{aligned} y_1(x) &= xp + \frac{1}{p} \\ &= x\frac{1}{\sqrt{x}} + \sqrt{x} \\ &= 2\sqrt{x} \qquad (3) \\ y_2(x) &= -x\sqrt{\frac{1}{x}} - \sqrt{x} \\ &= -2\sqrt{x} \qquad (4) \end{aligned}$$

Eq. (2) is the general solution and (3,4) are the singular solutions.

1.3.3.14 Example 14

$$\begin{aligned} x(y')^2 + (x - y)\,y' + 1 - y &= 0 \\ x(y')^2 + xy' - yy' + 1 - y &= 0 \\ y(-y' - 1) + x(y')^2 + xy' + 1 &= 0 \end{aligned}$$

Solving for y

$$\begin{aligned} y &= \frac{-x(y')^2 - xy' - 1}{-y' - 1} \\ &= \frac{-xp^2 - xp - 1}{-p - 1} \\ &= \frac{xp^2 + xp + 1}{p + 1} \\ &= x\left(\frac{p^2 + p}{p + 1}\right) + \frac{1}{1 + p} \\ &= xp + \frac{1}{1 + p} \\ &= xf + g \qquad (1) \end{aligned}$$

Where $f = p$ and $g = \frac{1}{1+p}$. Taking derivative of the above w.r.t. x gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xp + g(p)) \\
p &= p + (x + g'(p))\frac{dp}{dx} \\
0 &= (x + g'(p))\frac{dp}{dx}
\end{aligned}
$$

The general solution is given by

$$
\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}
$$

Substituting this in (1) gives the general solution

$$
y = c_1 x + \frac{1}{c_1 + 1} \tag{4}
$$

The term $(x + g'(p)) = 0$ is used to find singular solutions. But

$$
\begin{aligned}
x + g'(p) &= x + \frac{d}{dp}\left(\frac{1}{1+p}\right) \\
&= x - \frac{1}{(p+1)^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
x - \frac{1}{(p+1)^2} &= 0 \\
x(p+1)^2 - 1 &= 0 \\
(p+1)^2 &= \frac{1}{x} \\
p + 1 &= \pm\frac{1}{\sqrt{x}} \\
p &= \pm\frac{1}{\sqrt{x}} - 1
\end{aligned}
$$

Substituting these values into (1) gives

$$
\begin{aligned}
y_1 &= xp_1 + \frac{1}{1+p_1} \\
&= x\left(\frac{1}{\sqrt{x}} - 1\right) + \frac{1}{1+\left(\frac{1}{\sqrt{x}} - 1\right)} \\
&= \frac{x}{\sqrt{x}} - x + \sqrt{x} \\
&= \frac{x\sqrt{x}}{x} - x + \sqrt{x} \\
&= 2\sqrt{x} - x \qquad (5)
\end{aligned}
$$

And substituting p_2 into (1) gives

$$
\begin{aligned}
y_1 &= xp_1 + \frac{1}{1+p_1} \\
&= x\left(-\frac{1}{\sqrt{x}} - 1\right) + \frac{1}{1+\left(-\frac{1}{\sqrt{x}} - 1\right)} \\
&= -\frac{x}{\sqrt{x}} - x - \sqrt{x} \\
&= \frac{-x\sqrt{x}}{x} - x - \sqrt{x} \\
&= -2\sqrt{x} - x \qquad (6)
\end{aligned}
$$

There are 3 solutions given in (4,5,6). One is general and two are singular.

1.3.3.15 Example 15

$$
xyy' = y^2 + x\sqrt{4x^2 + y^2}
$$

Solving for y gives

$$
\begin{aligned}
y &= \operatorname{RootOf}\left(\_z^4 - 4 + \left(p^2 - 1\right)\_z^2 - 2\_z^3 p\right) x \\
y &= xf + g
\end{aligned}
$$

Where $f = \operatorname{RootOf}\left(\_z^4 - 4 + (p^2 - 1)\_z^2 - 2\_z^3 p\right)$ and $g = 0$. Taking derivative of the above w.r.t. x gives

$$
\begin{aligned}
p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= f + xf'\frac{dp}{dx} \\
p - f &= xf'\frac{dp}{dx}
\end{aligned}
$$

Using values for f the above simplifies to

$$p - \operatorname{RootOf}\left(\_z^4 - 4 + \left(p^2 - 1\right)\_z^2 - 2\_z^3 p\right) = \left(x \frac{d}{dp} \operatorname{RootOf}\left(\_z^4 - 4 + \left(p^2 - 1\right)\_z^2 - 2\_z^3 p\right)\right) \frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p = \operatorname{RootOf}\left(\_z^4 - 4 + \left(p^2 - 1\right)\_z^2 - 2\_$ Substituting this in (1) does not generate any real solutions (only 2 complex ones) hence will not be used.

The general solution is found by finding p from (2A). Since (2A) is not linear in p, then inversion is needed. Writing (2A) as

$$\begin{aligned} \frac{dx}{dp} &= \frac{xf}{p - f} \\ \frac{1}{x} dx &= \frac{f}{p - f} dp \end{aligned}$$

Due to complexity of result, one now needs to obtain explicit result for RootOf which makes the computation very complicated. So this is not practical to solve by hand. Will stop here. It is much easier to solve this ode as a homogeneous ode instead which gives the solution as

$$-\frac{\sqrt{4x^2 + y^2}}{x} + \ln(x) = c_1$$

1.3.3.16 Example 16

$$\ln(\cos y') + y' \tan y' = y$$

Solving for y gives

$$\begin{aligned} y &= \ln(\cos p) + p \tan p \quad &(1) \\ y &= xf + g \\ &= g \quad &(1A) \end{aligned}$$

Where $f = 0$ and $g(p) = \ln(\cos p) + p \tan p$. *Important note*: This ode has $f = 0$ which is strictly speaking is not of the form $y = xf(p) + g(p)$. But Maple says this is dAlembert. This is why it is included. I should make special case dAlmbert classification to handle this special case.

Taking derivative of (1A) w.r.t. x gives

$$
\begin{aligned}
p &= \frac{dg}{dp}\frac{dp}{dx} \\
p &= \left(-\frac{\sin p}{\cos p} + \tan p + p(1+\tan^2 p)\right)\frac{dp}{dx} \\
p &= (-\tan p + \tan p + p(1+\tan^2 p))\frac{dp}{dx} \\
p &= p(1+\tan^2 p)\frac{dp}{dx} \\
1 &= (1+\tan^2 p)\frac{dp}{dx} \qquad (1.1)
\end{aligned}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which does not result in solution.

The general solution is found by finding p from (2). Since (2) is not linear in p, then inversion is needed. Writing (1) as

$$
\begin{aligned}
\frac{dx}{dp} &= 1 + \tan^2 p \\
dx &= (1+\tan^2 p)\, dp
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x &= \tan p + c \\
p &= \arctan(x-c)
\end{aligned}
$$

Substituting the above in (1) gives the solution

$$
\begin{aligned}
y &= \ln(\cos p) + p\tan p \\
&= \ln(\cos(\arctan(x-c))) + (\arctan(x-c))\tan(\arctan(x-c)) \\
&= \ln(\cos(\arctan(x-c))) + (x-c)\arctan(x-c)
\end{aligned}
$$

This ode also have solution $y = 0$.

1.3.3.17 Extra example

This ode is an example where y does not appear explicitly in the ode so not possible to directly solve for y. It is given here to show possible problems with this method.

$$
y' = \sqrt{1+x+y} \qquad (1A)
$$

This ode is squared to first solve for y which gives

$$
(y')^2 = 1+x+y \qquad (2A)
$$

However, here care is needed. To get back to original ode (1A) then (2A) means two possible equations

$$y' = \pm\sqrt{1+x+y}$$

Hence the solutions obtained using (2A) can be the solution to one of these

$$y' = +\sqrt{1+x+y} \tag{B1}$$
$$y' = -\sqrt{1+x+y} \tag{B2}$$

Therefore the solution obtained by squaring both sides of (1A), which is done in order to solve for y, must be checked to see if it satisfies the original ode, else it will be extraneous solution resulting from squaring both sides of the ode.

Starting from (2A), in normal form (by replacing y' with p) it becomes

$$\begin{aligned} y &= -x-1+p^2 \\ &= xf+g \end{aligned} \tag{1}$$

Where $f=-1, g=-1+p^2$. Taking derivative w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p+1 = 2p\frac{dp}{dx} \tag{2}$$

Since $\frac{\partial \phi}{\partial x} = -1 \neq p$ then this is d'Alembert ode. The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p=-1$. Substituting this in (1) gives the singular solution

$$y(x) = -x \tag{3}$$

But this solution does not satisfy the ode, hence it is extraneous. The general solution is found by finding p from (2). Since (2) is nonlinear, then it is inverted which gives

$$\frac{p+1}{2p} = \frac{dp}{dx}$$
$$\frac{dx}{dp} = \frac{2p}{p+1}$$

Which is linear in x. Solving gives

$$x = 2p - 2\ln(p+1) + c_1 \tag{4}$$

Instead of inverting this to find p in terms of x, p is found from (1) which gives

$$y+x+1 = p^2$$
$$p = \pm\sqrt{y+x+1}$$

Substituting these solutions in (4) gives implicit solutions as

$$x = 2\sqrt{y+x+1} - 2\ln\left(1+\sqrt{y+x+1}\right) + c_1$$
$$x = -2\sqrt{y+x+1} - 2\ln\left(1-\sqrt{y+x+1}\right) + c_1$$

But only the first one above satisfies the ode. The second is extraneous. Therefore the final solution is

$$x = 2\sqrt{y+x+1} - 2\ln\left(1+\sqrt{y+x+1}\right) + c_1$$

And no singular solutions exist. If instead of doing the above, p was found from (4) using inversion, then it will be

$$p = -\operatorname{LambertW}\left(-c_1 e^{\frac{-x}{2}-1}\right) - 1$$

Substituting this in (1) gives

$$y = -x - 1 + \left(-\operatorname{LambertW}\left(-c_1 e^{\frac{-x}{2}-1}\right) - 1\right)^2$$

But this general solution does not satisfy the original ode. In general, it is best to avoid squaring both side of the ode in order to solve for y as this can generate extraneous solutions. Only use this method if the original ode is already given in the form where y shows explicitly.

1.3.4 references

1. An elementary treatise on differential equations. By Abraham Cohen. 1906.
2. Applied differential equations, N Curle. 1972
3. Ordinary differential equations, LB Jones. 1976.
4. Elementary differential equations, William Martin, Eric Reissner. second edition. 1961.
5. Differentialgleichungen, by E. Kamke, page 30.
6. Differential and integral calculus by N. Piskunov, Vol II

1.3.5 Special case. $(y')^{\frac{n}{m}} = f(x)\,g(y)$

ode internal name "first_order_nonlinear_p_but_separable"

For the special case of $(y')^{\frac{n}{m}} = F(x,y)$ where RHS is separable, i.e. $F(x,y) = f(x)\,g(y)$ then short cut method is described below. This only works if $F(x,y)$ is separable and if there is only one y' in the equation. For example, it will not work on $(y')^{\frac{3}{2}} + y' = yx$ and will not work on $(y')^{\frac{3}{2}} = y + x$ (see second special case below for the form $(y')^{\frac{n}{m}} = ax + by + c$)

If the form is $(y')^{\frac{n}{m}} = f(x)\,g(y)$ then we first write it as $(y')^n = (f(x)\,g(y))^m$ assuming $f(x)\,g(y) > 0$. Then find roots on unity for n. For example of $n = 2$ this gives

$$y' = \begin{cases} (f(x)\,g(y))^{\frac{m}{2}} \\ -(f(x)\,g(y))^{\frac{m}{2}} \end{cases}$$

And if $n = 3$ then

$$y' = \begin{cases} (f(x)\,g(y))^{\frac{m}{3}} \\ -(-1)^{\frac{1}{3}}\,(f(x)\,g(y))^{\frac{m}{3}} \\ (-1)^{\frac{2}{3}}\,(f(x)\,g(y))^{\frac{m}{3}} \end{cases}$$

And if $n = 4$ then

$$y' = \begin{cases} (f(x)\,g(y))^{\frac{m}{4}} \\ -i(f(x)\,g(y))^{\frac{m}{4}} \\ i(f(x)\,g(y))^{\frac{m}{4}} \\ -(f(x)\,g(y))^{\frac{m}{4}} \end{cases}$$

And so on. For works for positive or negative n, m integers. Now the ode are solved each as as separable. Examples given below.

1.3.5.1 Example 1

$$
\begin{aligned}
(y')^4 + f(x)(y-a)^3(y-b)^3(y-c)^2 &= 0 \\
(y')^4 &= -f(x)(y-a)^3(y-b)^3(y-c)^2 \\
\frac{(y')^4}{(y-a)^3(y-b)^3(y-c)^2} &= -f(x) \\
\left(\frac{y'}{\left((y-a)^3(y-b)^3(y-c)^2\right)^{\frac{1}{4}}}\right)^4 &= -f(x) \\
\frac{y'}{\left((y-a)^3(y-b)^3(y-c)^2\right)^{\frac{1}{4}}} &= (-f(x))^{\frac{1}{4}} \\
\frac{y'}{\left((y-a)(y-b)(y-c)^{\frac{2}{3}}\right)^{\frac{3}{4}}} &= (-f(x))^{\frac{1}{4}} \\
\frac{dy}{\left((y-a)(y-b)(y-c)^{\frac{2}{3}}\right)^{\frac{3}{4}}} &= (-f(x))^{\frac{1}{4}}\,dx \\
\int^{y(x)} \frac{1}{\left((z-a)(z-b)(z-c)^{\frac{2}{3}}\right)^{\frac{3}{4}}} dz &= \int^{x} (-f(\tau))^{\frac{1}{4}}\,d\tau + c_1
\end{aligned}
$$

1.3.5.2 Example 2

$$
\begin{aligned}
(y')^3 &= y\sin x \\
\frac{(y')^3}{y} &= \sin x \\
\left(\frac{y'}{y^{\frac{1}{3}}}\right)^3 &= \sin x
\end{aligned}
$$

Hence we have 3 solutions

$$\frac{y'}{y^{\frac{1}{3}}} = \begin{cases} \sin^{\frac{1}{3}} x \\ -(-1)^{\frac{1}{3}} \sin^{\frac{1}{3}} x \\ (-1)^{\frac{2}{3}} \sin^{\frac{1}{3}} x \end{cases}$$

$$\frac{dy}{y^{\frac{1}{3}}} = \begin{cases} \sin^{\frac{1}{3}} x dx \\ -(-1)^{\frac{1}{3}} \sin^{\frac{1}{3}} x dx \\ (-1)^{\frac{2}{3}} \sin^{\frac{1}{3}} x dx \end{cases}$$

$$\int \frac{dy}{y^{\frac{1}{3}}} = \begin{cases} \int \sin^{\frac{1}{3}} x dx \\ -(-1)^{\frac{1}{3}} \int \sin^{\frac{1}{3}} x dx \\ (-1)^{\frac{2}{3}} \int \sin^{\frac{1}{3}} x dx \end{cases}$$

$$\frac{3}{2} y^{\frac{2}{3}} = \begin{cases} \int \sin^{\frac{1}{3}} x dx + c_1 \\ -(-1)^{\frac{1}{3}} \int \sin^{\frac{1}{3}} x dx + c_1 \\ (-1)^{\frac{2}{3}} \int \sin^{\frac{1}{3}} x dx + c_1 \end{cases}$$

$$y^{\frac{2}{3}} = \begin{cases} \frac{2}{3} \int \sin^{\frac{1}{3}} x dx + c_1 \\ -\frac{2}{3}(-1)^{\frac{1}{3}} \int \sin^{\frac{1}{3}} x dx + c_1 \\ \frac{2}{3}(-1)^{\frac{2}{3}} \int \sin^{\frac{1}{3}} x dx + c_1 \end{cases}$$

$$y = \begin{cases} \left(\frac{2}{3} \int \sin^{\frac{1}{3}} x dx + c_1\right)^{\frac{3}{2}} \\ \left(-\frac{2}{3}(-1)^{\frac{1}{3}} \int \sin^{\frac{1}{3}} x dx + c_1\right)^{\frac{3}{2}} \\ \left(\frac{2}{3}(-1)^{\frac{2}{3}} \int \sin^{\frac{1}{3}} x dx + c_1\right)^{\frac{3}{2}} \end{cases}$$

1.3.5.3 Example 3

$$(y')^3 = yx$$
$$\frac{(y')^3}{y} = x$$
$$\left(\frac{y'}{y^{\frac{1}{3}}}\right)^3 = x$$

Hence we have 3 solutions

$$\frac{y'}{y^{\frac{1}{3}}} = \begin{cases} x^{\frac{1}{3}} \\ -(-1)^{\frac{1}{3}} x^{\frac{1}{3}} \\ (-1)^{\frac{2}{3}} x^{\frac{1}{3}} \end{cases}$$

$$\frac{dy}{y^{\frac{1}{3}}} = \begin{cases} x^{\frac{1}{3}} dx \\ -(-1)^{\frac{1}{3}} x^{\frac{1}{3}} x dx \\ (-1)^{\frac{2}{3}} x^{\frac{1}{3}} x dx \end{cases}$$

$$\int \frac{dy}{y^{\frac{1}{3}}} = \begin{cases} \int x^{\frac{1}{3}} dx \\ -(-1)^{\frac{1}{3}} \int x^{\frac{1}{3}} dx \\ (-1)^{\frac{2}{3}} \int x^{\frac{1}{3}} dx \end{cases}$$

$$\frac{3}{2} y^{\frac{2}{3}} = \begin{cases} \frac{3}{4} x^{\frac{4}{3}} + c_1 \\ -(-1)^{\frac{1}{3}} \left(\frac{3}{4} x^{\frac{4}{3}}\right) + c_1 \\ (-1)^{\frac{2}{3}} \left(\frac{3}{4} x^{\frac{4}{3}}\right) + c_1 \end{cases}$$

$$y^{\frac{2}{3}} = \begin{cases} \frac{1}{2} x^{\frac{4}{3}} + c_1 \\ -(-1)^{\frac{1}{3}} \left(\frac{1}{2} x^{\frac{4}{3}}\right) + c_1 \\ (-1)^{\frac{2}{3}} \left(\frac{1}{2} x^{\frac{4}{3}}\right) + c_1 \end{cases}$$

$$y = \begin{cases} \left(\frac{1}{2} x^{\frac{4}{3}} + c_1\right)^{\frac{3}{2}} \\ \left(-(-1)^{\frac{1}{3}} \left(\frac{1}{2} x^{\frac{4}{3}}\right) + c_1\right)^{\frac{3}{2}} \\ \left((-1)^{\frac{2}{3}} \left(\frac{1}{2} x^{\frac{4}{3}}\right) + c_1\right)^{\frac{3}{2}} \end{cases}$$

1.3.5.4 Example 4

$$(y')^{\frac{1}{3}} = yx$$

For this form, we write $y' = (yx)^3$ but this is always with the assumption that $yx > 0$.

$$\begin{aligned}
y' &= (yx)^3 \\
y' &= y^3x^3 \\
\frac{dy}{y^3} &= x^3 dx \\
-\frac{1}{2y^2} &= \frac{1}{4}x^4 + c_1 \\
2y^2 &= \frac{-1}{\frac{1}{4}x^4 + c_1} \\
y^2 &= \frac{1}{-\frac{1}{2}x^4 + c_2} \\
y &= \begin{cases} \sqrt{\frac{1}{-\frac{1}{2}x^4+c_2}} \\ -\sqrt{\frac{1}{-\frac{1}{2}x^4+c_2}} \end{cases} \\
&= \begin{cases} \sqrt{\frac{2}{-x^4+c_3}} \\ -\sqrt{\frac{2}{-x^4+c_3}} \end{cases} \\
&= \begin{cases} \frac{\sqrt{2}}{\sqrt{-x^4+c_3}} \\ -\frac{\sqrt{2}}{\sqrt{-x^4+c_3}} \end{cases}
\end{aligned}$$

1.3.5.5 Example 5

$$\begin{aligned}
(y')^2 &= \frac{1-y^2}{1-x^2} \\
\frac{(y')^2}{1-y^2} &= \frac{1}{1-x^2} \\
\left(\frac{y'}{(1-y^2)^{\frac{1}{2}}}\right)^2 &= \frac{1}{1-x^2}
\end{aligned}$$

Hence we have 2 solutions

$$\frac{y'}{\sqrt{(1-y^2)}} = \begin{cases} \sqrt{\frac{1}{1-x^2}} \\ -\sqrt{\frac{1}{1-x^2}} \end{cases}$$

$$\int \frac{dy}{\sqrt{(1-y^2)}} = \begin{cases} \int \sqrt{\frac{1}{1-x^2}} dx \\ -\int \sqrt{\frac{1}{1-x^2}} dx \end{cases}$$

$$= \begin{cases} \int \frac{1}{\sqrt{1-x^2}} dx \\ -\int \frac{1}{\sqrt{1-x^2}} dx \end{cases} \qquad -1 < x < 1$$

$$\arcsin(y) = \begin{cases} \arcsin(x) + c \\ -\arcsin(x) + c \end{cases} \qquad -1 < x < 1$$

$$y = \begin{cases} \sin(\arcsin(x) + c) \\ -\sin(\arcsin(x) + c) \end{cases} \qquad -1 < x < 1$$

1.3.5.6 Algorithm description to obtain the above solutions

Starting with

$$(y')^{\frac{n}{m}} = f(x)\, g(y)$$

Find the solution z of equation

$$z^{\frac{n}{m}} = fg$$

This will obtain number of solutions. For example for $n = 3, m = 1$

$$\begin{aligned} z_1 &= (fg)^{\frac{1}{3}} \\ z_2 &= -\frac{1}{2}(fg)^{\frac{1}{3}} + \frac{1}{2}i\sqrt{3}(fg)^{\frac{1}{3}} \\ z_3 &= -\frac{1}{2}(fg)^{\frac{1}{3}} - \frac{1}{2}i\sqrt{3}(fg)^{\frac{1}{3}} \end{aligned}$$

Now if we assume that $f > 0, g > 0$ then we can separate the f, g giving

$$\begin{aligned} z_1 &= f^{\frac{1}{3}} g^{\frac{1}{3}} \\ z_2 &= -\frac{1}{2} f^{\frac{1}{3}} g^{\frac{1}{3}} + \frac{1}{2}i\sqrt{3} f^{\frac{1}{3}} g^{\frac{1}{3}} \\ z_3 &= -\frac{1}{2} f^{\frac{1}{3}} g^{\frac{1}{3}} - \frac{1}{2}i\sqrt{3} f^{\frac{1}{3}} g^{\frac{1}{3}} \end{aligned}$$

or

$$
\begin{aligned}
z_1 &= f^{\frac{1}{3}} g^{\frac{1}{3}} \\
z_2 &= g^{\frac{1}{3}} \left(-\frac{1}{2} f^{\frac{1}{3}} + \frac{1}{2} i\sqrt{3} f^{\frac{1}{3}} \right) \\
z_3 &= g^{\frac{1}{3}} \left(-\frac{1}{2} f^{\frac{1}{3}} - \frac{1}{2} i\sqrt{3} f^{\frac{1}{3}} \right)
\end{aligned}
$$

This means

$$
\begin{aligned}
y' &= f^{\frac{1}{3}} g^{\frac{1}{3}} \\
y' &= g^{\frac{1}{3}} \left(-\frac{1}{2} f^{\frac{1}{3}} + \frac{1}{2} i\sqrt{3} f^{\frac{1}{3}} \right) \\
y' &= g^{\frac{1}{3}} \left(-\frac{1}{2} f^{\frac{1}{3}} - \frac{1}{2} i\sqrt{3} f^{\frac{1}{3}} \right)
\end{aligned}
$$

Which gives

$$
\begin{aligned}
\int \frac{dy}{g(y)^{\frac{1}{3}}} &= \int f(x)^{\frac{1}{3}}\, dx + c_1 \\
\int \frac{dy}{g(y)^{\frac{1}{3}}} &= \int \left(-\frac{1}{2} f^{\frac{1}{3}} + \frac{1}{2} i\sqrt{3} f^{\frac{1}{3}} \right) dx + c_1 \\
\int \frac{dy}{g(y)^{\frac{1}{3}}} &= \int \left(-\frac{1}{2} f^{\frac{1}{3}} - \frac{1}{2} i\sqrt{3} f^{\frac{1}{3}} \right) dx + c_1
\end{aligned}
$$

There is no need to evaluate the integrals unless needed. Without the assumption $f, g > 0$ we could not separate them. Since $(fg)^{\frac{n}{m}} = f^{\frac{n}{m}} g^{\frac{n}{m}}$ is true under this condition when $\frac{n}{m}$ is rational number. If $\frac{n}{m}$ is an integer, then this condition is not needed and we can always factor out f, g and separate them.

The assumption $f, g > 0$ might be too strict to use but without this assumption this method can not be used.

1.3.6 Special case. $(y')^{\frac{n}{m}} = ax + by + c$

ode internal name "first_order_nonlinear_p_but_linear_in_x_y"

For the special case of $(y')^{\frac{n}{m}} = F(x, y)$ where RHS is linear in both x and y, i.e. $F(x, y) = ax + by + c$ then a short cut method is described below using transformation $u = ax + by + c$. This makes it separable in u. This will not work if there is nonlinear x term, such as $(y')^{\frac{n}{m}} = by + x^2$ or nonlinear term in y such as $(y')^{\frac{n}{m}} = y^2 + x$.

Taking derivatives gives $u' = a + by'$ or $y' = \frac{u'-a}{b}$ and the ode becomes

$$
\begin{aligned}
\left(\frac{u'-a}{b}\right)^{\frac{n}{m}} &= u \\
\left(\frac{u'-a}{b}\right)^{n} &= u^m
\end{aligned}
$$

Here we need to find roots of unity for n. For example, for $n = 2$ we have

$$
\frac{u'-a}{b} = \begin{cases} (u)^{\frac{m}{2}} \\ -(u)^{\frac{m}{2}} \end{cases}
$$

And for $n = 3$

$$
\frac{u'-a}{b} = \begin{cases} (u)^{\frac{m}{3}} \\ -(-1)^{\frac{1}{3}} (u)^{\frac{m}{3}} \\ (-1)^{\frac{2}{3}} (u)^{\frac{m}{3}} \end{cases}
$$

And so on. From now on, this is solved as separable. For negative integer values n, we just replaced n by $-n$ in the above. For example, for $n = 3$

$$
\frac{u'-a}{b} = \begin{cases} (u)^{\frac{m}{-3}} \\ -(-1)^{\frac{1}{3}} (u)^{\frac{m}{-3}} \\ (-1)^{\frac{2}{3}} (u)^{\frac{m}{-3}} \end{cases}
$$

For symbolic values of n we can just leave the integral as is. For example for $(y')^r = ax + by$ we obtain

$$
\begin{aligned}
\left(\frac{u'-a}{b}\right)^{r} &= u \\
\frac{u'-a}{b} &= u^{\frac{1}{r}} \\
u' &= bu^{\frac{1}{r}} + a \\
\int \frac{du}{bu^{\frac{1}{r}} + a} &= \int dx + c_1 \\
\int^{ax+by(x)} \frac{dz}{bz^{\frac{1}{r}} + a} &= x + c_1
\end{aligned}
$$

1.3.6.1 Example 1

$$(y')^3 = 2y + 3x + 9$$

Let $u = 2y + 3x + 9$ then $u' = 2y' + 3$ then $y' = \frac{u'-3}{2}$ and the ode becomes

$$\begin{aligned}
\left(\frac{u'-3}{2}\right)^3 &= u \\
\frac{u'-3}{2} &= \begin{cases} (u)^{\frac{1}{3}} \\ -(-1)^{\frac{1}{3}} (u)^{\frac{1}{3}} \\ (-1)^{\frac{2}{3}} (u)^{\frac{1}{3}} \end{cases} \\
u' - 3 &= \begin{cases} 2(u)^{\frac{1}{3}} \\ -2(-1)^{\frac{1}{3}} (u)^{\frac{1}{3}} \\ 2(-1)^{\frac{2}{3}} (u)^{\frac{1}{3}} \end{cases} \\
u' &= \begin{cases} 2(u)^{\frac{1}{3}} + 3 \\ -2(-1)^{\frac{1}{3}} (u)^{\frac{1}{3}} + 3 \\ 2(-1)^{\frac{2}{3}} (u)^{\frac{1}{3}} + 3 \end{cases}
\end{aligned}$$

Each is now solved as separable.

$$\begin{aligned}
u' &= 2(u)^{\frac{1}{3}} + 3 \\
\frac{du}{2(u)^{\frac{1}{3}} + 3} &= dx \\
\int \frac{du}{2(u)^{\frac{1}{3}} + 3} &= \int dx \\
\int \frac{du}{2(u)^{\frac{1}{3}} + 3} &= x + c_1
\end{aligned}$$

Hence

$$\int^{2y(x)+3x+9} \frac{dz}{2z^{\frac{1}{3}} + 3} = x + c_1$$

For the second one $u' = -2(-1)^{\frac{1}{3}}(u)^{\frac{1}{3}} + 3$ results in

$$\frac{du}{-2\left(-1\right)^{\frac{1}{3}}\left(u\right)^{\frac{1}{3}}+3} = dx$$
$$\int \frac{du}{-2\left(-1\right)^{\frac{1}{3}}\left(u\right)^{\frac{1}{3}}+3} = \int dx$$
$$\int^{2y(x)+3x+9} \frac{dz}{-2\left(-1\right)^{\frac{1}{3}}\left(z\right)^{\frac{1}{3}}+3} = x + c_1$$

And for the third ode $u' = 2(-1)^{\frac{2}{3}}(u)^{\frac{1}{3}} + 3$

$$\frac{du}{2\left(-1\right)^{\frac{2}{3}}\left(u\right)^{\frac{1}{3}}+3} = dx$$
$$\int \frac{du}{2\left(-1\right)^{\frac{2}{3}}\left(u\right)^{\frac{1}{3}}+3} = \int dx$$
$$\int^{2y(x)+3x+9} \frac{dz}{2\left(-1\right)^{\frac{2}{3}}\left(z\right)^{\frac{1}{3}}+3} = x + c_1$$

Hence the three solutions are

$$\begin{cases} \int^{2y(x)+3x+9} \frac{dz}{2z^{\frac{1}{3}}+3} = x + c_1 \\ \int^{2y(x)+3x+9} \frac{dz}{-2(-1)^{\frac{1}{3}}(z)^{\frac{1}{3}}+3} = x + c_1 \\ \int^{2y(x)+3x+9} \frac{dz}{2(-1)^{\frac{2}{3}}(z)^{\frac{1}{3}}+3} = x + c_1 \end{cases}$$

1.3.6.2 Example 2

$$(y')^{\frac{3}{2}} = 2y + 3x + 9$$

Let $u = 2y + 3x + 9$ then $u' = 2y' + 3$ then $y' = \frac{u'-3}{2}$ and the ode becomes

$$\left(\frac{u'-3}{2}\right)^{\frac{3}{2}} = u$$
$$\left(\left(\frac{u'-3}{2}\right)^{\frac{1}{2}}\right)^{3} = u$$

Let $\left(\frac{u'-3}{2}\right)^{\frac{1}{2}} = Y$ then

$$Y^3 = u$$

$$Y = \begin{cases} u^{\frac{1}{3}} \\ u^{\frac{1}{3}}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{\frac{1}{3}} \\ u^{\frac{1}{3}}\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{\frac{1}{3}} \end{cases}$$

Hence

$$\left(\frac{u'-3}{2}\right)^{\frac{1}{2}} = \begin{cases} u^{\frac{1}{3}} \\ u^{\frac{1}{3}}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{\frac{1}{3}} \\ u^{\frac{1}{3}}\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{\frac{1}{3}} \end{cases}$$

$$\left(\frac{u'-3}{2}\right) = \begin{cases} u^{\frac{2}{3}} \\ u^{\frac{2}{3}}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{\frac{2}{3}} \\ u^{\frac{2}{3}}\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{\frac{2}{3}} \end{cases}$$

$$u' = \begin{cases} 2u^{\frac{2}{3}}+3 \\ 2u^{\frac{2}{3}}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{\frac{2}{3}}+3 \\ 2u^{\frac{2}{3}}\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{\frac{2}{3}}+3 \end{cases}$$

Each is solved as separable.

$$\begin{cases} \int \frac{du}{2u^{\frac{2}{3}}+3} = \int dx \\ \int \frac{du}{2u^{\frac{2}{3}}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{\frac{2}{3}}+3} = \int dx \\ \int \frac{du}{2u^{\frac{2}{3}}\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{\frac{2}{3}}+3} = \int dx \end{cases}$$

Hence the three solutions are

$$\begin{cases} \int^{2y(x)+3x+9} \frac{dz}{2z^{\frac{2}{3}}+3} = x + c_1 \\ \int^{2y(x)+3x+9} \frac{dz}{2z^{\frac{2}{3}}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{\frac{2}{3}}+3} = x + c_1 \\ \int^{2y(x)+3x+9} \frac{dz}{2z^{\frac{2}{3}}\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{\frac{2}{3}}+3} = x + c_1 \end{cases}$$

1.3.6.3 Algorithm description to obtain the above solutions

Starting with

$$(y')^{\frac{n}{m}} = ax + by + c$$

Find the solution z of equation

$$z^{\frac{n}{m}} = u$$

Where u now is a symbol. Lets say we found $s_1, s_2, \cdots$ solutions (depending on what n, m are). Then for each solution s_i change it to be

$$s_i = bs_i + a$$

Then write

$$\int \frac{du}{s_i} = x + c_1$$

Then replace each with letter u in each s_i by new letter say z (the integration variable). Now the solution becomes

$$\int^{ax+by+c} \frac{dz}{s_i} = x + c_1$$

This is basically what was done in the above examples. There is no need to find an explicit solution for the integral. But this can be done if needed afterwords.

1.4 System of first order ode's

1.4.1 Linear system of first order ode's

Currently the solver only supports first order system of odes, that are linear and not time varying.

ode internal name "system of linear ODEs"

System of linear first order ode's.

$$\boldsymbol{x}' = A\boldsymbol{x} + F(\boldsymbol{x})$$

Solved using both eigenvalues and eigenvectors method and also the matrix exponential method. Only linear ode's are supported. The following flow chart show the algorithm for two system of ode's.

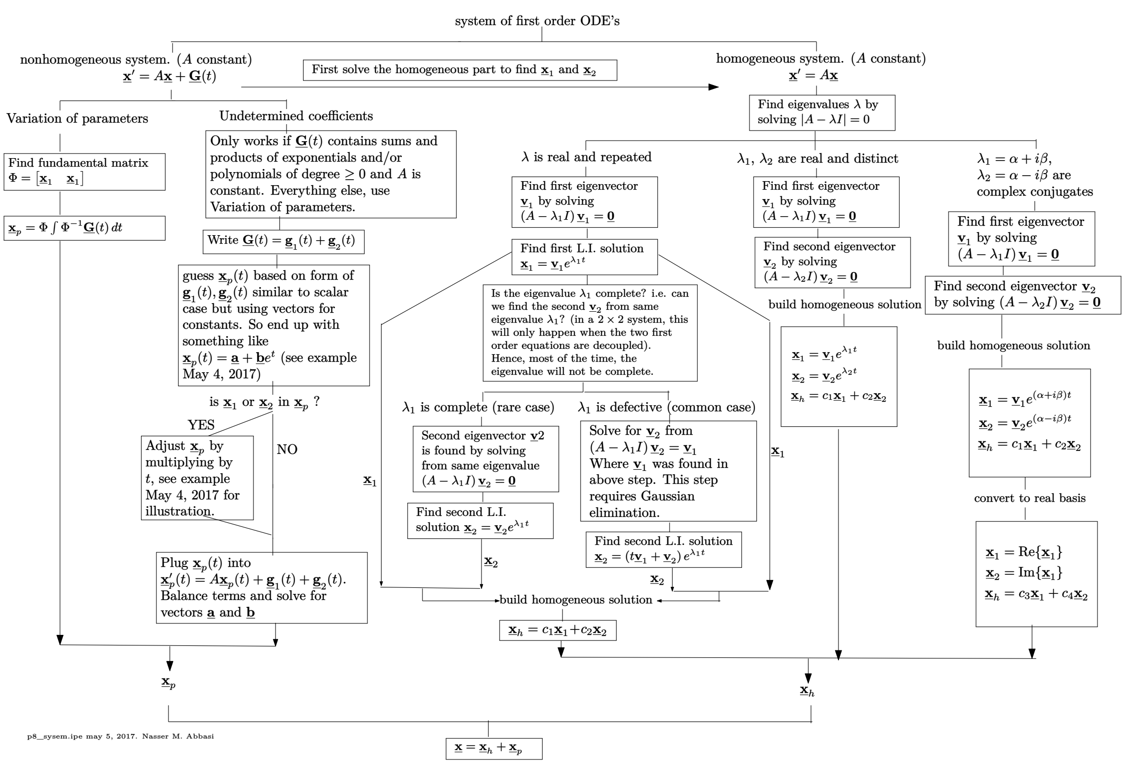

Figure 1.15: Flow chart for system of ode solver

These diagrams show the handling of repeated eigenvalues when a defective system is encountered.

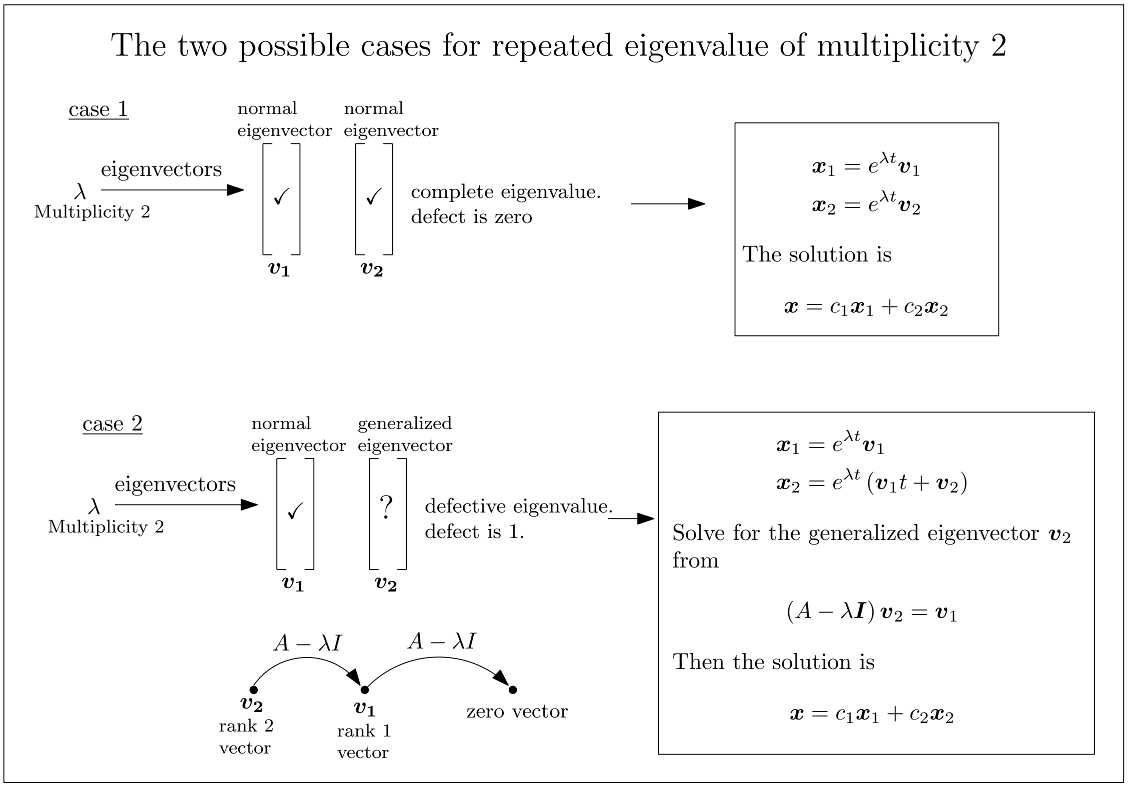

Figure 1.16: repeated eigenvalue of order 2

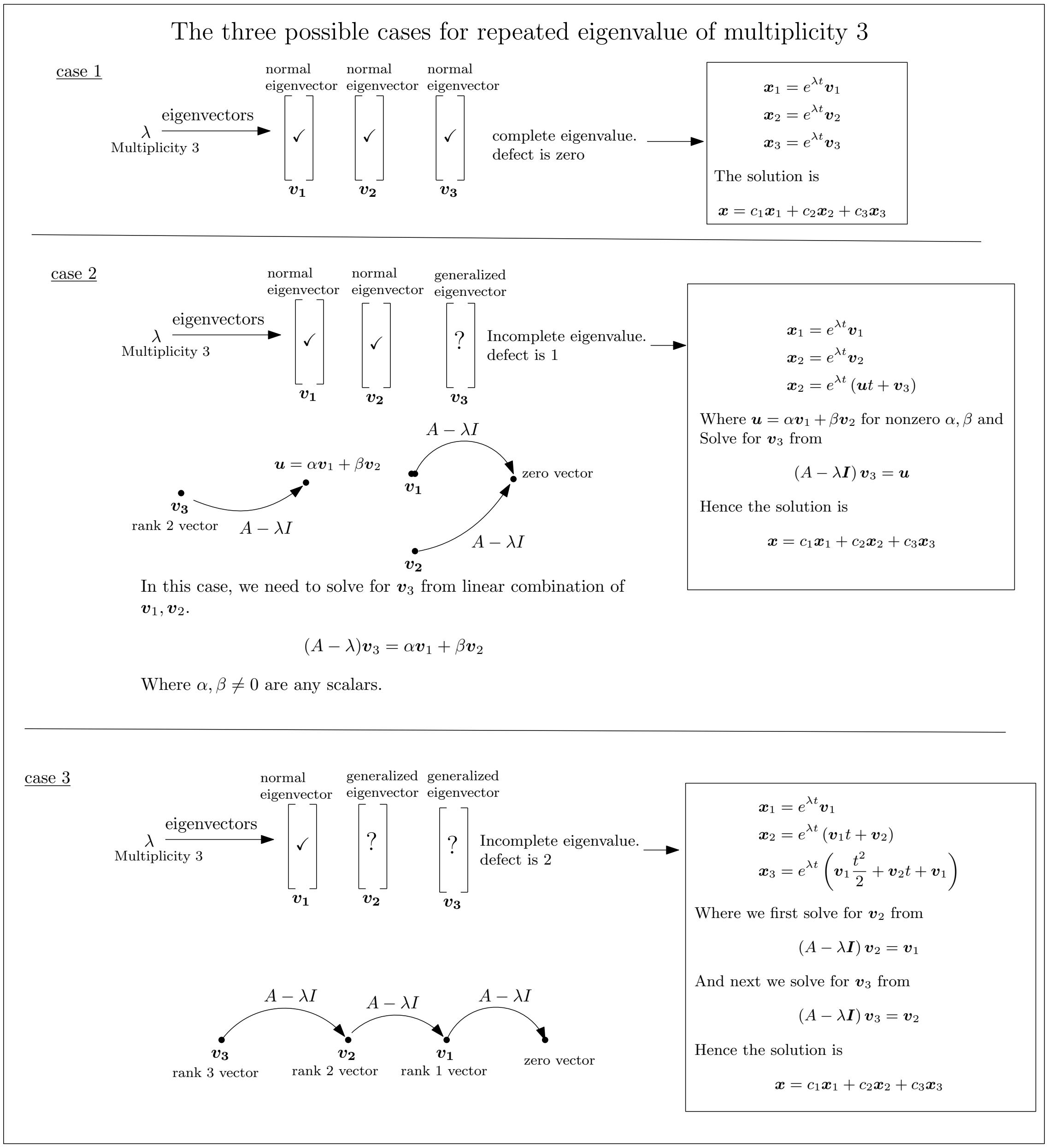

Figure 1.17: repeated eigenvalue of order 3

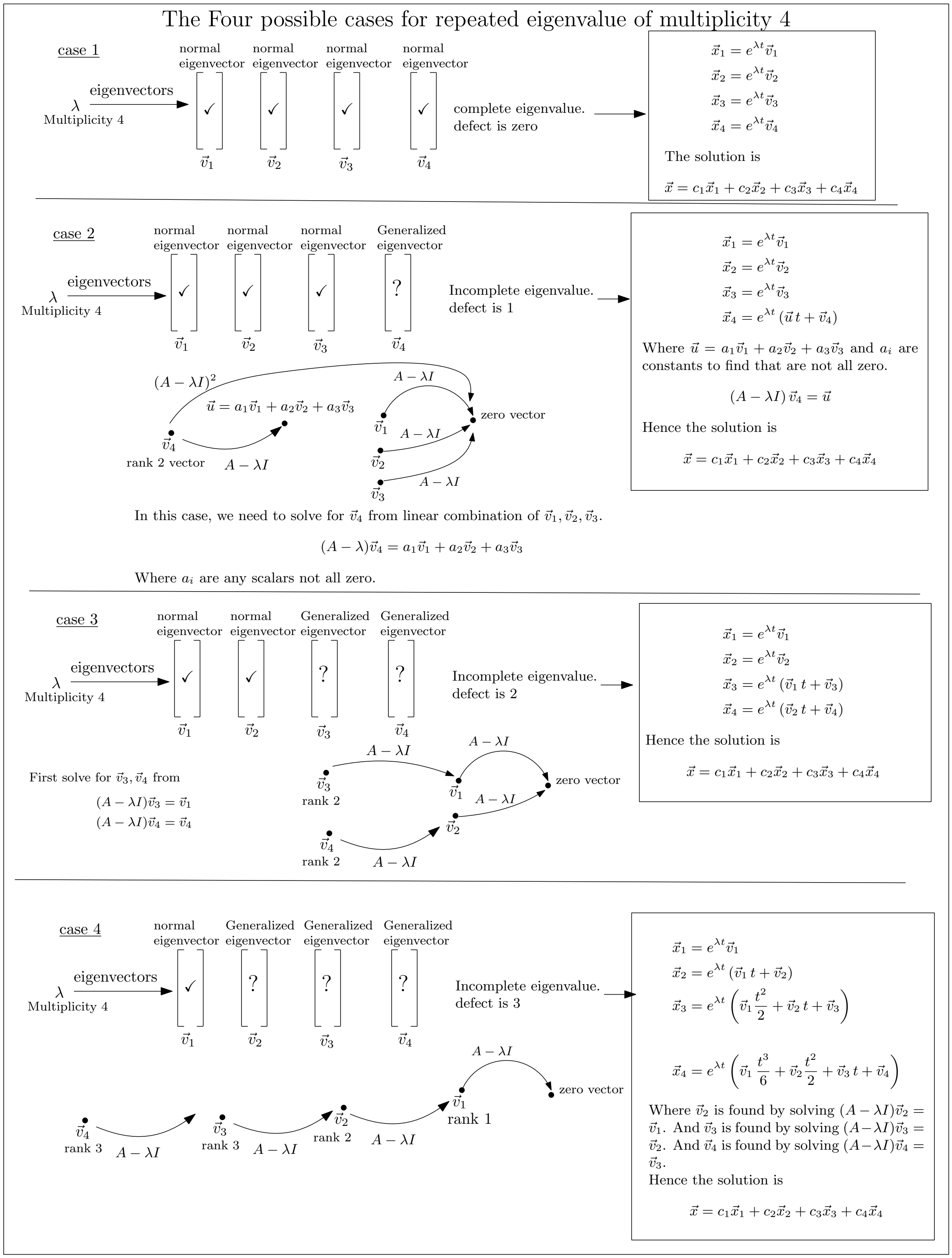

Figure 1.18: repeated eigenvalue of order 4

1.4.1.1 Examples

1.4.1.1.1 Example 1

$$x'(t)+y'(t)=x+y+t \tag{1}$$
$$x'(t)+y'(t)=2x+3y+e^t \tag{2}$$

Hence

$$\begin{aligned} x+y+t &= 2x+3y+e^t \\ y &= -\frac{1}{2}x+\frac{1}{2}t-\frac{1}{2}e^t \end{aligned} \tag{3}$$

Taking derivative w.r.t. t gives

$$y'=-\frac{x'}{2}+\frac{1}{2}-\frac{1}{2}e^t \tag{4}$$

Substituting (3,4) in (1) to eliminate y,y' gives

$$\begin{aligned} x'+\left(-\frac{x'}{2}-\frac{1}{2}e^t+\frac{1}{2}\right) &= x+\left(-\frac{x}{2}-\frac{1}{2}e^t+\frac{1}{2}t\right)+t \\ x' &= 3t+x-1 \end{aligned} \tag{5}$$

This is linear ode. Its solution is

$$x=c_1e^t-3t-2 \tag{6}$$

Substituting this in (3) gives

$$\begin{aligned} y &= -\frac{1}{2}\left(c_1e^t-3t-2\right)+\frac{1}{2}t-\frac{1}{2}e^t \\ &= 2t-\frac{1}{2}e^t-\frac{1}{2}c_1e^t+1 \end{aligned}$$

1.4.1.1.2 Example 2

$$x'(t) + y'(t) = x + y + t \tag{1}$$
$$2x'(t) + y'(t) = 2x + 3y + e^t \tag{2}$$

Let $x' = A, y' = B$ then

$$A + B = x + y + t \tag{1}$$
$$2A + B = 2x + 3y + e^t \tag{2}$$

From (1), $B = x + y + t - A$. Substituting in (2) gives

$$\begin{aligned} 2A + (x + y + t - A) &= 2x + 3y + e^t \\ A &= x - t + 2y + e^t \end{aligned} \tag{3}$$

Now we plugin the above in (1) which gives

$$\begin{aligned} \left(x - t + 2y + e^t\right) + B &= x + y + t \\ B &= 2t - y - e^t \end{aligned} \tag{4}$$

Hence we have the following two linear ode's of standard form now. These are (3,4)

$$\begin{aligned} x' &= x - t + 2y + e^t \\ y' &= 2t - y - e^t \end{aligned}$$

And now these can be solved using standard methods.

1.4.1.1.3 Example 3

$$
\begin{aligned}
x'(t) + y'(t) &= x + 2y + 2e^{t} && (1)\\
x'(t) + y'(t) &= 3x + 4y + e^{2t} && (2)
\end{aligned}
$$

Hence

$$
\begin{aligned}
x + 2y + 2e^{t} &= 3x + 4y + e^{2t}\\
y &= -x - \frac{1}{2}e^{2t} + e^{t} && (3)
\end{aligned}
$$

Taking derivative w.r.t. t gives

$$
y' = -x' - e^{2t} + e^{t} \tag{4}
$$

Substituting (3,4) in (1) to eliminate y, y' gives

$$
\begin{aligned}
x' + \left(-x' - e^{2t} + e^{t}\right) &= x + 2\left(-x - \frac{1}{2}e^{2t} + e^{t}\right) + 2e^{t}\\
x' - x' - e^{2t} + e^{t} &= x - 2x - e^{2t} + 2e^{t} + 2e^{t}\\
0 &= -x + 3e^{t}\\
x &= 3e^{t} && (5)
\end{aligned}
$$

Substituting this in (3) gives

$$
\begin{aligned}
y &= -3e^{t} - \frac{1}{2}e^{2t} + e^{t}\\
&= -2e^{t} - \frac{1}{2}e^{2t}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
x &= 3e^{t}\\
y &= -2e^{t} - \frac{1}{2}e^{2t}
\end{aligned}
$$

1.4.2 nonlinear system of first order ode's

Not currently supported.

CHAPTER 2

SECOND ORDER ODE $F(x, y, y', y'') = 0$

2.1 Flow charts

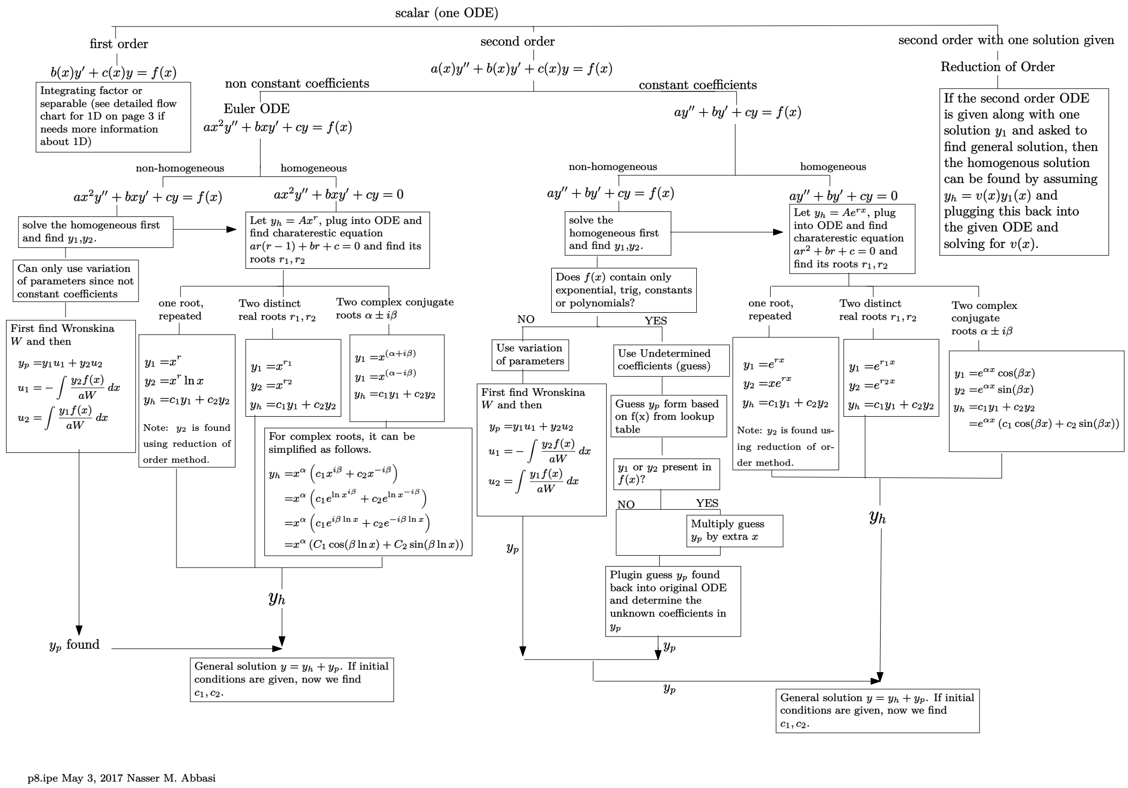

Figure 2.1: Flow chart for some of the supported ode types

2.2 Existence and uniqueness for second order ode

There are two theorems. One for linear second order ode and one for non-linear second order ode.

2.2.1 Existence and uniqueness for linear second order ode

Given linear second order ode

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

With initial conditions at x_0

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y'_0 \end{aligned}$$

If $p(x)\,,q(x)\,,f(x)$ are all continuous at x_0 then theorem guarantees that a solution exist and is unique on some interval than includes x_0. If this was not the case, (i.e. if any of p, q, f are not continuous at x_0) then the theorem does not apply. This means a solution could still exists and even be unique, but theory does not say anything about this.

2.2.1.1 Example

$$\begin{aligned} xy'' + y' + 3y &= \sin(x) \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

In standard form

$$y'' + \frac{1}{x}y' + \frac{3}{x}y = \frac{1}{x}\sin x$$

We see that $p(x) = \frac{1}{x}$ is not continuous at $x_0 = 0$. Hence theorem does not apply. It turns out that there is no solution to this ode with these initial conditions. Changing x_0 to 1 instead of zero, solution exists and is unique.

2.2.1.2 Example

$$
\begin{aligned}
y'' + \frac{1}{x-1}y' + 3y &= x \\
y(1) &= 0 \\
y'(1) &= 1
\end{aligned}
$$

In standard form

$$
y'' + py' + qy = f
$$

$p(x) = \frac{1}{x-1}$ is not continuous at $x_0 = 1$. Hence theorem does not apply. It turns out that there is no solution to this ode with these initial conditions. Changing x_0 to 0 instead then a solution exists and is unique.

2.2.2 Existence and uniqueness for non-linear second order ode

Now the ode is written in the form

$$
\begin{aligned}
y'' &= f(x,y,y') \\
y(x_0) &= y_0 \\
y'(x_0) &= y'_0
\end{aligned}
$$

Then if f is continuous at (x_0, y_0, y'_0) and f_y is also continuous at (x_0, y_0, y'_0) and also $f_{y'}$ is also continuous at (x_0, y_0, y'_0) then there is unique solution on interval that contains x_0.

2.2.2.1 Example

$$
\begin{aligned}
y'' &= 2yy' \\
y(0) &= 1 \\
y'(0) &= 2
\end{aligned}
$$

Hence $f(x,y,y') = 2yy'$. At $x = 0$ then $f = 4$ which is continuous. And $f_y = 2y'$ which at x_0 becomes 4. This is also continuous. And $f_{y'} = 2y$ which at x_0 becomes 4 which is also continuous. Hence solution exists and is unique on interval that contains $x = 0$. The solution can be found as follows

Let $y' = p(y)$ then $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p$. The ode becomes

$$
\begin{aligned}
\frac{dp}{dy}p &= 2yp \\
\frac{dp}{dy} &= 2y
\end{aligned}
$$

But at $x=0$ we have $y(0)=1$ and $y'(0)=p(y(0))=p(1)=2$. This is the initial condition used for solving the above quadrature ode. Integrating the above gives

$$
p = y^2 + c_1
$$

Applying IC $p(1)=2$ gives

$$
\begin{aligned}
2 &= 1 + c_1 \\
c_1 &= 1
\end{aligned}
$$

Hence $p = y^2+1$. But $y' = p$ or $y' = y^2+1$. This is separable with initial conditions $y(0)=1$. Integrating gives

$$
\begin{aligned}
\int \frac{dy}{y^2+1} &= \int dx \\
\arctan(y) &= x + c_2
\end{aligned}
$$

Applying IC

$$
\arctan(1) = c_2
$$

So $c_2 = \frac{\pi}{4}$. Hence the solution becomes

$$
\begin{aligned}
\arctan(y) &= x + \frac{\pi}{4} \\
y(x) &= \tan\left(x + \frac{\pi}{4}\right)
\end{aligned}
$$

2.2.2.2 Example

$$
\begin{aligned}
y'' + y &= \frac{1}{x} \\
y(0) &= 1 \\
y'(0) &= 2
\end{aligned}
$$

Here $f(x) = \frac{1}{x}$ is not continuous at $x=0$. Therefore theory does not apply. It turns out that no solution exists for this ode.

2.3 Linear second order ode

2.3.1 Linear ode with constant coefficients $Ay''+By'+Cy=f(x)$

2.3.1.1 Quadrature ode $y''=f(x)$

ode internal name "second order ode quadrature"

Solved by integration twice. $y'=\int f dx+c_1$ and $y=\int\left(\int f dx\right)dx+c_1x+c_2$

2.3.1.2 Solved by finding roots of characteristic equation

ode internal name "second order linear constant coeff"

These are solved by finding roots of characteristic equation. This is the standard method. Homogeneous and inhomogeneous. The method of Variation of parameters and the method of undetermined coefficients are both used to find the particular solution. If hint "laplace" is given, then the ODE is solved using Laplace transform method. If hint "series" is given then series method is used.

2.3.1.2.1 Example 1 (Variation of parameters)

$$4y''-y=e^{\frac{x}{2}}+6$$

Solution is $y=y_h+y_p$. The roots of the characteristic equation are $\pm\frac{1}{2}$,. hence y_h is

$$y_h=c_1e^{\frac{1}{2}x}+c_2e^{-\frac{1}{2}x}$$

The basis for y_h are $y_1=e^{\frac{1}{2}x},y_2=e^{-\frac{1}{2}x}$. Let

$$y_p=y_1u_1+y_2u_2$$

Where

$$u_1=-\int\frac{y_2f(x)}{aW}dx$$
$$u_2=\int\frac{y_1f(x)}{aW}dx$$

Where $a=4,f(x)=e^{\frac{x}{2}}+6$ and

$$W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=\begin{vmatrix}e^{\frac{1}{2}x} & e^{-\frac{1}{2}x}\\ \frac{1}{2}e^{\frac{1}{2}x} & -\frac{1}{2}e^{-\frac{1}{2}x}\end{vmatrix}=-\frac{1}{2}-\frac{1}{2}=-1$$

Hence

$$
\begin{aligned}
u_1 &= -\int \frac{e^{-\frac{1}{2}x}\left(e^{\frac{x}{2}}+6\right)}{-4} dx = \frac{1}{4}x - 3e^{-\frac{1}{2}x} \\
u_2 &= \int \frac{e^{\frac{1}{2}x}\left(e^{\frac{x}{2}}+6\right)}{-4} dx = -\frac{1}{4}e^{\frac{1}{2}x}\left(e^{\frac{1}{2}x}+12\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
y_p &= y_1u_1 + y_2u_2 \\
&= e^{\frac{1}{2}x}\left(\frac{1}{4}x - 3e^{-\frac{1}{2}x}\right) + e^{-\frac{1}{2}x}\left(-\frac{1}{4}e^{\frac{1}{2}x}\left(e^{\frac{1}{2}x}+12\right)\right) \\
&= \frac{1}{4}xe^{\frac{1}{2}x} - \frac{1}{4}e^{\frac{1}{2}x} - 6
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1e^{\frac{1}{2}x} + c_2e^{-\frac{1}{2}x} + \frac{1}{4}xe^{\frac{1}{2}x} - \frac{1}{4}e^{\frac{1}{2}x} - 6
\end{aligned}
$$

Or by combining terms into new constant, the above becomes

$$
y = c_3e^{\frac{1}{2}x} + c_2e^{-\frac{1}{2}x} + \frac{1}{4}xe^{\frac{1}{2}x} - 6
$$

2.3.1.3 Solved using Laplace transform

ode internal name "second order laplace"

These are solved using Laplace transform. These are only solved using this method if 'hint'="laplace" is given.

2.3.1.3.1 Example 1

$$
\begin{aligned}
y'' + 2y' + y &= 0 \\
y(1) &= 2 \\
y'(0) &= 2
\end{aligned}
$$

Taking Laplace transform gives

$$
\begin{aligned}
\left(s^2Y - sy(0) - y'(0)\right) + 2(sY - y(0)) + Y &= 0 \\
\left(s^2Y - sy(0) - 2\right) + (2sY - 2y(0)) + Y &= 0
\end{aligned}
$$

Let $y(0) = c$

$$\begin{aligned}
\left(s^2Y - sc - 2\right) + (2sY - 2c) + Y &= 0 \\
Y\left(s^2 + 2s + 1\right) - sc - 2 - 2c &= 0 \\
Y &= \frac{sc + 2 + 2c}{s^2 + 2s + 1}
\end{aligned}$$

Applying inverse Laplace transform gives

$$y(t) = (c + 2t + ct)\, e^{-t} \tag{1}$$

But $y(1) = 2$ hence

$$\begin{aligned}
2 &= (c + 2 + c)\, e^{-1} \\
2e &= 2c + 2 \\
c &= e - 1
\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}
y(t) &= (e - 1 + 2t + (e - 1)\, t)\, e^{-t} \\
&= e^{-t}(-1 + e + t + et)
\end{aligned}$$

2.3.1.3.2 Example 2

$$\begin{aligned}
y'' - 2y' - 3y &= 0 \\
y(4) &= -3 \\
y'(4) &= -17
\end{aligned}$$

Taking Laplace transform gives

$$\left(s^2Y - sy(0) - y'(0)\right) - 2(sY - y(0)) - 3Y = 0$$

Since given initial conditions are not at $t = 0$, then let $y(0) = c_1, y'(0) = c_2$ and the above becomes

$$\begin{aligned}
\left(s^2Y - sc_1 - c_2\right) - 2(sY - c_1) - 3Y &= 0 \\
Y\left(s^2 - 2s - 3\right) - sc_1 - c_2 + 2c_1 &= 0 \\
Y &= \frac{sc_1 + c_2 - 2c_1}{s^2 - 2s - 3}
\end{aligned}$$

Taking inverse Laplace gives

$$y(t) = \frac{1}{4} e^{-t}\left(c_2\left(e^{4t} - 1\right) + c_1\left(3 + e^{4t}\right)\right) \tag{1}$$

Hence

$$y'(t)=\frac{1}{4}e^{-t}(4c_1e^{-4t}+4c_2e^{4t})-\frac{1}{4}e^{-t}(c_2(-1+e^{4t})+c_1(3+e^{4t})) \tag{2}$$

At $t=4$ then (1,2) become

$$\begin{aligned}-3&=\frac{1}{4}e^{-4}\left(c_2\left(e^{16}-1\right)+c_1\left(3+e^{16}\right)\right)\\-17&=\frac{1}{4}e^{-4}(4c_1e^{-16}+4c_2e^{16})-\frac{1}{4}e^{-4}(c_2(-1+e^{16})+c_1(3+e^{16}))\end{aligned}$$

Solving the above for c_1,c_2 gives

$$\begin{aligned}c_1&=\frac{-5+2e^{16}}{e^{12}}\\c_2&=\frac{-15-2e^{16}}{e^{12}}\end{aligned}$$

Hence the solution (1) becomes

$$\begin{aligned}y(t)&=\frac{1}{4}e^{-t}\left(\frac{-15-2e^{16}}{e^{12}}\left(e^{4t}-1\right)+\frac{-5+2e^{16}}{e^{12}}\left(3+e^{4t}\right)\right)\\&=-e^{3t}\left(5e^{-12}-2e^{4}e^{-4t}\right)\\&=-5e^{3t-12}+2e^{4-t}\end{aligned}$$

2.3.1.3.3 Example 3

$$\begin{aligned}y''+2y'+5y&=50t-100\\y(2)&=-4\\y'(2)&=14\end{aligned}$$

Taking Laplace transform gives

$$\left(s^2Y-sy(0)-y'(0)\right)+2(sY-y(0))+5Y=\frac{50}{s^2}-\frac{100}{s}$$

Since given initial conditions are not at $t=0$, then let $y(0)=c_1, y'(0)=c_2$ and the above becomes

$$\begin{aligned}\left(s^2Y-sc_1-c_2\right)+2(sY-c_1)+5Y&=\frac{50}{s^2}-\frac{100}{s}\\Y\left(s^2+2s+5\right)-sc_1-c_2-2c_1&=\frac{50}{s^2}-\frac{100}{s}\\Y&=\frac{sc_1+c_2+2c_1+\frac{50}{s^2}-\frac{100}{s}}{s^2+2s+5}\end{aligned}$$

Taking inverse Laplace gives

$$y(t) = -24 + 10t + (24 + c_1)\, e^{-t} \cos(2t) + (14 + c_1 + c_2)\, e^{-t} \cos t \sin t \tag{1}$$

Hence

$$y'(t) = e^{-t}\big(10e^t + (c_2 - 10)\cos(2t) - (110 + 5c_1 + c_2)\cos t \sin t\big) \tag{2}$$

At $t = 2$ then (1,2) become

$$\begin{aligned} -4 &= -24 + 20 + (24 + c_1)\, e^{-2} \cos(4) + (14 + c_1 + c_2)\, e^{-2} \cos 2 \sin 2 \\ 14 &= e^{-2}\big(10e^2 + (c_2 - 10)\cos(4) - (110 + 5c_1 + c_2)\cos 2 \sin 2\big) \end{aligned}$$

Solving the above for c_1, c_2 gives

$$\begin{aligned} c_1 &= -2\big(12 + e^2 \sin 4\big) \\ c_2 &= 2\big(5 + e^2(2\cos 4 + \sin 4)\big) \end{aligned}$$

Hence the solution (1) becomes

$$y(t) = -24+10t+\big(24 - 2\big(12 + e^2 \sin 4\big)\big)\, e^{-t} \cos(2t)+\big(14 - 2\big(12 + e^2 \sin 4\big) + 2\big(5 + e^2(2\cos 4 + \sin 4)\big)\big)$$

Which simplifies to

$$y(t) = -24 + 10t - 2e^{2-t} \sin(4 - 2t)$$

2.3.1.3.4 Example 4

$$\begin{aligned} y'' + 2y' + 10y &= \delta(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Taking Laplace transform gives

$$\big(s^2 Y - s y(0) - y'(0)\big) + 2(sY - y(0)) + 10Y = 1$$

Since given initial conditions then the above becomes

$$\begin{aligned} s^2 Y + 2sY + 10Y &= 1 \\ Y &= \frac{1}{s^2 + 2s + 10} \\ &= \frac{1}{(s+2)(s+5)} \end{aligned}$$

Taking inverse Laplace transform gives

$$\begin{aligned}
y &= \frac{1}{6}ie^{(-1-3i)t} - \frac{1}{6}ie^{(-1+3i)t} \\
&= \frac{1}{6}ie^{-t}e^{-3it} - \frac{1}{6}ie^{-t}e^{3it} \\
&= \frac{1}{6}ie^{-t}\left(e^{-3it} - e^{3it}\right) \\
&= \frac{1}{6}ie^{-t}(\cos 3t - i\sin 3t - (\cos 3t + i\sin 3t)) \\
&= \frac{1}{6}ie^{-t}(-i\sin 3t - i\sin 3t) \\
&= \frac{1}{6}ie^{-t}(-2i\sin 3t) \\
&= \frac{1}{3}e^{-t}\sin 3t
\end{aligned}$$

Which is the same as

$$y = \left(\frac{1}{3}e^{-t}\sin(3t)\right)U(t)$$

Where $U(t)$ is Heaviside function which is one for $t>0$. Note that it seems one should not give IC at same point of application of $\delta(t)$ as in this problem. So this problem might be ill posed. Need to look more into this.

2.3.1.4 Solved using series method

2.3.1.4.1 Ordinary point using Taylor series method

ode internal name "second_order_taylor_series_method_ordinary_point"

This is the same as section below under non-constant coefficient.

2.3.1.4.2 Ordinary point using power series method

ode internal name "second_order_power_series_method_ordinary_point"

This is the same as section below under non-constant coefficient.

2.3.2 Linear ode with non-constant coefficients $A(x)\, y'' + B(x)\, y' + C(x)\, y = f(x)$

2.3.2.1 Euler ode $x^2y''+xy'+y=f(x)$

ode internal name "second order euler ode"

Solved by substitution $y=x^r$ and solving for r. Solution will be $y=c_1x^{r_1}+c_2x^{r_2}$ where r_1,r_2 are the roots of the characteristic equation. For repeated root, the second solution is multiplied by extra $\ln(x)$ and not extra x as is the case with standard constant coefficient ode. The particular solution is found in the same way using variation of parameters. Can not use undetermined coefficient method since this is not constant coefficients ode. The basis functions here are x^{r_1},x^{r_2} if not repeated roots, else the basis are $x^{r_1},\ln(x)\,x^{r_2}$.

2.3.2.2 Kovacic type

ode internal name "kovacic"

These are ode that are solvable using Kovacic algorithm. See my paper on arxiv on this with algorithm description.

2.3.2.3 Method of conversion to first order Riccati

ode internal name This is currently not implemented.

Given linear second order ode $A(x)\,y''+B(x)\,y'+C(x)\,y=0$ then using the transformation $v(x)=-\frac{y'}{y}$ converts the second order ode to a first order Riccati

$$
\begin{aligned}
v' &= \frac{-yy''+(y')^2}{y^2}\\
&= \frac{-y\left(-\frac{B}{A}y'-\frac{C}{A}y\right)+(y')^2}{y^2}\\
&= \frac{\frac{B}{A}yy'+\frac{C}{A}y^2+(y')^2}{y^2}\\
&= \frac{B}{A}\frac{y'}{y}+\frac{C}{A}+\frac{(y')^2}{y^2}\\
&= \frac{C}{A}+\frac{B}{A}v+v^2
\end{aligned}
$$

Which is Riccati of the form $v'=f_0(x)+f_1(x)\,v+f_2v^2$. where $f_0=\frac{C}{A},f_1=\frac{B}{A},f_2=1$. Lets say we can now find the solution to this Riccati $v(x)$ (see section earlier on Riccati for algorithm). Then the solution to the second order ode is found from $y'=-yv$ by solving this first order ode. The solution is

$$
y=e^{-\int v(x)dx}+c_2
$$

Notice there is also a second constant of integration inside $v(x)$. This method of course works only if we can solve the generated Riccati ode which does not have a general method for solving and only for specific cases it can be solved. So this will be tried as last resort.

We want to look for reduced Riccati generated from the above, which is $v' = f_0 + f_2 v^2$. Which means $f_1 = 0$ or $B = 0$ in the hope of solving the Riccati. This means ode of the form $A(x)\, y'' + C(x)\, y = 0$ will have hope of solving using this Riccati conversion method. See Riccati section why that is.

2.3.2.4 Airy ode $y'' \pm k^2 xy = f(x)$

ode internal name "second order airy"

Full solution now implemented.

2.3.2.5 Solved using series method

function SOLVE_SECOND_ORDER_ODE_SERIES($y''=f(x,y,y')$)
if $f(x,y,y')$ analytic at expansion point x_0 **then**
This means x_0 is an ordinary point. Apply Taylor series defintion directly to find the series expansion. Let $y_0=y(x_0), y'(x_0)=y'_0$ and

$$y=y_0+y'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n(x,y)\bigg|_{\substack{x_0\\y_0\\y'_0}}$$

Where

$$\begin{aligned}F_0&=f(x,y,y')\\F_n&=\frac{d}{dx}(F_{n-1})\\&=\frac{\partial}{\partial x}F_{n-1}+\left(\frac{\partial F_{n-1}}{\partial y}\right)y'+\left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\&=\frac{\partial}{\partial x}F_{n-1}+\left(\frac{\partial F_{n-1}}{\partial y}\right)y'+\left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0\end{aligned}$$

return y as the solution
else
if $f(x,y,y')$ not linear in $y(x)$ or not linear in $y'(x)$ **then**
return Not supported.
else
if expansion point x_0 is not regular singular point **then**
return Not supported.
else
This is a regular singular point. Determine the roots of the indicial equation. Let roots be r_1,r_2.
if Roots r_1,r_2 are complex (they will conjugate) **then**
Example is Euler ode $x^2y''+xy+y=0$
Use Frobenius series as is for each basis solution y_1,y_2 where

$$\begin{aligned}y_1&=\sum_{n=0}^{\infty}a_nx^{n+r_1}\\y_2&=\sum_{n=0}^{\infty}b_nx^{n+r_2}\end{aligned}$$

Where a_n,b_n above are found from the recurrence relation using each r_i root.
else if Roots r_1,r_2 differ by non-integer **then** ▷ Ex. $2x^2y''+3xy-xy=0$
Use Frobenius series as is for each basis solution y_1,y_2 as above case.
else if Roots r_1,r_2 are repeated. This means one root r, a double root **then**
An example ode is $x^2y''+xy'+xy=0$
y_1 is found use Frobenius series as above. For y_2 a modification is needed. Let $y_2=y_1\ln(x)+\sum_{n=1}^{\infty}b_nx^{n+r}$ where $b_n=\frac{d}{dr}a_n(r)$ after finding $a_n(r)$ evaluated at the root.
else if Roots r_1,r_2 differ by an integer **then**
if Both roots r_1,r_2 are good **then** ▷ Ex. $(x-x^2)y''+3y'+3y=0$
Called the lucky case. This means the recurrence equation and all a_n are defined for all n for both r_1,r_2. In this case both solutions y_1,y_2 are found using standard Frobenius series and no modification is needed.

Figure 2.2: Series method for second order ode algorithm

Ordinary point and regular singular point are supported. irregular singular point support will be added in the future. Expansion around point other than zero is also supported, including initial conditions. All three cases of regular point are supported, these are when the roots on indicial equation are repeated, or differ by an integer, or differ by non integer. case of Complex roots of indicial equation is also supported. Only second order and first order series solution is supported. Higher order ode support will be added in the future.

2.3.2.5.1 Ordinary point using Taylor series method

ode internal name "second_order_taylor_series_method_ordinary_point"

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \qquad (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \qquad (2.1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (2.2)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \qquad (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \qquad (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$.

2.3.2.5.2 Ordinary point using power series method

ode internal name "second_order_power_series_method_ordinary_point"

Expansion point is an ordinary point. Using standard power series. For an ordinary point, and for inhomogeneous. ode, always generate the full solution directly from the summation. Do not split the problem into y_h, y_p. To be able to do this, we have to express the RHS as Taylor series (expand it around the same expansion point). If the RHS is already a polynomial in x then there is nothing to do as it is already in Taylor series form. Examples below show how to do this. When the RHS is not zero, do not attempt to find recurrence relation as the RHS will get in the way, If the RHS is zero, then find recurrence relation.

$$y'' = f(x,y,y')$$

In this method, we let Let $y = \sum_{n=0}^{\infty} a_n x^n$ and replace this in the above ode and solve for a_n using recurrence relation. Examples below show how these methods work.

2.3.2.5.2.1 Example 1

Solved using Taylor series method.

$$\begin{aligned}
y'' + xy' + y &= 2x + x^2 + x^4 \\
y'' &= -xy' - y + 2x + x^2 + x^4 \\
y'' &= f(x,y,y')
\end{aligned}$$

Hence

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+1)!} F_n|_{x_0,y_0,y'_0}$$

Where

$$\begin{aligned}
F_0 &= f(x,y,y') \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0
\end{aligned}$$

Hence

$$
\begin{aligned}
F_1 &= \frac{\partial(-xy'-y+2x+x^2+x^4)}{\partial x} + \frac{\partial(-xy'-y+2x+x^2+x^4)}{\partial y}y' + \frac{\partial(-xy'-y+2x+x^2+x^4)}{\partial y'}y'' \\
&= \left(4x^3+2x-y'+2\right)-y'-xy'' \\
&= 2x-2y'-xy''+4x^3+2
\end{aligned}
$$

But $y''=f(x,y,y')$, the above becomes

$$
F_1 = 2x-2y'+x^2y'+xy-2x^2+3x^3-x^5+2
$$

And

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \\
&= \frac{\partial}{\partial x}\left(2x-2y'+x^2y'+xy-2x^2+3x^3-x^5+2\right)+ \\
&\quad + \left(\frac{\partial}{\partial y}\left(2x-2y'+x^2y'+xy-2x^2+3x^3-x^5+2\right)\right)y' \\
&\quad + \left(\frac{\partial}{\partial y'}\left(2x-2y'+x^2y'+xy-2x^2+3x^3-x^5+2\right)\right)y'' \\
&= \left(y-4x+2xy'+9x^2-5x^4+2\right)+xy'+\left(-2+x^2\right)y'' \\
&= y-4x-2y''+3xy'+x^2y''+9x^2-5x^4+2
\end{aligned}
$$

But $y''=f(x,y,y')$, the above becomes

$$
\begin{aligned}
F_2 &= y-4x-2\left(-xy'-y+2x+x^2+x^4\right)+3xy'+x^2\left(-xy'-y+2x+x^2+x^4\right)+9x^2-5x^4+2 \\
&= 3y-8x+5xy'-x^2y-x^3y'+7x^2+2x^3-6x^4+x^6+2
\end{aligned}
$$

And

$$
\begin{aligned}
F_3 &= \frac{d}{dx}(F_2) \\
&= \frac{\partial}{\partial x}F_2 + \left(\frac{\partial F_2}{\partial y}\right)y' + \left(\frac{\partial F_2}{\partial y'}\right)y'' \\
&= \frac{\partial}{\partial x}\left(3y-8x+5xy'-x^2y-x^3y'+7x^2+2x^3-6x^4+x^6+2\right) \\
&\quad + \left(\frac{\partial}{\partial y}\left(3y-8x+5xy'-x^2y-x^3y'+7x^2+2x^3-6x^4+x^6+2\right)\right)y' \\
&\quad + \left(\frac{\partial}{\partial y'}\left(3y-8x+5xy'-x^2y-x^3y'+7x^2+2x^3-6x^4+x^6+2\right)\right)y'' \\
&= 14x+5y'-3x^2y'-2xy+6x^2-24x^3+6x^5-8+\left(3-x^2\right)y'+\left(5x-x^3\right)y''
\end{aligned}
$$

But $y''=f(x,y,y')$, the above becomes

$$
\begin{aligned}
F_3 &= 14x+5y'-3x^2y'-2xy+6x^2-24x^3+6x^5-8+\left(3-x^2\right)y'+\left(5x-x^3\right)\left(-xy'-y+2x+x^2+
\right.\\
&= 14x+8y'+x^3y-9x^2y'+x^4y'-7xy+16x^2-19x^3-2x^4+10x^5-x^7-8
\end{aligned}
$$

And so on. Evaluating each of the above at $x=0, y=y_0, y'=y'_0$ gives

$$
\begin{aligned}
F_0 &= \left(-xy'-y+2x+x^2+x^4\right)_{x=0,y_0,y'_0} = -y_0\\
F_1 &= \left(2x-2y'+x^2y'+xy-2x^2+3x^3-x^5+2\right)_{x=0,y_0,y'_0} = \left(-2y'_0+2\right)\\
F_2 &= 3y-8x+5xy'-x^2y-x^3y'+7x^2+2x^3-6x^4+x^6+2 = 3y_0+2\\
F_3 &= 14x+8y'+x^3y-9x^2y'+x^4y'-7xy+16x^2-19x^3-2x^4+10x^5-x^7-8 = 8y'_0-8
\end{aligned}
$$

Hence

$$
\begin{aligned}
y(x) &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0}\\
&= y_0+xy'_0+\frac{x^2}{2}F_0+\frac{x^3}{6}F_1+\frac{x^4}{24}F_2+\frac{x^5}{5!}F_3+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}(-y_0)+\frac{x^3}{6}(-2y'_0+2)+\frac{x^4}{24}(3y_0+2)+\frac{x^5}{5!}(8y'_0-8)+\cdots\\
&= y_0\left(1-\frac{x^2}{2}+\frac{1}{8}x^4+\cdots\right)+y'_0\left(x-\frac{x^3}{3}+\frac{1}{15}x^4\cdots\right)+\left(\frac{1}{3}x^3+\frac{1}{12}x^4-\frac{1}{15}x^4\right)\\
&= c_1\left(1-\frac{x^2}{2}+\frac{1}{8}x^4+\cdots\right)+c_2\left(x-\frac{x^3}{3}+\frac{1}{15}x^4\cdots\right)+\left(\frac{1}{3}x^3+\frac{1}{12}x^4-\frac{1}{15}x^4\right)
\end{aligned}
$$

Solved using power series method.

$$
y''+xy'+y=2x+x^2+x^4
$$

Comparing the homogenous ode to $y''+p(x)\,y'+q(x)\,y=0$ shows that $p(x)=x, q(x)=1$. These are defined everywhere. Let the expansion point be $x_0=0$. This is ordinary point since $p(x)\,,q(x)$ are defined at x_0. Let $y=\sum_{n=0}^{\infty}a_nx^n$. Hence $y'=\sum_{n=0}^{\infty}na_nx^{n-1}=\sum_{n=1}^{\infty}na_nx^{n-1}$ and $y''=\sum_{n=1}^{\infty}(n)\,(n-1)\,a_nx^{n-2}=\sum_{n=2}^{\infty}(n)\,(n-1)\,a_nx^{n-2}$. The homogenous ode becomes

$$
\sum_{n=2}^{\infty}(n)\,(n-1)\,a_nx^{n-2}+x\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n=2x+x^2+x^4
$$

$$
\sum_{n=2}^{\infty}(n)\,(n-1)\,a_nx^{n-2}+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=2x+x^2+x^4
$$

Adjust all sums to lowest power on x gives

$$\sum_{n=2}^{\infty}(n)(n-1)a_n x^{n-2}+\sum_{n=3}^{\infty}(n-2)a_{n-2}x^{n-2}+\sum_{n=2}^{\infty}a_{n-2}x^{n-2}=2x+x^2+x^4$$

$n=2$ gives x^0 on the LHS with no match on the RHS. Hence

$$\begin{aligned}2a_2+a_0&=0\\ a_2&=-\frac{1}{2}a_0\end{aligned}$$

$n=3$ gives x^1 on the LHS with one match on the RHS. Hence

$$\begin{aligned}6a_3+2a_1&=2\\ a_3&=\frac{2-2a_1}{6}\\ &=\frac{1}{3}-\frac{1}{3}a_1\end{aligned}$$

$n=4$ gives x^2 on the LHS with one match on the RHS. Hence

$$\begin{aligned}12a_4+3a_2&=1\\ a_4&=\frac{1-3a_2}{12}\\ &=\frac{1-3\left(-\frac{1}{2}a_0\right)}{12}\\ &=\frac{1}{8}a_0+\frac{1}{12}\end{aligned}$$

$n=5$ gives x^3 on the LHS with no match on the RHS. Hence

$$\begin{aligned}20a_5+4a_3&=0\\ a_5&=\frac{-4a_3}{20}\\ &=\frac{-4\left(\frac{1}{3}-\frac{1}{3}a_1\right)}{20}\\ &=\frac{1}{15}a_1-\frac{1}{15}\end{aligned}$$

$n=6$ gives x^4 on the LHS with one match on the RHS. Hence

$$\begin{aligned}30a_6+5a_4&=1\\ a_6&=\frac{1-5a_4}{30}\\ &=\frac{1-5\left(\frac{1}{8}a_0+\frac{1}{12}\right)}{30}\\ &=\frac{7}{360}-\frac{1}{48}a_0\end{aligned}$$

And for $n \geq 7$ we have recurrence relation

$$
\begin{aligned}
(n)\,(n-1)\,a_n + (n-2)\,a_{n-2} + a_{n-2} = 0 \\
a_n = -\frac{n-1}{n\,(n-1)}a_{n-2}
\end{aligned}
$$

Hence for $n=7$

$$
\begin{aligned}
a_7 &= -\frac{6}{42}a_5 \\
&= -\frac{6}{42}\left(\frac{1}{15}a_1 - \frac{1}{15}\right) \\
&= \frac{1}{105} - \frac{1}{105}a_1
\end{aligned}
$$

For $n=8$

$$
\begin{aligned}
a_8 &= -\frac{7}{(8)\,(7)}a_6 \\
&= -\frac{7}{(8)\,(7)}\left(\frac{7}{360} - \frac{1}{48}a_0\right) \\
&= \frac{1}{384}a_0 - \frac{7}{2880}
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= a_0 + a_1 x - \frac{1}{2}a_0 x^2 + \left(\frac{1}{3} - \frac{1}{3}a_1\right)x^3 + \left(\frac{1}{8}a_0 + \frac{1}{12}\right)x^4 + \left(\frac{1}{15}a_1 - \frac{1}{15}\right)x^5 + \left(\frac{7}{360} - \frac{1}{48}a_0\right)x^6 + \left(\frac{}{1} \right. \\
&= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \cdots\right) + a_1\left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7 - \cdots\right) + \left(\frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{1}{15}x^5 + \right.
\end{aligned}
$$

Which is the same answer given using the Taylor series method. We see that the Taylor series method is much simpler, but requires using the computer to calculate the derivatives as they become very complicated as more terms are needed.

Even though the expansion point is ordinary, we can also solve this using Frobenius series as follows. Comparing the ode $y'' + xy' + y = 0$ to

$$
y'' + p(x)\,y' + q(x)\,y = 0
$$

Hence $p(x)=x, q(x)=1$. Therefore $p_0=\lim_{x\to 0} xp(x)=\lim_{x\to 0} x^2=0$ and $q_0=\lim_{x\to 0} x^2 q(x)=\lim_{x\to 0} x^2=0$. Hence the indicial equation is

$$
\begin{aligned}
r(r-1)+p_0 r+q_0 &= 0\\
r(r-1) &= 0\\
r &= 1,0
\end{aligned}
$$

Hence $r_1=1, r_2=0$. All ordinary points will have the same roots. Let

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r}\\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0\\
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0
\end{aligned}
$$

Reindex to lowest powers gives

$$
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + \sum_{n=2}^{\infty} (n+r-2)\, a_{n-2} x^{n+r-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-2} = 0 \quad (1)
$$

For $n=0$

$$
r(r-1)\, a_0 x^{r-2} = 0
$$

The homogenous ode therefore satisfies

$$
y'' + xy' + y = r(r-1)\, a_0 x^r \tag{2}
$$

For $n=1$, Eq (1) gives

$$
(1+r)\,(r)\, a_1 = 0
$$

For $r=1$ we see that $a_1=0$. But for $r=0$ then the above gives $0b_1=0$. This means b_1 can be any value and we choose $b_1=0$ in this case.

For $n \geq 2$ we obtain the recurrence relation

$$(n+r)(n+r-1)a_n + (n+r-2)a_{n-2} + a_{n-2} = 0$$

$$a_n = \frac{-(n+r-2)a_{n-2} - a_{n-2}}{(n+r)(n+r-1)} = \frac{-(n+r-1)a_{n-2}}{(n+r)(n+r-1)} \tag{3}$$

Now we find y_1 which is associated with $r=1$. From (3) and for $r=1$ it becomes

$$a_n = -\frac{n}{(n+1)n}a_{n-2} = -\frac{1}{n+1}a_{n-2} \tag{4}$$

For $n=2$ and using $a_0=1$

$$a_2 = -\frac{1}{3}a_0 = -\frac{1}{3}$$

For $n=3$

$$a_3 = -\frac{1}{4}a_1 = 0$$

All odd a_n will be zero. For $n=4$

$$a_4 = -\frac{1}{5}a_2 = -\frac{1}{5}\left(-\frac{1}{3}\right) = \frac{1}{15}$$

And so on. Hence

$$\begin{aligned} y_1 &= \sum a_n x^{n+r_1} \\ &= x\sum a_n x^n \\ &= x\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \\ &= x\left(1 - \frac{1}{2}x^2 + \frac{1}{10}x^4 - \cdots\right) \\ &= x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \cdots \end{aligned}$$

Now we find y_2 associated with $r=0$. From (3) this becomes (using b instead of a) and $r=0$

$$\begin{aligned} b_n &= \frac{-(n+r-1)b_{n-2}}{(n+r)(n+r-1)} \\ &= \frac{-(n-1)b_{n-2}}{(n)(n-1)} \\ &= -\frac{b_{n-2}}{n} \end{aligned} \tag{5}$$

From above, we found that $b_1 = 0$. Now we use (5) to find all b_n for $n \geq 2$. For $n = 2$

$$b_2 = -\frac{b_0}{2} = -\frac{1}{2}$$

For $n = 3$

$$b_3 = -\frac{b_1}{3} = 0$$

For $n = 4$

$$b_4 = -\frac{b_2}{4} = \frac{1}{8}$$

And so on. Hence

$$\begin{aligned} y_2 &= \sum b_n x^{n+r_2} \\ &= \sum b_n x^n \\ &= \left(b_0 + b_1 x + b_2 x^2 + \cdots\right) \\ &= \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots\right) \end{aligned}$$

Hence the solution y_h is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \cdots\right) + c_2\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots\right) \end{aligned}$$

We see this is the same y_h obtained using standard power series. This shows that we can also use Frobenius series to solve for ordinary point. The roots will always be $r_1 = 1, r_2 = 0$. But this requires more work than using standard power series. The main advantage of using Frobenius series for ordinary point comes in when the RHS has no series expansion at $x = 0$. For example, if the RHS in this ode was say $\sqrt{x}$ then we must use Frobenius to be able to solve it as standard power series will fail, since$\sqrt{x}$ has no series representation at $x = 0$. Examples below shows how to do this.

2.3.2.5.2.2 Example 2

$$\frac{1}{x^5}y'' + y' + y = 0$$

Solved using Taylor series method.

$$\begin{aligned} y'' &= -x^5(y' + y) \\ &= -x^5 y - x^5 y' \\ y'' &= f(x, y, y') \end{aligned}$$

Hence

$$y(x)=y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+1)!}F_n|_{x_0,y_0,y_0'}$$

Where

$$\begin{aligned}F_0&=f(x,y,y')\\F_n&=\frac{d}{dx}(F_{n-1})\\&=\frac{\partial}{\partial x}F_{n-1}+\left(\frac{\partial F_{n-1}}{\partial y}\right)y'+\left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\&=\frac{\partial}{\partial x}F_{n-1}+\left(\frac{\partial F_{n-1}}{\partial y}\right)y'+\left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0\end{aligned}$$

Hence

$$\begin{aligned}F_1&=\frac{\partial(-x^5y-x^5y')}{\partial x}+\frac{\partial(-x^5y-x^5y')}{\partial y}y'+\frac{\partial(-x^5y-x^5y')}{\partial y'}y''\\&=(-5x^4y-5x^4y')-x^5y'-x^5y''\end{aligned}$$

But $y''=f(x,y,y')$, the above becomes

$$\begin{aligned}F_1&=(-5x^4y-5x^4y')-x^5y'-x^5(-x^5y-x^5y')\\&=x^{10}y-5x^4y-5x^4y'-x^5y'+x^{10}y'\end{aligned}$$

And

$$\begin{aligned}F_2&=\frac{d}{dx}(F_{n-1})\\&=\frac{\partial}{\partial x}F_1+\left(\frac{\partial F_1}{\partial y}\right)y'+\left(\frac{\partial F_1}{\partial y'}\right)y''\\&=\frac{\partial}{\partial x}(x^{10}y-5x^4y-5x^4y'-x^5y'+x^{10}y')+\\&+\left(\frac{\partial}{\partial y}(x^{10}y-5x^4y-5x^4y'-x^5y'+x^{10}y')\right)y'\\&+\left(\frac{\partial}{\partial y'}(x^{10}y-5x^4y-5x^4y'-x^5y'+x^{10}y')\right)y''\\&=(10x^9y-20x^3y-20x^3y'-5x^4y'+10x^9y')+x^4(x^6-5)y'+(-5x^4-x^5+x^{10})y''\end{aligned}$$

But $y''=f(x,y,y')$, the above becomes

$$\begin{aligned}F_2&=(10x^9y-20x^3y-20x^3y'-5x^4y'+10x^9y')+x^4(x^6-5)y'+(-5x^4-x^5+x^{10})(-x^5(y'+y))\\&=-x^3(20y+20y'+10xy'-15x^6y-x^7y+x^{12}y-15x^6y'-2x^7y'+x^{12}y')\end{aligned}$$

And

$$
\begin{aligned}
F_3 &= \frac{d}{dx}(F_2)\\
&= \frac{\partial}{\partial x}F_2 + \left(\frac{\partial F_2}{\partial y}\right)y' + \left(\frac{\partial F_2}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}\left(-x^3\left(20y+20y'+10xy'-15x^6y-x^7y+x^{12}y-15x^6y'-2x^7y'+x^{12}y'\right)\right)\\
&+\left(\frac{\partial}{\partial y}\left(-x^3\left(20y+20y'+10xy'-15x^6y-x^7y+x^{12}y-15x^6y'-2x^7y'+x^{12}y'\right)\right)\right)y'\\
&+\left(\frac{\partial}{\partial y'}\left(-x^3\left(20y+20y'+10xy'-15x^6y-x^7y+x^{12}y-15x^6y'-2x^7y'+x^{12}y'\right)\right)\right)y''\\
&= -5x^2\left(12y+12y'+8xy'-27x^6y-2x^7y+3x^{12}y-27x^6y'-4x^7y'+3x^{12}y'\right)+x^3\left(-x^{12}+x^7+15x
\end{aligned}
$$

But $y''=f(x,y,y')$, the above becomes

$$
\begin{aligned}
F_3 &= -5x^2\left(12y+12y'+8xy'-27x^6y-2x^7y+3x^{12}y-27x^6y'-4x^7y'+3x^{12}y'\right)+x^3\left(-x^{12}+x^7+15x\right.\\
&= -x^2\left(60y+60y'+60xy'-155x^6y-20x^7y+30x^{12}y+2x^{13}y-x^{18}y-155x^6y'-45x^7y'-x^8y'+30\right.
\end{aligned}
$$

And so on. Since the derivatives become very complicated, the result was done on the computer which results in (Evaluating each of the above at $x=0, y=y_0, y'=y'_0$)

$$
\begin{aligned}
F_0 &= 0\\
F_1 &= 0\\
F_2 &= 0\\
F_3 &= 0\\
F_4 &= 0\\
F_5 &= -120y'_0-120y_0\\
F_6 &= -720y'_0\\
F_7 &= 0\\
F_8 &= 0\\
F_9 &= 0\\
F_{10} &= 0\\
F_{11} &= 6652800y'_0+6652800y_0\\
F_{12} &= 79833600y'_0+11404800y_0\\
F_{13} &= 111196800y'_0\\
F_{14} &= 0\\
&\vdots
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y(x) &= y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \\
&= y_0 + xy_0' + \frac{x^7}{7!}(-120y_0' - 120y_0) - \frac{x^8}{8!}(720y_0') + \frac{x^{13}}{13!}(6652800y_0' + 6652800y_0) \\
&+ \frac{x^{14}}{14!}(79833600y_0' + 11404800y_0) + \frac{x^{15}}{15!}(111196800y_0') + \cdots \\
&= y_0\left(1 - \frac{120}{7!}x^7 + \frac{6652800}{13!}x^{13} + \frac{11404800}{14!}x^{14} - \cdots\right) + y_0'\left(x - \frac{120}{7!}x^7 - \frac{720}{8!}x^8 + \frac{6652800}{13!}x^{13} + \right. \\
&= y_0\left(1 - \frac{1}{42}x^7 + \frac{1}{936}x^{13} + \frac{1}{7644}x^{14} + \cdots\right) + y_0'\left(x - \frac{1}{42}x^7 - \frac{1}{56}x^8 + \frac{1}{936}x^{13} + \frac{1}{1092}x^{14} + \frac{1}{11\,760}\right.
\end{aligned}
$$

Solved using power series method

Expansion around $x=0$. This is ordinary point. Since RHS is zero, we will find recurrence relation.

Let $y=\sum_{n=0}^{\infty} a_n x^n$. Hence $y'=\sum_{n=0}^{\infty} n a_n x^{n-1}=\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''=\sum_{n=1}^{\infty}(n)(n-1)a_n x^{n-2}=\sum_{n=2}^{\infty}(n)(n-1)a_n x^{n-2}$. The ode becomes

$$x^{-5}y''+y'+y=0$$

Hence

$$
\begin{aligned}
\sum_{n=2}^{\infty}(n)(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0 \\
\sum_{n=2}^{\infty}(n)(n-1)a_n x^{n-7} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0
\end{aligned}
$$

Reindex so all powers start at lowest powers $n-7$

$$\sum_{n=2}^{\infty}(n)(n-1)a_n x^{n-7} + \sum_{n=7}^{\infty}(n-6)a_{n-6}x^{n-7} + \sum_{n=7}^{\infty} a_{n-7}x^{n-7} = 0 \tag{1}$$

For $n=2,3,4,5,6$ it generates $a_2=0, a_3=0, a_4=0, a_5=0, a_6=0$ since there is only one term in each one of these and the RHS is zero.

For $n\geq 7$ we have the recurrence relation

$$(n)(n-1)a_n + (n-6)a_{n-6} + a_{n-7} = 0 \tag{2}$$

$$a_n = -\frac{(n-6)a_{n-6}+a_{n-7}}{(n+2)(n+1)}$$

Hence for $n=7$

$$a_7 = -\frac{a_1+a_0}{42}$$

For $n=8$

$$a_8 = -\frac{2a_2+a_1}{(6+2)(6+1)} = \frac{-a_1}{56}$$

For $n=9$

$$a_9 = -\frac{(7-4)a_3+a_2}{(7+2)(7+1)} = 0$$

For $n=10$

$$a_{10} = -\frac{(8-4)a_4+a_3}{(8+2)(8+1)} = 0$$

For $n=11$

$$a_{11} = -\frac{(9-4)a_5+a_4}{(9+2)(9+1)} = 0$$

For $n=12$

$$a_{12} = -\frac{(n-4)a_6+a_5}{(n+2)(n+1)} = 0$$

For $n=13$

$$a_{13} = -\frac{(11-4)a_7+a_6}{(11+2)(11+1)} = -\frac{(11-4)a_7}{(11+2)(11+1)} = -\frac{7}{156}a_7 = -\frac{7}{156}\left(-\frac{a_1+a_0}{42}\right) = \frac{1}{936}a_0+\frac{1}{936}a_1$$

And so on. Hence

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_7 x^7 + a_{13} x^{13} + \cdots \end{aligned}$$

Notice that all terms $a_n = 0$ for $n = 2\cdots 6$. The above becomes

$$\begin{aligned} y &= a_0 + a_1 x + \left(-\frac{1}{42}a_0 - \frac{1}{42}a_1\right)x^7 + \left(\frac{1}{936}a_0 + \frac{1}{936}a_1\right)x^{13} + \cdots \\ &= a_0\left(1 - \frac{1}{42}x^7 + \frac{1}{936}x^{13} + \cdots\right) + a_1\left(x - \frac{1}{42}x^7 + \frac{1}{936}x^{13} + \cdots\right) \end{aligned}$$

2.3.2.5.2.3 Example 3

$$\frac{1}{x^2}y''+y'+y=\sin x$$

Expansion around $x=0$. This is ordinary point. Since RHS is not zero, do not find recurrence relation. Let $y=\sum_{n=0}^{\infty}a_nx^n$. Hence $y'=\sum_{n=0}^{\infty}na_nx^{n-1}=\sum_{n=1}^{\infty}na_nx^{n-1}$ and $y''=\sum_{n=1}^{\infty}(n)(n-1)a_nx^{n-2}=\sum_{n=2}^{\infty}(n)(n-1)a_nx^{n-2}$. The ode becomes

$$y''+x^2y'+x^2y=x^2\sin x$$

Hence

$$\sum_{n=2}^{\infty}(n)(n-1)a_nx^{n-2}+x^2\sum_{n=1}^{\infty}na_nx^{n-1}+x^2\sum_{n=0}^{\infty}a_nx^n=x^2\sin x$$
$$\sum_{n=2}^{\infty}(n)(n-1)a_nx^{n-2}+\sum_{n=1}^{\infty}na_nx^{n+1}+\sum_{n=0}^{\infty}a_nx^{n+2}=x^2\sin x$$

Reindex so all powers to start from n. This results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=x^2\sin x$$

To be able to continue, we have to expand $\sin x$ as Taylor series around x. The above becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=x^2\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\cdots\right)$$
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=x^3-\frac{1}{6}x^5+\frac{1}{120}x^7-\frac{1}{5040}x^9+\cdots$$

For $n=0$

$$2a_2=0$$
$$a_2=0$$

For $n=1$

$$(3)(2)a_3=0$$
$$a_3=0$$

For $n=2$

$$(2+2)(2+1)a_4+(2-1)a_1+a_0=0$$
$$12a_4+a_1+a_0=0$$
$$a_4=\frac{-a_1-a_0}{12}$$

For $n=3$ (now we pick one term from the RHS which match on x^3)

$$\begin{aligned}20a_5+2a_2+a_1&=1\\a_5&=\frac{1-a_1}{20}\end{aligned}$$

For $n=4$

$$\begin{aligned}30a_6+3a_3+a_2&=0\\a_6&=0\end{aligned}$$

For $n=5$

$$\begin{aligned}42a_7+4a_4+a_3&=-\frac{1}{6}\\a_7&=\frac{-\frac{1}{6}-4a_4}{42}=\frac{-\frac{1}{6}-4\left(\frac{-a_1-a_0}{12}\right)}{42}=\frac{1}{126}a_0+\frac{1}{126}a_1-\frac{1}{252}\end{aligned}$$

And so on. Hence

$$\begin{aligned}y&=\sum_{n=0}^{\infty}a_nx^n\\&=a_0+a_1x+a_2x^2+a_3x^3+\cdots\\&=a_0+a_1x+\left(\frac{-a_1-a_0}{12}\right)x^4+\left(\frac{1-a_1}{20}\right)x^5+\left(\frac{1}{126}a_0+\frac{1}{126}a_1-\frac{1}{252}\right)x^7+\cdots\\&=a_0\left(1-\frac{1}{12}x^4+\frac{1}{126}x^7+\cdots\right)+a_1\left(x-\frac{1}{12}x^4-\frac{1}{20}x^5+\frac{1}{126}x^7+\cdots\right)+\left(\frac{1}{20}x^5-\frac{1}{252}x^7+\cdots\right)\end{aligned}$$

2.3.2.5.3 Regular singular point using Frobenius series method.

expansion point is regular singular point. Four sub methods depending on type of roots of the indicial equations.

2.3.2.5.3.1 Roots of indicial equation are complex

ode internal name "second_order_series_method_regular_singular_point_complex_roots"

In this case the solution is

$$y=c_1y_1+c_2y_2$$

Where

$$\begin{aligned}y_1&=\sum_{n=0}^{\infty}a_nx^{n+r_1}\\y_2&=\sum_{n=0}^{\infty}b_nx^{n+r_2}\end{aligned}$$

Where r_1, r_2 are roots of the indicial equation. a_0, b_0 are set to 1 as arbitrary.

Example 1

$$x^2y'' + xy' + y = 1$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{1}{x}, q(x) = \frac{1}{x^2}$. There is one singular point at $x_0 = 0$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} 1 = 1$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) + r + 1 &= 0 \\ r^2 + 1 &= 0 \\ r &= \pm i \end{aligned}$$

Hence $r_1 = i, r_2 = -i$. Expansion around $x = 0$. This is regular singular point. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} \end{aligned}$$

Solving first for the homogenous ode.

$$\begin{aligned} x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \end{aligned}$$

For $n = 0$

$$(r(r-1) + r + 1)\,a_0 x^r = 0 \tag{1}$$

Since $a_0 \neq 0$, then $(r(r-1) + r + 1) = 0$ or $r^2 + 1 = 0$. Therefore $r = \pm i$ as was found above. The homogenous ode therefore satisfies

$$x^2y'' + xy' + y = \left(r^2 + 1\right) a_0 x^r$$

Since when $r = \pm i$, the RHS is zero. For $n \geq 1$ the recurrence relation is

$$\begin{aligned}(n+r)(n+r-1)a_n + (n+r)a_n + a_n &= 0\\((n+r)(n+r-1) + (n+r) + 1)a_n &= 0\\\left(n^2 + 2nr + r^2 + 1\right)a_n &= 0\end{aligned} \tag{2}$$

Let $a_0 = 1$. For $r = i$. For $n = 1$

$$(1 + 2i - 1 + 1)a_1 = 0$$

Hence $a_1 = 0$. Similarly all $a_n = 0$ for $n \geq 1$. Hence

$$\begin{aligned}y_1 &= \sum_{n=0}^{\infty} a_n x^{n+i}\\&= x^i(a_0 + a_1 x + \cdots)\\&= a_0 x^i\\&= x^i\end{aligned}$$

For $r = -i$. For $n = 1$ and using b instead of a, we obtain (also using $b_0 = 1$)

$$(1 - 2i + 1 + 1)b_n = 0$$

Hence $b_1 = 0$. Similarly all $b_n = 0$ for $n \geq 1$. Hence

$$\begin{aligned}y_2 &= \sum_{n=0}^{\infty} b_n x^{n-i}\\&= x^{-i}(b_0 + b_1 x + \cdots)\\&= b_0 x^{-i}\\&= x^{-i}\end{aligned}$$

Therefore

$$\begin{aligned}y_h &= c_1 y_1 + c_2 y_2\\&= c_1 x^i + c_2 x^{-i}\end{aligned}$$

To find y_p since the ode satisfies

$$x^2 y'' + xy' + y = \left(r^2 + 1\right)a_0 x^r$$

Relabel $r = m, a_0 = c_0$ to avoid confusion with terms used above, then we balance RHS, hence

$$\left(m^2 + 1\right)c_0 x^m = 1$$

This implies $m = 0$ and $c_0 = 1$. Therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the recurrence relation (2) found above, but now using the values found $m = 0$ and $c_0 = 1$, then (2) becomes

$$\begin{aligned} \left(n^2 + 2nm + m^2 + 1\right) c_n &= 0 \\ \left(n^2 + 1\right) c_n &= 0 \end{aligned}$$

Hence all $c_n = 0$ except for c_0. Therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 \\ &= 1 \end{aligned}$$

Hence the solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 x^i + c_2 x^{-i} + 1 \end{aligned}$$

2.3.2.5.3.2 Roots of indicial equation differ by non integer

ode internal name "second_order_series_method_regular_singular_point_difference_not_integer"

If one of the roots is an integer, and the ode is inhomogeneous. ode, then we do not need to split the solution into y_h, y_p and can use the integer root to find y_p directly. If both roots are non-integer, we have to split the problem into y_h, y_p. This is because it will not be possible to match powers on x from the left side to the right side. Because the RHS will be polynomial in x, but the LHS will not be polynomial in x because of the non integer powers on x.In this case the solution is

$$y = c_1 y_1 + c_2 y_2$$

Where

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\ y_2 &= \sum_{n=0}^{\infty} b_n x^{n+r_2} \end{aligned}$$

And r_1, r_2 are roots of the indicial equation. a_0, b_0 are set to 1 as arbitrary.

Example 1

$$2x^2y'' + 3xy' - xy = x^2 + 2x$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{3}{2x}, q(x) = \frac{-1}{2x}$. There is one singular point at $x = 0$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}$ and $q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} -\frac{x}{2} = 0$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0r + q_0 &= 0 \\ r(r-1) + \frac{3}{2}r + 0 &= 0 \\ r(2r+1) &= 0 \\ r &= 0, -\frac{1}{2} \end{aligned}$$

Therefore $r_1 = 0, r_2 = -\frac{1}{2}$.

Expansion around $x = x_0 = 0$. This is regular singular point. Hence Frobenius is needed. First we find y_h. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, hence

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} \end{aligned}$$

The ode becomes

$$\begin{aligned} 2x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} + 3x \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} - x \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\,a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)\,a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0 \end{aligned}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\,a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)\,a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

When $n=0$

$$\begin{aligned}2(r)(r-1)\,a_0x^r+3(r)\,a_0x^r&=0\\(r(2r+1))\,a_0x^r&=0\end{aligned}$$

Since $a_0\neq 0$ then $r(2r+1)=0$ and $r=0, r=-\frac{1}{2}$ as was found above. Therefore the homogenous ode satisfies

$$2x^2y''+3xy'-xy=(r(2r+1))\,a_0x^r$$

Where the RHS will be zero when $r=0$ or $r=-\frac{1}{2}$. For $n\geq 1$ the recurrence relation is

$$\begin{aligned}2(n+r)(n+r-1)\,a_n+3(n+r)\,a_n-a_{n-1}&=0\\a_n&=\frac{a_{n-1}}{2\,(n+r)\,(n+r-1)+3\,(n+r)}\\&=\frac{a_{n-1}}{2n^2+4nr+n+2r^2+r}\end{aligned}\tag{1}$$

For $r=0$ the above becomes

$$a_n=\frac{a_{n-1}}{2n^2+n}$$

For $n=1$ and letting $a_0=1$

$$a_1=\frac{1}{3}$$

For $n=2$

$$a_2=\frac{a_1}{8+2}=\frac{a_1}{10}=\frac{1}{30}$$

For $n=3$

$$a_3=\frac{a_2}{18+3}=\frac{a_2}{21}=\frac{1}{21\,(30)}=\frac{1}{630}$$

And so on. Hence

$$\begin{aligned}y_1&=\sum_{n=0}^{\infty}a_nx^{n+r}=\sum_{n=0}^{\infty}a_nx^n\\&=a_0+a_1x+a_2x^2+\cdots\\&=1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\cdots\end{aligned}$$

And for $r=-\frac{1}{2}$ the recurrence relation (2) becomes, and using b instead of a

$$b_n=\frac{b_{n-1}}{2n^2+4n\left(-\frac{1}{2}\right)+n+\frac{1}{2}-\frac{1}{2}}=-\frac{b_{n-1}}{n-2n^2}$$

For $n=1$ and using $b_0=1$

$$b_1=-\frac{b_0}{1-2}=1$$

For $n=2$

$$b_2 = -\frac{b_1}{2-8} = -\frac{1}{2-8} = \frac{1}{6}$$

For $n=3$

$$b_3 = -\frac{b_2}{3-18} = -\frac{\frac{1}{6}}{3-18} = \frac{1}{90}$$

And so on. Hence

$$\begin{aligned}
y_2 &= \sum_{n=0}^{\infty} b_n x^{n+r_2} \\
&= \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} b_n x^n \\
&= \frac{1}{\sqrt{x}} (b_0 + b_1 x + b_2 x^2 + \cdots) \\
&= \frac{1}{\sqrt{x}} \left(1 + x + \frac{1}{6}x^2 + \frac{1}{90}x^3 + \cdots\right)
\end{aligned}$$

Hence

$$\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(1 + \frac{1}{3}x + \frac{1}{30}x^2 + \frac{1}{630}x^3 + \cdots\right) + c_2 \frac{1}{\sqrt{x}} \left(1 + x + \frac{1}{6}x^2 + \frac{1}{90}x^3 + \cdots\right)
\end{aligned}$$

Now we find y_p. Since ode satisfies

$$2x^2y'' + 3xy' - xy = (r(2r+1))\, a_0 x^r$$

To find y_p, and relabeling r as m and a as c so not to confuse terms used for y_h. Then the above becomes

$$2x^2y'' + 3xy' - xy = (m(2m+1))\, c_0 x^m$$

The RHS is $x^2 + 2x$. We balance each term at a time, this finds a particular solution for each term on the RHS, then these particular solutions are added at the end. For the input $2x$ the balance equation is

$$(m(2m+1))\, c_0 x^m = 2x$$

This implies that

$$m = 1$$

Therefore $(m(2m+1))\, c_0 = 2$, or $c_0(1(2+1)) = 2$ or $3c_0 = 2$ or

$$c_0 = \frac{2}{3}$$

The recurrence relation now becomes (using m for r and c_0 for a_0)

$$c_n = \frac{c_{n-1}}{2n^2+4nm+n+2m^2+m}$$

For $m=1$ the above becomes

$$c_n = \frac{c_{n-1}}{2n^2+5n+3}$$

For $n=1$ and using $c_0=\frac{2}{3}$

$$c_1 = \frac{\frac{2}{3}}{2+5+3} = \frac{1}{15}$$

For $n=2$

$$c_2 = \frac{c_1}{8+10+3} = \frac{\frac{1}{15}}{8+10+3} = \frac{1}{315}$$

For $n=3$

$$c_3 = \frac{c_2}{18+15+3} = \frac{\frac{1}{315}}{18+15+3} = \frac{1}{11\,340}$$

And so on. Hence

$$\begin{aligned}
y_{p_1} &= \sum_{n=0}^{\infty} c_n x^{n+m} = x\sum_{n=0}^{\infty} c_n x^n \\
&= x\left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \\
&= x\left(\frac{2}{3} + \frac{1}{15}x + \frac{1}{315}x^2 + \frac{1}{11\,340}x^3 + \cdots\right) \\
&= \left(\frac{2}{3}x + \frac{1}{15}x^2 + \frac{1}{315}x^3 + \frac{1}{11\,340}x^4 + \cdots\right)
\end{aligned}$$

The second term x^2 is now balanced x^2. The balance equation is

$$(m(2m+1))\, c_0 x^m = x^2$$

Therefore $m=2$ and $(m(2m+1))\, c_0 = 1$. Hence

$$\begin{aligned}
(2(4+1))\, c_0 &= 1 \\
c_0 &= \frac{1}{10}
\end{aligned}$$

The recurrence relation becomes for $m=2$

$$c_n = \frac{c_{n-1}}{2n^2+4nm+n+2m^2+m}$$

For $m=2$ the above becomes

$$c_n = \frac{c_{n-1}}{2n^2+9n+10}$$

For $n=1$ and using $c_0=\frac{1}{10}$

$$c_1=\frac{\frac{1}{10}}{2+9+10}=\frac{1}{210}$$

For $n=2$

$$c_2=\frac{c_1}{8+18+10}=\frac{\frac{1}{210}}{8+18+10}=\frac{1}{7560}$$

For $n=3$

$$c_3=\frac{c_2}{18+27+10}=\frac{\frac{1}{7560}}{18+27+10}=\frac{1}{415\,800}$$

And so on. Hence

$$\begin{aligned}
y_{p_2}&=\sum_{n=0}^{\infty}c_nx^{n+m}=x^2\sum_{n=0}^{\infty}c_nx^n\\
&=x^2\left(c_0+c_1x+c_2x^2+\cdots\right)\\
&=x^2\left(\frac{1}{10}+\frac{1}{210}x+\frac{1}{7560}x^2+\frac{1}{415\,800}x^3+\cdots\right)\\
&=\left(\frac{1}{10}x^2+\frac{1}{210}x^3+\frac{1}{7560}x^4+\frac{1}{415\,800}x^5+\cdots\right)
\end{aligned}$$

The particular solution is the sum of all the particular solutions found above, which is

$$\begin{aligned}
y_p&=y_{p_1}+y_{p_2}\\
&=\left(\frac{2}{3}x+\frac{1}{15}x^2+\frac{1}{315}x^3+\frac{1}{11\,340}x^4+\cdots\right)+\left(\frac{1}{10}x^2+\frac{1}{210}x^3+\frac{1}{7560}x^4+\frac{1}{415\,800}x^5+\cdots\right)\\
&=\frac{2}{3}x+\left(\frac{1}{15}+\frac{1}{10}\right)x^2+\left(\frac{1}{315}+\frac{1}{210}\right)x^3+\left(\frac{1}{11\,340}+\frac{1}{7560}\right)x^4+\cdots\\
&=\frac{2}{3}x+\frac{1}{6}x^2+\frac{1}{126}x^3+\frac{1}{4536}x^4+\cdots
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
y&=y_h+y_p\\
&=c_1\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\cdots\right)+c_2\frac{1}{\sqrt{x}}\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\cdots\right)+\frac{2}{3}x+\frac{1}{6}x^2+\frac{1}{126}x^3+\frac{1}{45
\end{aligned}$$

Example 2

$$2xy''+(x+1)\,y'+3y=5$$

Comparing the ode to

$$y''+p(x)\,y'+q(x)\,y=0$$

Hence $p(x)=\frac{(x+1)}{2x}, q(x)=\frac{3}{2x}$. There is one singular point at $x=0$. Therefore $p_0=\lim_{x\to 0} xp(x)=\lim_{x\to 0}\frac{(x+1)}{2}=\frac{1}{2}$ and $q_0=\lim_{x\to 0} x^2q(x)=\lim_{x\to 0}\frac{3x}{2}=0$. Hence the indicial equation is

$$
\begin{aligned}
r(r-1)+p_0r+q_0&=0\\
r(r-1)+\frac{1}{2}r+0&=0\\
r(2r-1)&=0\\
r&=0,\frac{1}{2}
\end{aligned}
$$

Therefore $r_1=0, r_2=\frac{1}{2}$.

Expansion around $x=x_0=0$. This is regular singular point. Hence Frobenius is needed. Let

$$
\begin{aligned}
y&=\sum_{n=0}^{\infty} a_n x^{n+r}\\
y'&=\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\\
y''&=\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}
\end{aligned}
$$

The homogenous ode becomes

$$
2x\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}+(x+1)\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+3\sum_{n=0}^{\infty}a_n x^{n+r}=0
$$

$$
\sum_{n=0}^{\infty}2(n+r)\,(n+r-1)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}3a_n x^{n+r}=0
$$

Re indexing to lowest powers on x gives

$$
\sum_{n=0}^{\infty}2(n+r)\,(n+r-1)\,a_n x^{n+r-1}+\sum_{n=1}^{\infty}(n+r-1)\,a_{n-1}x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=1}^{\infty}3a_{n-1}x^{n+r-1}=
$$

For $n=0$

$$
\begin{aligned}
(2(r)\,(r-1)\,a_0+ra_0)\,x^{r-1}&=0\\
(2r(r-1)+r)\,a_0&=0
\end{aligned}
$$

Since $a_0 \neq 0$ then the first term above will vanish only when $2r(r-1)+r=0$ or $r(2r-1)=0$. Hence $r=0, r=\frac{1}{2}$ as was found above. For $n \geq 1$

$$2(n+r)(n+r-1)a_n + (n+r-1)a_{n-1} + (n+r)a_n + 3a_{n-1} = 0$$

$$a_n = -\frac{n+r+2}{(n+r)(2r+2n-1)}a_{n-1} \tag{1}$$

Therefore the differential equation satisfies

$$2xy'' + (x+1)y' + 3y = r(2r-1)a_0x^{r-1} \tag{2}$$

The RHS above will be zero when $r=0$ or $r=\frac{1}{2}$. When $r=0$ the recurrence relation (1) becomes

$$a_n = -\frac{n+2}{(n)(2n-1)}a_{n-1}$$

Which gives (for $a_0 = 1$) (working out few terms using the above)

$$y_1 = 1 - 3x + 2x^2 - \frac{2}{3}x^3 + \cdots$$

And when $r=\frac{1}{2}$ the recurrence relation is (using b in place of a and letting $b_0 = 1$ also)

$$b_n = -\frac{n+\frac{5}{2}}{\left(n+\frac{1}{2}\right)(1+2n-1)}b_{n-1}$$

Which gives (working out few terms)

$$y_2 = \sqrt{x}\left(1 - \frac{7x}{6} + 21\frac{x^2}{40} + \cdots\right)$$

Hence the solution is

$$\begin{aligned}
y_h &= c_1y_1 + c_2y_2 \\
&= c_1\left(1 - 3x + 2x^2 - \frac{2}{3}x^3 + \cdots\right) + c_2\left(\sqrt{x}\left(1 - \frac{7x}{6} + 21\frac{x^2}{40} + \cdots\right)\right)
\end{aligned}$$

Now we find y_p. From (2), and relabeling r as m and a as c so not to confuse terms used

$$2xy'' + (x+1)y' + 3y = m(2m-1)c_0x^{m-1}$$

Therefore we need to balance $m(2m-1)c_0x^{m-1} = 5$ since the RHS is 5. This implies $m-1=0$ or $m=1$. Therefore $m(2m-1)c_0 = 5$ or $(2-1)c_0 = 5$ which gives $c_0 = 5$.

Hence

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= x \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

To find c_n, the same recurrence relation (1) is used by with r replaced by m and a replaced by c. This gives

$$
c_n = -\frac{n+m+2}{(n+m)(2m+2n-1)} c_{n-1}
$$

For $m=1$ the above becomes

$$
c_n = -\frac{n+3}{(n+1)(1+2n)} c_{n-1}
$$

For $n=1$

$$
c_1 = -\frac{1+3}{(1+1)(1+2)} c_0 = -\frac{2}{3} c_0 = -\frac{2}{3}(5) = -\frac{10}{3}
$$

For $n=2$

$$
c_2 = -\frac{2+3}{(2+1)(1+4)} c_1 = -\frac{1}{3} c_1 = -\frac{1}{3}\left(-\frac{10}{3}\right) = \frac{10}{9}
$$

For $n=3$

$$
c_1 = -\frac{3+3}{(3+1)(1+6)} c_2 = -\frac{3}{14}\left(\frac{10}{9}\right) = -\frac{2}{3}(5) = -\frac{5}{21}
$$

And so on. Hence

$$
\begin{aligned}
y_p &= x \sum_{n=0}^{\infty} c_n x^n \\
&= x\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right) \\
&= x\left(5 - \frac{10}{3} x + \frac{10}{9} x^2 - \frac{5}{21} x^3 + \cdots\right) \\
&= \left(5x - \frac{10}{3} x^2 + \frac{10}{9} x^3 - \frac{5}{21} x^4 + \cdots\right)
\end{aligned}
$$

Hence the final solution

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1\left(1 - 3x + 2x^2 - \frac{2}{3} x^3 + \cdots\right) + \sqrt{x} c_2 \left(1 - \frac{7x}{6} + 21\frac{x^2}{40} + \cdots\right) + \left(5x - \frac{10}{3} x^2 + \frac{10}{9} x^3 - \frac{5}{21} x^4 + \cdots
\end{aligned}
$$

Example 3

$$2xy'' + (x+1)\,y' + 3y = x$$

This is the same problem as above but different RHS. As shown above, we obtained that the differential equation satisfies

$$2xy'' + (x+1)\,y' + 3y = r(2r-1)\,a_0 x^{r-1}$$

To find y_p, and using m in place of r and c in place of a so not to confuse terms with the y_h terms, then the above becomes

$$2xy'' + (x+1)\,y' + 3y = m(2m-1)\,c_0 x^{m-1}$$

The RHS above will be zero when $m=0$ or $m=\frac{1}{2}$. We now need to balance the RHS against given RHS which is x. Hence

$$m(2m-1)\,c_0 x^{m-1} = x$$

To balance this we need $m-1=1$ or $m=2$. Hence $2(4-1)\,c_0=1$ or $c_0=\frac{1}{6}$. Using the recurrence relation we found above, which is for $n\geq 1$ (again, calling r as m so not to confuse y_h terms with y_p terms), we obtain

$$c_n = -\frac{n+m+2}{(n+r)\,(2m+2n-1)}c_{n-1}$$

But now using $m=2$

$$c_n = -\frac{n+4}{(n+2)\,(4+2n-1)}c_{n-1}$$

Hence for $n=1$

$$\begin{aligned}
c_1 &= -\frac{1+4}{(1+2)\,(4+2-1)}c_0\\
&= -\frac{1}{3}c_0\\
&= -\frac{1}{3}\left(\frac{1}{6}\right) = -\frac{1}{18}
\end{aligned}$$

for $n=2$

$$\begin{aligned}
c_2 &= -\frac{6}{(2+2)\,(4+4-1)}c_1\\
&= -\frac{3}{14}c_1 = -\frac{3}{14}\left(-\frac{1}{18}\right) = \frac{1}{84}
\end{aligned}$$

For $n=3$

$$\begin{aligned} c_3 &= -\frac{3+4}{(3+2)(4+6-1)}c_2 \\ &= -\frac{7}{45}c_2 = -\frac{7}{45}\left(\frac{1}{84}\right) = -\frac{1}{540} \end{aligned}$$

For $n=4$

$$\begin{aligned} c_4 &= -\frac{4+4}{(4+2)(4+8-1)}c_3 \\ &= -\frac{4}{33}c_3 = -\frac{4}{33}\left(-\frac{1}{540}\right) = \frac{1}{4455} \end{aligned}$$

And so on. Hence

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \\ &= x^2\left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \\ &= x^2\left(\frac{1}{6} - \frac{1}{18}x + \frac{1}{84}x^2 - \frac{1}{540}x^3 + \frac{1}{4455}x^4 + \cdots\right) \end{aligned}$$

Hence the solution is (y_h was found in the earlier problem)

$$\begin{aligned} y &= y_h + y_p \\ &= c_1\left(1 - 3x + 2x^2 - \frac{2}{3}x^3 + \cdots\right) + c_2\left(\sqrt{x}\left(1 - \frac{7x}{6} + 21\frac{x^2}{40} + \cdots\right)\right) + x^2\left(\frac{1}{6} - \frac{1}{18}x + \frac{1}{84}x^2 - \frac{1}{540}x^3\right. \end{aligned}$$

Example 4

$$x^2 y'' + (x+1)\,y' + y = 5$$

Expansion around $x = x_0 = 0$. This is regular singular point. Hence Frobenius is needed. Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{x+1}{x^2}, q(x) = \frac{1}{x^2}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0}\frac{x+1}{x}$ which is not defined. Hence not possible to solve this using series solution.

Example 5

$$2x^2 y'' - xy' + \left(1 - x^2\right) y = x^2$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{-x}{2x^2} = -\frac{1}{2x}, q(x) = \frac{(1-x^2)}{2x^2}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{-1}{2} = \frac{-1}{2}$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{(1-x^2)}{2} = \frac{1}{2}$. Hence the indicial equation is

$$
\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0 \\
r(r-1) - \frac{1}{2}r + \frac{1}{2} &= 0 \\
r^2 - \frac{3}{2}r + \frac{1}{2} &= 0 \\
r &= 1, \frac{1}{2}
\end{aligned}
$$

Therefore $r_1 = 0, r_2 = -\frac{1}{2}$. Expansion around $x = x_0 = 0$. This is regular singular point. Hence Frobenius is needed. First we find y_h. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, hence

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

The homogenous ode becomes

$$
\begin{aligned}
2x^2 y'' - xy' + \left(1 - x^2\right) y &= 0 \\
2x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \left(1 - x^2\right) \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} &= 0
\end{aligned}
$$

Re indexing to lowest powers on x gives

$$
\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0
$$

When $n = 0$

$$
\begin{aligned}
(2(n+r)\,(n+r-1)\, a_0 - (n+r)\, a_0 + a_0)\, x^r &= 0 \\
(2r(r-1) - r + 1)\, a_0 x^r &= 0 \\
\left(2r^2 - 3r + 1\right) a_0 x^r &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then $2r^2-3r+1=0$, hence $r=1, r=\frac{1}{2}$ as was found above. Therefore the homogenous ode satisfies

$$2x^2y''-xy'+\left(1-x^2\right)y=\left(2r^2-3r+1\right)a_0x^r$$

Where the RHS will be zero when $r=1, r=\frac{1}{2}$. When $n=1$

$$\begin{aligned}
2(1+r)(1+r-1)a_1-(1+r)a_1+a_1&=0\\
(2(1+r)(1+r-1)-(1+r)+1)a_1&=0\\
r(2r+1)a_1&=0
\end{aligned}$$

Hence $a_1=0$. For $n\geq 2$ the recurrence relation is

$$\begin{aligned}
2(n+r)(n+r-1)a_n-(n+r)a_n+a_n-a_{n-2}=0\\
a_n&=\frac{a_{n-2}}{2(n+r)(n+r-1)-(n+r)+1}\\
&=\frac{a_{n-2}}{2(n+r)(n+r-1)-(n+r)+1}
\end{aligned} \tag{1}$$

For $r=1$ the above becomes

$$a_n=\frac{a_{n-2}}{n(2n+1)}$$

For $n=2$ and letting $a_0=1$

$$a_2=\frac{a_0}{2(4+1)}=\frac{1}{10}$$

For $n=3$

$$a_3=\frac{a_1}{n(2n+1)}=0$$

For $n=4$

$$a_4=\frac{a_2}{4(8+1)}=\frac{\frac{1}{10}}{4(8+1)}=\frac{1}{360}$$

And so on. Hence

$$\begin{aligned}
y_1&=\sum_{n=0}^{\infty}a_nx^{n+r}=x\sum_{n=0}^{\infty}a_nx^n\\
&=x\left(a_0+a_1x+a_2x^2+\cdots\right)\\
&=x\left(1+\frac{x^2}{10}+\frac{x^4}{360}+\cdots\right)
\end{aligned}$$

And for $r=\frac{1}{2}$ the recurrence relation (1) becomes, and using b instead of a

$$\begin{aligned}
b_n &== \frac{b_{n-2}}{2(n+r)(n+r-1)-(n+r)+1}\\
&= \frac{b_{n-2}}{2\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}-1\right)-\left(n+\frac{1}{2}\right)+1}\\
&= \frac{b_{n-2}}{n(2n-1)}
\end{aligned}$$

Notice also that $b_1=0$ just like $a_1=0$ from above. Now, for $n=2$ and using $b_0=1$

$$b_2=\frac{b_0}{2(4-1)}=\frac{1}{6}$$

For $n=3$

$$b_2=-\frac{b_1}{2-8}=-\frac{1}{2-8}=\frac{1}{6}$$

For $n=3$

$$b_3=\frac{b_1}{n(2n-1)}=0$$

For $n=4$

$$b_n=\frac{b_2}{4(8-1)}=\frac{\frac{1}{6}}{4(8-1)}=\frac{1}{168}$$

And so on. Hence

$$\begin{aligned}
y_2 &= \sum_{n=0}^{\infty} b_n x^{n+r_2}\\
&= \sqrt{x}\sum_{n=0}^{\infty} b_n x^n\\
&= \sqrt{x}\left(b_0+b_1x+b_2x^2+\cdots\right)\\
&= \sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\cdots\right)
\end{aligned}$$

Hence

$$\begin{aligned}
y_h &= c_1y_1+c_2y_2\\
&= c_1\left(x\left(1+\frac{x^2}{10}+\frac{x^4}{360}+\cdots\right)\right)+c_2\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\cdots\right)\\
&= c_1\left(x+\frac{x^3}{10}+\frac{x^5}{360}+\cdots\right)+c_2\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\cdots\right)
\end{aligned}$$

Now we find y_p. Since ode satisfies

$$2x^2y''-xy'+\left(1-x^2\right)y=\left(2r^2-3r+1\right)a_0x^r$$

To find y_p, and relabeling r as m and a as c so not to confuse terms used for y_h. Then the above becomes

$$2x^2y'' - xy' + (1-x^2)\,y = (2m^2 - 3m + 1)\,c_0x^m$$

The RHS is x^2. Hence the balance equation is

$$(2m^2 - 3m + 1)\,c_0x^m = x^2$$

This implies that

$$m = 2$$

Therefore $(2m^2 - 3m + 1)\,c_0 = 1$, or $(8 - 6 + 1)\,c_0 = 1$ or

$$c_0 = \frac{1}{3}$$

The recurrence relation (1) from above now becomes (using m for r and c_0 for a_0)

$$c_n = \frac{c_{n-2}}{2\,(n+m)\,(n+m-1) - (n+m) + 1}$$

For $m = 2$ the above becomes

$$\begin{aligned} c_n &= \frac{c_{n-2}}{2\,(n+2)\,(n+1) - (n+2) + 1} \\ &= \frac{c_{n-2}}{2n^2 + 5n + 3} \end{aligned}$$

For $n = 1$ we use $c_1 = 0$ the same as was found for a_1, b_1. For $n \geq 2$ the above is used. Hence for $n = 2$

$$c_2 = \frac{c_0}{8+10+3} = \frac{\frac{1}{3}}{8+10+3} = \frac{1}{63}$$

For $n = 3$

$$c_3 = \frac{c_1}{18+15+3} = 0$$

For $n = 4$

$$c_4 = \frac{c_2}{32+20+3} = \frac{\frac{1}{63}}{32+20+3} = \frac{1}{3465}$$

And so on. Hence

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} = x^2 \sum_{n=0}^{\infty} c_n x^n \\ &= x^2\left(c_0 + c_1x + c_2x^2 + \cdots\right) \\ &= x^2\left(\frac{1}{3} + \frac{1}{63}x^2 + \frac{1}{3465}x^4 + \cdots\right) \\ &= \frac{1}{3}x^2 + \frac{1}{63}x^4 + \frac{1}{3465}x^6 + \cdots \end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
y &= y_h + y_p\\
&= c_1\left(x+\frac{x^3}{10}+\frac{x^5}{360}+\cdots\right)+c_2\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\cdots\right)+\left(\frac{1}{3}x^2+\frac{1}{63}x^4+\frac{1}{3465}x^6+\cdots\right)
\end{aligned}
$$

Alternative way to find y_p is the the following. Let $y_p = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ then $y_p' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$ and $y_p'' = 2c_2 + 6c_3 x + \cdots$. Hence the ode becomes

$$
\begin{aligned}
2x^2(2c_2+6c_3x+\cdots) - x(c_1+2c_2x+3c_3x^2+\cdots) + (1-x^2)(c_0+c_1x+c_2x^2+c_3x^3+\cdots) &= x^2\\
c_0 + x(-c_1+c_1) + x^2(4c_2-2c_2+c_2-c_0) + x^3(\cdots) &= x^2
\end{aligned}
$$

Hence $c_0=0, 4c_2-2c_2+c_2-c_0=1$ or $3c_2-c_0=1$ or $c_2=\frac{1}{3}$. We need to keep adding more equations and solving them simultaneously. This method is not as easy to use as the method used above, which uses the balance equation to find to y_p. Also this method could fail, since in practice we should not use undetermined coefficients method (which is what this does) on an ode with variable coefficients. So I will not use this any more.

Example 6

$$
2xy'' + y' + y = 0
$$

Comparing the ode to

$$
y'' + p(x)\,y' + q(x)\,y = 0
$$

Hence $p(x)=\frac{1}{2x}, q(x)=\frac{1}{2x}$. Therefore $p_0=\lim_{x\to 0} xp(x) = \lim_{x\to 0}\frac{1}{2}=\frac{1}{2}$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0}\frac{x}{2}=0$. Hence the indicial equation is

$$
\begin{aligned}
r(r-1)+p_0 r+q_0 &= 0\\
r(r-1)+\frac{1}{2}r &= 0\\
r(2r-1) &= 0\\
r &= 0,\frac{1}{2}
\end{aligned}
$$

Therefore $r_1=0, r_2=\frac{1}{2}$. Expansion around $x=x_0=0$. This is regular singular point. Hence Frobenius is needed. First we find y_h. Let $y=\sum_{n=0}^{\infty} a_n x^{n+r}$, hence

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r}\\
y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
xy'' + y' + y &= 0 \\
2x\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty}2(n+r)(n+r-1)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0
\end{aligned}
$$

Re indexing to lowest powers on x gives

$$
\sum_{n=0}^{\infty}2(n+r)(n+r-1)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0
$$

When $n=0$

$$
\begin{aligned}
2(r)(r-1)\,a_0 x^{r-1} + r a_0 x^{r-1} &= 0 \\
(2r(r-1)+r)\,a_0 x^{r-1} &= 0 \\
(r(2r-1))\,a_0 x^{r-1} &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then $r(2r-1)=0$, hence $r=0, r=\frac{1}{2}$ as was found above. Therefore the homogenous ode satisfies

$$
2xy'' + y' + y = (r(2r-1))\,a_0 x^{r-1}
$$

Where the RHS will be zero when $r=1, r=\frac{1}{2}$. For $n \geq 1$ the recurrence relation is

$$
\begin{aligned}
2(n+r)(n+r-1)\,a_n + (n+r)\,a_n &= -a_{n-1} \\
a_n &= \frac{-a_{n-1}}{2(n+r)(n+r-1)+(n+r)} \\
&= \frac{-a_{n-1}}{2n^2+4nr-n+2r^2-r}
\end{aligned} \tag{1}
$$

For $r=0$ the above becomes

$$
a_n = \frac{-a_{n-1}}{n(2n-1)}
$$

For $n=1$ and using $a_0=1$

$$
a_1 = \frac{-a_0}{n(2n-1)} = -1
$$

For $n=2$

$$
a_2 = \frac{-a_1}{2(3)} = \frac{1}{6}
$$

For $n=3$

$$
a_3 = \frac{-a_2}{3(5)} = \frac{-\frac{1}{6}}{15} = -\frac{1}{90}
$$

And so on. Hence

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\ &= a_0 + a_1 x + a_2 x^2 + \cdots \\ &= 1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots \end{aligned}$$

To find y_2, using (1) but replacing a by b and using $r = \frac{1}{2}$ and letting $b_0 = 1$ and following the above process gives

$$b_n = \frac{-b_{n-1}}{2n^2 + 4n\left(\frac{1}{2}\right) - n + 2\left(\frac{1}{2}\right)^2 - \frac{1}{2}} = -\frac{b_{n-1}}{2n^2 + n}$$

For $n = 1$

$$b_1 = -\frac{b_0}{3} = -\frac{1}{3}$$

For $n = 2$

$$b_2 = -\frac{b_1}{8+2} = -\frac{b_1}{10} = -\frac{-\frac{1}{3}}{10} = \frac{1}{30}$$

And so on. Hence we obtain

$$\begin{aligned} y_2 &= \sqrt{x} \sum_{n=0}^{\infty} b_n x^n \\ &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + \cdots\right) \\ &= \sqrt{x}\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 + \cdots\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_1 \\ &= c_1\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots\right) + c_2\left(\sqrt{x}\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 + \cdots\right)\right) \end{aligned}$$

Example 7

$$4xy'' + 3y' + 3y = \sqrt{x}$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{3}{4x}, q(x) = \frac{3}{4x}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{3}{4} = \frac{3}{4}$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{3x}{4} = 0$. Hence $x = 0$ is regular singular point. The indicial equation is

$$\begin{aligned} r(r-1)+p_0r+q_0 &= 0 \\ r(r-1)+\frac{3}{4}r+0 &= 0 \\ r(r-1)+\frac{3}{4}r &= 0 \\ r &= \frac{1}{4},0 \end{aligned}$$

Frobenius is now used. Roots differ by non integer. First we find y_h. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \end{aligned}$$

The homogenous ode becomes

$$\begin{aligned} 4x\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + 3\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + 3\sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} 4(n+r)\,(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} 3a_n x^{n+r} &= 0 \end{aligned}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} 4(n+r)\,(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1} + \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1} = 0$$

When $n=0$

$$\begin{aligned} 4(n+r)\,(n+r-1)\, a_n x^{n+r-1} + 3(n+r)\, a_n x^{n+r-1} &= 0 \\ 4r(r-1)\, a_0 + 3ra_0 &= 0 \\ (4r(r-1)+3r)\, a_0 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then $4r(r-1)+3r=0$, hence $r=0, r=\frac{1}{4}$ as was found above. Therefore the homogenous ode satisfies

$$4xy''+3y'+3y = (4r(r-1)+3r)\, a_0 x^{r-1}$$

Hence the balance equation is that we will use to find the particular solution is

$$(4m(m-1)+3m)\,c_0x^{m-1}=\sqrt{x}$$

We will get back to the above after finding y_h. Going over the same steps as before, we find the recurrence relation

$$a_n=-\frac{3a_{n-1}}{4n^2+8nr+4r^2-n-r}$$

For $r=\frac{1}{4}, n>0$ and similarly

$$b_n=-\frac{3a_{n-1}}{4n^2+8nr+4r^2-n-r}$$

For $r=0, n>0$. Finding few terms using the above gives the solution as

$$\begin{aligned}y_h&=c_1y_1(x)+c_2y_2(x)\\&=c_1x^{\frac{1}{4}}\left(1-\frac{3}{5}x+\frac{1}{10}x^2-\frac{1}{130}x^3+\cdots\right)+c_2\left(1-x+\frac{3}{14}x^2-\frac{3}{154}x^3+\cdots\right)\end{aligned}$$

Now we need to find y_p. From the balance equation

$$(4m(m-1)+3m)\,c_0x^{m-1}=\sqrt{x}$$

Hence $m-1=\frac{1}{2}$ or $m=\frac{3}{2}$. And $(4m(m-1)+3m)\,c_0=1$, hence $\left(4\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)+3\left(\frac{3}{2}\right)\right)c_0=1$, which gives $c_0=\frac{2}{15}$. Therefore

$$\begin{aligned}y_p&=x^m\sum_{n=0}^{\infty}c_nx^n\\&=x^{\frac{3}{2}}\left(c_0+c_1x+c_2x^2+\cdots\right)\\&=x^{\frac{3}{2}}\left(\frac{2}{15}+c_1x+c_2x^2+\cdots\right)\end{aligned}$$

We now just need to determined c_n for $n>0$. For this we use the same recurrence relation as found above. We can use a_n or b_n as they are the same, but change a_n to c_n and r to c (so not to confuse notations). This gives

$$c_n=-\frac{3c_{n-1}}{4n^2+8nm+4m^2-n-m}$$

For $n>0$ and $m=\frac{3}{2}$. Hence for $n=1$ the above gives

$$\begin{aligned}c_1&=-\frac{3c_0}{4+8\left(\frac{3}{2}\right)+4\left(\frac{3}{2}\right)^2-1-\frac{3}{2}}\\&=-\frac{3\left(\frac{2}{15}\right)}{4+8\left(\frac{3}{2}\right)+4\left(\frac{3}{2}\right)^2-1-\frac{3}{2}}\\&=-\frac{4}{225}\end{aligned}$$

For $n=2$

$$
\begin{aligned}
c_1 &= -\frac{3c_1}{4\left(2\right)^2+8\left(2\right)\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2-2-\left(\frac{3}{2}\right)} \\
&= -\frac{3\left(-\frac{4}{225}\right)}{4\left(2\right)^2+8\left(2\right)\left(\frac{3}{2}\right)+4\left(\frac{3}{2}\right)^2-2-\left(\frac{3}{2}\right)} \\
&= \frac{8}{6825}
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y_p &= x^{\frac{3}{2}}\left(\frac{2}{15}+c_1x+c_2x^2+\cdots\right) \\
&= x^{\frac{3}{2}}\left(\frac{2}{15}-\frac{4}{225}x+\frac{8}{6825}x^2-\frac{16}{348075}x^3+\cdots\right)
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
y &= y_h+y_p \\
&= c_1x^{\frac{1}{4}}\left(1-\frac{3}{5}x+\frac{1}{10}x^2-\frac{1}{130}x^3+\cdots\right)+c_2\left(1-x+\frac{3}{14}x^2-\frac{3}{154}x^3+\cdots\right)+x^{\frac{3}{2}}\left(\frac{2}{15}-\frac{4}{225}x+\frac{8}{6825}\right.
\end{aligned}
$$

2.3.2.5.3.3 Roots of indicial equation differ by integer. Good case

ode internal name "second_order_series_method_regular_singular_point_difference_is_integer_good_case".

In this case the solution is

$$
y = c_1y_1+c_2y_2
$$

There are two sub cases that show up when roots differ by integer. First sub case is when the second solution y_2 is obtained similar to how y_1 is obtained. i.e. using standard Frobenius series but with the second root. The second sub case is the harder one, this is when y_2 fails to be obtained using the standard method due to b_N being undefined where N is the difference between the roots. In this sub case we need to use a modified Frobenius series method where, which is explained more using examples below. Therefore for sub case one (called the good case) we have

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\
y_2 &= \sum_{n=0}^{\infty} b_n x^{n+r_2}
\end{aligned}
$$

For the second subcase (called the bad case) first we will find the bad root r of the indicial equation which causes the recurrence relation to become undefined at some n. Call it r_{bad}, then we first find $\overline{y}$ defined as

$$\overline{y} = x^r \sum_{n=0}^{\infty} (r - r_{bad})\, a_n x^n$$

Where a_n is found using the recurrence relation (but r is kept symbolic). y_1 is then found from by evaluating it $r = r_{bad}$

$$y_1 = \overline{y}_{r=r_{bad}}$$

And also setting $a_0 = 1$. Note that some terms will vanish above but not all, since there will be cancellation of $(r - r_{bad})$ during the process. y_2 is next found using

$$\begin{aligned} y_2 &= \left(\frac{d}{dr}\overline{y}\right)_{r=r_{bad}} \\ &= y_1 \ln(x) + x^{r_{bad}} \sum_{n=0}^{\infty} \left(\frac{d}{dr}\left((r - r_{bad})\, a_n x^n\right)\right)_{r=r_{bad}} \end{aligned}$$

Example 1

$$\left(x - x^2\right) y'' + 3y' + 2y = 3x^2$$

Comparing the above to $y'' + p(x)\, y' + q(x)\, y = 0$ shows that $p(x) = \frac{3}{x(1-x)}, q(x) = \frac{2}{x(x-1)}$. Hence there are two singular points, one at $x = 0$ and one at $x = 1$. Let the expansion be around $x = 0$. This means the solution will define up to $x = 1$, which is the next nearest singular point.

$$p_0 = \lim_{x\to 0} x p(x) = \lim_{x\to 0} x \frac{3}{x(1-x)} = 3$$

And

$$q_0 = \lim_{x\to 0} x^2 \frac{2}{x(1-x)} = 0$$

Hence $x_0 = 0$ is a regular singular point. The indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) + 3r &= 0 \\ r^2 - r + 3r &= 0 \\ r^2 + 2r &= 0 \\ r(r+2) &= 0 \end{aligned}$$

Therefore $r=0, r=-2$. They differ by an integer $N=2$. Therefore two linearly independent solutions can be constructed using

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$$
$$y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

Where C above can be zero depending on a condition given below. Now we will work out the solution for a general r. Let

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$
$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

The homogeneous ode becomes

$$(x-x^2)\, y'' + 3y' + 2y$$
$$(x-x^2) \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + 3\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r}$$
$$\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r-1)\,(n+r-2)\, a_{n-1} x^{n+r-1} + \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_n \tag{1A}$$

For $n=0$

$$\begin{aligned}(n+r)\,(n+r-1)\, a_n x^{n+r-1} + 3(n+r)\, a_n x^{n+r-1} &= 0\\ (r(r-1)+3r)\, a_0 x^{r-1} &=\\ (r^2+2r)\, a_0 x^{r-1} &= 0 \end{aligned} \tag{1B}$$

Since $a_0 \neq 0$, then $r=0, r=-2$ as was found above. Hence $N=2$ which is the difference between the two roots. The homogenous ode therefore satisfies

$$(x-x^2)\, y'' + 3y' + 2y = (r^2+2r)\, a_0 x^{r-1}$$

Since when $r=0, r=-2$ the RHS is zero. The term on the right of the above is important as it will be used to determine the particular solution. The recurrence relation is when $n\geq 1$ from (1A) and is given by

$$(n+r)(n+r-1)a_n-(n+r-1)(n+r-2)a_{n-1}+3(n+r)a_n+2a_{n-1}=0$$

Keeping larger a_n on the left and all lower a_n on the right gives

$$\begin{aligned} a_n &= \frac{-2+(n+r-1)(n+r-2)}{(n+r)(n+r-1)+3(n+r)}a_{n-1} \\ a_n &= \frac{n+r-3}{n+r+2}a_{n-1} \end{aligned} \tag{1}$$

Now we find $y_h=c_1y_1+c_2y_2$. For $r=0$ then (1) becomes

$$a_n=\frac{n-3}{n+2}a_{n-1} \tag{2}$$

For $n=1$ and letting $a_0=1$ then (2) gives

$$a_1=\frac{1-3}{1+2}a_0=\frac{-2}{3}$$

For $n=2$ Eq. (2) gives

$$a_2=\frac{2-3}{2+2}a_1=\frac{2-3}{2+2}\left(\frac{-2}{3}\right)=\frac{1}{6}$$

For $n=3$ Eq. (2) gives

$$a_3=\frac{3-3}{3+2}a_2=0$$

And all other higher $a_n=0$. Hence

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty}a_nx^n \\ &= a_0+a_1x+a_2x^2 \\ &= 1-\frac{2}{3}x+\frac{1}{6}x^2 \end{aligned}$$

Now we need to find y_2. We first check if y_2 can be found using standard method as was done above for y_1. For this we calculate $b_N=b_2$ using same recurrence relation (1) to see if it is defined or not. If it is defined, then we continue, else we have to use the modified Frobenius method. From (1) and using b instead of a and using $r=r_2=-2$

gives

$$\begin{aligned}b_n &= \frac{n+r-3}{n+r+2}b_{n-1}\\ &= \frac{n-2-3}{n-2+2}b_{n-1}\\ &= \frac{n-5}{n}b_{n-1}\end{aligned}$$

Hence for $n=1$ and using $b_0=1$ as we did for a_0 gives

$$b_1 = -4b_0 = -4$$

For $n=N=2$

$$b_n = \frac{-3}{2}b_1 = 6$$

Since b_N is defined, we can continue and y_2 is found using same recurrence relation. Hence this is subcase one. For $n=3$

$$b_3 = \frac{-2}{3}b_2 = -4$$

For $n=4$

$$b_4 = \frac{-1}{4}b_3 = 1$$

And so on. Hence

$$\begin{aligned}y_2 &= \frac{1}{x^2}\sum_{n=0}^{\infty} b_n x^n\\ &= \frac{1}{x^2}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4\right)\\ &= \frac{1}{x^2}\left(1 - 4x + 6x^2 - 4x^3 + x^4\right)\end{aligned}$$

Therefore

$$\begin{aligned}y_h &= c_1y_1 + c_2y_2\\ &= c_1\left(1 - \frac{2}{3}x + \frac{1}{6}x^2\right) + c_2\left(\frac{1}{x^2}\left(1 - 4x + 6x^2 - 4x^3 + x^4\right)\right)\end{aligned}$$

Now we find y_p. From earlier we found in (1B) the balance equation which gives

$$\left(x-x^2\right)y'' + 3y' + 2y = \left(r^2+2r\right)a_0x^{r-1}$$

Relabeling r as m and a as c so not to confuse terms used in finding y_h the above becomes

$$\left(x-x^2\right)y'' + 3y' + 2y = \left(m^2+2m\right)c_0x^{m-1}$$

Therefore we need to balance $(m^2 + 2m)\, c_0 x^{m-1} = 3x^2$. This implies $m - 1 = 2$ or $m = 3$. Therefore $(m^2 + 2m)\, c_0 = 3$ or $(9 + 6)\, c_0 = 3$ which gives $c_0 = \frac{3}{15} = \frac{1}{5}$. Hence

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= x^3 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

To find c_n, the same recurrence relation given in (1) is used again but now r is replaced by m and a replaced by c. This gives the recurrence relation to find coefficients of the particular solution as

$$c_n = \frac{n + m - 3}{n + m + 2} c_{n-1}$$

For $m = 3$ the above becomes

$$\begin{aligned} c_n &= \frac{n + 3 - 3}{n + 3 + 2} c_{n-1} \\ &= \frac{n}{n + 5} c_{n-1} \end{aligned}$$

For $n = 1$

$$c_1 = \frac{1}{6} c_0 = \frac{1}{6}\left(\frac{1}{5}\right) = \frac{1}{30}$$

For $n = 2$

$$c_2 = \frac{2}{2 + 5} c_1 = \frac{2}{7}\left(\frac{1}{30}\right) = \frac{1}{105}$$

And so on. Hence

$$\begin{aligned} y_p &= x^3 \sum_{n=0}^{\infty} c_n x^n \\ &= x^3 \left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \\ &= x^3 \left(\frac{1}{5} + \frac{1}{30} x + \frac{1}{105} x^2 + \cdots\right) \end{aligned}$$

Hence the final solution

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 \left(1 - \frac{2}{3} x + \frac{1}{6} x^2\right) + c_2 \left(\frac{1}{x^2}\left(1 - 4x + 6x^2 - 4x^3 + x^4\right)\right) + \left(\frac{1}{5} x^3 + \frac{1}{30} x^4 + \frac{1}{105} x^5 + \cdots\right) \end{aligned}$$

If we try to find y_p by assuming $y_p = \sum_{n=0}^{\infty} c_n x^n$ and substituting into the ode and try to match coefficients, we can not always be successful. The above method using the balance equation always works and that is what I am using in my solver.

Example 2

$$4x^2y'' + 4xy' + (4x^2 - 1)\,y = 0$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{1}{x}, q(x) = \frac{4x^2-1}{4x^2}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{4x^2-1}{4} = -\frac{1}{4}$. Hence the indicial equation is

$$\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0\\
r(r-1) + r - \frac{1}{4} &= 0\\
r^2 - \frac{1}{4} &= 0\\
r &= -\frac{1}{2}, \frac{1}{2}
\end{aligned}$$

Therefore $r_1 = \frac{1}{2}, r_2 = -\frac{1}{2}$.

Expansion around $x=0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r}\\
y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
4x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} + 4x \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + 4x^2 \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} &= 0\\
\sum_{n=0}^{\infty} 4(n+r)\,(n+r-1)\,a_n x^{n+r} + \sum_{n=0}^{\infty} 4(n+r)\,a_n x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r} &= 0\\
\sum_{n=0}^{\infty} \left(4(n+r)\,(n+r-1) + 4(n+r) - 1\right) a_n x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r+2} &= 0\\
\sum_{n=0}^{\infty} \left(4n^2 + 8nr + 4r^2 - 1\right) a_n x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r+2} &= 0\\
\sum_{n=0}^{\infty} \left(4(n+r)^2 - 1\right) a_n x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r+2} &= 0
\end{aligned} \tag{1}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty}\left(4(n+r)^2-1\right)a_n x^{n+r}+\sum_{n=2}^{\infty}4a_{n-2}x^{n+r}=0 \tag{2}$$

$n=0$ gives

$$\left(4r^2-1\right)a_0x^r=0$$

Since $a_0\neq 0$, then $r_1=\frac{1}{2}, r_2=-\frac{1}{2}$ as was found above. The ode therefore satisfies

$$4x^2y''+4xy'+\left(4x^2-1\right)y=\left(4r^2-1\right)a_0x^r$$

Since when $r_1=\frac{1}{2}, r_2=-\frac{1}{2}$ the RHS is zero. When $n=1$ then (2) gives

$$\left(4(1+r)^2-1\right)a_1=0 \tag{3}$$

The recurrence relation is when $n\geq 2$ from (2) is given by

$$\begin{aligned}\left(4(n+r)^2-1\right)a_n+4a_{n-2}&=0\\ a_n&=\frac{-4}{4\left(n+r\right)^2-1}a_{n-2}\end{aligned} \tag{4}$$

Since roots differ by an integer $N=1$ then there two linearly independent solutions can be constructed using

$$\begin{aligned}y_1&=x^{r_1}\sum_{n=0}^{\infty}a_nx^n\\ y_2&=Cy_1\ln\left(x\right)+x^{r_2}\sum_{n=0}^{\infty}b_nx^n\end{aligned}$$

C above can come out to be zero. We start by finding y_1 (the one with the larger r).

Now, using $r=\frac{1}{2}$. For $n=1$ and from (3)

$$\begin{aligned}\left(4\left(1+\frac{1}{2}\right)^2-1\right)a_1&=0\\ 8a_1&=0\\ a_1&=0\end{aligned}$$

From $n=2$ from (4) and using $r=\frac{1}{2}$ it becomes

$$\begin{aligned}a_n&=\frac{-4}{4\left(n+\frac{1}{2}\right)^2-1}a_{n-2}\\ &=-\frac{1}{n^2+n}a_{n-2}\end{aligned} \tag{5}$$

For $n=2$ then (5) gives (and using $a_0=1$)

$$\begin{aligned}a_2 &= -\frac{1}{6}a_0\\ &= -\frac{1}{6}\end{aligned}$$

For $n=3$ Eq (5) gives

$$\begin{aligned}a_3 &= -\frac{1}{12}a_1\\ &= 0\end{aligned}$$

For $n=4$ Eq (5) gives

$$\begin{aligned}a_4 &= -\frac{1}{20}a_2\\ &= -\frac{1}{20}\left(-\frac{1}{6}\right) = \frac{1}{120}\end{aligned}$$

And so on. Hence

$$\begin{aligned}y_1 &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\\ &= x^{\frac{1}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)\\ &= \sqrt{x}\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \cdots\right)\end{aligned}$$

Now we need to find y_2. We first check if y_2 can be found using standard method as was done above for y_1. For this we calculate $b_N = b_1$ using same recurrence relation (1) to see if it is defined or not. If it is defined, then we continue, else we have to use the modified Frobenius method. From (1) and using b instead of a and using $r = r_2 = -\frac{1}{2}$ gives

$$\begin{aligned}\left(4\left(1-\frac{1}{2}\right)^2 - 1\right)b_1 &= 0\\ 0b_1 &= 0\end{aligned}$$

Hence b_1 is arbitrary. Let $b_1=0$. Since $b_N = b_1$ is defined, we can continue and y_2 is found using same recurrence relation. Hence this is subcase one. From (4) and using $r=-\frac{1}{2}$ it becomes

$$\begin{aligned}b_n &= \frac{-4}{4\left(n-\frac{1}{2}\right)^2 - 1}b_{n-2}\\ &= -\frac{1}{n\left(n-1\right)}b_{n-2} \qquad (6)\end{aligned}$$

For $n=2$ Eq (6) gives (and using $b_0=1$)

$$\begin{aligned} b_2 &= -\frac{1}{2(2-1)}b_0 \\ &= -\frac{1}{2} \end{aligned}$$

For $n=3$ Eq (6) gives

$$\begin{aligned} b_3 &= -\frac{1}{3(3-1)}b_1 \\ &= 0 \end{aligned}$$

For $n=4$ Eq (6) gives

$$\begin{aligned} b_4 &= -\frac{1}{4(4-1)}b_2 \\ &= --\frac{1}{12}\left(-\frac{1}{2}\right) = \frac{1}{24} \end{aligned}$$

And so on. Hence

$$\begin{aligned} y_2 &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \\ &= \frac{1}{\sqrt{x}}\left(b_0 + b_1 x + b_2 x^2 + \cdots\right) \\ &= \frac{1}{\sqrt{x}}\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) \end{aligned}$$

Therefore the final solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\sqrt{x}\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \cdots\right) + c_2\frac{1}{\sqrt{x}}\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) \end{aligned}$$

Example 3

$$y'' + y' + y = \sqrt{x}$$

This ode is here because the RHS has no series expansion at $x=0$. Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x)=1, q(x)=1$. Therefore $p_0=\lim_{x\to 0} xp(x)=\lim_{x\to 0} x=0$ and $q_0=\lim_{x\to 0} x^2 q(x)=\lim_{x\to 0} x^2=0$. Hence the indicial equation is

$$\begin{aligned} r(r-1) &= 0 \\ r &= 0,1 \end{aligned}$$

Therefore $r_1=1, r_2=0$.

Expansion around $x=0$. This is regular singular point (due to the RHS not having series expansion). Hence Frobenius is needed. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \end{aligned}$$

The ode becomes

$$\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{1}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + \sum_{n=1}^{\infty} (n+r-1)\, a_{n-1} x^{n+r-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-2} = 0 \tag{2}$$

$n=0$ gives

$$r(r-1)\, a_0 x^{r-2} = 0$$

Since $a_0 \neq 0$, then $r_1=1, r_2=0$ as was found above. The ode therefore satisfies

$$y'' + y' + y = r(r-1)\, a_0 x^{r-2}$$

When $n=1$ then (2) gives

$$\begin{aligned} (1+r)\,(r)\, a_1 + r a_0 &= 0 \\ a_1 &= \frac{-a_0}{1+r} \end{aligned} \tag{3}$$

The recurrence relation is when $n \geq 2$ from (2) is given by

$$\begin{aligned} (n+r)\,(n+r-1)\, a_n + (n+r-1)\, a_{n-1} + a_{n-2} &= 0 \\ a_n &= \frac{-(n+r-1)\, a_{n-1} - a_{n-2}}{(n+r)\,(n+r-1)} \end{aligned} \tag{4}$$

Since roots differ by an integer $N=1$ then there two linearly independent solutions can be constructed using

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$$
$$y_2 = C y_1 \ln(x) + x^{r_2}\sum_{n=0}^{\infty} b_n x^n$$

C above can come out to be zero. We start by finding y_1 (the one with the larger r).

Now, using $r=1$. For $n=1$ and from (3) and using $a_0=1$ gives

$$a_1 = \frac{-a_0}{2}$$
$$a_1 = \frac{-1}{2}$$

From $n=2$ from (4) and using $r=1$ it becomes

$$a_2 = \frac{-2a_1 - a_0}{(2+1)(2)} = \frac{-2a_1 - a_0}{6} = \frac{-2\left(\frac{-1}{2}\right) - 1}{6} = 0$$

For $n=3$ then (5) gives

$$a_3 = \frac{-(3)\,a_2 - a_1}{(3+1)(3)} = \frac{-a_1}{12} = \frac{-\left(\frac{-1}{2}\right)}{12} = \frac{1}{24}$$

And so on. Hence

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right) \\ &= x\left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{120}x^3 + \cdots\right) \end{aligned}$$

Now we need to find y_2. We first check if y_2 can be found using standard method as was done above for y_1. For this we look at $a_1 = \frac{-a_0}{1+r}$ and see this is defined for $r=0$. Next we look at the recurrence relation $a_n = \frac{-(n+r-1)a_{n-1} - a_{n-2}}{(n+r)(n+r-1)}$ and see this is also defined for $r=1$. Hence $C=0$ and we can find y_2 using same series expansion and using $b_0=1$.

$$b_1 = \frac{-b_0}{1+r} = \frac{-1}{1} = -1$$

For $n \geq 2$ we have

$$b_n = \frac{-(n+r-1)\,b_{n-1} - b_{n-2}}{(n+r)(n+r-1)}$$

Which for $r=0$ becomes

$$b_n = \frac{-(n-1)\,b_{n-1} - b_{n-2}}{n\,(n-1)} \tag{5}$$

For $n=2$

$$b_2 = \frac{-(2-1)\,b_1 - b_0}{2} = \frac{-(2-1)\,(-1) - 1}{2} = 0$$

For $n=3$

$$b_3 = \frac{-(3-1)\,b_2 - b_1}{3\,(3-1)} = \frac{1}{6}$$

For $n=4$

$$b_4 = \frac{-(3)\,b_3 - b_2}{4\,(3)} = \frac{-(3)\left(\frac{1}{6}\right)}{4\,(3)} = -\frac{1}{24}$$

And so on. Hence

$$\begin{aligned}
y_2 &= \sum_{n=0}^{\infty} b_n x^{n+0} \\
&= \left(b_0 + b_1 x + b_2 x^2 + \cdots\right) \\
&= 1 - x + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots
\end{aligned}$$

Therefore y_h

$$\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1 x\left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{120}x^3 + \cdots\right) + c_2\left(1 - x + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots\right)
\end{aligned}$$

Now we find y_p. From above $y'' + y' + y = r(r-1)\,a_0 x^{r-2}$, and relabeling r as m and a as c so not to confuse terms used

$$y'' + y' + y = m(m-1)\,c_0 x^{m-2}$$

Therefore we need to balance $m(m-1)\,c_0 x^{m-2} = x^{\frac{1}{2}}$ since the RHS is $\sqrt{x}$. This implies $m-2 = \frac{1}{2}$ or $m = \frac{5}{2}$. Therefore $m(m-1)\,c_0 = 1$ or $\frac{5}{2}\left(\frac{5}{2}-1\right)c_0 = 1$, $c_0 = \frac{4}{15}$. Hence

$$\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= x^{\frac{5}{2}} \sum_{n=0}^{\infty} c_n x^n
\end{aligned}$$

To find c_n, the same recurrence relation (4) is used by with r replaced by m and a replaced by c. This gives

$$\begin{aligned} c_n &= \frac{-(n+m-1)\,c_{n-1}-c_{n-2}}{(n+m)\,(n+m-1)} \\ &= \frac{-\left(n+\frac{5}{2}-1\right)c_{n-1}-c_{n-2}}{\left(n+\frac{5}{2}\right)\left(n+\frac{5}{2}-1\right)} \\ &= -4\frac{\frac{3}{2}c_{n-1}+c_{n-2}+nc_{n-1}}{(2n+3)\,(2n+5)} \end{aligned} \tag{6}$$

The above is only for $n\geq 2$. For $n=1$, using $a_1=\frac{-a_0}{1+r}$ and replacing a by c and r by m gives

$$c_1=\frac{-c_0}{1+m}=\frac{-\frac{4}{15}}{1+\left(\frac{5}{2}\right)}=-\frac{8}{105}$$

For $n=2$ from (6)

$$c_2=-4\frac{\frac{3}{2}c_1+c_0+2c_1}{(4+3)\,(4+5)}=-4\left(\frac{\frac{3}{2}\left(-\frac{8}{105}\right)+\frac{4}{15}+2\left(-\frac{8}{105}\right)}{(4+3)\,(4+5)}\right)=0$$

For $n=3$

$$c_3=-4\frac{\frac{3}{2}c_2+c_1+3c_2}{(6+3)\,(6+5)}=-4\left(\frac{-\frac{8}{105}}{(6+3)\,(6+5)}\right)=\frac{32}{10\,395}$$

And so on. Hence

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+\frac{5}{2}} \\ &= x^{\frac{5}{2}} \sum_{n=0}^{\infty} c_n x^n \\ &= x^{\frac{5}{2}}\left(c_0+c_1x+c_2x^2+\cdots\right) \\ &= x^{\frac{5}{2}}\left(\frac{4}{15}-\frac{8}{105}x+\frac{32}{10\,395}x^3+\cdots\right) \end{aligned}$$

Hence the final solution

$$\begin{aligned} y &= y_h+y_p \\ &= c_1x\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{120}x^3+\cdots\right)+c_2\left(1-x+\frac{1}{6}x^3-\frac{1}{24}x^4+\cdots\right)+x^{\frac{5}{2}}\left(\frac{4}{15}-\frac{8}{105}x+\frac{32}{10\,395}x\right. \end{aligned}$$

2.3.2.5.3.4 Roots of indicial equation differ by integer. Bad case

ode internal name "second_order_series_method_regular_singular_point_difference_is_integer_bad_case".

The description is given above. Only examples are given below.

Example 1

$$x^2y'' + xy' + (x^2 - 4)\, y = 0$$

Comparing the ode to

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Hence $p(x) = \frac{1}{x}, q(x) = \frac{x^2-4}{x^2}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2 - 4 = -4$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) + r - 4 &= 0 \\ r^2 - 4 &= 0 \\ r &= 2, -2 \end{aligned}$$

Therefore $r_1 = 2, r_2 = -2$. Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \end{aligned}$$

The ode becomes

$$\begin{aligned} x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + (x^2-4) \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} - 4 \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} 4a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} ((n+r)\,(n+r-1) + (n+r) - 4)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} &= 0 \end{aligned}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty}\left((n+r)\left(n+r-1\right)+\left(n+r\right)-4\right)a_n x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0 \tag{2}$$

$n=0$ gives

$$\begin{aligned}(r(r-1)+r-4)\,a_0x^r&=0\\ \left(r^2-4\right)a_0x^r&=0\end{aligned}$$

Since $a_0\neq 0$, then $r^2=4$ or $r_1=2, r_2=-2$ as was found above. The ode therefore satisfies

$$x^2y''+xy'+\left(x^2-4\right)y=\left(r^2-4\right)a_0x^r$$

Since when $r_1=2$ or $r_2=-2$ then the RHS is zero. When $n=1$ then (2) gives

$$\begin{aligned}((1+r)\,r+(1+r)-4)\,a_1&=0\\ \left(r^2+2r-3\right)a_1&=0\end{aligned}$$

Hence

$$a_1=0$$

The recurrence relation is when $n\geq 2$ from (2) is given by

$$((n+r)\,(n+r-1)+(n+r)-4)\,a_n+a_{n-2}=0$$

$$a_n=\frac{-a_{n-2}}{((n+r)\,(n+r-1)+(n+r)-4)} \tag{4}$$

We check first if this is subcase one or two. To do this, we check if the recurrence relation is defined for both roots for all $n\geq 2$. The above for $r=2$ gives

$$a_n=\frac{-a_{n-2}}{((n+2)\,(n+2-1)+(n+2)-4)}=-\frac{1}{n}\frac{a_{n-2}}{n+4}$$

We see that it is defined for all $n\geq 2$. Now we check the other root $r_2=-2$. (4) now becomes

$$a_n=\frac{-a_{n-2}}{((n-2)\,(n-3)+(n-2)-4)}=-\frac{1}{n}\frac{a_{n-2}}{n-4}$$

We see that this is the difficult root as at $n=4$ it is *not defined* as it gives $1/0$ error. Hence

$$r_{bad}=-2$$

Therefore this is subcase two. For this case we do the following. We first find the solution using symbolic r using (4), and at the end replace a_0 by $(r-r_{bad})\,b_0=(r+2)\,b_0$. From (4) and for $n=2$

$$a_2=\frac{-a_0}{((2+r)(1+r)+(2+r)-4)}=-\frac{1}{r}\frac{a_0}{r+4}$$

Since $a_1=0$ then all odd $a_n=0$. For $n=4$

$$a_4=\frac{-a_2}{((4+r)(3+r)+(4+r)-4)}=-\frac{a_2}{(r+6)(r+2)}=-\frac{-\frac{1}{r}\frac{a_0}{r+4}}{(r+6)(r+2)}=\frac{1}{r}\frac{a_0}{(r+4)(r+6)(r+2)}$$

For $n=6$

$$\begin{aligned}a_6&=\frac{-a_4}{((6+r)(5+r)+(6+r)-4)}=-\frac{a_4}{(r+8)(r+4)}\\&=-\frac{\frac{1}{r}\frac{a_0}{(r+4)(r+6)(r+2)}}{(r+8)(r+4)}\\&=-\frac{1}{r}\frac{a_0}{(r+8)(r+4)(r+4)(r+6)(r+2)}\end{aligned}$$

And so on. Hence

$$\begin{aligned}\overline{y}&=x^r\left(a_0+a_2x^2+a_4x^4+\cdots\right)\\&=x^ra_0\left(1-\frac{1}{r}\frac{1}{r+4}x^2+\frac{1}{r}\frac{1}{(r+4)(r+6)(r+2)}x^4-\frac{1}{r}\frac{1}{(r+8)(r+4)(r+4)(r+6)(r+2)}x^6+\cdots\right)\end{aligned}$$

Replacing a_0 by $b_0(r-r_{bad})$ or $b_0(r+2)$ the above becomes

$$\overline{y}=x^rb_0\left((r+2)-\frac{1}{r}\frac{(r+2)}{r+4}x^2+\frac{1}{r}\frac{(r+2)}{(r+4)(r+6)(r+2)}x^4-\frac{1}{r}\frac{(r+2)}{(r+8)(r+4)(r+4)(r+6)(r+2)}x^6\right.\tag{5}$$

Now

$$\begin{aligned}y_1&=\overline{y}_{r=r_{bad}}\\&=\overline{y}_{r=-2}\\&=x^{-2}b_0\left((r+2)-\frac{1}{r}\frac{(r+2)}{r+4}x^2+\frac{1}{r}\frac{(r+2)}{(r+4)(r+6)(r+2)}x^4-\frac{1}{r}\frac{(r+2)}{(r+8)(r+4)(r+4)(r+6)(r+2)}x\right.\\&=x^{-2}b_0\left(\frac{1}{r}\frac{1}{(r+4)(r+6)}x^4-\frac{1}{r}\frac{1}{(r+8)(r+4)(r+4)(r+6)}x^6+\cdots\right)_{r=-2}\\&=x^{-2}b_0\left(-\frac{1}{16}x^4+\frac{1}{192}x^6-\cdots\right)\end{aligned}$$

But $b_0 = 1$. Hence

$$
\begin{aligned}
y_1 &= \left(-\frac{1}{16}x^2 + \frac{1}{192}x^4 - \cdots\right) \\
&= -\frac{1}{16}\left(x^2 - \frac{1}{12}x^4 - \cdots\right)
\end{aligned}
$$

We can removing the leading $-\frac{1}{16}$ since it will be absorbed by the c_1 constant. Hence

$$
y_1 = c_1\left(x^2 - \frac{1}{12}x^4 - \cdots\right)
$$

Now we find y_2 using

$$
y_2 = \left(\frac{d\overline{y}}{dr}\right)_{r=r_{bad}}
$$

Notice the derivative is evaluated also at the bad root $r = r_{bad} = -2$ same as for y_1. Hence, and using $b_0 = 1$ and using (5) the above gives

$$
\begin{aligned}
y_2 &= \frac{d}{dr}\left(x^r\left((r+2) - \frac{1}{r}\frac{(r+2)}{r+4}x^2 + \frac{1}{r}\frac{(r+2)}{(r+4)(r+6)(r+2)}x^4 - \frac{1}{r}\frac{(r+2)}{(r+8)(r+4)(r+4)(r+6)(r+2}\right.\right. \\
&= \overline{y}_{r=-2}\ln x + x^r\frac{d}{dr}\left((r+2) - \frac{1}{r}\frac{(r+2)}{r+4}x^2 + \frac{1}{r}\frac{(r+2)}{(r+4)(r+6)(r+2)}x^4 - \frac{1}{r}\frac{(r+2)}{(r+8)(r+4)(r+4)(r}\right.
\end{aligned}
$$

But

$$
y_1 = \overline{y}_{r=-2}
$$

Therefore, evaluating all the derivatives gives

$$
\begin{aligned}
y_2 &= y_1\ln x + x^r\left(1 + \frac{(r^2+4r+8)}{r^2(r+4)^2}x^2 - \frac{1}{r^2}\frac{3r^2+20r+24}{(r^2+10r+24)^2}x^4 + \frac{(5r^3+68r^2+256r+192)}{r^2(r+4)^3(r^2+14r+48)^2}x^6 + \cdots\right)_{r=} \\
&= y_1\ln x + x^{-2}\left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{11}{2304}x^6 + \cdots\right)
\end{aligned}
$$

Hence

$$
y_2 = y_1\ln x + \left(\frac{1}{4} + \frac{1}{x^2} + \frac{1}{64}x^2 - \frac{11}{2304}x^4 + \cdots\cdots\right)
$$

Therefore the final solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x^2 - \frac{1}{12}x^4 - \cdots\right) \\
&\quad + c_2\left(\ln(x)\left(x^2 - \frac{1}{12}x^4 - \cdots\right) + \left(\frac{1}{4} + \frac{1}{x^2} + \frac{1}{64}x^2 - \frac{11}{2304}x^4 + \cdots\cdots\right)\right)
\end{aligned}
$$

Example 2

$$xy'' - 3y' + xy = 0$$

Comparing the ode to

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Hence $p(x) = \frac{-3}{x}, q(x) = 1$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0}(-3) = -3$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 = 0$. Hence the indicial equation is

$$\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0\\
r(r-1) - 3r &= 0\\
r^2 - 4r &= 0\\
r(r-4) &= 0\\
r &= 0, 4
\end{aligned}$$

Therefore $r_1 = 4, r_2 = 0$. Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r}\\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
xy'' - 3y' + xy &= 0\\
x\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} - 3\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + x\sum_{n=0}^{\infty} a_n x^{n+r} &= 0\\
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-1} - \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0
\end{aligned}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-1} - \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0 \tag{2}$$

$n = 0$ gives

$$\begin{aligned}
r(r-1)\, a_0 x^{r-1} - 3r a_0 x^{r-1} &= 0\\
(r(r-4))\, a_0 x^{r-1} &= 0
\end{aligned}$$

Since $a_0 \neq 0$, then $r(r-4)=0$ or $r_1=0, r_2=4$ as was found above. The ode therefore satisfies

$$xy''-3y'+xy=(r(r-4))\,a_0x^{r-1}$$

Since when $r_1=4$ or $r_2=0$ then the RHS is zero. When $n=1$ then (2) gives

$$\begin{aligned}(1+r)\,(r)\,a_1-3(1+r)\,a_1&=0\\ \left(r^2-2r-3\right)a_1&=0\end{aligned}$$

Hence

$$a_1=0$$

The recurrence relation is when $n\geq 2$ from (2) is given by

$$\begin{aligned}(n+r)\,(n+r-1)\,a_n-3(n+r)\,a_n+a_{n-2}&=0\\ a_n&=\frac{-a_{n-2}}{(n+r)\,(n+r-1)-3\,(n+r)}\end{aligned}\tag{4}$$

We check first if this is subcase one or two. To do this, we check if the recurrence relation is defined for both roots for all $n\geq 2$. The above for $r=4$ gives

$$a_n=\frac{-a_{n-2}}{(n+4)\,(n+3)-3\,(n+4)}=-\frac{1}{n}\frac{a_{n-2}}{n+4}$$

Which is defined for all $n\geq 2$. Checking the second root $r=0$ gives

$$a_n=\frac{-a_{n-2}}{(n+0)\,(n+0-1)-3\,(n+0)}=-\frac{1}{n}\frac{a_{n-2}}{n-4}$$

Which is not defined for $n=4$. Hence this is subcase two, where y_2 does not exist using standard method. Hence

$$r_{bad}=0$$

For this case we do the following. We find the solution using symbolic r and replace a_0 by $(r-r_{bad})\,b_0$. From (4) and for $n=2$

$$a_2=\frac{-a_0}{(2+r)\,(1+r)-3\,(2+r)}=-\frac{a_0}{r^2-4}$$

Since $a_1=0$ then all odd $a_n=0$. For $n=4$

$$a_4=\frac{-a_2}{(4+r)\,(4+r-1)-3\,(4+r)}=\frac{\frac{a_0}{r^2-4}}{r\,(r+4)}=\frac{a_0}{r\,(r+4)\,(r^2-4)}$$

For $n=6$

$$\begin{aligned}a_6&=\frac{-a_4}{(6+r)\,(5+r)-3\,(6+r)}=\frac{-\frac{a_0}{r(r+4)(r^2-4)}}{r^2+8r+12}\\ &=\frac{-a_0}{(r^2+8r+12)\,r\,(r+4)\,(r^2-4)}\end{aligned}$$

And so on. Hence

$$\begin{aligned}\overline{y} &= x^r\left(a_0 + a_2x^2 + a_4x^4 + \cdots\right)\\ &= x^r a_0\left(1 - \frac{1}{r^2-4}x^2 + \frac{1}{r\left(r+4\right)\left(r^2-4\right)}x^4 - \frac{1}{\left(r^2+8r+12\right)r\left(r+4\right)\left(r^2-4\right)}x^6 + \cdots\right)\end{aligned}$$

Replacing a_0 by $b_0(r-r_2)$ or $b_0 r$ since $r_2=0$, the above becomes

$$\overline{y} = x^r b_0\left(r - \frac{r}{r^2-4}x^2 + \frac{1}{\left(r+4\right)\left(r^2-4\right)}x^4 - \frac{1}{\left(r^2+8r+12\right)\left(r+4\right)\left(r^2-4\right)}x^6 + \cdots\right) \tag{5}$$

Now

$$\begin{aligned}y_1 &= \overline{y}_{r=r_{bad}}\\ &= \overline{y}_{r=0}\\ &= b_0\left(\frac{1}{\left(4\right)\left(-4\right)}x^4 - \frac{1}{\left(12\right)\left(4\right)\left(-4\right)}x^6 + \cdots\right)\\ &= b_0\left(-\frac{1}{16}x^4 + \frac{1}{192}x^6 + \cdots\right)\end{aligned}$$

But $b_0=1$. Hence

$$\begin{aligned}y_1 &= \left(-\frac{1}{16}x^4 + \frac{1}{192}x^6 + \cdots\right)\\ &= -\frac{1}{16}\left(x^4 - \frac{1}{12}x^6 + \cdots\right)\end{aligned}$$

We can removing the leading $-\frac{1}{16}$ since it will be absorbed by the c_1 constant. Hence

$$\begin{aligned}y_1 &= c_1\left(x^4 - \frac{1}{12}x^6 + \cdots\right)\\ &= x^4 c_1\left(1 - \frac{1}{12}x^2 + \cdots\right)\end{aligned}$$

Now we find y_2 using

$$y_2 = \left(\frac{d\overline{y}}{dr}\right)_{r=r_{bad}}$$

Notice the derivative is evaluated also at root $r=r_{bad}=0$ the same as for y_1. Hence, and using $b_0 = 1$ and using (5) the above gives

$$
\begin{aligned}
y_2 &= \frac{d}{dr}\left(x^r\left(r - \frac{r}{r^2-4}x^2 + \frac{1}{(r+4)(r^2-4)}x^4 - \frac{1}{(r^2+8r+12)(r+4)(r^2-4)}x^6 + \cdots\right)\right)_{r=0} \\
&= \overline{y}_{r=0}\ln x + x^r\frac{d}{dr}\left(r - \frac{r}{r^2-4}x^2 + \frac{1}{(r+4)(r^2-4)}x^4 - \frac{1}{(r^2+8r+12)(r+4)(r^2-4)}x^6 + \cdots\right)_{r=0} \\
&= \overline{y}_{r=0}\ln x + x^0\left(1 + \frac{(r^2+4)}{(r^2-4)^2}x^2 - \frac{3r^2+8r-4}{(r^3+4r^2-4r-16)^2}x^4 + \frac{1}{(r+2)^3}\frac{5r^3+38r^2+44r-88}{(r^3+8r^2+4r-48)^2}x^6 - \cdots\right. \\
&= \overline{y}_{r=0}\ln x + \left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{11}{2304}x^6 + \cdots\right)
\end{aligned}
$$

But

$$
\overline{y}_{r=0} = y_1
$$

Therefore

$$
y_2 = y_1\ln x + \left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{11}{2304}x^6 + \cdots\right)
$$

The complete solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= x^4c_1\left(1 - \frac{1}{12}x^2 + \cdots\right) \\
&\quad + c_2\left(\ln x\left(x^4\left(1 - \frac{1}{12}x^2 + \cdots\right)\right) + \left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{11}{2304}x^6 + \cdots\right)\right)
\end{aligned}
$$

Example 3

$$
x^2y'' + (x^2-2x)\,y' + 2y = 0
$$

Comparing the ode to

$$
y'' + p(x)\,y' + q(x)\,y = 0
$$

Show that $p(x) = \frac{(x^2-2x)}{x^2} = \frac{(x-2)}{x}, q(x) = \frac{2}{x^2}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0}(x-2) = -2$ and $q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} 2 = 2$. Hence the indicial equation is

$$
\begin{aligned}
r(r-1) + p_0r + q_0 &= 0 \\
r(r-1) - 2r + 2 &= 0 \\
r^2 - 3r + 2 &= 0 \\
r &= 2, 1
\end{aligned}
$$

Therefore $r_1 = 2, r_2 = 1$. Expansion around $x = 0$. This is regular singular point. Hence

Frobenius is needed. Let

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$
$$y'=\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}$$
$$y''=\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r-2}$$

The ode becomes

$$x^2y''+\left(x^2-2x\right)y'+2y=0$$
$$x^2\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r-2}+\left(x^2-2x\right)\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}+2\sum_{n=0}^{\infty}a_nx^{n+r}=0$$
$$\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r}+x^2\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}-2x\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}+\sum_{n=0}^{\infty}2a_nx^{n+r}=0$$
$$\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r+1}-\sum_{n=0}^{\infty}2(n+r)\,a_nx^{n+r}+\sum_{n=0}^{\infty}2a_nx^{n+r}=0$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r}+\sum_{n=1}^{\infty}(n+r-1)\,a_{n-1}x^{n+r}-\sum_{n=0}^{\infty}2(n+r)\,a_nx^{n+r}+\sum_{n=0}^{\infty}2a_nx^{n+r}=0 \tag{2}$$

$n=0$ gives

$$r(r-1)\,a_0x^r-2ra_0x^r+2a_0x^r=0$$
$$(r(r-1)-2r+2)\,a_0x^r=0$$
$$\left(r^2-3r+2\right)a_0x^r=0$$

Since $a_0\neq 0$, then $r^2-3r+2=0$,or $r_1=2,r_2=1$ as was found above. The ode therefore satisfies

$$x^2y''+\left(x^2-2x\right)y'+2y=\left(r^2-3r+2\right)a_0x^r$$

Recurrence relation is when $n\geq 1$. From (2)

$$(n+r)\,(n+r-1)\,a_n+(n+r-1)\,a_{n-1}-2(n+r)\,a_n+2a_n=0$$

Therefore

$$
\begin{aligned}
a_n &= -\frac{(n+r-1)}{(n+r)(n+r-1)-2(n+r)+2}a_{n-1} \\
&= -\frac{1}{n+r-2}a_{n-1}
\end{aligned} \tag{3}
$$

We check first if this is subcase one or two. To do this, we check if the above recurrence relation is defined for both roots for all $n \geq 1$. The above for $r = r_1 = 2$ gives

$$
a_n = -\frac{1}{n}a_{n-1}
$$

Which is defined for all $n \geq 1$. Checking the second root $r = 1$ gives

$$
a_n = -\frac{1}{n-1}a_{n-1}
$$

Which is not defined for $n = 1$. Hence this is subcase two, where y_2 does not exist using standard method. Hence

$$
r_{bad} = 1
$$

For this case we do the following. We find the solution using symbolic r and replace a_0 by $(r - r_{bad})\, b_0$. From (3) and for $n = 1$

$$
a_1 = -\frac{1}{r-1}a_0
$$

For $n = 2$

$$
a_2 = -\frac{1}{r}a_1 = \frac{1}{(r)(r-1)}a_0
$$

For $n = 3$

$$
a_3 = -\frac{1}{r+1}a_2 = -\frac{a_0}{(r)(r-1)(r+1)}
$$

For $n = 4$

$$
a_4 = -\frac{1}{2+r}a_3 = \frac{a_0}{(r)(r-1)(r+1)(r+2)}
$$

And so on. Hence

$$
\begin{aligned}
\overline{y} &= x^r\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \\
&= x^r a_0\left(1 - \frac{1}{r-1}x + \frac{1}{(r)(r-1)}x^2 - \frac{1}{(r)(r-1)(r+1)}x^3 + \frac{1}{(r)(r-1)(r+1)(r+2)}x^4 - \cdots\right)
\end{aligned}
$$

Replacing a_0 by $b_0(r - r_{bad})$ or $b_0(r-1)$ since $r_{bad} = 1$, the above becomes

$$
\begin{aligned}
\overline{y} &= x^r b_0\left((r-1) - \frac{(r-1)}{r-1}x + \frac{(r-1)}{(r)(r-1)}x^2 - \frac{(r-1)}{(r)(r-1)(r+1)}x^3 + \frac{(r-1)}{(r)(r-1)(r+1)(r+2)}x^4 - \cdots\right. \\
&= x^r b_0\left((r-1) - x + \frac{1}{r}x^2 - \frac{1}{r(r+1)}x^3 + \frac{1}{r(r+1)(r+2)}x^4 - \cdots\right)
\end{aligned} \tag{5}
$$

Now

$$\begin{aligned}
y_1 &= \overline{y}_{r=r_{bad}} \\
&= \overline{y}_{r=1} \\
&= xb_0\left(-x+x^2-\frac{1}{2}x^3+\frac{1}{(1)\,(2)\,(3)}x^4-\cdots\right) \\
&= xb_0\left(-x+x^2-\frac{1}{2}x^3+\frac{1}{6}x^4-\cdots\right)
\end{aligned}$$

But $b_0 = 1$. Hence

$$\begin{aligned}
y_1 &= x\left(-x+x^2-\frac{1}{2}x^3+\frac{1}{6}x^4-\cdots\right) \\
&= -x^2+x^3-\frac{1}{2}x^4+\frac{1}{6}x^5-\cdots
\end{aligned}$$

Now we find y_2 using

$$y_2 = \left(\frac{d\overline{y}}{dr}\right)_{r=r_{bad}}$$

Notice the derivative is evaluated also at root $r = r_{bad} = 1$, the same as for y_1. Hence, and using $b_0 = 1$ and using (5) the above gives

$$\begin{aligned}
y_2 &= \frac{d}{dr}\left(x^r b_0\left((r-1)-x+\frac{1}{r}x^2-\frac{1}{r\,(r+1)}x^3+\frac{1}{r\,(r+1)\,(r+2)}x^4-\cdots\right)\right)_{r=1} \\
&= \overline{y}_{r=1}\ln x + x^{r=1}\frac{d}{dr}\left((r-1)-x+\frac{1}{r}x^2-\frac{1}{r\,(r+1)}x^3+\frac{1}{r\,(r+1)\,(r+2)}x^4-\cdots\right)_{r=1} \\
&= y_1\ln x + x\frac{d}{dr}\left((r-1)-x+\frac{1}{r}x^2-\frac{1}{r\,(r+1)}x^3+\frac{1}{r\,(r+1)\,(r+2)}x^4-\cdots\right)_{r=1} \\
&= y_1\ln x + x\left(1-\frac{1}{r^2}x^2+\frac{1}{r^2}\frac{2r+1}{(r+1)^2}x^3-\frac{1}{r^2}\frac{3r^2+6r+2}{(r^2+3r+2)^2}x^4-\cdots\right)_{r=1} \\
&= y_1\ln x + x\left(1-x^2+\frac{3}{4}x^3-\frac{11}{36}x^4-\cdots\right)
\end{aligned}$$

Therefore

$$y_2 = y_1\ln x + \left(x-x^3+\frac{3}{4}x^4-\frac{11}{36}x^5-\cdots\right)$$

The complete solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(-x^2+x^3-\frac{1}{2}x^4+\frac{1}{6}x^5-\cdots\right) \\
&\quad + c_2\left(\ln x\left(-x^2+x^3-\frac{1}{2}x^4+\frac{1}{6}x^5-\cdots\right)+\left(x-x^3+\frac{3}{4}x^4-\frac{11}{36}x^5-\cdots\right)\right)
\end{aligned}$$

Example 4

$$(x-1)\,y''+xy'+\frac{y}{x}=0$$

Comparing the ode to

$$y''+p(x)\,y'+q(x)\,y=0$$

Hence $p(x)=\frac{x}{x-1}, q(x)=\frac{1}{x(x-1)}$. There is a singular point at $x=0$ and at $x=1$. For $x=0$, $p_0=\lim_{x\to 0}xp(x)=0$ and $q_0=\lim_{x\to 0}x^2q(x)=0$. Hence the indicial equation is

$$\begin{aligned} r(r-1)+p_0r+q_0&=0\\ r(r-1)&=0\\ r&=0,1 \end{aligned}$$

For expansion around $x=0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned} y&=\sum_{n=0}^{\infty}a_nx^{n+r}\\ y'&=\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}\\ y''&=\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r-2} \end{aligned}$$

The ode becomes

$$(x-1)\,y''+xy'+\frac{y}{x}$$

$$(x-1)\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r-2}+x\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}+x^{-1}\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r-1}-\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r-1}$$

Re indexing to lowest powers on x gives

$$\sum_{n=1}^{\infty}(n+r-1)\,(n+r-2)\,a_{n-1}x^{n+r-2}-\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r-2}+\sum_{n=2}^{\infty}(n+r-2)\,a_{n-2}x^{n+r-2}+\sum_{n=}$$

(2)

$n=0$ gives

$$(r(r-1))\,a_0=0$$

Since $a_0 \neq 0$, then $r_1 = 0, r_2 = 1$ as was found above. For $n = 1$

$$(r)(r-1)a_0 - (1+r)(r)a_1 + a_0 = 0$$

$$a_1 = \frac{a_0 + (r)(r-1)a_0}{(1+r)(r)} = \frac{1+(r)(r-1)}{(1+r)(r)}a_0$$

For $r = 0$ the above is not defined. Therefore this falls into case two (difficult case). Hence $r_{bad} = 0$. For $r = 1$ we see a_1 is defined.

For this case we do the following. We find the solution using symbolic r and replace a_0 by $(r - r_{bad})\,b_0 = rb_0$. For $n = 1$

$$a_1 = \frac{1+(r)(r-1)}{(1+r)(r)}a_0$$

For $n \geq 2$, the recurrence relation is

$$(n+r-1)(n+r-2)a_{n-1} - (n+r)(n+r-1)a_n + (n+r-2)a_{n-2} + a_{n-1} = 0$$

Or

$$a_n = \frac{(n+r-1)(n+r-2)+1}{(n+r)(n+r-1)}a_{n-1} + \frac{(n+r-2)}{(n+r)(n+r-1)}a_{n-2} \tag{3}$$

For $n = 2$

$$\begin{aligned}
a_2 &= \frac{(1+r)(r)+1}{(2+r)(1+r)}a_1 + \frac{r}{(2+r)(1+r)}a_0 \\
&= \frac{r(1+r)+1}{(2+r)(1+r)}\left(\frac{1+r(r-1)}{(1+r)(r)}a_0\right) + \frac{r}{(2+r)(1+r)}a_0 \\
&= \left(\frac{r(1+r)+1}{(2+r)(1+r)}\frac{1+r(r-1)}{r(1+r)} + \frac{r}{(2+r)(1+r)}\right)a_0 \\
&= \left(\frac{(r(1+r)+1)(1+r(r-1))}{(2+r)(1+r)(1+r)(r)} + \frac{r}{(2+r)(1+r)}\right)a_0
\end{aligned}$$

For $n = 3$

$$\begin{aligned}
a_3 &= \frac{(2+r)(1+r)+1}{(3+r)(2+r)}a_2 + \frac{(1+r)}{(3+r)(2+r)}a_1 \\
&= \frac{(2+r)(1+r)+1}{(3+r)(2+r)}\left(\left(\frac{(r(1+r)+1)(1+r(r-1))}{(2+r)(1+r)(1+r)(r)} + \frac{r}{(2+r)(1+r)}\right)a_0\right) + \frac{(1+r)}{(3+r)(2+r)}\left(\frac{1+(}{(1}\right. \\
&= \left[\frac{(2+r)(1+r)+1}{(3+r)(2+r)}\left(\frac{(r(1+r)+1)(1+r(r-1))}{(2+r)(1+r)(1+r)(r)} + \frac{r}{(2+r)(1+r)}\right) + \frac{(1+r)}{(3+r)(2+r)}\frac{1+(r)(r-}{(1+r)(r}\right.
\end{aligned}$$

And so on. Hence

$$\begin{aligned}
\overline{y} &= x^r\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\right) \\
&= x^r a_0\left(1 + \frac{1+(r)(r-1)}{(1+r)(r)}x + \left(\frac{(r(1+r)+1)(1+r(r-1))}{(2+r)(1+r)(1+r)(r)} + \frac{r}{(2+r)(1+r)}\right)x^2 + \cdots\right)
\end{aligned}$$

Replacing a_0 by $b_0(r - r_{bad})$ or $b_0 r$ since $r_{bad} = 0$, the above becomes

$$\begin{aligned}\overline{y} &= x^r b_0 \left(r + r\frac{1+(r)(r-1)}{(1+r)(r)}x + r\left(\frac{(r(1+r)+1)(1+r(r-1))}{(2+r)(1+r)(1+r)(r)} + \frac{r}{(2+r)(1+r)}\right)x^2 + \cdots\right)\\ &= x^r b_0 \left(r + \frac{1+(r)(r-1)}{(1+r)}x + \left(\frac{(r(1+r)+1)(1+r(r-1))}{(2+r)(1+r)(1+r)} + \frac{r^2}{(2+r)(1+r)}\right)x^2 + \cdots\right)\end{aligned} \tag{5}$$

Now

$$\begin{aligned}y_1 &= \overline{y}_{r=r_{bad}}\\ &= \overline{y}_{r=0}\\ &= x^0 b_0 \left(x + \left(\frac{1}{(2)(1)(1)}\right)x^2 + \left[\frac{(2)(1)+1}{(3)(2)}\left(\frac{(1)(1)}{(2)(1)(1)}\right) + \frac{(1)}{(3)(2)}\right]x^3 \cdots\right)\\ &= b_0\left(x + \frac{1}{2}x^2 + \frac{5}{12}x^3 + \cdots\right)\end{aligned}$$

But $b_0 = 1$. Hence

$$y_1 = x + \frac{1}{2}x^2 + \frac{5}{12}x^3 + \cdots$$

y_2 is found using

$$y_2 = \left(\frac{d\overline{y}}{dr}\right)_{r=r_{bad}}$$

Notice the derivative is evaluated also at root $r = r_{bad} = 0$, the same as for y_1. Hence, and using $b_0 = 1$ and using (5) the above gives

$$\begin{aligned}y_2 &= \frac{d}{dr}\left(x^r\left(r + \frac{1+(r)(r-1)}{(1+r)}x + \left(\frac{(r(1+r)+1)(1+r(r-1))}{(2+r)(1+r)(1+r)} + \frac{r^2}{(2+r)(1+r)}\right)x^2 + \cdots\right)\right)_{r=0}\\ &= \overline{y}_{r=0}\ln x + x^{r=0}\frac{d}{dr}\left(r + \frac{1+(r)(r-1)}{(1+r)}x + \left(\frac{(r(1+r)+1)(1+r(r-1))}{(2+r)(1+r)(1+r)} + \frac{r^2}{(2+r)(1+r)}\right)x^2 + \cdot\right.\end{aligned}$$

But $\overline{y}_{r=0} = y_1$. The above becomes

$$y_2 = y_1 \ln x + \frac{d}{dr}\left(r + \frac{1+(r)(r-1)}{(1+r)}x + \left(\frac{(r(1+r)+1)(1+r(r-1))}{(2+r)(1+r)(1+r)} + \frac{r^2}{(2+r)(1+r)}\right)x^2 + \cdots\right)_{r=0}$$

Carrying out the derivatives gives

$$y_2 = y_1 \ln x + \left(1 + \frac{1}{(r+1)^2}\left(r^2 + 2r - 2\right)x + \left(\frac{(r^5 + 7r^4 + 10r^3 + 8r^2 + 5r - 5)}{(r+1)^3(r+2)^2}\right)x^2 + \cdots\right)_{r=0}$$

Evaluating at $r = 0$

$$y_2 = y_1 \ln x + \left(1 - 2x - \frac{5}{4}x^2 + \cdots\right)$$

Therefore the complete solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x + \frac{1}{2}x^2 + \frac{5}{12}x^3 + \cdots\right) \\ &+ c_2\left(\ln x\left(x + \frac{1}{2}x^2 + \frac{5}{12}x^3 + \cdots\right) + \left(1 - 2x - \frac{5}{4}x^2 + \cdots\right)\right) \end{aligned}$$

Example 5

$$x^2 y'' + x y' + \left(x^2 - 1\right) y = 0$$

Comparing the ode to

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Hence $p(x) = \frac{1}{x}, q(x) = \frac{x^2-4}{x^2}$. Therefore $p_0 = \lim_{x\to 0} x p(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \left(x^2 - 1\right) = -1$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) + r - 1 &= 0 \\ r^2 - 1 &= 0 \\ r &= 1, -1 \end{aligned}$$

Therefore $r_1 = 1, r_2 = -1$. Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \end{aligned}$$

The ode becomes

$$\begin{aligned} x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \left(x^2 - 1\right) \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \end{aligned}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r}=0 \quad (2)$$

$n=0$ gives

$$\begin{aligned}
r(r-1)a_0x^r+ra_0x^r-a_0&=0\\
(r(r-1)+r-1)a_0x^r&=0\\
(r^2-1)a_0x^r&=0
\end{aligned}$$

Since $a_0\neq 0$, then $r^2=1$ or $r_1=,r_2=-1$ as was found above. The ode therefore satisfies

$$x^2y''+xy'+(x^2-1)y=(r^2-1)a_0x^r \tag{2A}$$

When $n=1$ then (2) gives

$$\begin{aligned}
(1+r)(r)a_1+(1+r)a_1-a_1&=0\\
((1+r)(r)+(1+r)-1)a_1&=0\\
(r(r+2))a_1&=0
\end{aligned}$$

Hence

$$a_1=0$$

The recurrence relation is when $n\geq 2$ from (2) is given by

$$(n+r)(n+r-1)a_n+(n+r)a_n+a_{n-2}-a_n=0$$
$$a_n=\frac{-a_{n-2}}{(n+r)(n+r-1)+(n+r)-1} \tag{4}$$

We check first if this is subcase one or two. To do this, we check if the recurrence relation is defined for both roots for all $n\geq 2$. The above for $r=1$ gives

$$a_n=\frac{-a_{n-2}}{(n+1)n+n}$$

We see that it is defined for all $n\geq 2$. Now we check the other root $r_2=-1$. (4) now becomes

$$a_n=\frac{-a_{n-2}}{(n-1)(n-2)+(n-2)}$$

We see that this is the difficult root as at $n=2$ it is *not defined* as it gives $1/0$ error. Hence

$$r_{bad}=-1$$

Therefore this is subcase two. For this case we do the following. We first find the solution using symbolic r using (4), and at the end replace a_0 by $(r-r_{bad})\,b_0=(r+1)\,b_0$. From (4) and for $n=2$

$$a_2=\frac{-a_0}{((2+r)(1+r)+(2+r)-1)}=\frac{-a_0}{(r+1)(r+3)}$$

Since $a_1=0$ then all odd $a_n=0$. For $n=4$

$$a_4=\frac{-a_2}{((4+r)(3+r)+(4+r)-1)}=-\frac{a_2}{(r+5)(r+3)}=-\frac{\frac{-a_0}{(r+1)(r+3)}}{(r+5)(r+3)}=\frac{a_0}{(r+5)(r+3)(r+1)(r+}$$

For $n=6$

$$\begin{aligned}a_6&=\frac{-a_4}{((6+r)(5+r)+(6+r)-1)}=-\frac{a_4}{(r+7)(r+5)}=-\frac{\frac{a_0}{(r+5)(r+3)(r+1)(r+3)}}{(r+7)(r+5)}\\&=-\frac{a_0}{(r+7)(r+5)(r+5)(r+3)(r+1)(r+3)}\end{aligned}$$

And so on. Hence

$$\begin{aligned}\overline{y}&=x^r\left(a_0+a_2x^2+a_4x^4+\cdots\right)\\&=x^ra_0\left(1-\frac{1}{(r+1)(r+3)}x^2+\frac{1}{(r+5)(r+3)(r+1)(r+3)}x^4-\frac{1}{(r+7)(r+5)(r+5)(r+3)(r+1}\right.\end{aligned}$$

Replacing a_0 by $b_0(r-r_{bad})$ or $b_0(r+1)$ the above becomes

$$\begin{aligned}\overline{y}&=x^rb_0\left((r+1)-\frac{(r+1)}{(r+1)(r+3)}x^2+\frac{(r+1)}{(r+5)(r+3)(r+1)(r+3)}x^4-\frac{(r+1)}{(r+7)(r+5)(r+5)(r+3)}\right.\\&=x^rb_0\left((r+1)-\frac{1}{(r+3)}x^2+\frac{1}{(r+5)(r+3)(r+3)}x^4-\frac{1}{(r+7)(r+5)(r+5)(r+3)(r+3)}x^6+\cdots\right.\end{aligned}\tag{5}$$

Now

$$\begin{aligned}y_1&=\overline{y}_{r=r_{bad}}\\&=\overline{y}_{r=-1}\\&=x^{-1}b_0\left(-\frac{1}{(r+3)}x^2+\frac{1}{(r+5)(r+3)(r+3)}x^4-\frac{1}{(r+7)(r+5)(r+5)(r+3)(r+3)}x^6+\cdots\right)_{r=-}\\&=x^{-1}b_0\left(-\frac{1}{(-1+3)}x^2+\frac{1}{(-1+5)(-1+3)(-1+3)}x^4-\frac{1}{(-1+7)(-1+5)(-1+5)(-1+3)(-1+}\right.\\&=x^{-1}b_0\left(-\frac{1}{2}x^2+\frac{1}{16}x^4-\frac{1}{384}x^6+\cdots\right)\end{aligned}$$

$b_0 = 1$. Hence

$$
\begin{aligned}
y_1 &= \frac{1}{x}\left(-\frac{1}{2}x^2 + \frac{1}{16}x^4 - \frac{1}{384}x^6 + \cdots\right) \\
&= \left(-\frac{1}{2}x + \frac{1}{16}x^3 - \frac{1}{384}x^5 + \cdots\right) \\
&= -\frac{1}{2}\left(x - \frac{1}{8}x^3 + \frac{1}{192}x^5 + \cdots\right)
\end{aligned}
$$

We can remove the leading $-\frac{1}{2}$ since it will be absorbed by the c_1 constant. Hence

$$
y_1 = \left(x - \frac{1}{8}x^3 + \frac{1}{192}x^5 + \cdots\right)
$$

Now we find y_2 using

$$
y_2 = \left(\frac{d\overline{y}}{dr}\right)_{r=r_{bad}}
$$

Notice the derivative is evaluated also at the bad root $r = r_{bad} = -2$ same as for y_1. Hence, and using $b_0 = 1$ and using (5) the above gives

$$
\begin{aligned}
y_2 &= \frac{d}{dr}\left(x^r b_0\left((r+1) - \frac{1}{(r+3)}x^2 + \frac{1}{(r+5)(r+3)(r+3)}x^4 - \frac{1}{(r+7)(r+5)(r+5)(r+3)(r+3)}x^6 \right.\right. \\
&= \overline{y}_{r=-1}\ln x + x^r\frac{d}{dr}\left((r+1) - \frac{1}{(r+3)}x^2 + \frac{1}{(r+5)(r+3)(r+3)}x^4 - \frac{1}{(r+7)(r+5)(r+5)(r+3)(\cdots}\right.
\end{aligned}
$$

But

$$
y_1 = \overline{y}_{r=-2}
$$

Therefore, evaluating all the derivatives gives

$$
\begin{aligned}
y_2 &= y_1\ln x + x^{-1}\frac{d}{dr}\left((r+1) - \frac{1}{(r+3)}x^2 + \frac{1}{(r+5)(r+3)(r+3)}x^4 - \frac{1}{(r+7)(r+5)(r+5)(r+3)(r\cdots}\right. \\
&= y_1\ln x + x^{-1}\left(1 + \frac{1}{(r+3)^2}x^2 - \frac{3r+13}{(r+3)^3(r+5)^2}x^4 + \frac{1}{(r+7)^2}\frac{5r^2+52r+127}{(r^2+8r+15)^3}x^6 + \cdots\right)_{r=-1} \\
&= y_1\ln x + x^{-1}\left(1 + \frac{1}{4}x^2 - \frac{5}{64}x^4 + \frac{5}{1152}x^6 + \cdots\right)
\end{aligned}
$$

Hence

$$
y_2 = y_1\ln x + \left(\frac{1}{x} + \frac{1}{4}x - \frac{5}{64}x^3 + \frac{5}{1152}x^5 + \cdots\right)
$$

Therefore the final solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2\\
&= c_1\left(x-\frac{1}{8}x^3+\frac{1}{192}x^5+\cdots\right)\\
&\quad + c_2\left(\ln(x)\left(x-\frac{1}{8}x^3+\frac{1}{192}x^5+\cdots\right)+\left(\frac{1}{x}+\frac{1}{4}x-\frac{5}{64}x^3+\frac{5}{1152}x^5+\cdots\right)\right)
\end{aligned}
$$

Example 6

$$x^2y''+xy'+\left(x^2-1\right)y=1$$

This is same example as above but with non zero in the RHS. So we can use the solution for y_h obtained above, but need to find y_p here and add these to obtain the general solution. From above we found that

$$
\begin{aligned}
y_h &= c_1\left(x-\frac{1}{8}x^3+\frac{1}{192}x^5+\cdots\right)\\
&\quad + c_2\left(\ln(x)\left(x-\frac{1}{8}x^3+\frac{1}{192}x^5+\cdots\right)+\left(\frac{1}{x}+\frac{1}{4}x-\frac{5}{64}x^3+\frac{5}{1152}x^5+\cdots\right)\right)
\end{aligned}
$$

And from (2A) in the above example we also found the balance equation, which is always the starting point to finding y_p, which is

$$x^2y''+xy'+\left(x^2-1\right)y=\left(r^2-1\right)a_0x^r$$

Therefore, and as we did all the time, relabel r as m and a as c so not to confuse notations. Therefore we have

$$\left(m^2-1\right)c_0x^m=1$$

Hence

$$m=0$$

This implies $\left(m^2-1\right)c_0=1$ or

$$c_0=-1$$

Now we find y_p using the same recursive relation found when finding y_h terms but using $r=m=0$ now and using $a_0=c_0=-1$ (instead of $a_0=1$ as is always done when finding y_h). Also let $c_1=0$ as that is the same as a_1. Now we get to the recurrence relation (4) in last example which is

$$a_n=\frac{-a_{n-2}}{(n+r)(n+r-1)+(n+r)-1}$$

Using c in place of a and using m in place r it becomes for $n\geq 2$

$$c_n=\frac{-c_{n-2}}{(n+m)(n+m-1)+(n+m)-1}$$

But $m=0$

$$c_n = \frac{-c_{n-2}}{n(n-1)+(n-1)}$$

For $n=2$

$$c_2 = \frac{-c_0}{2+1} = -\frac{c_0}{3}$$

But $c_0=-1$. The above becomes

$$c_2 = \frac{-c_0}{2+1} = \frac{1}{3}$$

For $n=4$ (since all odd $c_n=0$)

$$c_4 = \frac{-c_2}{4(3)+(3)} = \frac{-\frac{1}{3}}{4(3)+(3)} = -\frac{1}{45}$$

For $n=6$

$$c_6 = \frac{-c_4}{6(5)+(5)} = \frac{\frac{1}{45}}{6(5)+(5)} = \frac{1}{1575}$$

And so on. Hence

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_2 x^2 + c_4 x^4 + \cdots \\ &= -1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \frac{1}{1575}x^6 + \cdots \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1\left(x - \frac{1}{8}x^3 + \frac{1}{192}x^5 + \cdots\right) + \\ &c_2\left(\ln(x)\left(x - \frac{1}{8}x^3 + \frac{1}{192}x^5 + \cdots\right) + \left(\frac{1}{x} + \frac{1}{4}x - \frac{5}{64}x^3 + \frac{5}{1152}x^5 + \cdots\right)\right) \\ &+ \left(-1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \frac{1}{1575}x^6 + \cdots\right) \end{aligned}$$

Example 7

$$x^2y'' + xy' + (x^2-1)\,y = \frac{1}{x}$$

This is same example as above but with $\frac{1}{x}$ instead of 1 in the RHS to show that there will not be a series solution in this. From (2A) in the above example we found the balance equation, which is always the starting point to finding y_p, which is

$$x^2y'' + xy' + (x^2-1)\,y = (r^2-1)\,a_0 x^r$$

Therefore, and as we did all the time, relabel r as m and a as c so not to confuse notations. Therefore we have

$$\left(m^2-1\right)c_0x^m = x^{-1}$$

Hence

$$m=-1$$

This implies $(m^2-1)\,c_0 = 1$ or

$$\begin{aligned}\left((-1)^2-1\right)c_0 &= 1\\ 0c_0 &= 1\end{aligned}$$

Therefore no solution exists. This is why there is no series solution for this ode. If we try to solve this using Maple, will will get no answer and the above explains why.

2.3.2.5.3.5 Roots of indicial equation are repeated

ode internal name "second_order_series_method_regular_singular_point_repeated_root".

In this case the solution is

$$y = c_1y_1 + c_2y_2$$

Where

$$\begin{aligned}y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1}\\ y_2 &= y_1\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_2}\end{aligned}$$

r_1, r_2 are roots of the indicial equation. a_0, b_0 are set to 1 as arbitrary. The coefficients b_n are not found from the recurrence relation but found using using $b_n = \frac{d}{dr}a_n(r)$ after finding a_n first, and the result evaluated at root r_2. (notice that $r = r_1 = r_2$ in this case). Notice there is no C term in from of the ln in this case as when root differ by an integer and the sum on b_n starts at 1 since b_0 is always zero due to $\frac{d}{dr}a_0(r) = 0$ always as $a_0 = 1$ by default.

Example 1

$$x^2y'' + xy' + xy = 0$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{1}{x}, q(x) = \frac{1}{x}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x = 0$. Hence the indicial equation is

$$
\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0 \\
r(r-1) + r &= 0 \\
r^2 &= 0 \\
r &= 0, 0
\end{aligned}
$$

Therefore $r_1 = 0, r_2 = 0$.

Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0
\end{aligned}
$$

Re indexing to lowest powers on x gives

$$
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0 \tag{1}
$$

The indicial equation is obtained from $n = 0$. The above reduces to

$$
\begin{aligned}
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} &= 0 \\
(n+r)\,(n+r-1)\, a_n + (n+r)\, a_n &= 0 \\
(r)\,(r-1)\, a_0 + r a_0 &= 0 \\
a_0\left((r^2 - r) + r\right) &= 0 \\
a_0 r^2 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then

$$r^2 = 0$$

Hence $r_1 = 0, r_2 = 0$. Since the roots are repeated then two linearly independent solutions can be constructed using

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2 = y_1 \ln(x) + x^{r_2} \sum_{n=1}^{\infty} b_n x^n = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^n$$

For $n \geq 1$ the recurrence relation is

$$(n+r)(n+r-1) a_n + (n+r) a_n + a_{n-1} = 0$$
$$\begin{aligned} a_n &= -\frac{a_{n-1}}{(n+r)(n+r-1)+(n+r)} \\ &= -\frac{a_{n-1}}{(n+r)^2} \end{aligned} \tag{1}$$

Starting with y_1. From (1) with $r=0$ gives

$$a_n = -\frac{a_{n-1}}{n^2}$$

For $n=1$ and using $a_0 = 1$

$$a_1 = -1$$

For $n=2$

$$a_2 = -\frac{a_1}{4} = \frac{1}{4}$$

And so on. Hence

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= 1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \cdots \end{aligned}$$

In the case of duplicate roots, b_n is found using $b_n = \frac{d}{dr} a_n(r)$. And this is evaluated at $r = r_0 = 0$ in this case since $r_0 = 0$ here. So we need to find $a_n(r)$. This is done from (1). For $n=1$

$$b_1 = \frac{d}{dr}(a_1(r))$$
$$b_1 = \frac{d}{dr}\left(-\frac{a_0}{(1+r)^2}\right) = \frac{d}{dr}\left(-\frac{1}{(1+r)^2}\right) = \frac{2}{(r+1)^3}$$

Evaluated at $r = 0$ gives

$$b_1 = 2$$

For $n = 2$ then (2) becomes

$$b_2 = \frac{d}{dr}(a_2(r))$$

$$b_2 = \frac{d}{dr}\left(-\frac{a_1}{(2+r)^2}\right) = \frac{d}{dr}\left(-\frac{-\frac{1}{(1+r)^2}}{(2+r)^2}\right) = \frac{d}{dr}\left(\frac{1}{(r+1)^2(r+2)^2}\right) = -2\frac{2r+3}{(r^2+3r+2)^3}$$

At $r = 0$ the above becomes

$$b_2 = -2\frac{3}{(2)^3} = -\frac{3}{4}$$

And so on. Just remember when replacing the a_n in the above, is to use the original $a_n(r)$ as function of r and not the actual a_n values from above. It has to be function of r first before taking derivatives, Hence

$$\begin{aligned} y_2 &= y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^n \\ &= y_1 \ln(x) + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ &= y_1 \ln(x) + 2x - \frac{3}{4}x^2 + \cdots \\ &= y_1 \ln(x) + \left(2x - \frac{3}{4}x^2 + \cdots\right) \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \cdots\right) + c_2\left(y_1 \ln(x) + \left(2x - \frac{3}{4}x^2 + \cdots\right)\right) \end{aligned}$$

Example 2

$$x^2 y'' + xy' + xy = 1$$

The homogenous ode was solved up, so we just need to find y_p. To find y_p, and using m in place of r and c in place of a so not to confuse terms with the y_h terms, then from the above problem, we found the indicial equation. Hence the balance equation is

$$c_0 m^2 x^m = 1$$

To balance this we need $m = 0$. Hence $0c_0 = 1$ which is not possible. Hence no particular solution exists. No solution in series exists.

Example 3

$$x^2y''+xy'+xy=\frac{1}{x}$$

This is the same ode as above but with different RHS. So we will go directly to finding y_p. From above we found that the balance equation is

$$x^2y''+xy'+xy=m^2c_0x^m$$

Hence

$$m^2c_0x^m=x^{-1}$$

Which implies $m=-1$ and therefore $m^2c_0=1$ or $c_0=1$. Using the recurrence equation (1) in the above problem using using c_n in place of a_n and m in place or r gives

$$c_n=-\frac{c_{n-1}}{(n+m)^2}$$

For $m=-1$

$$c_n=-\frac{c_{n-1}}{(n-1)^2}$$

Hence

$$\begin{aligned}y_p&=\sum_{n=0}^{\infty}c_nx^{n+m}\\&=\frac{1}{x}\sum_{n=0}^{\infty}c_nx^n\end{aligned}$$

Now to find few c_n terms. For $n=1$

$$c_1=-\frac{c_0}{(1-1)^2}$$

Which is not defined. Hence no y_p exist. There is no solution in terms of series solution.

Example 4

$$x^2y''+xy'+xy=x$$

This is the same ode as above, where we found y_h but with different RHS. So we will go directly to finding y_p. From above we found that the balance equation is

$$x^2y''+xy'+xy=m^2c_0x^m$$

Hence

$$m^2 c_0 x^m = x$$

Which implies $m=1$ and therefore $m^2 c_0 = 1$ or $c_0 = 1$. Using the recurrence equation (1) in the above problem and using c_n in place of a_n and m in place or r gives

$$c_n = -\frac{c_{n-1}}{(n+m)^2}$$

For $m=1$

$$c_n = -\frac{c_{n-1}}{(n+1)^2}$$

Hence

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= x\sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Now to find few c_n terms. For $n=1$

$$c_1 = -\frac{c_0}{(2)^2} = -\frac{1}{4}$$

For $n=2$

$$c_2 = -\frac{c_1}{(2+1)^2} = \frac{\frac{1}{4}}{9} = \frac{1}{36}$$

For $n=3$

$$c_3 = -\frac{c_2}{(3+1)^2} = -\frac{\frac{1}{36}}{16} = -\frac{1}{576}$$

And so on. Hence

$$\begin{aligned} y_p &= x\sum_{n=0}^{\infty} c_n x^n \\ &= x\left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \\ &= x\left(1 - \frac{1}{4}x + \frac{1}{36}x^2 - \frac{1}{576}x^3 + \cdots\right) \\ &= \left(x - \frac{1}{4}x^2 + \frac{1}{36}x^3 - \frac{1}{576}x^4 + \cdots\right) \end{aligned}$$

Using y_h found in the above problem since that does not change, then the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \frac{1}{576}x^4 + \cdots\right) \\
&+ c_2\left(\ln(x)\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \frac{1}{576}x^4 + \cdots\right) + \left(2x - \frac{3}{4}x^2 + \frac{14}{108}x^3 + \cdots\right)\right) \\
&+ \left(x - \frac{1}{4}x^2 + \frac{1}{36}x^3 - \frac{1}{576}x^4 + \cdots\right)
\end{aligned}
$$

Example 5

$$
xy'' + y' - xy = 0
$$

Comparing the ode to

$$
y'' + p(x)\, y' + q(x)\, y = 0
$$

Hence $p(x) = \frac{1}{x}, q(x) = -1$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 = 0$. Hence the indicial equation is

$$
\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0 \\
r(r-1) + r &= 0 \\
r^2 &= 0 \\
r &= 0, 0
\end{aligned}
$$

Therefore $r_1 = 0, r_2 = 0$. Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
x\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} - x\sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0
\end{aligned}
$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$
$$\sum_{n=0}^{\infty}((n+r)(n+r-1)+(n+r))\,a_n x^{n+r-1} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$
$$\sum_{n=0}^{\infty}(n+r)^2\,a_n x^{n+r-1} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0 \quad (1)$$

The indicial equation is obtained from $n = 0$. The above reduces to

$$r^2 a_0 x^{n+r-1} = 0$$

Since $a_0 \neq 0$ then

$$r^2 = 0$$

Hence $r_1 = 0, r_2 = 0$ as found earlier. Since the roots are repeated then two linearly independent solutions can be constructed using

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2 = y_1 \ln(x) + x^{r_2}\sum_{n=1}^{\infty} b_n x^n = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^n$$

$n = 1$ gives

$$(1+r)(r)\,a_1 + (1+r)\,a_1 = 0$$
$$(r+1)^2\,a_1 = 0$$

Hence $a_1 = 0$. The recurrence relation is obtained for $n \geq 2$. From (1)

$$n + r(n+r-1)\,a_n + (n+r)\,a_n - a_{n-2} = 0$$
$$a_n = \frac{a_{n-2}}{(n+r)^2} \quad (1)$$

Since we need to differentiate y_1 to obtain y_2 and the differentiation is w.r.t r, we will carry the calculations with r in place and at the end replace r by its value (which happened to be *zero* in this example). We do this only in the case of repeated roots.

For $n = 2$

$$a_2 = \frac{a_0}{(2+r)^2} = \frac{1}{(2+r)^2}$$

For $n = 3$

$$a_3 = \frac{a_1}{(3+r)^2} = 0$$

For $n = 4$

$$a_4 = \frac{a_2}{(4+r)^2} = \frac{\frac{1}{(2+r)^2}}{(4+r)^2} = \frac{1}{(2+r)^2(4+r)^2}$$

For $n = 5$, we will find $a_5 = 0$ (for all odd n this is the case). For $n = 6$

$$a_6 = \frac{a_4}{(6+r)^2} = \frac{1}{(2+r)^2(4+r)^2(6+r)^2}$$

And so on. We see that n^{th} term is $a_n = \Pi_{j=1}^{k} \frac{1}{(2j+r)^2}$. Now we can substitute the $r = 0$ value into the above to obtain

$$\begin{aligned} a_2 &= \frac{1}{4} \\ a_4 &= \frac{1}{64} \\ a_6 &= \frac{1}{2304} \end{aligned}$$

Hence

$$\begin{aligned} y_1 &= \sum a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + \cdots \\ &= 1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \frac{1}{2304}x^6 + \cdots \end{aligned}$$

To find y_2 we use $b_n = \frac{d}{dr}a_n$ and evaluate this at $r = r_2$ which in this case is zero. Hence

$$\begin{aligned} b_2 &= \frac{d}{dr}a_2 = \frac{d}{dr}\left(\frac{1}{(2+r)^2}\right) = \left(-\frac{2}{(r+2)^3}\right)_{r=0} = -\frac{2}{8} = -\frac{1}{4} \\ b_4 &= \frac{d}{dr}a_4 = \frac{d}{dr}\left(\frac{1}{(2+r)^2(4+r)^2}\right) = \left(-4\frac{r+3}{(r^2+6r+8)^3}\right)_{r=0} = \left(-4\frac{3}{(8)^3}\right) = -\frac{3}{128} \\ b_6 &= \frac{d}{dr}a_6 \\ &= \frac{d}{dr}\left(\frac{1}{(2+r)^2(4+r)^2(6+r)^2}\right) \\ &= \left(-2\frac{3r^2+24r+44}{(r^3+12r^2+44r+48)^3}\right)_{r=0} \\ &= -2\frac{44}{(48)^3} \\ &= -\frac{11}{13\,824} \end{aligned}$$

And so on. Hence

$$\begin{aligned}
y_1 &= y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_2} \\
&= y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n} \\
&= y_1 \ln(x) + \left(b_2 x^2 + b_4 x^4 + b_6 x^6 + \cdots\right) \\
&= y_1 \ln(x) + \left(-\frac{1}{4}x^2 - \frac{3}{128}x^4 + -\frac{11}{13\,824}x^6 + \cdots\right)
\end{aligned}$$

Therefore the complete solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \frac{1}{2304}x^6 + \cdots\right) \\
&+ c_2\left(\ln(x)\left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \frac{1}{2304}x^6 + \cdots\right) + \left(-\frac{1}{4}x^2 - \frac{3}{128}x^4 + -\frac{11}{13\,824}x^6 + \cdots\right)\right)
\end{aligned}$$

Example 6

$$\sin(x)\, y'' + y' + y = 0$$

Comparing the ode to

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Hence $p(x) = \frac{1}{\sin(x)}, q(x) = \frac{1}{\sin x}$. Therefore $p_0 = \lim_{x\to 0} x p(x) = \lim_{x\to 0} \frac{x}{x - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots} = \frac{1}{1 - \frac{x}{3!} + \frac{x^4}{5!} -} = 1$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{x^2}{x - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots} = \frac{x}{1 - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots} = 0$. Hence the indicial equation is

$$\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0 \\
r(r-1) + r &= 0 \\
r^2 &= 0 \\
r &= 0, 0
\end{aligned}$$

Therefore $r_1 = 0, r_2 = 0$. Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}
\end{aligned}$$

The ode becomes

$$\sin(x)\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Using $O(x^7)$ terms as the Order of the series (if more terms are needed we will use more terms from the $\sin x$ series). This means we have to now only expand up to $n=7$ as that is the order used for the series of $\sin x$. The above becomes

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}-\frac{x^3}{3!}\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}$$
$$+\frac{x^5}{5!}\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Which becomes

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-1}-\sum_{n=0}^{\infty}\frac{1}{6}(n+r)(n+r-1)\,a_n x^{n+r+1}$$
$$+\sum_{n=0}^{\infty}\frac{1}{120}(n+r)(n+r-1)\,a_n x^{n+r+3}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-1}-\sum_{n=2}^{\infty}\frac{1}{6}(n+r-2)(n+r-3)\,a_{n-2}x^{n+r-1}$$
$$+\sum_{n=4}^{\infty}\frac{1}{120}(n+r-4)(n+r-5)\,a_{n-4}x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0$$

Simplifying gives

$$\sum_{n=0}^{\infty}(n+r)^2\,a_n x^{n+r-1}-\sum_{n=2}^{\infty}\frac{(n+r-2)(n+r-3)}{6}a_{n-2}x^{n+r-1}+\sum_{n=4}^{\infty}\frac{(n+r-4)(n+r-5)}{120}a_{n-4}x^{n+r-1}+\sum \tag{1}$$

The indicial equation is obtained from $n=0$. The above reduces to

$$r^2 a_0 x^{r-1}=0$$

Since $a_0\neq 0$ then

$$r^2=0$$

Hence $r_1=0, r_2=0$ as found earlier. Since the roots are repeated then two linearly independent solutions can be constructed using

$$
\begin{aligned}
y_1 &= x^{r_1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n \\
y_2 &= y_1 \ln(x) + x^{r_2}\sum_{n=1}^{\infty} b_n x^n = y_1\ln(x) + \sum_{n=1}^{\infty} b_n x^n
\end{aligned}
$$

$n=1$ gives from (1) and by taking $a_0=1$

$$
\begin{aligned}
(1+r)^2 a_1 + a_0 &= 0 \\
a_1 &= -\frac{a_0}{(1+r)^2} \\
&= -\frac{1}{(1+r)^2}
\end{aligned}
$$

For $n=2$ gives from (1)

$$
\begin{aligned}
(2+r)^2 a_2 - \frac{(r)(r-1)}{6} a_0 + a_1 = 0 \\
(2+r)^2 a_2 &= -a_1 + \frac{(r)(r-1)}{6} a_0 \\
a_2 &= \frac{1}{(1+r)^2(2+r)^2} + \frac{(r)(r-1)}{6(2+r)^2}
\end{aligned}
$$

For $n=3$

$$
\begin{aligned}
(3+r)^2 a_3 - \frac{(1+r)(r)}{6} a_1 + a_2 = 0 \\
a_3 &= -\frac{a_2}{(3+r)^2} + \frac{(1+r)(r)}{6(3+r)^2} a_1 \\
&= -\frac{\frac{1}{(1+r)^2(2+r)^2} + \frac{(r)(r-1)}{6(2+r)^2}}{(3+r)^2} - \frac{(1+r)(r)}{6(3+r)^2}\frac{1}{(1+r)^2} \\
&= -\frac{(r^4+r^3-r^2-r+6)}{6(r+3)^2(r^2+3r+2)^2} - \frac{(1+r)(r)}{6(3+r)^2(1+r)^2}
\end{aligned}
$$

For $n\geq 4$ the recurrence relation is

$$
(n+r)^2 a_n - \frac{(n+r-2)(n+r-3)}{6} a_{n-2} + \frac{(n+r-4)(n+r-5)}{120} a_{n-4} + a_{n-1} = 0
$$

Or

$$
a_n = -\frac{a_{n-1}}{(n+r)^2} + \frac{(n+r-2)(n+r-3)}{6(n+r)^2} a_{n-2} - \frac{(n+r-4)(n+r-5)}{120(n+r)^2} a_{n-4} \tag{2}
$$

Since we need to differentiate y_1 to obtain y_2 and the differentiation is w.r.t r, we will carry the calculations with r in place and at the end replace r by its value (which happened to be *zero* in this example). We do this only in the case of repeated roots.

For $n=4$ then (2) gives

$$
\begin{aligned}
a_4 &= -\frac{a_3}{(4+r)^2} + \frac{(2+r)(1+r)}{6(4+r)^2}a_2 - \frac{(r)(-1+r)}{120(4+r)^2}a_0 \\
&= -\frac{-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}}{(4+r)^2} + \frac{(2+r)(1+r)}{6(4+r)^2}a_2 - \frac{(r)(-1+r)}{120(4+r)^2}a_0 \\
&= \frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2} + \frac{(2+r)(1+r)}{6(4+r)^2}\frac{1}{(r+1)^2(r+2)^2} - \frac{(r)(-1+r)}{120(4+r)^2}
\end{aligned}
$$

And so on. Now we replace $r=0$ to find y_1. Just remember not to use anything over $n=5$ since we cut off the series for $\sin(x)$ at x^5.

Using $r=0$, then the above values for a_i found become

$$
\begin{aligned}
a_1 &= -\frac{1}{(1+r)^2} = -1 \\
a_2 &= \frac{1}{(1+r)^2(2+r)^2} + \frac{(r)(r-1)}{6(2+r)^2} = \frac{1}{4} \\
a_3 &= -\frac{(r^4+r^3-r^2-r+6)}{6(r+3)^2(r^2+3r+2)^2} - \frac{(1+r)(r)}{6(3+r)^2(1+r)^2} = -\frac{1}{(2)^2(3)^2} = -\frac{1}{36} \\
a_4 &= \frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2} + \frac{(2+r)(1+r)}{6(4+r)^2}\frac{1}{(r+1)^2(r+2)^2} - \frac{(r)(-1+r)}{120(4+r)^2} \\
&= \frac{1}{(2)^2(3)^2(4)^2} + \frac{(2)}{6(4)^2}\frac{1}{(2)^2} \\
&= \frac{1}{144}
\end{aligned}
$$

Let find one more term. For $n=5$ then (2) gives

$$
\begin{aligned}
a_5 &= -\frac{a_4}{(5+r)^2} + \frac{(3+r)(2+r)}{6(5+r)^2}a_3 - \frac{(1+r)(r)}{120(5+r)^2}a_1 \\
&= -\frac{\frac{1}{144}}{5^2} + \frac{(3)(2)}{6(5)^2}\left(-\frac{1}{36}\right) \\
&= -\frac{1}{720}
\end{aligned}
$$

For $n=6$ the above recurrence relation gives

$$\begin{aligned}
a_6 &= -\frac{a_5}{(6+r)^2} + \frac{(4+r)(3+r)}{6(6+r)^2}a_4 - \frac{(2+r)(1+r)}{120(6+r)^2}a_2 \\
&= -\frac{-\frac{1}{720}}{6^2} + \frac{(4)(3)}{6(6)^2}\frac{1}{144} - \frac{(2)}{120(6)^2}\frac{1}{4} \\
&= \frac{1}{3240}
\end{aligned}$$

For $n=7$

$$\begin{aligned}
a_7 &= -\frac{a_6}{(7+r)^2} + \frac{(5+r)(4+r)}{6(7+r)^2}a_5 - \frac{(3+r)(2+r)}{120(7+r)^2}a_3 \\
&= -\frac{\frac{1}{3240}}{(7)^2} + \frac{(5)(4)}{6(7)^2}\left(-\frac{1}{720}\right) - \frac{(3)(2)}{120(7)^2}\left(-\frac{1}{36}\right) \\
&= -\frac{23}{317\,520}
\end{aligned}$$

For $n=8$

$$\begin{aligned}
a_8 &= -\frac{a_7}{(8+r)^2} + \frac{(6+r)(5+r)}{6(8+r)^2}a_6 - \frac{(4+r)(3+r)}{120(8+r)^2}a_4 \\
&= -\frac{\left(-\frac{23}{317\,520}\right)}{(8)^2} + \frac{(6)(5)}{6(8)^2}\left(\frac{1}{3240}\right) - \frac{(4)(3)}{120(8)^2}\left(\frac{1}{144}\right) \\
&= \frac{13}{903\,168}
\end{aligned}$$

Which is now the wrong value. It should be $\frac{1}{62720}$. So using 3 terms from $\sin x$ we obtain up to a_7 correct terms. Hence

$$\begin{aligned}
y_1 &= \sum a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + \cdots \\
&= 1 - \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{144}x^4 - \frac{1}{720}x^5 + \frac{1}{3240}x^6 - \frac{23}{317\,520}x^7 + \cdots
\end{aligned}$$

What would have happened if we expanded $\sin(x)$ only for two terms? Lets find out. The ode becomes

$$\sin(x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\left(x - \frac{x^3}{3!} + \cdots\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

The above becomes

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}-\frac{x^3}{3!}\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_n$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}-\sum_{n=0}^{\infty}\frac{1}{6}(n+r)(n+r-1)a_nx^{n+r+1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_n$$

Reindex

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}-\sum_{n=2}^{\infty}\frac{1}{6}(n+r-2)(n+r-3)a_{n-2}x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=1}^{\infty}$$

$$\sum_{n=0}^{\infty}(n+r)^2a_nx^{n+r-1}-\sum_{n=2}^{\infty}\frac{1}{6}(n+r-2)(n+r-3)a_{n-2}x^{n+r-1}+\sum_{n=1}^{\infty}$$

For $n=0$ we obtain the indicial equation as we did above. For $n=1$

$$\begin{aligned}\left(1+r^2\right)a_1+a_0&=0\\ a_1&=-\frac{a_0}{(1+r^2)}=-\frac{1}{(1+r^2)}\end{aligned}$$

For $r=0$ this gives

$$a_1=-1$$

$n\geq 2$ gives

$$\begin{aligned}(n+r)^2a_n-\frac{1}{6}(n+r-2)(n+r-3)a_{n-2}+a_{n-1}&=0\\ a_n&=-\frac{a_{n-1}}{(n+r)^2}+\frac{1}{6}\frac{(n+r-2)(n+r-3)}{(n+r)^2}a_{n-2}\end{aligned} \tag{2A}$$

Hence for $n=2$

$$\begin{aligned}a_2&=-\frac{a_1}{(2+r)^2}+\frac{1}{6}\frac{r(-1+r)}{(2+r)^2}a_0\\ &=-\frac{-\frac{1}{(1+r^2)}}{(2+r)^2}+\frac{1}{6}\frac{r(-1+r)}{(2+r)^2}\end{aligned}$$

For $r=0$ the above gives

$$a_2=-\frac{-\frac{1}{(1)}}{(2)^2}=\frac{1}{4}$$

$n=3$ gives

$$\begin{aligned}a_3 &= -\frac{a_2}{(3+r)^2}+\frac{1}{6}\frac{(1+r)(r)}{(3+r)^2}a_1\\&=-\frac{\frac{1}{4}}{(3+r)^2}-\frac{1}{6}\frac{(1+r)(r)}{(3+r)^2}\end{aligned}$$

For $r=0$

$$a_3=-\frac{\frac{1}{4}}{(3)^2}=-\frac{1}{36}$$

For $n=4$

$$\begin{aligned}a_4 &= -\frac{a_3}{(4+r)^2}+\frac{1}{6}\frac{(2+r)(1+r)}{(4+r)^2}a_2\\&=-\frac{a_3}{(4)^2}+\frac{1}{6}\frac{(2)(1)}{(4)^2}a_2\\&=-\frac{\left(-\frac{1}{36}\right)}{(4)^2}+\frac{1}{6}\frac{(2)(1)}{(4)^2}\left(\frac{1}{4}\right)\\&=\frac{1}{144}\end{aligned}$$

For $n=5$

$$\begin{aligned}a_5 &= -\frac{a_4}{(5+r)^2}+\frac{1}{6}\frac{(3+r)(2+r)}{(5+r)^2}a_3\\&=-\frac{\frac{1}{144}}{(5)^2}+\frac{1}{6}\frac{(3)(2)}{(5)^2}\left(-\frac{1}{36}\right)=-\frac{1}{720}\end{aligned}$$

For $n=6$

$$\begin{aligned}a_6 &= -\frac{a_5}{(6+r)^2}+\frac{1}{6}\frac{(6+r-2)(6+r-3)}{(6+r)^2}a_4\\&=-\frac{\left(-\frac{1}{720}\right)}{(6)^2}+\frac{1}{6}\frac{(4)(3)}{6^2}\frac{1}{144}\\&=\frac{11}{25\,920}\end{aligned}$$

Which is the wrong value. We see that using two terms only from the $\sin(x)$ gave up correct a_n values up to a_5. What if we used only one term? Lets find out.

$$\begin{aligned}
\sin(x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r}&=0\\
(x+\cdots)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r}&=0\\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r}&=0\\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}&=0\\
\sum_{n=0}^{\infty}(n+r)^2a_nx^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}&=0
\end{aligned}$$

$n=0$ gives the indicial equation. For $n\geq 1$ the recurrence relation is

$$\begin{aligned}
(n+r)^2a_n+a_{n-1}&=0\\
a_n&=-\frac{a_{n-1}}{(n+r)^2}
\end{aligned}$$

For $n=1$

$$\begin{aligned}
a_1&=-\frac{a_0}{(1+r)^2}\\
&=-\frac{1}{(1+r)^2}
\end{aligned}$$

For $r=0$

$$a_1=-1$$

For $n=2$

$$a_2=-\frac{a_1}{(2+r)^2}=\frac{1}{(2+r)^2}$$

For $r=0$

$$a_2=\frac{1}{4}$$

For $n=3$

$$a_3=-\frac{a_2}{(3+r)^2}=-\frac{\frac{1}{4}}{(3+r)^2}$$

For $r=0$

$$a_3=-\frac{\frac{1}{4}}{(3)^2}=-\frac{1}{36}$$

For $n=4$

$$a_4 = -\frac{a_3}{(4+r)^2} = -\frac{-\frac{1}{36}}{(4+r)^2}$$

For $r=0$

$$a_4 = -\frac{-\frac{1}{36}}{(4)^2} = \frac{1}{576}$$

We see that this is the wrong value. So when using one term only we obtain correct a_n up to a_3. What do we learn from all the above? It is that if we expand $f(x)$ up to $O(x^n)$ order, then we can only determine correct terms up to a_n and no more. In the above when we used $\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)$ then we obtained correct terms up to a_7. And when we used $\sin(x) = x - \frac{x^3}{6} + O(x^5)$ then we obtained correct terms up to a_5 and when we used $\sin(x) = x + O(x^3)$ then we obtained correct terms up to a_3. So we should keep this in mind from now on,.

To find y_2 we use $b_n = \frac{d}{dr}a_n$ and evaluate this at $r=r_2$ which in this case is zero. Hence

$$b_1 = \frac{d}{dr}a_1 = \frac{d}{dr}\left(-\frac{1}{(1+r)^2}\right)_{r=0} = \frac{2}{(r+1)^3} = 2$$

$$b_2 = \frac{d}{dr}a_2 = \frac{d}{dr}\left(\frac{1}{(1+r)^2(2+r)^2} + \frac{(r)(r-1)}{6(2+r)^2}\right) = \left(\frac{5r^4+13r^3+9r^2-25r-38}{6(r^2+3r+2)^3}\right)_{r=0} = \frac{-38}{6(2)^3} = -\frac{19}{24}$$

$$\begin{aligned}
b_3 &= \frac{d}{dr}a_3 \\
&= \frac{d}{dr}\left(-\frac{(r^4+r^3-r^2-r+6)}{6(r+3)^2(r^2+3r+2)^2} - \frac{(1+r)(r)}{6(3+r)^2(1+r)^2}\right) \\
&= \left(\frac{(4r^6+18r^5+20r^4-15r^3-18r^2+93r+114)}{6(r^3+6r^2+11r+6)^3}\right)_{r=0} \\
&= \frac{114}{6(6)^3} \\
&= \frac{19}{216}
\end{aligned}$$

And so on. Hence

$$\begin{aligned}
y_1 &= y_1\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_2} \\
&= y_1\ln(x) + \sum_{n=1}^{\infty} b_n x^n \\
&= y_1\ln(x) + \left(2x - \frac{19}{24}x^2 + \frac{19}{216}x^3 + \cdots\right)
\end{aligned}$$

Therefore the complete solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(1 - \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{144}x^4 + \cdots\right) \\
&+ c_2 \left(\left(1 - \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{144}x^4 + \cdots\right) \ln(x) + \left(2x - \frac{19}{24}x^2 + \frac{19}{216}x^3 + \cdots\right)\right)
\end{aligned}
$$

2.3.2.5.4 irregular singular point

ode internal name "second_order_series_method_irregular_singular_point"

expansion point is irregular singular point. Not supported.

2.3.2.6 Reduction of order

ode internal name "reduction_of_order"

This is second order ode where on solution is known. The second solution is found using reduction of order.

2.3.2.7 Transformation to a constant coefficient ODE methods

2.3.2.7.1 Introduction

Starting with a second order linear ode in the following normal form

$$
y'' + p(x)\,y' + q(x)\,y = r(x) \tag{A}
$$

The goal is to find a transformation that converts this ode to one with constant coefficients which is then easily solved. There are two transformations to try. One uses transformation on the independent variable x and the second is on the dependent variable y. The transformation on the independent variable uses $\tau = g(x)$ and the one on the dependent variable uses $y = v(x)\,z(x)$ and $y = v(x)\,x^n$ as special case.

2.3.2.7.2 Flow diagram

The following is diagram of the algorithms.

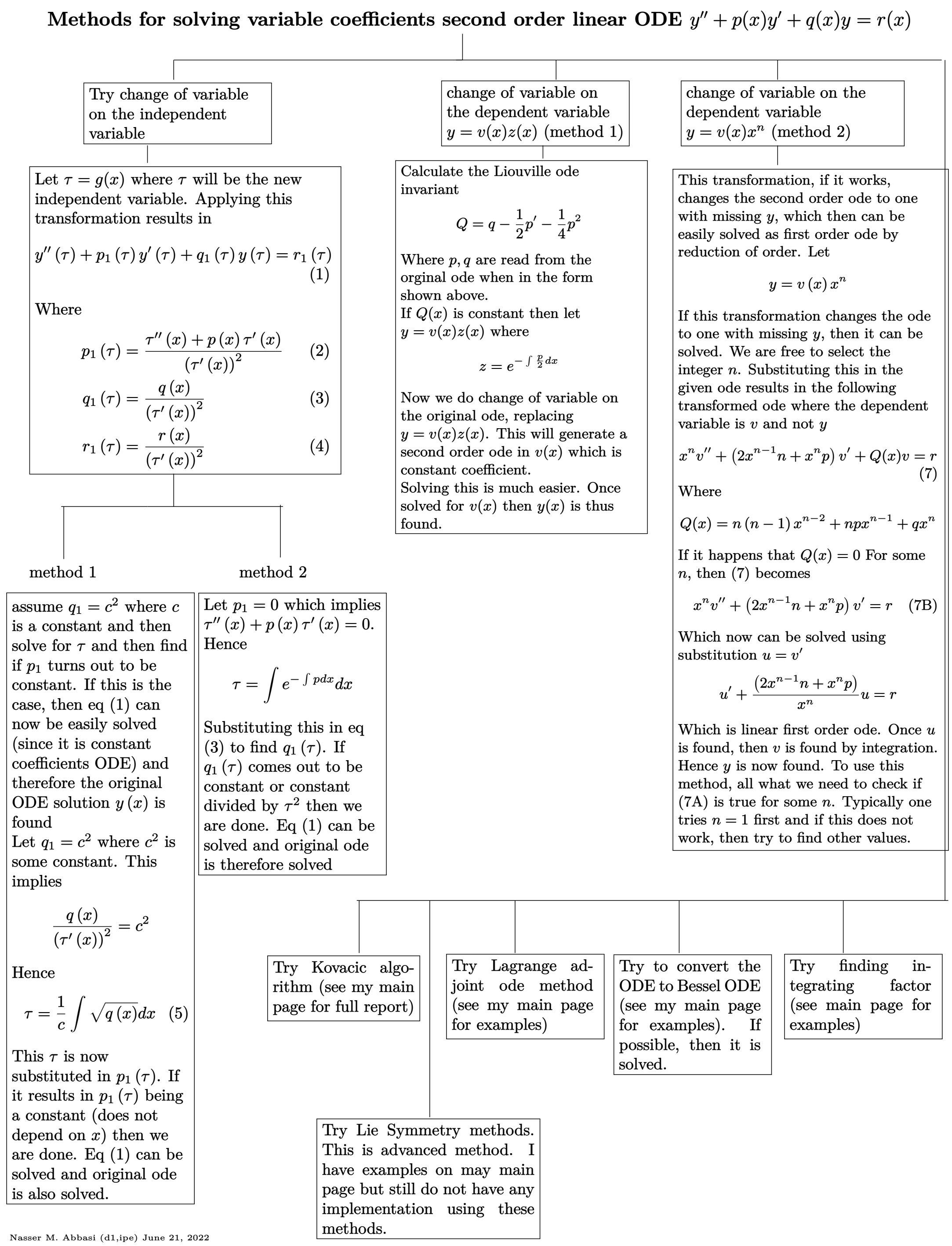

Figure 2.3: Algorithm diagram

2.3.2.7.3 Transformation on the independent variable x method 1

ode internal name "second_order_change_of_variable_on_x_method_1"

Given ode

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{A}$$

Let $\tau=g(x)$ where τ is the new independent variable. Applying this to (A) results in (details not shown)

$$y''(\tau)+p_1(\tau)\,y'(\tau)+q_1(\tau)\,y(\tau)=r_1(\tau) \tag{1}$$

Where

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\left(\tau'\left(x\right)\right)^2} \tag{2}$$

$$q_1(\tau)=\frac{q(x)}{\left(\tau'\left(x\right)\right)^2} \tag{3}$$

$$r_1(\tau)=\frac{r(x)}{\left(\tau'\left(x\right)\right)^2} \tag{4}$$

The idea of the transformation is to determine if ode (1) can be solved instead of (A).

Let $q_1=c^2$ where c is a constant then from (2)

$$\frac{q(x)}{\left(\tau'\left(x\right)\right)^2}=c^2$$

$$\tau'=\frac{1}{c}\sqrt{q\left(x\right)} \tag{5}$$

$$\tau''=\frac{1}{2c}\frac{q'(x)}{\sqrt{q\left(x\right)}} \tag{5A}$$

Substituting (5,5A) in (2) finds $p_1(\tau)$. If $p_1(\tau)$ is a constant (does not depend on x) then (1) can be solved for $y(\tau)$ and (A) is therefore solved for $y(x)$.

2.3.2.7.4 Transformation on the independent variable x method 2

ode internal name "second_order_change_of_variable_on_x_method_2"

Given ode

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{A}$$

Let $\tau=g(x)$ where τ is the new independent variable. Applying this to (A) results in (details not shown)

$$y''(\tau)+p_1(\tau)\,y'(\tau)+q_1(\tau)\,y(\tau)=r_1(\tau) \tag{1}$$

Where

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{(\tau'(x))^2} \tag{2}$$

$$q_1(\tau) = \frac{q(x)}{(\tau'(x))^2} \tag{3}$$

$$r_1(\tau) = \frac{r(x)}{(\tau'(x))^2} \tag{4}$$

The idea of the transformation is to determine if ode (1) can be solved instead of (A). Let $p_1 = 0$ then τ is solved for from $\tau''(x) + p(x)\,\tau'(x) = 0$.

$$\tau = \int e^{-\int p dx} dx$$

If this solution $\tau(x)$ results in q_1 above being a constant, then (1) can now be easily solved.

2.3.2.7.5 Transformation on the dependent variable (method 1) $y = v(x)\,z(x)$

ode internal name "second_order_change_of_variable_on_y_method_1"

This is also called Liouville transformation. Book by Einar Hille, ordinary differential equations in the complex domain. Page 179. This method assumes that

$$y = v(x)\,z(x)$$

Substituting this into (A) results in the following ode where the dependent variable is v and not y

$$v''(x) + \left(p + \frac{2}{z} z'(x)\right) v'(x) + \frac{1}{z}(z''(x) + pz'(x) + qz(x))\,v(x) = \frac{r}{z} \tag{6}$$

Assuming that coefficient of v' in (6) zero implies

$$p + \frac{2}{z} z'(x) = 0$$

Solving gives (where constant of integration is taken as one)

$$z = e^{-\int \frac{p}{2} dx} \tag{6A}$$

With this choice (6) becomes

$$v'' + \frac{1}{z}(z'' + pz' + qz)\,v = \frac{r}{z}$$

Substituting z from (6A) into the above reduces it to (after some algebra) to

$$v''+q_1v=r_1 \tag{6B}$$

Where

$$\begin{aligned} q_1 &= q-\frac{1}{2}p'-\frac{1}{4}p^2 \\ r_1 &= \frac{r}{z} \\ &= re^{\frac{1}{2}\int p dx} \end{aligned}$$

q_1 is called the Liouville ode invariant. If q_1 is constant, then the substitution $y = v(x)\,z(x)$ is used in the original original ode which will result in a constant coefficient ode. In $y=v(x)\,z(x)$ the $z(x)$ term is known from 6A and $v(x)$ is the new unknown dependent variable.

The new ode will be in $v(x)$ but with constant coefficients. Solving it for $v(x)$ gives y.

2.3.2.7.5.1 Example 1

$$y''+\frac{2}{x}y'+y=\frac{1}{x} \tag{1}$$

In the form $y''+p(x)\,y'+q(x)\,y=r(x)$ then $p=\frac{2}{x}, q=1, r=\frac{1}{x}$. Hence (6A) is

$$\begin{aligned} z &= e^{-\int \frac{p}{2}dx} \\ &= e^{-\int \frac{1}{x}dx} \\ &= e^{-\ln x} \\ &= \frac{1}{x} \end{aligned}$$

Now we check if q_1 is constant or a constant divided by x^2.

$$\begin{aligned} q_1 &= q-\frac{1}{2}p'-\frac{1}{4}p^2 \\ &= 1-\frac{1}{2}\left(\frac{2}{x}\right)'-\frac{1}{4}\left(\frac{2}{x}\right)^2 \\ &= 1-\left(-\frac{1}{x^2}\right)-\frac{1}{4}\frac{4}{x^2} \\ &= 1-\left(-\frac{1}{x^2}\right)-\frac{1}{4}\frac{4}{x^2} \\ &= 1+\frac{1}{x^2}-\frac{1}{x^2} \\ &= 1 \end{aligned}$$

Since q_1 is constant, then we can use the change of the variable $y = v(x)\, z(x)$ which is

$$y = \frac{v}{x}$$

Since $z = \frac{1}{x}$. Substituting the above into the original ODE (1) gives

$$\begin{aligned}
\left(\frac{v}{x}\right)'' + \left(\frac{2}{x}\left(\frac{v}{x}\right)'\right) + \frac{v}{x} &= \frac{1}{x}\\
\left(\frac{v'}{x} - \frac{v}{x^2}\right)' + \frac{2}{x}\left(\frac{v'}{x} - \frac{v}{x^2}\right) + \frac{v}{x} &= \frac{1}{x}\\
\left(\frac{v''}{x} - \frac{v'}{x^2} - \left(\frac{v'}{x^2} - 2\frac{v}{x^3}\right)\right) + \frac{2}{x}\left(\frac{v'}{x} - \frac{v}{x^2}\right) + \frac{v}{x} &= \frac{1}{x}\\
\frac{v''}{x} - \frac{v'}{x^2} - \frac{v'}{x^2} + 2\frac{v}{x^3} + \frac{2v'}{x^2} - \frac{2v}{x^3} + \frac{v}{x} &= \frac{1}{x}\\
\frac{v''}{x} - \frac{v'}{x^2} - \frac{v'}{x^2} + \frac{2v'}{x^2} + \frac{v}{x} &= \frac{1}{x}\\
\frac{v''}{x} + \frac{v}{x} &= \frac{1}{x}\\
v'' + v &= 1
\end{aligned}$$

This is constant coefficient ODE which is easily solved. If the ode in $v(x)$ did not come to be constant coefficient then we made a mistake. The solution is

$$v = c_1 \cos x + c_2 \sin x + 1$$

Hence

$$\begin{aligned}
y &= \frac{v}{x}\\
&= c_1 \frac{\cos x}{x} + c_2 \frac{\sin x}{x} + \frac{1}{x}
\end{aligned}$$

2.3.2.7.5.2 Example 2

$$\begin{aligned}
x^2 y'' - x(x+2)\, y' + (x+2)\, y &= 2x^3\\
y'' - \frac{x+2}{x} y' + \frac{x+2}{x^2} y &= 2x \qquad (1)
\end{aligned}$$

In the form $y'' + p(x)\, y' + q(x)\, y = r(x)$ then $p = -\frac{x+2}{x}, q = \frac{(x+2)}{x^2}, r = 2x$. Hence (6A) is

$$\begin{aligned}
z &= e^{-\int \frac{p}{2} dx}\\
&= e^{\int \frac{x+2}{2x} dx}\\
&= x e^{\frac{x}{2}}
\end{aligned}$$

Now we check if Liouville ode invariant q_1 is constant or a constant divided by x^2.

$$\begin{aligned}
q_1 &= q - \frac{1}{2}p' - \frac{1}{4}p^2 \\
&= \frac{(x+2)}{x^2} - \frac{1}{2}\left(xe^{\frac{x}{2}}\right)' - \frac{1}{4}\left(-\frac{x+2}{x}\right)^2 \\
&= -\frac{1}{4}
\end{aligned}$$

Since q_1 is constant, then we can use the change of the variable $y = v(x)\,z(x)$ which is

$$\begin{aligned}
y &= v(x)\,z(x) \\
&= v\left(xe^{\frac{x}{2}}\right)
\end{aligned}$$

Substituting the above into the original ODE (1) gives

$$\begin{aligned}
y'' - \frac{x+2}{x}y' + \frac{x+2}{x}y &= 2x \\
\left(v\left(xe^{\frac{x}{2}}\right)\right)'' - \frac{x+2}{x}\left(v\left(xe^{\frac{x}{2}}\right)\right)' + \frac{x+2}{x^2}v\left(xe^{\frac{x}{2}}\right) &= 2x
\end{aligned}$$

Carrying out the simplification gives

$$4v'' - v = 8e^{-\frac{x}{2}}$$

Which is constant coefficient ode. This is easily solved giving the solution

$$v = c_1 \sinh\left(\frac{x}{2}\right) + c_2 \cosh\left(\frac{x}{2}\right) - 2xe^{\frac{-x}{2}}$$

Hence

$$\begin{aligned}
y &= v(x)\,z(x) \\
&= \left(c_1 \sinh\left(\frac{x}{2}\right) + c_2 \cosh\left(\frac{x}{2}\right) - 2xe^{\frac{-x}{2}}\right) xe^{\frac{x}{2}}
\end{aligned}$$

2.3.2.7.5.3 Example 3

$$y'' - 4xy' + \left(4x^2 - 2\right)y = 0 \tag{1}$$

In the form $y'' + p(x)\,y' + q(x)\,y = r(x)$ then $p = -4x, q = (4x^2 - 2)\,, r = 0$. Hence (6A) is

$$\begin{aligned}
z &= e^{-\int \frac{p}{2}dx} \\
&= e^{\int 2x dx} \\
&= e^{x^2}
\end{aligned}$$

Now we check if Liouville ode invariant q_1 is constant or a constant divided by x^2.

$$
\begin{aligned}
q_1 &= q - \frac{1}{2}p' - \frac{1}{4}p^2 \\
&= \left(4x^2-2\right) - \frac{1}{2}(-4x)' - \frac{1}{4}(-4x)^2 \\
&= \left(4x^2-2\right) + 2 - \frac{1}{4}\left(16x^2\right) \\
&= 4x^2 - 2 + 2 - 4x^2 \\
&= 0
\end{aligned}
$$

Since q_1 is constant, then we can use the change of the variable $y = v(x)\,z(x)$ which is

$$
\begin{aligned}
y &= v(x)\,z(x) \\
&= v\left(e^{x^2}\right)
\end{aligned}
$$

Substituting the above into the original ODE (1) gives

$$
\begin{aligned}
y'' - 4xy' + \left(4x^2-2\right)y &= 0 \\
\left(ve^{x^2}\right)'' - 4x\left(ve^{x^2}\right)' + \left(4x^2-2\right)ve^{x^2} &= 0
\end{aligned}
$$

Carrying out the simplification gives

$$
v'' = 0
$$

Which is constant coefficient ode. This is easily solved giving the solution

$$
v = c_1 + c_2 x
$$

Hence

$$
\begin{aligned}
y &= v(x)\,z(x) \\
&= (c_1 + c_2 x)\,e^{x^2}
\end{aligned}
$$

2.3.2.7.5.4 Example 4

$$
x^2y'' + 3xy' + y = 0 \tag{1}
$$

This is of course Euler ode, and we do not need to try this method as solving it as Euler ode is much simpler. But this is just for illustration for the case when the Liouville ode invariant comes out not a constant. In the form $y'' + p(x)\,y' + q(x)\,y = r(x)$ then

$$
y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0 \tag{1A}
$$

Where now $p = \frac{3}{x}, q = \frac{1}{x^2}, r = 0$. Hence (6A) is

$$
\begin{aligned}
z &= e^{-\int \frac{p}{2} dx} \\
&= e^{\frac{-3}{2} \int \frac{1}{x} dx} \\
&= \frac{1}{x^{\frac{3}{2}}}
\end{aligned}
$$

Now we check if Liouville ode invariant q_1 is constant.

$$
\begin{aligned}
q_1 &= q - \frac{1}{2}p' - \frac{1}{4}p^2 \\
&= \left(\frac{1}{x^2}\right) - \frac{1}{2}\left(\frac{3}{x}\right)' - \frac{1}{4}\left(\frac{3}{x}\right)^2 \\
&= \left(\frac{1}{x^2}\right) - \frac{3}{2}\left(\frac{-1}{x^2}\right) - \frac{1}{4}\left(\frac{9}{x^2}\right) \\
&= \frac{1}{x^2} + \frac{3}{2x^2} - \frac{9}{4x^2} \\
&= \frac{1}{4x^2}
\end{aligned}
$$

Since q_1 is not constant then the ode can not not converted to an ode in $v(x)$ with constant coefficient.

2.3.2.7.5.5 Example 5

$$
xy'' + 2y' - xy = 0 \tag{1}
$$

In the form $y'' + p(x)\,y' + q(x)\,y = r(x)$ then

$$
y'' + \frac{2}{x}y' - y = 0 \tag{1A}
$$

Where now $p = \frac{2}{x}, q = -1, r = 0$. Hence (6A) is

$$
\begin{aligned}
z &= e^{-\int \frac{p}{2} dx} \\
&= e^{-\int \frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

Now we check if Liouville ode invariant q_1 is constant.

$$\begin{aligned}
q_1 &= q - \frac{1}{2}p' - \frac{1}{4}p^2 \\
&= (-1) - \frac{1}{2}\left(\frac{2}{x}\right)' - \frac{1}{4}\left(\frac{2}{x}\right)^2 \\
&= -1 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= -1 + \frac{1}{x^2} - \frac{1}{x^2} \\
&= -1
\end{aligned}$$

Since q_1 is constant, then we can use the change of the variable $y = v(x)\,z(x)$ which is

$$\begin{aligned}
y &= v(x)\,z(x) \\
&= v\frac{1}{x}
\end{aligned}$$

Substituting the above into the original ODE (1A) gives

$$\begin{aligned}
y'' + \frac{2}{x}y' - y &= 0 \\
\left(v\frac{1}{x}\right)'' + \frac{2}{x}\left(v\frac{1}{x}\right)' - v\frac{1}{x} &= 0
\end{aligned}$$

Carrying out the simplification gives

$$v'' - v = 0$$

Which is constant coefficient ode. This is easily solved giving the solution

$$v = c_1 e^{x} + c_2 e^{-x}$$

Hence

$$\begin{aligned}
y &= v(x)\,z(x) \\
&= \left(c_1 e^{x} + c_2 e^{-x}\right)\frac{1}{x}
\end{aligned}$$

2.3.2.7.5.6 Example 6

$$y'' - \frac{1}{\sqrt{x}}y' + \left(\frac{1}{4x} + \frac{1}{4x^{\frac{3}{2}}} - \frac{2}{x^2}\right)y = 0 \tag{1}$$

In the form $y'' + p(x)\,y' + q(x)\,y = r(x)$ then $p = -\frac{1}{\sqrt{x}}, q = \left(\frac{1}{4x} + \frac{1}{4x^{\frac{3}{2}}} - \frac{2}{x^2}\right), r = 0$. Hence (6A) is

$$\begin{aligned} z &= e^{-\int \frac{p}{2} dx} \\ &= e^{\int \frac{1}{\sqrt{x}} dx} \\ &= e^{2\sqrt{x}} \end{aligned}$$

Now we check if Liouville ode invariant q_1 is constant.

$$\begin{aligned} q_1 &= q - \frac{1}{2}p' - \frac{1}{4}p^2 \\ &= \left(\frac{1}{4x} + \frac{1}{4x^{\frac{3}{2}}} - \frac{2}{x^2}\right) - \frac{1}{2}\left(-\frac{1}{\sqrt{x}}\right)' - \frac{1}{4}\left(-\frac{1}{\sqrt{x}}\right)^2 \\ &= -\frac{2}{x^2} \end{aligned}$$

Not constant. Stop here. This can be solved using Kovacic algorithm.

2.3.2.7.6 Transformation on the dependent variable (method 2) $y = v(x)\,x^n$

<u>ode internal name</u> "second_order_change_of_variable_on_y_method_2"

This transformation, if it works, changes the second order ode to an one with missing y, which then can be solved as first order ode by reduction of order. This transformation does not necessarily changes the second order ode to one with constant coefficient like the above general transformation. But to an ode with missing y.

This method assumes

$$y = v(x)\,x^n$$

If this transformation changes the ode to one with missing y, then it can be used. Substituting this in (A) results in the following ode where the dependent variable is now v and not y

$$\begin{aligned} x^n v'' + \left(2x^{n-1}n + x^n p\right) v' + \left(n(n-1)\,x^{n-2} + npx^{n-1} + qx^n\right) v &= r \\ v'' + \left(2\frac{n}{x} + p\right) v' + \left(n(n-1)\,x^{-2} + npx^{-1} + q\right) v &= \frac{r}{x^n} \end{aligned} \tag{7}$$

If it happens that

$$n(n-1)\,x^{-2} + npx^{-1} + q = 0 \tag{7A}$$

For some integer or rational number n, then (7) becomes

$$v''+\left(2\frac{n}{x}+p\right)v'=\frac{r}{x^n} \tag{7B}$$

Which now can be solved using substitution $u=v'$.

$$u'+\left(2\frac{n}{x}+p\right)u=\frac{r}{x^n}$$

Which is linear first order ode. Once u is found, then v is by found integration. Hence y is now found. To use this method, all what we need is to check if (7A) is true for some number n. Typically one tries $n=\pm 1$ first and if this does not work, then try to find other values. Example below shows how to apply this method.

2.3.2.7.7 Worked Examples on all above 4 methods

2.3.2.7.7.1 Example 1. $xy''+2y'-xy=0$

Trying change of variable on independent variable first. Let $\tau=g(x)$ where z will be the new independent variable. Writing the ode in normal form gives

$$\begin{aligned} y''+py'+qy&=r\\ p&=\frac{2}{x}\\ q&=-1\\ r&=0 \end{aligned}$$

Applying $\tau=g(x)$ transformation on the above ode gives

$$y''(\tau)+p_1(\tau)\,y'(\tau)+q_1(\tau)\,y(\tau)=r_1(\tau) \tag{1}$$

Where

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{(\tau'(x))^2} \tag{2}$$

$$q_1(\tau)=\frac{q(x)}{(\tau'(x))^2} \tag{3}$$

$$r_1(\tau)=\frac{r(x)}{(\tau'(x))^2} \tag{4}$$

Approach 1. Let $q_1=c^2$ where c^2 is some constant. This implies

$$\frac{q}{(\tau'(x))^2}=c^2$$

$$\tau'=\frac{1}{c}\sqrt{q} \tag{5}$$

If p_1 is constant using this τ then (1) is a second order constant coefficient ode which can be solved easily. This ode has $q=-1$, therefore from (3)

$$\tau' = \frac{1}{c}\sqrt{-1}$$

Hence p_1 becomes using (2)

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x)+p\tau'(x)}{(\tau'(x))^2} \\ &= \frac{0+(2x^{-1})\frac{1}{c}\sqrt{-1}}{\frac{-1}{c^2}} \\ &= -2x^{-1}\sqrt{-1}c \end{aligned}$$

Which is <u>not a constant</u>. So this transformation failed.

<u>Approach 2</u> Let $p_1=0$. If with this choice now q_1 becomes constant or a constant divided by τ^2 then (2) can be integrated. $p_1=0$ implies from (2) that

$$\begin{aligned} \tau''+p\tau' &= 0 \\ \tau &= \int e^{-\int p\,dx}dx \\ &= \int e^{-\int 2x^{-1}dx}dx \\ &= \int x^{-2}dx \\ &= \frac{-1}{x} \end{aligned}$$

Using this then q_1 becomes

$$\begin{aligned} q_1 &= \frac{q}{(\tau'(x))^2} \\ &= \frac{-1}{\left(\frac{1}{x^2}\right)^2} \\ &= -x^4 \\ &= -\frac{1}{\tau^4} \end{aligned}$$

Which is not constant and nor a constant divided by τ^2. So this transformation did not work.

Trying change of variables on the dependent variable transformation (first method). This method assumes

$$y = v(x)\,z(x)$$

Substituting this in the given ode results in new ode where the dependent variable is v and not y which can be found to be

$$v''(x)+\left(p+\frac{2}{z}z'(x)\right)v'(x)+\frac{1}{z}(z''(x)+pz'(x)+qz(x))\,v(x)=\frac{r}{z}$$

Let $p+\frac{2}{z}z'(x)=0$. Solving gives $z=e^{-\int \frac{p}{2}dx}$. With this choice the above ode becomes

$$v''+\frac{1}{z}(z''+pz'+qz)\,v=\frac{r}{z}$$

Applying $z=e^{-\int \frac{p}{2}dx}$ to the above reduces it to

$$v''+q_1v=r_1 \tag{6}$$

Where

$$\begin{aligned} q_1 &= q-\frac{1}{2}p'-\frac{1}{4}p^2 \\ r_1 &= re^{\frac{1}{2}\int pdx} \end{aligned}$$

If q_1 turns out to be constant or a constant divided by x^2 with this choice of z, then v is solved for from (6) and the solution to the original ode is obtained. Applying this method on the given ode gives

$$\begin{aligned} z &= e^{-\int \frac{p}{2}dx} \\ &= e^{-\int x^{-1}dx} \\ &= e^{-\ln x} \\ &= x^{-1} \end{aligned}$$

Hence

$$\begin{aligned} q_1 &= q-\frac{1}{2}p'-\frac{1}{4}p^2 \\ &= -1+\frac{2}{2}x^{-2}-\frac{1}{4}\left(2x^{-1}\right)^2 \\ &= -1+x^{-2}-x^{-2} \\ &= -1 \end{aligned}$$

Since q_1 is constant, then this transformation works. Eq (6) now becomes

$$v''-v=0$$

The solution is

$$v=c_1e^{-x}+c_2e^{x}$$

Therefore, since $z=x^{-1}$ then

$$
\begin{aligned}
y &= v(x)\,z(x)\\
&= \frac{1}{x}\left(c_1 e^{-x}+c_2 e^{x}\right)
\end{aligned}
$$

This example shows that change of variable on the independent variable did not work, but change of variable on the dependent variable (general case) worked.

Trying change of variable on the dependent variable (second method). This method assumes that

$$
y = v(x)\,x^n
$$

For some n, This transformation changes the ode to an ode with a missing y, which can be easily solved as two first order ode's. Substituting this in (A) results in the following ode where the dependent variable is v and not y

$$
x^n v'' + \left(2x^{n-1}n + x^n p\right) v' + \left(n(n-1)\,x^{n-2} + npx^{n-1} + qx^n\right) v = r \tag{7}
$$

If it happens that

$$
\left(n(n-1)\,x^{n-2} + npx^{n-1} + qx^n\right) = 0 \tag{7A}
$$

For some n, then (7) becomes

$$
x^n v'' + \left(2x^{n-1}n + x^n p\right) v' = r \tag{7B}
$$

Which can be solved using substitution $u=v'$ to give

$$
u' + \frac{(2x^{n-1}n + x^n p)}{x^n} u = r
$$

Applying (7A) on this example ode gives

$$
\begin{aligned}
\left(n(n-1)\,x^{n-2} + n\left(\frac{2}{x}\right)x^{n-1} + (-1)\,x^n\right) &= 0\\
n(n-1)\,x^{n-2} + 2nx^{n-2} - x^n &= 0\\
\left(n+n^2\right)x^{n-2} - x^n &= 0
\end{aligned}
$$

It is clear that there exists no integer or rational number n which makes the LHS above zero. Hence this special transformation did not work.

This is an example where only the change of variable on the dependent variable (general case) worked.

2.3.2.7.7.2 Example 2. Euler ODE $x^2y''(x)+xy'(x)+y(x)=0$

One way to solve Euler ODE

$$x^2y''(x)+xy'(x)+y(x)=0 \tag{A}$$

Putting it in normal form gives

$$y''(x)+\frac{1}{x}y'(x)+\frac{1}{x^2}y(x)=0$$

Hence

$$\begin{aligned}p&=\frac{1}{x}\\ q&=\frac{1}{x^2}\\ r&=0\end{aligned}$$

Trying change of variable on the independent variable. Let $\tau=g(x)$ where τ will be the new independent variable. Applying this transformation results in

$$y''+p_1y'+q_1y=r_1 \tag{1}$$

Where

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{(\tau'(x))^2} \tag{2}$$

$$q_1(\tau)=\frac{q(x)}{(\tau'(x))^2} \tag{3}$$

$$r_1(\tau)=\frac{r(x)}{(\tau'(x))^2} \tag{4}$$

Approach 1. Let $q_1=c^2$ where c^2 is some constant. This implies

$$\frac{q}{(\tau'(x))^2}=c^2$$

$$\tau=\frac{1}{c}\int\sqrt{q}dx \tag{5}$$

If with this τ, then p_1 turns out to be constant, then (1) is now a second order constant coefficient ode which is easily solved. Applying (5) on the given ode gives

$$\begin{aligned}\tau&=\frac{1}{c}\int\sqrt{x^{-2}}dx\\ &=\frac{1}{c}\ln x\end{aligned}$$

Using the above on (2) gives

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x)+p\tau'(x)}{(\tau'(x))^2} \\ &= 0 \end{aligned}$$

Which is a constant. Hence this transformation worked. Therefore(1) becomes (using $q_1 = c^2$ which is a constant c^2)

$$\begin{aligned} y''(\tau)+p_1 y'(\tau)+q_1 y(\tau) &= r_1 \\ y''(\tau)+c^2 y(\tau) &= 0 \end{aligned}$$

The solution is

$$y(\tau) = A\cos(c\tau) + B\sin(c\tau)$$

But $\tau = \frac{1}{c}\ln x$. Hence the above becomes

$$y(x) = A\cos(\ln x) + B\sin(\ln x)$$

In practice, this longer method is not needed to solve Euler ode $x^2y''(x)+xy'(x)+y(x)=0$ as that the substitution $y=x^r$ works more easily. But the above method is more general. For example, using $y=x^r$, then $x^2y''(x)+xy'(x)+y(x)=0$ becomes $r(r-1)+r+1=0$. The roots r are $i,-i$. Then the solution is linear combination of the basis solutions given by

$$\begin{aligned} y &= Ax^i + Bx^{-i} \\ &= Ae^{\ln x^i} + Be^{\ln x^{-i}} \\ &= Ae^{i\ln x} + Be^{-i\ln x} \\ &= A\cos(\ln x) + B\sin(\ln x) \end{aligned}$$

Where the last step used Euler relation to do the conversion. Another known transformation for Euler (which is not as simple as the above) is to use $x=e^t$. Using this gives

$$\frac{dx}{dt} = e^t \tag{2}$$

But $\ln x = t$, hence

$$\frac{dt}{dx} = \frac{1}{x} \tag{3}$$

To do this change of variable and obtain a new ode where now $y(x)$ becomes $y(t)$, then $y'(x)$ is changed to $y'(t)$ and $y''(x)$ is changed $y''(t)$. Using

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} \tag{4}$$

Substituting (3) into (4) gives

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{1}{x}$$

But $\frac{1}{x}=e^{-t}$. The above becomes

$$\frac{dy}{dx}=e^{-t}\frac{dy}{dt} \tag{5}$$

Now $y''(x)$ needs to change to $y''(t)$. Since

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Substituting (5) into the above gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right)$$

Dividing the numerator and denominator of $\frac{d}{dx}$ by dt gives

$$\begin{aligned}\frac{d^2y}{dx^2}&=\frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(e^{-t}\frac{dy}{dt}\right)\\&=\frac{dt}{dx}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\end{aligned}$$

But from (3) $\frac{dt}{dx}=\frac{1}{x}=e^{-t}$. Hence the above becomes

$$\frac{d^2y}{dx^2}=e^{-t}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)$$

Using the the product rule gives

$$\begin{aligned}\frac{d^2y}{dx^2}&=e^{-t}\left(-e^{-t}\frac{dy}{dt}+e^{-t}\frac{d^2y}{dt^2}\right)\\&=e^{-2t}\left(-\frac{dy}{dt}+\frac{d^2y}{dt^2}\right)\\&=e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\end{aligned} \tag{6}$$

Now $y'(x)$ and $y''(x)$ have been converted to $y'(t),y''(t)$. Substituting (5,6) in the gives ode gives

$$\begin{aligned}x^2y''(x)+xy'(x)+y(x)&=0\\x^2e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+xe^{-t}\frac{dy}{dt}+y(t)&=0\end{aligned}$$

But $x=e^t$ and $x^2=e^{2t}$. The above becomes

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}+\frac{dy}{dt}+y(t)=0$$
$$\frac{d^2y}{dt^2}+y(t)=0$$

This is now constant coefficient ODE. The solution is

$$y(t)=A\cos(t)+B\sin(t)$$

Since $\ln x=t$, then the above becomes

$$y(x)=A\cos(\ln x)+B\sin(\ln x)$$

This completes the solution.

2.3.2.7.7.3 Example 3. $y''\sin^2(2x)+y'\sin(4x)-4y=0$

Writing the ode in normal form gives

$$y''+p(x)\,y'+q(x)\,y=r$$
$$p=\frac{\sin(4x)}{\sin^2(2x)} \qquad \sin(2x)\neq 0$$
$$q=-\frac{4}{\sin^2(2x)}$$

Trying change of variable on the independent variable as above. Let $\tau=g(x)$ where τ will be the new independent variable. Applying this transformation results in

$$y''+p_1y'+q_1y=r_1 \tag{1}$$

Where

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{(\tau'(x))^2} \tag{2}$$
$$q_1(\tau)=\frac{q(x)}{(\tau'(x))^2} \tag{3}$$
$$r_1(\tau)=\frac{r(x)}{(\tau'(x))^2} \tag{4}$$

Approach 1. Let $q_1=c^2$ where c is some constant. This implies

$$\frac{q}{(\tau'(x))^2}=c^2$$
$$\tau=\frac{1}{c}\int\sqrt{q}dx \tag{5}$$

If with this τ, then p_1 turns out to be constant, then it means (1) is second order constant coefficient ode. Applying this on the given ode (5) becomes

$$\begin{aligned}
\tau &= \frac{1}{c}\int \sqrt{-\frac{4}{\sin^2(2x)}}dx \\
&= \frac{2i}{c}\int \frac{1}{\sin(2x)}dx \\
&= \frac{i}{c}\ln(\csc(2x)-\cot(2x))
\end{aligned}$$

Eq (2) now becomes

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x)+p\tau'(x)}{(\tau'(x))^2} \\
&= 0
\end{aligned}$$

Which is constant. Hence this transformation worked. Therefore (1) becomes (since $q_1 = c^2$ is constant c^2)

$$\begin{aligned}
y''(\tau)+p_1 y'(\tau)+q_1 y(\tau) &= r_1 \\
y''+c^2 y &= 0
\end{aligned}$$

This gives

$$y(\tau) = A\cos(c\tau)+B\sin(c\tau)$$

Using $\tau = \frac{i}{c}\ln(\csc(2x)-\cot(2x))$ the above becomes

$$y(x) = A\cos(i\ln(\csc(2x)-\cot(2x)))+B\sin(i\ln(\csc(2x)-\cot(2x)))$$

Simplifying using trig identities gives

$$\begin{aligned}
y(x) &= \frac{-iB\cos(2x)+A}{\sin(2x)} \\
&= \frac{B_0\cos(2x)}{\sin(2x)}+\frac{A}{\sin(2x)} \\
&= B_0\cot(2x)+A\csc(2x)
\end{aligned}$$

Approach 2 Let $p_1 = 0$. If with this choice now q_1 becomes constant or a constant divided by τ^2 then (2) can be integrated. $p_1 = 0$ implies from (2) that

$$\begin{aligned}
\tau''+p\tau' &= 0 \\
\tau &= \int e^{-\int p dx}dx \\
\tau &= \int e^{-\int \frac{\sin(4x)}{\sin^2(2x)}dx}dx \\
\tau &= \int \frac{1}{\sin(2x)}dx
\end{aligned}$$

Using this gives

$$\begin{aligned} q_1 &= \frac{q}{(\tau'(x))^2} \\ &= \frac{-\frac{4}{\sin^2(2x)}}{-\frac{1}{\frac{1}{\sin^2(2x)}}} \\ &= -4 \end{aligned}$$

Which is a constant. Hence this transformation also works. Eq (1) now becomes

$$\begin{aligned} y'' + p_1 y' + q_1 y &= r_1 \\ y''(\tau) - 4y(\tau) &= 0 \\ y(\tau) &= A e^{-2\tau} + B e^{2\tau} \end{aligned}$$

But $\tau = \int \frac{1}{\sin(2x)} dx = \frac{1}{2} \ln(\csc(2x) - \cot(2x))$, hence

$$\begin{aligned} y(x) &= A e^{-2\frac{1}{2}\ln(\csc(2x)-\cot(2x))} + B e^{2\frac{1}{2}\ln(\csc(2x)-\cot(2x))} \\ &= A e^{-\ln(\csc(2x)-\cot(2x))} + B e^{\ln(\csc(2x)-\cot(2x))} \\ &= \frac{A}{\csc(2x) - \cot(2x)} + B \csc(2x) - \cot(2x) \end{aligned}$$

Which can be simplified to same solution shown in approach 1. This was an example where both sub methods of change of variable on the independent variable worked.

2.3.2.7.7.4 Example 4. $(1-x^2)\, y'' - x y' + y = 0$

Writing the ode in normal form gives

$$\begin{aligned} y'' + p(x)\, y' + q(x)\, y &= r \\ p &= \frac{-x}{(1-x^2)} \qquad x \neq 1, x \neq -1 \\ q &= \frac{1}{(1-x^2)} \end{aligned}$$

Trying change of variable on the independent variable as above. Let $\tau = g(x)$ where τ will be the new independent variable. Applying this transformation results in

$$y'' + p_1 y' + q_1 y = r_1 \tag{1}$$

Where

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{(\tau'(x))^2} \tag{2}$$

$$q_1(\tau) = \frac{q(x)}{(\tau'(x))^2} \tag{3}$$

$$r_1(\tau) = \frac{r(x)}{(\tau'(x))^2} \tag{4}$$

Approach 1. Let $q_1 = c^2$ where c^2 is some constant. This implies

$$\frac{q}{(\tau'(x))^2} = c^2$$

$$\tau = \frac{1}{c}\int \sqrt{q}dx \tag{5}$$

If with this τ, then p_1 turns out to be constant, then it means (1) is second order constant coefficient ode which is easily solved. Using the given ode (5) becomes

$$\begin{aligned}\tau &= \frac{1}{c}\int \sqrt{\frac{1}{(1-x^2)}}dx\\ &= \frac{i}{c}\ln\left(x+\sqrt{x^2-1}\right)\end{aligned}$$

Hence (2) now becomes

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p\tau'(x)}{(\tau'(x))^2}\\ &= 0\end{aligned}$$

Which is constant. Hence this transformation worked. Therefore the ode (1) becomes (since $q_1 = c^2$ is constant c^2)

$$\begin{aligned}y''(\tau) + p_1 y'(\tau) + q_1 y(\tau) &= r_1\\ y'' + c^2 y &= 0\end{aligned}$$

The solution is

$$y(\tau) = A\cos(c\tau) + B\sin(c\tau)$$

Using $\tau = \frac{i}{c}\ln\left(x+\sqrt{x^2-1}\right)$ the above becomes

$$y(x) = A\cos\left(i\ln\left(x+\sqrt{x^2-1}\right)\right) + B\sin\left(i\ln\left(x+\sqrt{x^2-1}\right)\right)$$

Approach 2 Let $p_1 = 0$. If with this choice now q_1 becomes constant or a constant divided by τ^2 then (2) can be integrated. $p_1 = 0$ implies from (2) that

$$\begin{aligned}
\tau'' + p\tau' &= 0\\
\tau &= \int e^{-\int p dx} dx\\
\tau &= \int e^{\int \frac{x}{(1-x^2)} dx} dx\\
\tau &= \int \frac{1}{\sqrt{x-1}\sqrt{x+1}} dx
\end{aligned}$$

Therefore

$$\begin{aligned}
q_1 &= \frac{q}{(\tau'(x))^2}\\
&= \frac{\frac{1}{(1-x^2)}}{\left(\frac{1}{\sqrt{x-1}\sqrt{x+1}}\right)^2}\\
&= \frac{\frac{1}{(1-x^2)}}{\frac{1}{(x-1)(x+1)}}\\
&= \frac{\frac{1}{(1-x^2)}}{\frac{1}{x^2-1}}\\
&= -1
\end{aligned}$$

Which is a constant. This transformation also worked. Eq (1) becomes

$$\begin{aligned}
y'' + p_1 y' + q_1 y &= r_1\\
y''(\tau) - y(\tau) &= 0\\
y(\tau) &= Ae^{-\tau} + Be^{\tau}
\end{aligned}$$

Using $\tau = \int \frac{1}{\sqrt{x-1}\sqrt{x+1}} dx = \ln\left(x + \sqrt{x^2-1}\right)$, $(x > 1)$ the above

$$\begin{aligned}
y(x) &= Ae^{-\tau} + Be^{\tau}\\
&= Ae^{-\ln\left(x+\sqrt{x^2-1}\right)} + Be^{\ln\left(x+\sqrt{x^2-1}\right)}\\
&= \frac{A}{x+\sqrt{x^2-1}} + B\left(x + \sqrt{x^2-1}\right)
\end{aligned}$$

This solution looks different from the solution found above using approach 1, but can be shown to be the same. This was an example where both methods of change of variable on the independent variable work.

2.3.2.7.7.5 Example 5. $x^2y''-xy'+\left(-x^2-\frac{1}{4}\right)y=0$

Writing the ode in normal form gives

$$y''+p(x)\,y'+q(x)\,y=r$$

$$p=\frac{-1}{x}\qquad x\neq 0$$

$$q=-\frac{x^2+\frac{1}{4}}{x^2}$$

$$r=0$$

Trying change of variable on the independent variable as above. Let $\tau=g(x)$ where τ will be the new independent variable. Applying this transformation results in

$$y''+p_1y'+q_1y=r_1 \tag{1}$$

Where

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\left(\tau'\left(x\right)\right)^2} \tag{2}$$

$$q_1(\tau)=\frac{q(x)}{\left(\tau'\left(x\right)\right)^2} \tag{3}$$

$$r_1(\tau)=\frac{r(x)}{\left(\tau'\left(x\right)\right)^2} \tag{4}$$

Approach 1. Let $q_1=c^2$ where c^2 is some constant. This implies

$$\frac{q}{\left(\tau'\left(x\right)\right)^2}=c^2$$

$$\tau=\frac{1}{c}\int\sqrt{q}dx \tag{5}$$

If with this τ, then p_1 turns out to be constant, then it means (1) is second order constant coefficient ode which is easily solved. Applying this on the given ode then (5)

$$\tau=\frac{1}{c}\int\sqrt{-\frac{x^2+\frac{1}{4}}{x^2}}dx$$

$$=\frac{1}{2c}\sqrt{-4x^2-1}+\arctan\left(\frac{1}{\sqrt{-4x^2-1}}\right)$$

Hence (2) now becomes

$$p_1(\tau)=\frac{\tau''(x)+p\tau'(x)}{\left(\tau'\left(x\right)\right)^2}$$

$$=\frac{\left(8x^2+4\right)c}{\left(-4x^2-1\right)^{\frac{3}{2}}}$$

Which is not constant. Therefore this transformation did not work.

Approach 2 Let $p_1 = 0$. If with this choice now q_1 becomes constant or a constant divided by τ^2 then (2) can be integrated. $p_1 = 0$ implies from (2) that

$$
\begin{aligned}
\tau'' + p\tau' &= 0 \\
\tau &= \int e^{-\int p dx} dx \\
&= \int e^{\int \frac{1}{x} dx} dx \\
&= \int e^{\ln x} dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned}
$$

Using this then q_1 becomes

$$
\begin{aligned}
q_1 &= \frac{q}{(\tau'(x))^2} \\
&= \frac{-\frac{x^2+\frac{1}{4}}{x^2}}{x^2} \\
&= -\frac{1}{x^4}\left(x^2 + \frac{1}{4}\right)
\end{aligned}
$$

Which is not constant. Trying change of variable on the dependent variable (first method). This method assumes

$$
y = v(x)\, z(x)
$$

The Liouville ode invariant is

$$
\begin{aligned}
q_1 &= q - \frac{1}{2}p' - \frac{1}{4}p^2 \\
&= -\frac{x^2+\frac{1}{4}}{x^2} - \frac{1}{2}\frac{d}{dx}\left(\frac{-1}{x}\right) - \frac{1}{4}\left(\frac{-1}{x}\right)^2 \\
&= -\frac{1}{x^2}\left(x^2+1\right)
\end{aligned}
$$

Which is not constant. Hence this method does not work. One way to solve this is as a Bessel ODE. I have many examples how to do this on my main page.

2.3.2.7.7.6 Example 6. $(x^2-1)\,y''-2xy'+2y=0$

Writing the ode in normal form gives

$$
\begin{aligned}
y''+p(x)\,y'+q(x)\,y&=r\\
p&=\frac{-2x}{x^2-1}\qquad x\neq\pm1\\
q&=\frac{2}{x^2-1}\\
r&=0
\end{aligned}
$$

Trying change of variable on the independent variable as above. Let $\tau=g(x)$ where τ will be the new independent variable. Applying this transformation results in

$$
y''+p_1y'+q_1y=r_1 \tag{1}
$$

Where

$$
\begin{aligned}
p_1(\tau)&=\frac{\tau''(x)+p(x)\,\tau'(x)}{(\tau'(x))^2} &(2)\\
q_1(\tau)&=\frac{q(x)}{(\tau'(x))^2} &(3)\\
r_1(\tau)&=\frac{r(x)}{(\tau'(x))^2} &(4)
\end{aligned}
$$

Approach 1. Let $q_1=c^2$ where c^2 is some constant. This implies

$$
\begin{aligned}
\frac{q}{(\tau'(x))^2}&=c^2\\
\tau&=\frac{1}{c}\int\sqrt{q}dx &(5)
\end{aligned}
$$

If with this τ, then p_1 turns out to be constant, then it means (1) is second order constant coefficient ode. Applying this on the given ode (5) becomes

$$
\begin{aligned}
\tau&=\frac{1}{c}\int\sqrt{\frac{2}{x^2-1}}dx\\
&=\frac{1}{c}\sqrt{2}\ln\left(x+\sqrt{x^2-1}\right)
\end{aligned}
$$

Hence (2) reduces to

$$
\begin{aligned}
p_1(\tau)&=\frac{\tau''(x)+p\tau'(x)}{(\tau'(x))^2}\\
&=-\frac{3\sqrt{2}cx}{\sqrt{\frac{1}{x^2-1}}\,(2x^2-2)}
\end{aligned}
$$

Which is not constant. This transformation did not work.

Approach 2 Let $p_1 = 0$. If with this choice now q_1 becomes constant or a constant divided by τ^2 then (2) can be easily integrated. $p_1 = 0$ implies from (2) that

$$\begin{aligned}
\tau'' + p\tau' &= 0 \\
\tau &= \int e^{-\int p dx} dx \\
&= \int e^{\int \frac{2x}{x^2-1} dx} dx \\
&= \int (x^2 - 1) \, dx
\end{aligned}$$

Hence q_1 becomes

$$\begin{aligned}
q_1 &= \frac{q}{(\tau'(x))^2} \\
&= \frac{\frac{2}{x^2-1}}{(x^2 - 1)^2} \\
&= \frac{2}{(x^2 - 1)^3}
\end{aligned}$$

Which is not constant. This transformation did not work.

Trying change of variable on the dependent variable (first method). This method assumes that

$$y = v(x) \, z(x)$$

The Liouville ode invariant is

$$\begin{aligned}
q_1 &= q - \frac{1}{2}p' - \frac{1}{4}p^2 \\
&= \left(\frac{2}{x^2 - 1}\right) - \frac{1}{2}\frac{d}{dx}\left(\frac{-2x}{x^2 - 1}\right) - \frac{1}{4}\left(\frac{-2x}{x^2 - 1}\right)^2 \\
&= -\frac{3}{(x^2 - 1)^2}
\end{aligned}$$

Which is not constant and not constant divided by x^2. Hence this transformation also did not work.

Trying the Lagrange adjoint ode method. From above the adjoint ode is

$$z'' - \frac{d(zp)}{dx} + zq = 0$$

For some unknown function $z(x)$. Hence it becomes

$$
\begin{aligned}
z'' - \frac{d}{dx}\left(z\left(\frac{-2x}{x^2-1}\right)\right) + z\left(\frac{2}{x^2-1}\right) &= 0 \\
z'' - \left(-\frac{2z'x}{x^2-1} + \frac{4zx^2}{(x^2-1)^2} - \frac{2z}{x^2-1}\right) + z\left(\frac{2}{x^2-1}\right) &= 0 \\
z'' + \frac{2x}{x^2-1}z' - \frac{4x^2+4(x^2-1)}{(x^2-1)^2}z &= 0
\end{aligned}
$$

Clearly this is just as hard to solve as the original ode So this method does it work.

Trying integrating factor method. For this to work the condition is that $\frac{1}{2}\left(p' + \frac{1}{2}p^2\right) = q$. Applying this on the current ode gives

$$
\begin{aligned}
\frac{1}{2}\left(p' + \frac{1}{2}p^2\right) &= q \\
\frac{1}{2}\left(\frac{d}{dx}\left(\frac{-2x}{x^2-1}\right) + \frac{1}{2}\left(\frac{-2x}{x^2-1}\right)^2\right) &= \frac{2}{x^2-1} \\
\frac{(2x^2+1)}{(x^2-1)^2} &= \frac{2}{x^2-1} \\
\frac{2x^2+1}{x^2-1} &= 2
\end{aligned}
$$

Which is not true. Hence there is no integrating factor.

Trying transformation on the dependent variable (second method). This method assumes

$$
y = v(x)\,x^n
$$

This works only if (7A) given in the introduction is satisfied.

$$
\left(n(n-1)\,x^{n-2} + npx^{n-1} + qx^n\right) = 0 \tag{7A}
$$

Applying this on the current ode example gives

$$
\left(n(n-1)\,x^{n-2} + n\left(\frac{-2x}{x^2-1}\right)x^{n-1} + \left(\frac{2}{x^2-1}\right)x^n\right) = 0
$$

Trying $n = 1$ the above becomes

$$
\left(\left(\frac{-2x}{x^2-1}\right) + \left(\frac{2}{x^2-1}\right)x\right) = 0
$$

Hence this transformation works for $n=1$. Therefore $y=v(x)\,x$. eq (7) in the introduction now reduces to

$$x^n v'' + \left(2x^{n-1}n + x^n p\right) v' + \left(n(n-1)\,x^{n-2} + npx^{n-1} + qx^n\right) v = r \tag{7}$$
$$v'' + \frac{(xp+2)}{x} v' = 0$$

Which now can be solved using substitution $u=v'$.

$$u' + \frac{(xp+2)}{x} u = r$$

Which is linear first order ode. Once u is found, then v is found by integration. Hence y is now found. Hence

$$u' - \frac{2}{x^3-x} u = 0$$

Which has the solution $u = c_1 \frac{x^2}{x^2-1}$. Hence $v' = c_1 \frac{x^2}{x^2-1}$. Integrating gives $v = c_1\left(x+\frac{1}{x}\right) + c_2$. Therefore $y = xv = c_1(x^2+1) + c_2 x$

This was an example where only the transformation on the dependent second method $y=v(x)\,x^n$ worked.

2.3.2.7.7.7 Example 7. $xy'' + (x^2-1)\,y' + x^3 y = 0$

Writing the ode in normal form gives

$$\begin{aligned}
y'' + p(x)\,y' + q(x)\,y &= r\\
p &= \frac{x^2-1}{x} \qquad x \neq 0\\
q &= x^2\\
r &= 0
\end{aligned}$$

Trying change of variable on the independent variable as above. Let $\tau = g(x)$ where τ will be the new independent variable. Applying this transformation results in

$$y'' + p_1 y' + q_1 y = r_1 \tag{1}$$

Where

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\left(\tau'(x)\right)^2} \tag{2}$$
$$q_1(\tau) = \frac{q(x)}{\left(\tau'(x)\right)^2} \tag{3}$$
$$r_1(\tau) = \frac{r(x)}{\left(\tau'(x)\right)^2} \tag{4}$$

Approach 1. Let $q_1 = c^2$ where c^2 is some constant. This implies

$$\begin{aligned}\frac{q}{\left(\tau'(x)\right)^2} &= c^2\\ \tau' &= \frac{1}{c}\sqrt{q}\end{aligned} \tag{5}$$

If p_1 turns out to be constant with this τ then it implies (1) is second order constant coefficient ode. Eq (5) becomes

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{x^2}\\ \tau''(x) &= \frac{1}{2c}\frac{2x}{\sqrt{x^2}}\end{aligned}$$

Hence from (2)

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p\tau'(x)}{\left(\tau'(x)\right)^2}\\ &= \frac{\frac{1}{2c}\frac{2x}{\sqrt{x^2}} + \left(\frac{x^2-1}{x}\right)\frac{1}{c}\sqrt{x^2}}{\left(\frac{1}{c}\sqrt{x^2}\right)^2}\\ &= c\end{aligned}$$

Which is a constant. Then (1) becomes second order of constant coefficient

$$y''(\tau) + cy'(\tau) + c^2 y(\tau) = 0$$

Which has the solution

$$y(\tau) = e^{-\frac{c\tau}{2}}\left(A\sin\left(\frac{c\sqrt{3}\tau}{2}\right) + B\sin\left(\frac{c\sqrt{3}\tau}{2}\right)\right)$$

But from earlier $\tau = \frac{x^2}{2c}$. Hence the above becomes

$$\begin{aligned}y(x) &= Ae^{-\frac{c\frac{x^2}{2c}}{2}}\sin\left(\frac{c\sqrt{3}\frac{x^2}{2c}}{2}\right) + Be^{-\frac{c\frac{x^2}{2c}}{2}}\sin\left(\frac{c\sqrt{3}\frac{x^2}{2c}}{2}\right)\\ &= e^{-\frac{x^2}{4}}\left(A\sin\left(\frac{\sqrt{3}x^2}{4}\right) + B\sin\left(\frac{\sqrt{3}x^2}{4}\right)\right)\end{aligned}$$

Approach 2

Let $p_1 = 0$. If with this choice now q_1 becomes constant or a constant divided by τ^2 then (2) can be easily integrated. $p_1 = 0$ implies from (2) that

$$
\begin{aligned}
\tau'' + p\tau' &= 0 \\
\tau &= \int e^{-\int p dx} dx \\
&= \int e^{-\int \frac{x^2-1}{x} dx} dx \\
&= \int x e^{-\frac{x^2}{2}} dx \\
&= -e^{-\frac{x^2}{2}}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
q_1 &= \frac{q}{(\tau'(x))^2} \\
&= \frac{x^2}{\left(x e^{-\frac{x^2}{2}}\right)^2} \\
&= e^{x^2}
\end{aligned}
$$

Which is not constant. Now it is checked to see if it is constant divided by τ^2. Since $\tau^2 = \left(-e^{-\frac{x^2}{2}}\right)^2 = e^{-x^2}$ then $q_1 = \frac{1}{\tau^2}$. Therefore this approach also worked.

Eq (2) becomes

$$
\begin{aligned}
y'' + p_1 y' + q_1 y &= 0 \qquad (1) \\
y'' + \frac{1}{\tau^2} y &= 0 \\
\tau^2 y'' + y &= 0
\end{aligned}
$$

Which is standard Euler ode which can be solved easily. Giving

$$
y(\tau) = A\sqrt{\tau} \cos\left(\frac{\sqrt{3}}{2} \ln(\tau)\right) + B\sqrt{\tau} \sin\left(\frac{\sqrt{3}}{2} \ln(\tau)\right)
$$

But $\tau = -e^{-\frac{x^2}{2}}$. Hence the above becomes

$$
y(x) = A\sqrt{-e^{-\frac{x^2}{2}}} \cos\left(\frac{\sqrt{3}}{2} \ln\left(-e^{-\frac{x^2}{2}}\right)\right) + B\sqrt{-e^{-\frac{x^2}{2}}} \sin\left(\frac{\sqrt{3}}{2} \ln\left(-e^{-\frac{x^2}{2}}\right)\right)
$$

This looks different from the solution obtained in approach 1, but it verifies also as correct solution. This is an example where change of independent variable using $q_1 = c^2$ works and also change of independent variable using $p_1 = 0$ works as well.

2.3.2.7.7.8 Example 8. $4x^2\sin(x)\,y''+(-4x^2\cos x-4x\sin x)\,y'+(2x\cos x+3\sin x)\,y=0$

Writing the ode in normal form gives

$$
\begin{aligned}
y''+p(x)\,y'+q(x)\,y&=0\\
p&=\frac{-4x^2\cos x-4x\sin x}{4x^2\sin(x)} \qquad x\neq 0,\pi,2\pi,\cdots\\
q&=\frac{2x\cos x+3\sin x}{4x^2\sin(x)}\\
r&=0
\end{aligned}
$$

Applying transformation on the dependent variable second method $y=v(x)\,x^n$ results in

$$
\begin{aligned}
x^nv''+\left(2nx^{n-1}+px^n\right)v'+\left(n(n-1)\,x^{n-2}+px^{n-1}n+qx^n\right)v&=0\\
v''+\frac{\left(2nx^{n-1}+px^n\right)}{x^n}v'+\left(\frac{n(n-1)\,x^{n-2}+px^{n-1}n+qx^n}{x^n}\right)v&=0\\
v''+\left(2nx^{-1}+p\right)v'+\left(n(n-1)\,x^{-2}+px^{-1}n+q\right)v&=0\\
v''+\left(2nx^{-1}+p\right)v'+\left(pnx^{-1}+q+\left(n^2-n\right)x^{-2}\right)v&=0 \qquad (1)
\end{aligned}
$$

Assuming the coefficient of $v(x)$ above is zero. This gives

$$
pnx^{-1}+q+\left(n^2-n\right)x^{-2}=0
$$

Substituting the values for p,q in the above gives

$$
\left(\frac{-4x^2\cos x-4x\sin x}{4x^2\sin(x)}\right)nx^{-1}+\frac{2x\cos x+3\sin x}{4x^2\sin(x)}+\left(n^2-n\right)x^{-2}=0
$$

Solving for n shows that $n=\frac{1}{2}$. Hence (1) now reduces to

$$
\begin{aligned}
v''+\left(x^{-1}+p\right)v'&=0\\
v''+\left(\frac{1}{x}+\frac{-4x^2\cos x-4x\sin x}{4x^2\sin(x)}\right)v'&=0\\
v''+\left(\frac{4x\sin x-4x^2\cos x-4x\sin x}{4x^2\sin x}\right)v'&=0\\
v''+\left(\frac{-4x^2\cos x}{4x^2\sin x}\right)v'&=0\\
v''-\frac{\cos x}{\sin x}v'&=0
\end{aligned}
$$

Let $v'=u$, the above becomes

$$u' - \frac{\cos x}{\sin x}u = 0$$

Which is linear first order ode. It has the solution $u = c_1 \sin(x)$. Hence

$$v' = c_1 \sin(x)$$

Integrating gives

$$v = -c_1 \cos(x) + c_2$$

Therefore

$$\begin{aligned} y &= v\sqrt{x} \\ &= (-c_1 \cos(x) + c_2)\sqrt{x} \end{aligned}$$

This can also be written as

$$y = (c_3 \cos(x) + c_2)\sqrt{x}$$

2.3.2.7.7.9 Example 9 $x^2y'' - (2a-1)xy' + a^2y = 0$

The above is standard Euler ode. But below shows how to apply these transformations if one did not know this.

Trying change of variable on independent variable first. Let $\tau = g(x)$ where z will be the new independent variable. Writing the ode in normal form gives

$$\begin{aligned} y'' + py' + qy &= r \\ p &= \frac{(1-2a)}{x} \\ q &= \frac{a^2}{x^2} \\ r &= 0 \end{aligned}$$

Applying $\tau = g(x)$ transformation on the above ode gives

$$y''(\tau) + p_1(\tau)\,y'(\tau) + q_1(\tau)\,y(\tau) = r_1(\tau) \tag{1}$$

Where

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{(\tau'(x))^2} \tag{2}$$

$$q_1(\tau) = \frac{q(x)}{(\tau'(x))^2} \tag{3}$$

$$r_1(\tau) = \frac{r(x)}{(\tau'(x))^2} \tag{4}$$

Approach 1. Let $q_1 = c^2$ where c^2 is some constant. This implies

$$\frac{q}{(\tau'(x))^2} = c^2$$
$$\tau' = \frac{1}{c}\sqrt{q} \tag{5}$$

If p_1 is constant using this τ then (1) is a second order constant coefficient ode which can be solved easily. This ode has $q = \frac{a^2}{x^2}$, therefore from (5) assuming positive

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{\frac{a^2}{x^2}}\\ &= \frac{a}{cx}\end{aligned}$$

Hence p_1 becomes using (2)

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p\tau'(x)}{(\tau'(x))^2}\\ &= \frac{(1-2a)\,c}{x}\end{aligned}$$

Which is not a constant. So this transformation failed.

Approach 2 Let $p_1 = 0$. If with this choice q_1 becomes a constant or a constant divided by τ^2 then (2) can be integrated. $p_1 = 0$ implies from (2) that

$$\begin{aligned}\tau'' + p\tau' &= 0\\ \tau &= \int e^{-\int p dx} dx\\ &= \int e^{-\int \frac{(1-2a)}{x} dx} dx\\ &= \int x^{2a-1} dx\\ &= \frac{x^{2a}}{2a}\end{aligned}$$

Using this then q_1 becomes

$$\begin{aligned}q_1 &= \frac{q}{(\tau'(x))^2}\\ &= \frac{\left(\frac{a^2}{x^2}\right)}{(x^{2a-1})^2}\\ &= \frac{a^2}{x^2 x^{4a-2}}\\ &= \frac{a^2}{x^{4a}}\end{aligned} \tag{6}$$

Which is not constant. But $\tau^2 = \left(\frac{x^{2a}}{2a}\right)^2 = \frac{x^{4a}}{4a^2}$. Hence $q_1 = \frac{1}{4}\frac{1}{\tau^2}$. Hence this transformation works. Eq (2) becomes

$$\begin{aligned} y'' + p_1 y' + q_1 y &= 0 \quad (1) \\ y'' + \frac{1}{4}\frac{1}{\tau^2} y &= 0 \\ 4\tau^2 y'' + y &= 0 \end{aligned}$$

Which is standard Euler ode which can be solved easily. Giving

$$y(\tau) = A\sqrt{\tau} + B\sqrt{\tau}\ln(\tau)$$

But $\tau = \frac{x^{2a}}{2a}$. Hence the above becomes

$$\begin{aligned} y(x) &= A\sqrt{\frac{x^{2a}}{2a}} + B\sqrt{\frac{x^{2a}}{2a}}\ln\left(\frac{x^{2a}}{2a}\right) \\ &= A\sqrt{\frac{x^{2a}}{2a}} + B\sqrt{\frac{x^{2a}}{2a}}\ln\left(\frac{x^{2a}}{2a}\right) \\ &= A_1 x^a + B_1 x^a \ln\left(\frac{x^{2a}}{2a}\right) \end{aligned}$$

2.3.2.7.7.10 Example 10. Bessel ODE

Given the ode

$$y''(x) + \left(1 - \frac{3}{4x^2}\right) y(x) = 0 \quad \text{(A)}$$

Trying change of variables on the dependent variable (first method). In this method we assume

$$y = v(x)\, z(x)$$

The ode is $y'' + py' + qy = 0$. Hence $p = 0, q = \left(1 - \frac{3}{4x^2}\right)$. Therefore the Liouville ode invariant is

$$\begin{aligned} q_1 &= q - \frac{1}{2}p' - \frac{1}{4}p^2 \\ &= \left(1 - \frac{3}{4x^2}\right) \end{aligned}$$

Since q_1 is not constant, then this method does not work.

Trying change of variable on independent variable.

Let $z=g(x)$ where z will be the new independent variable. In general, given an ode of the form

$$y''(x)+p(x)\,y'(x)+q(x)\,y(x)=r(x)$$

Then applying this transformation results in

$$y''(z)+p_1(z)\,y'(z)+q_1(z)\,y(z)=r_1(z) \tag{1}$$

Where

$$p_1(z)=\frac{z''(x)+pz'(x)}{\left(z'(x)\right)^2} \tag{2}$$

$$q_1(z)=\frac{q}{\left(z'(x)\right)^2} \tag{3}$$

$$r_1(z)=\frac{r}{\left(z'(x)\right)^2} \tag{4}$$

Approach 1. Let $q_1=c^2$ where c^2 is some constant. This implies

$$\frac{q}{\left(z'(x)\right)^2}=c^2$$

$$z=\frac{1}{c}\int\sqrt{q}dx \tag{5}$$

If with this z, then p_1 turns out to be constant, then it means (1) is second order constant coefficient ode. Applying this on current ode then (5) becomes

$$\begin{aligned}z&=\frac{1}{c}\int\sqrt{\left(1-\frac{3}{4x^2}\right)}dx\\&=\frac{1}{2c}\left(\sqrt{4x^2-3}+\sqrt{3}\arctan\left(\frac{\sqrt{3}}{\sqrt{4x^2-3}}\right)\right)\end{aligned}$$

Hence (2) becomes

$$\begin{aligned}p_1(z)&=\frac{z''(x)+pz'(x)}{\left(z'(x)\right)^2}\\&=\frac{6c}{(4x^2-3)^{\frac{3}{2}}}\end{aligned}$$

Which is not a constant. So this transformation did not work. So change of variable on both the dependent and independent variable does not work for this ode to convert it to one with constant coefficient. Trying converting it to standard Bessel ODE. Using this change of variable on the dependent variable

$$y = ux^{\frac{1}{2}}$$

To transform (A) to standard Bessel ODE

$$x^2u'' + xu' + \left(x^2 - 1\right)u = 0$$

Since $y = ux^{\frac{1}{2}}$ then

$$\frac{dy}{dx} = \frac{du}{dx}x^{\frac{1}{2}} + u\frac{x^{\frac{-1}{2}}}{2} \tag{2A}$$

And

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{du}{dx}x^{\frac{1}{2}} + u\frac{x^{\frac{-1}{2}}}{2}\right)\\
&= \frac{d}{dx}\left(\frac{du}{dx}x^{\frac{1}{2}}\right) + \frac{d}{dx}\left(u\frac{x^{\frac{-1}{2}}}{2}\right)\\
&= \frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{1}{2}\frac{du}{dx}x^{-\frac{1}{2}} + \frac{1}{2}\frac{du}{dx}x^{\frac{-1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}}\\
&= \frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}} \qquad (3A)
\end{aligned}$$

Substituting (2A,3A) into (A) gives

$$\begin{aligned}
\frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}} + \left(1 - \frac{3}{4x^2}\right)ux^{\frac{1}{2}} &= 0\\
\frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}} + ux^{\frac{1}{2}} - \frac{3}{4}ux^{-\frac{3}{2}} &= 0\\
\frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - ux^{-\frac{3}{2}} + ux^{\frac{1}{2}} &= 0
\end{aligned}$$

Multiplying both side by $x^{\frac{3}{2}}$ gives

$$\begin{aligned}
x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - u + ux^2 &= 0\\
x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - \left(1 - x^2\right)u &= 0\\
x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \left(x^2 - 1\right)u &= 0
\end{aligned}$$

Which is Bessel ode where order is $n = 1$. This has known standard solution. Once $u(x)$ is known, then $y(x)$ which is the solution to the original ODE (A) is now known also. There is a more general method and better method to find if second order ode can be transformed to Bessel ODE. See my main page for examples and description.

2.3.2.8 Exact linear second order ode $p(x)\,y''+q(x)\,y'+r(x)\,y=0$

ode internal name "exact_linear_second_order_ode"

Given the ode

$$p(x)\,y''+q(x)\,y'+r(x)\,y=0 \tag{1}$$

We want to first find the condition for exactness. This is the same as saying the above ode has a corresponding adjoint ode, which is $(py'+B(x)\,y)'=0$. i.e. if an ode $p(x)\,y''+q(x)\,y'+r(x)\,y=0$ can be written in the form $(py'+B(x)\,y)'=0$ for some $B(x)$ then the ode $(py'+By)'=0$ is called the adjoint of $py''+qy'+ry=0$ which is the same thing as saying the ode $p(x)\,y''+q(x)\,y'+r(x)\,y=0$ is exact. i.e. it has complete differential.

The goal therefore is to determine if a linear second order ode has a corresponding adjoint ODE or not of the form $(py'+B(x)\,y)'=0$. If so, then it is exact and we can solve it by solving the adjoint ODE instead since it is much simpler to solve as it is a first order ODE. Lets see how to find the adjoint ODE.

Let

$$py''+q(x)\,y'+r(x)\,y=(py'+By)'$$

Expanding gives

$$py''+qy'+ry=p'y'+py''+B'y+By'$$

Comparing coefficients

$$\begin{aligned} q &= p'+B \\ r &= B' \end{aligned}$$

Differentiating the first ode gives $q'=p''+B'$. Using the second ode gives $q'=p''+r$ or

$$p''-q'+r=0 \tag{2}$$

This is the condition for exactness. i.e. if the input ODE (1) satisfies (2) then the ODE is exact and has an adjoint ODE of the form $(py'+By)'=0$ which we now can be easily solve since it is complete differential.

$$\begin{aligned} (py'+B(x)\,y)' &= 0 \\ (py'+(q-p')\,y)' &= 0 \end{aligned} \tag{3}$$

We see that solving (3) is much simpler than (1) since (3) is first order. Integrating this once gives

$$py'+(q-p')\,y=c$$

This is first order ode. This is also called the first integral equation of (1). In summary, given an ode $py''+qy'+ry=0$ which is exact, then its first integral is $py'+(q-p')\,y=c$ and the solution to this is the solution to the original second order ode.

2.3.2.8.1 Example 1

$$x^2y''+xy'-y=x^4$$

Then $p=x^2, q=x, r=-1, f(x)=x^4$. Condition (2) becomes

$$\begin{aligned}p''-q'+r&=0\\2-1-1&=0\\0&=0\end{aligned}$$

Hence it is second order exact. Therefore the adjoint ode (3) is

$$\begin{aligned}\left(x^2y'+(x-2x)\,y\right)'&=x^4\\x^2y'+(x-2x)\,y&=\int x^4dx+c\\x^2y'-xy&=\frac{x^5}{5}+c\end{aligned}$$

The first integral is

$$\begin{aligned}x^2y'+(x-2x)\,y&=\int x^4dx+c\\x^2y'-xy&=\frac{x^5}{5}+c\end{aligned}$$

This is linear ode. Solving this ode gives

$$y=\frac{x^4}{15}-\frac{c}{2x}+c_2x$$

Note that this is also a Euler ode.

2.3.2.8.2 Example 2

$$y''+xy'+y=0$$

Here $p=1, q=x, r=1$. Let $F(x,y,y',y'')=y''+xy'+y$. The condition for exactness is

$$p''-q'+r=0$$

Hence the above becomes

$$\begin{aligned} 0-1+1&=0\\ 0&=0 \end{aligned}$$

The ode is already exact. i.e. no integrating factor is needed. The solution becomes

$$\begin{aligned} (py'+(q-p')\,y)'&=0\\ (y'+xy)'&=0 \end{aligned}$$

The first integral is

$$y'+xy=c_1$$

Solving this gives

$$\begin{aligned} \frac{d}{dx}(Iy)&=Ic_1\\ \frac{d}{dx}\left(ye^{\int x dx}\right)&=e^{\int x dx}c_1\\ ye^{\int x dx}&=\int e^{\int x dx}c_1 dx+c_2\\ y&=e^{\int -x dx}\left(\int e^{\int x dx}c_1 dx\right)+c_2e^{\int -x dx}\\ &=c_1e^{\frac{-x^2}{2}}\left(\int e^{\frac{x^2}{2}}dx\right)+c_2e^{\frac{-x^2}{2}dx}\\ &=e^{\frac{-x^2}{2}}\left(c_1\int e^{\frac{x^2}{2}}dx+c_2\right) \end{aligned}$$

2.3.2.9 Linear second order not exact but solved by finding $\mu(x)$ integrating factor.

ode internal name "linear_second_order_ode_solved_by_mu_integrating_factor"

(**not implemented yet**).

As mentioned above, an exact ode is one which has a corresponding adjoint ODE. In the case when the ode was exact, we did not use an integrating factor (this is the same as saying the integrating factor was 1), i.e. $\mu(x)=1$.

But if the ode is not exact, then we look for integrating factor $\mu(x)$ that when multiplied by the ode makes it exact and hence will have an adjoint ODE. Given

$$py''+q(x)\,y'+r(x)\,y=f(x) \tag{1}$$

Which is assumed not to be exact. Multiplying both sides by $\mu(x)$ gives $\mu(py''+q(x)\,y'+r(x)\,y)=\mu f(x)$. Let

$$\mu(py''+q(x)\,y'+r(x)\,y)=(\mu py'+By)' \tag{2}$$

Expanding gives

$$\begin{aligned}\mu(py''+q(x)\,y'+r(x)\,y)&=\mu'py'+\mu p'y'+\mu py''+B'y+By'\\ \mu py''+\mu qy'+\mu ry&=\mu py''+y'(\mu'p+\mu p'+B)+yB'\end{aligned}$$

Comparing coefficients gives the following 2 equations

$$\mu q=\mu'p+\mu p'+B \tag{2A}$$
$$\mu r=B' \tag{2B}$$

Taking derivative of (2A) gives

$$\mu'q+\mu q'=\mu''p+\mu'p'+\mu'p'+\mu p''+B'$$

Substituting for B' from (2B) into the above gives

$$\mu'q+\mu q'=\mu''p+\mu'p'+\mu'p'+\mu p''+\mu r \tag{3}$$

Arranging

$$\mu''p+\mu'(2p'-q)+\mu(p''-q'+r)=0 \tag{4}$$

The integrating factor μ is the solution to the above ODE (called the adjoint ode also). Note that in (4), the term $p''-q'+r$ will not be zero, as this is the condition for exactness, and this ode is not exact (else we will not need an integrating factor to start with).

We can obtain (4) directly from $py''+qy'+ry=0$. Since the relation between an ode and its adjoint ode is the following: given

$$py''+qy'+ry=0$$

Its adjoint ode is

$$\begin{aligned}\left((p\mu)'-q\mu\right)'+r\mu&=0\\ (p\mu)''-(q\mu)'+r\mu&=0\\ (p'\mu+p\mu')'-(q'\mu+q\mu')+r\mu&=0\\ p''\mu+p'\mu'+p'\mu'+p\mu''-q'\mu-q\mu'+r\mu&=0\\ p\mu''+\mu'(2p'-q)+\mu(p''-q'+r)&=0\end{aligned}$$

We see this is the same as (4). In summary, an ode $py''+qy'+ry=0$ has adjoint ode $(p\mu)''-(q\mu)'+r\mu=0$ where the solution to the adjoint ode makes the first ode exact. Once the integrating factor μ is found then the first integral is given by

$$py''+qy'+ry=(\mu py'+By)'$$

Where

$$\begin{aligned}B&=\mu q-\mu' p-\mu p'\\&=\mu(q-p')-\mu' p\end{aligned}$$

Hence

$$py''+qy'+ry=(\mu py'+(\mu(q-p')-\mu' p)\,y)' \tag{5}$$

There is a known relation between an ode and its adjoint ode given by

$$\mu(py''+qy'+ry)-y\overline{(py''+qy'+ry)}=\frac{d}{dx}(P(y,u))$$

Where the bar above the ode means its complex conjugate. The function $P(y,u)$ is called the bilinear concomitant (see Murphy book, page 93). And is given by

$$P(y,u)=p(y'\mu-y\mu')+(q-p')\,y\mu$$

Unfortunately, all this does not help us in solving the adjoint ode (4) in order to find the integrating factor μ. Since it will also be a second order ode which can be as hard to solve as the original ode. So this method is not practical as far as I can see unless the adjoint ODE comes out very simple to solve, but in all the examples I looked at, this was not the case.

2.3.2.9.1 Example 1

$$y''-4xy'+\left(4x^2-2\right)y=0$$

$p=1,q=-4x,r=(4x^2-2)$. Let us first check if the ode is exact or not as is. The condition for exactness is

$$p''-q'+r=0$$

Therefore the above becomes

$$0+4+\left(4x^2-2\right)=0$$

The LHS is not zero. This means the ode is not exact. Therefore we need to try to find an integration factor $\mu(x)$ to make the ode exact. (4) becomes

$$\begin{aligned}\mu''p+\mu'(2p'-q)+\mu(p''-q'+r)&=0\\ \mu''+\mu'(4x)+\mu\left(4+\left(4x^2-2\right)\right)&=0\\ \mu''+4x\mu'+\mu\left(2+4x^2\right)&=0\end{aligned}$$

We see in practice that finding the integrating factor leads to yet another second order ode which is as hard to solve as the original ode. The solution to this ode can be found to be e^{-x^2}, xe^{-x^2}. We only need one integrating factor. Hence let

$$\mu(x) = e^{-x^2}$$

Multiplying this by the given ode now makes it exact

$$e^{-x^2}y'' - 4xe^{-x^2}y' + \left(4x^2-2\right)e^{-x^2}y = 0$$

To see this let us check the condition again now. Here $p = e^{-x^2}, q = -4xe^{-x^2}, r = \left(4x^2-2\right)e^{-x^2}$. Hence

$$\begin{aligned} p'' - q' + r &= 0 \\ \left(4e^{-x^2}x^2 - 2e^{-x^2}\right) - \left(8e^{-x^2}x^2 - 4e^{-x^2}\right) + \left(4x^2-2\right)e^{-x^2} &= 0 \\ 0 &= 0 \end{aligned}$$

We see that it is now exact. Hence it has adjoint ODE of the form (5)

$$\left(\mu p y' + \left(\mu(q-p') - \mu' p\right)y\right)' = 0$$

Hence the first integral is

$$\mu p y' + \left(\mu(q-p') - \mu' p\right)y = c$$

Using $\mu = e^{-x^2}, p = 1, q = -4x$ the above becomes

$$\begin{aligned} e^{-x^2}y' + \left(-4xe^{-x^2} - \left(-2xe^{-x^2}\right)\right)y &= c \\ e^{-x^2}y' - 2xe^{-x^2}y &= c \\ y' - 2xy &= ce^{x^2} \end{aligned}$$

This is linear first ode whose solution is

$$y = e^{x^2}(cx + c_2)$$

2.3.2.9.2 Example 2

$$y'' + \frac{1}{x}y' + \frac{1}{x}y = 0$$

Here $p = 1, q = \frac{1}{x}, r = \frac{1}{x}, f(x) = 0$. The condition of exactness is

$$\begin{aligned} p'' - q' + r &= 0 \\ 0 - \left(-\frac{1}{x^2}\right) + \frac{1}{x} &= 0 \end{aligned}$$

Is not satisfied. Hence the ode is not exact. The adjoint ode (4) to find the integrating factor becomes

$$
\begin{aligned}
\mu'' p + \mu'(p' - q) + \mu(p'' - q' + r) &= 0 \\
\mu'' + \mu'\left(-\frac{1}{x}\right) + \mu\left(-\frac{1}{x^2} + \frac{1}{x}\right) &= 0 \\
\mu'' - \frac{1}{x}\mu' - \mu\left(\frac{1-x}{x^2}\right) &= 0 \\
x^2\mu'' - x\mu' - (1-x)\,\mu &= 0
\end{aligned}
$$

Which has solutions μ as bessel functions. We see that trying to find an integrating factor using this method is not practical, as it leads to an ode just as hard to solve as the original one. We could just have solved $y'' + \frac{1}{x}y' + \frac{1}{x}y = 0$ directly, since this is Bessel ode. Unless there is a short cut to solving the ODE to find the integrating factor, this method is not practical. See section below for simpler method

The main difficulty when second order is not exact, is in finding the integrating factor $\mu(x)$ which itself requires solving another second order ode. The whole point of an ODE being exact is that it is a complete differential which means the order is reduced by one to make it easier to solve. This means solving a second order ode becomes solving a first order ode when the ode is exact.

2.3.2.10 Linear second order not exact but solved by finding an M integrating factor.

ode internal name "linear_second_order_ode_solved_by_an_M_integrating_factor"

This is another method to find integrating factor method for the second order ode. This method of finding an integrating factor is not a general one like the above using $\mu(x)$ but it is easier to check. This is tried first and if this does not work, then the above will be tried.

Given the ode, normalized so that the coefficient of y'' is one

$$
y'' + Q(x)\,y' + R(x)\,y = f(x) \tag{1}
$$

Let there exists an integrating factor $M(x)$ such that

$$
(M(x)\,y)'' = M(x)\,f(x) \tag{2}
$$

Then it can be integrated twice and solved. To find M, the above becomes

$$\begin{aligned}
(M'y+My')' &= Mf\\
M''y+M'y'+M'y'+My'' &= Mf\\
My''+y'(2M')+M''y &= Mf\\
y''+y'\left(2\frac{M'}{M}\right)+\frac{M''}{M}y &= f \qquad (2A)
\end{aligned}$$

Comparing (2A) to (1) gives

$$\begin{aligned}
2\frac{M'}{M} &= Q\\
\frac{M''}{M} &= R
\end{aligned}$$

Or

$$M'-\frac{1}{2}MQ=0 \qquad (3)$$

$$M''-MR=0 \qquad (4)$$

Starting with (3) gives $M=e^{\frac{1}{2}\int Q dx}$. If this also satisfies (4), then M is found by integration. If not, then this method did not work.

2.3.2.10.1 Example 1

$$y''-4xy'+\left(4x^2-2\right)y=0$$

Hence $Q=-4x$ and $R=(4x^2-2)$, $f(x)=0$. Eq(3) becomes

$$M'-\frac{1}{2}MQ=0$$

Therefore

$$\begin{aligned}
M &= e^{\frac{1}{2}\int Q dx}\\
&= e^{\frac{1}{2}\int -4x dx}\\
&= e^{-x^2}
\end{aligned}$$

Now we much check that equation (4) is verified with such M.

$$\begin{aligned}
M' &= -2xe^{-x^2}\\
M'' &= -2e^{-x^2}-2x\left(-2e^{-x^2}\right)\\
&= -2e^{-x^2}+4xe^{-x^2}
\end{aligned}$$

Substituting these in (4) gives

$$\begin{aligned}\left(-2e^{-x^2}+4xe^{-x^2}\right)-e^{-x^2}\left(4x^2-2\right)&=0\\ -2e^{-x^2}+4xe^{-x^2}+2e^{-x^2}-4x^2e^{-x^2}&=0\\ 0&=0\end{aligned}$$

M is satisfied. Therefore the integrating factor is

$$M=e^{-x^2}$$

Eq (2) now becomes

$$\begin{aligned}(My)''&=0\\ My'&=c_1\\ My&=c_1x+c_2\\ y&=\frac{c_1x+c_2}{M}\\ &=(c_1x+c_2)\,e^{x^2}\end{aligned}$$

Which is the same answer found using the more general method of $\mu(x)$ in the above section but this is simpler when it works since it does not involve solving another ode (the adjoint ode) to find an integrating factor.

2.3.2.10.2 Example 2

Here is an example where the method of integrating factor does not work.

$$y''+\frac{1}{x}y'+\frac{1}{x}y=0$$

Here $p=1,q=\frac{1}{x},r=\frac{1}{x},f(x)=0$. The condition of exactness is

$$\begin{aligned}p''-q'+r&=0\\ 0-\left(-\frac{1}{x^2}\right)+\frac{1}{x}&=0\end{aligned}$$

Is not satisfied. Hence the ode is not exact. Therefore let us try to find M. Using

$$\begin{aligned}M&=e^{\frac{1}{2}\int q\,dx}\\ &=e^{\frac{1}{2}\ln x}\\ &=\sqrt{x}\end{aligned}$$

Therefore $M' = \frac{1}{2}x^{\frac{-1}{2}}$ and $M'' = -\frac{1}{4}x^{\frac{-3}{2}}$. Substituting these in (4) to verify gives (using $r = x^{-1}$)

$$
\begin{aligned}
-\frac{1}{4}x^{\frac{-3}{2}} - x^{\frac{1}{2}}\left(x^{-1}\right) &= 0 \\
-\frac{1}{4}x^{\frac{-3}{2}} - x^{-\frac{1}{2}} &= 0
\end{aligned}
$$

Which does not verify as the LHS is not zero. Therefore the integrating method did not work on this ode.

An easier method to find if an M integrating factor exists is the following. Since $M = e^{\frac{1}{2}\int q dx}$ then substituting this into (2A) gives

$$
y'' + y'\left(2\frac{M'}{M}\right) + \frac{M''}{M}y = f(x)
$$

Since $M' = \frac{1}{2}qM$ and since

$$
\begin{aligned}
M'' &= \frac{1}{2}(q'M + qM') \\
&= \frac{1}{2}\left(q'M + \frac{1}{2}q^2M\right)
\end{aligned}
$$

Then (2A) now becomes

$$
\begin{aligned}
y'' + y'\left(2\frac{\frac{1}{2}qM}{M}\right) + \frac{\frac{1}{2}\left(q'M + \frac{1}{2}q^2M\right)}{M}y &= f \\
y'' + qy' + \frac{1}{2}\left(q' + \frac{1}{2}q^2\right)y &= f
\end{aligned}
$$

By comparing the above to the given ode in normal form shows that for M to exist the condition is

$$
r = \frac{1}{2}\left(q' + \frac{1}{2}q^2\right)
$$

if the above is true, then M exists and is given by

$$
M = e^{\frac{1}{2}\int q dx}
$$

Using this method on the first example above $y'' - 4xy' + (4x^2-2)\,y = 0$, where $q = -4x$ and $r = (4x^2-2)$. Checking if $(4x^2-2) = \frac{1}{2}\left(q' + \frac{1}{2}q^2\right)$, then $\frac{1}{2}\left(-4 + \frac{1}{2}(16x^2)\right) = 4x^2 - 2 = r$. Hence M exists. This is a much faster method to determine if M exists or not.

The second example $y'' + \frac{1}{x}y' + \frac{1}{x}y = 0$ where $q = \frac{1}{x}, r = \frac{1}{x}$, then $\frac{1}{2}\left(q' + \frac{1}{2}q^2\right) = \frac{1}{2}\left(-x^{-2} + \frac{1}{2}x^{-2}\right) = -\frac{1}{4x^2} \neq r$. Therefore no M exists and the integration factor does not exist for this ode. Note this does not mean there is no integrating factor. It just means this short cut method which I call the M integrating factor does not work.

2.3.2.11 Solved using Lagrange adjoint equation method.

<u>ode internal name</u> "second order ode lagrange adjoint equation method"

This method is used when hint is "adjoint". This transformation does not use change of variables. It was discovered by Lagrange in his Miscellanea Taurensis paper. It reduces the order of the ode by one, assuming the so called adjoint ode can be solved. This is also described in section 1.5.1 on page 14 of the "book Change and Variations A History of Differential Equations to 1900" by Jeremy Gray. This method will only work if adjoint equation turns out to be simple and can be solved. It is now only used by the program if the hint "adjoint" is detected or if all the other methods were first tried and they all fail to solve the ode. So this method works if the adjoint ode can be solved. But the adjoint ode itself is second order non constant ode. So we need to solve a second order non-constant ode in order to reduce the order by one of the original ode. Luckily the adjoint ode turns out to be possible to solve by change of variables when the original one is not, and that is why this method is tried.

Given the ode

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{1}$$

This method starts by multiplying the ode by some unknown function $z\equiv z(x)$ which gives

$$zy''+zpy'+zqy=zr \tag{2}$$

Integrating gives

$$\int zy''dx+\int zpy'dx+\int zqydx=\int zrdx \tag{3}$$

Using integration by parts on $\int zpy'dx$ using $\int udv=uv-\int vdu$ where $u=zp$ and $dv=y'$, hence $v=y$ and $du=\frac{d}{dx}(zp)$. Therefore

$$\int zpy'dx=zpy-\int y\frac{d(zp)}{dx}dx$$

Using integration by parts on $\int zy''dx$ using $\int udv=uv-\int vdu$ where $u=z$ and $dv=y''$, hence $v=y'$ and $du=z'$. Therefore

$$\int zy''dx=zy'-\int y'z'dx$$

Eq (3) becomes

$$\left(zy'-\int y'z'dx\right)+\left(zpy-\int y\frac{d(zp)}{dx}dx\right)+\int zqydx=\int zrdx \tag{4}$$

Integrating by part again the term $\int y'z'dx$ using $\int udv = uv - \int vdu$ where $u = z'$ and $dv = y'$, hence $v = y$ and $du = z''$. Therefore

$$\int y'z'dx = yz' - \int yz''dx$$

Substituting this in (4) gives

$$\begin{aligned}
\left(zy' - \left(yz' - \int yz''dx\right)\right) + \left(zpy - \int y\frac{d(zp)}{dx}dx\right) + \int zqydx &= \int zrdx \\
zy' - yz' + \int yz''dx + zpy - \int y\frac{d(zp)}{dx}dx + \int zqydx &= \int zrdx \\
zy' - yz' + zpy + \int \left(yz'' - y\frac{d(zp)}{dx} + zqy\right) dx &= \int zrdx \\
zy' - yz' + zpy + \int y\left(z'' - \frac{d(zp)}{dx} + zq\right) dx &= \int zrdx \\
zy' + (zp - z')\, y + \int y\left(z'' - \frac{d(zp)}{dx} + zq\right) dx &= \int zrdx \qquad (5)
\end{aligned}$$

The adjoint ode is the term inside the integral above given by

$$z'' - \frac{d(zp)}{dx} + zq = 0 \qquad (6)$$

If this can be solved, where the solution $z_{sol}(x) \neq 0$, then (5) reduces to

$$\begin{aligned}
z_{sol}y' + \left(z_{sol}p - (z_{sol})'\right) y &= \int zrdx \\
y' + y\left(p - \frac{(z_{sol})'}{z_{sol}}\right) &= \frac{1}{z}\int zrdx
\end{aligned}$$

Which is first order ode in $y(x)$ which can be easily solved for $y(x)$. Equation (6) is called the Lagrange adjoint equation. This method of course works only if the adjoint ode can be solved for $z(x)$ and the solution is not zero.

2.3.2.12 Solved By transformation on $B(x)$ for ODE $Ay''(x) + By'(x) + C(x)\,y(x) = 0$

ode internal name "second_order_ode_non_constant_coeff_transformation_on_B"

This method is tried to reduce the order ode the ODE by one, by doing direct transformation on $B(x)$ for the ode

$$A(x)\,y''(x) + B(x)\,y'(x) + C(x)\,y(x) = 0$$

Let

$$y = Bv$$

Then $y' = B'v + v'B$ and $y'' = B''v + B'v' + v''B + v'B' = v''B + 2v'B' + B''v$ then the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv = 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v = 0 \end{aligned}$$

Now we check if $AB'' + BB' + CB = 0$ or not. If it is zero, then this method works and we can now solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

Using $u = v'$ which reduces the order to one.

$$ABu' + \left(2AB' + B^2\right)u = 0$$

This is first order ode now. Solved for u gives v' which is solved for v as first order ode. Then $y = Bv$ and we are done. This method only works of course if $AB'' + BB' + CB = 0$ comes out to be zero. Here is an example

2.3.2.12.1 Example 1

$$xy'' + (-1 - x)y' + y = 0$$

Here $A = x, B = (-1 - x)$ and $C = 1$, hence $B' = -1, B'' = 0$ and therefore

$$\begin{aligned} AB'' + BB' + CB &= 0 + (-1 - x)(-1) + (-1 - x) \\ &= 1 + x - 1 - x \\ &= 0 \end{aligned}$$

It works. Hence the reduces ode becomes

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

Let $u=v'$ then

$$
\begin{aligned}
ABu' + \left(2AB' + B^2\right)u &= 0\\
x((-1-x))\,u' + \left(-2x + (-1-x)^2\right)u &= 0\\
u - xu' + ux^2 - x^2u' &= 0\\
u'\left(-x - x^2\right) + u\left(1+x^2\right) &= 0\\
u' - \frac{(1+x^2)}{(x+x^2)}u &= 0
\end{aligned}
$$

This is linear first order ode solved using integrating factor which gives

$$
u = c_1 \frac{xe^x}{(1+x)^2}
$$

Hence since $v'=u$ then

$$
v' = c_1 \frac{xe^x}{(1+x)^2}
$$

This is quadrature. Solving gives

$$
v = c_2 + c_1 \frac{e^x}{1+x}
$$

Therefore

$$
\begin{aligned}
y &= Bv\\
&= (-1-x)\left(c_2 + c_1\frac{e^x}{1+x}\right)\\
&= c_2(1+x) + c_1e^x
\end{aligned}
$$

Note that this method is sensitive to the ODE is written. If we divide the ode by A is becomes

$$
y'' + \frac{(-1-x)}{x} + \frac{1}{x}y = 0
$$

And now $A=1, B=\frac{(-1-x)}{x}$ and $C=\frac{1}{x}$, hence $B'=-\frac{1}{x}+\frac{1+x}{x^2}$ and $B''=\frac{2}{x^2}-\frac{2}{x^3}(1+x)$ then

$$
\begin{aligned}
AB''+BB'+CB &= \left(\frac{2}{x^2}-\frac{2}{x^3}(1+x)\right)+\left(\frac{(-1-x)}{x}\right)\left(-\frac{1}{x}+\frac{1+x}{x^2}\right)+\frac{1}{x}\left(\frac{(-1-x)}{x}\right)\\
&= -\frac{1}{x^3}\left(x^2+2x+3\right)\\
&\neq 0
\end{aligned}
$$

So this method now fails to reduce the ode order by one. So in practice, I try first on the ode as given, and then try again by normalizing it so that B is not rational function and try again. In other words, given an ode $y''+\frac{(-1-x)}{x}+\frac{1}{x}y=0$ then try with $B=\frac{(-1-x)}{x}$ and if this fails, try again after multiplying the ode by x so now $B=(-1-x)$ and $A=x$ and $C=1$ and see if this works or not. This method of course only works when B is not zero.

2.3.2.13 Bessel type ode $x^2y''+xy'+(x^2-n^2)\,y=f(x)$

ode internal name "second order bessel ode"

Solves Besself ode or an ode which can be converted to bessel ode.

2.3.2.13.1 Introduction

This gives examples of converting (when possible) a second order linear ode to Bessel form. Bessel ODE is

$$
x^2y''+xy'+\left(x^2-n^2\right)y=0 \tag{A}
$$

Where n is the order which can be integer or non-integer. This comes out when doing separation of variables for the Laplace and Helmholtz PDE in spherical and cylindrical coordinates. n is integer for cylindrical coordinates and half integer values ($n=\frac{1}{2}+\mathbb{Z}$), for spherical coordinates. n can also be any other real value. The case $n=\frac{1}{2}+\mathbb{Z}$ is special in that the solution of the ode is reducible to standard trigonometric functions and complex exponential function. In all other cases, the solution remains in terms of Bessel functions.

The solution to (A) is known to be

$$
y(x)=c_1J_n(x)+c_2Y_n(x)
$$

Where $J_n(x)$ is Bessel function of first kind (order n). And $Y_n(x)$ Bessel function of second kind (order n).

There is also the modified Bessel ODE which differ by a sign

$$x^2y'' + xy' - (x^2+n^2)\,y = 0 \tag{B}$$

There is however a generalized form of (A,B). Which will be used below. (Bowman 1958). This form is

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right) y = 0 \tag{C}$$

Which is obtained by applying the transformation $\eta = \frac{y}{x^\alpha}, \xi = \beta x^\gamma$ to (A). The above has the solution

$$y(x) = x^\alpha(c_1J_n(\beta x^\gamma) + c_2Y_n(\beta x^\gamma)) \qquad \text{integer } n \tag{C1}$$
$$y(x) = x^\alpha(c_1J_n(\beta x^\gamma) + c_2J_{-n}(\beta x^\gamma)) \qquad \text{noninteger } n \tag{C2}$$

2.3.2.13.2 Collection of transformations

This section shows number of transformations applied to second order linear ode in order to make it of the form (A) or (B) if it is not already in that form. Once the ode is in form A or B, then its solution is now known using Bowman transformation.

2.3.2.13.2.1 Example $x^2y'' + xy' + (ax^2-n^2)\,y = 0$

$$x^2y'' + xy' + (ax^2 - n^2)\,y = 0 \tag{1}$$

Comparing (1) to (C) shows that

$$\begin{aligned}(1-2\alpha) &= 1\\ 2\gamma &= 2\\ a &= \beta^2\gamma^2\\ \gamma^2 &= 1\\ \alpha &= 0\end{aligned}$$

Solving shows that $\gamma = 1, \beta = \sqrt{a}$. Hence the solution from (C1) can now be written directly as

$$y(x) = c_1J_n(\sqrt{a}x) + c_2Y_n(\sqrt{a}x)$$

Another way to obtain this solution is to use the transformation

$$x = \frac{1}{\sqrt{a}}z$$

Which converts (1) to

$$z^2y'' + zy' + \left(x^2 - v^2\right)y = 0 \tag{2}$$

This is now in standard form (A) which has solution

$$y(z) = c_1 J_v(z) + c_2 Y_v(z)$$

Replacing back $z = \sqrt{a}x$ in the above gives

$$y(x) = c_1 J_v\left(\sqrt{a}x\right) + c_2 Y_v\left(\sqrt{a}x\right)$$

So the rule is that, the term is $(ax^2 - n^2)\,y$ then we can just replace $J_n(x)$ and $Y_n(x)$ in the standard solution with $J_n\left(\sqrt{a}x\right)$ and $Y_n\left(\sqrt{a}x\right)$. For example $x^2y'' + xy' + (4x^2 - 9)\,y = 0$ will have the solution $y(x) = c_1 J_3(2x) + c_2 Y_3(2x)$.

2.3.2.13.2.2 Example $x^2y'' + xy' + xy = 0$

$$x^2y'' + xy' + xy = 0 \tag{1}$$

Comparing (1) to (C) shows that

$$\begin{aligned} (1-2\alpha) &= 1 \qquad (2)\\ \left(\beta^2\gamma^2x^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right) &= x \end{aligned}$$

Hence

$$\begin{aligned} \beta^2\gamma^2x^{2\gamma} &= x\\ \left(n^2\gamma^2 - \alpha^2\right) &= 0 \qquad (3) \end{aligned}$$

Which implies

$$2\gamma = 1 \tag{4}$$

$$\beta^2\gamma^2 = 1 \tag{5}$$

(2) gives $\alpha = 0$. (4) gives $\gamma = \frac{1}{2}$. Substituting these into (3) gives

$$n = 0$$

And (5) gives $\beta^2 = 4$ or $\beta = \pm 2$. Therefore from (C1) the solution is

$$\begin{aligned} y(x) &= x^\alpha(c_1 J_n(\beta x^\gamma) + c_2 Y_n(\beta x^\gamma))\\ &= c_1 J_0\left(2\sqrt{x}\right) + c_2 Y_n\left(2\sqrt{x}\right) \end{aligned}$$

2.3.2.13.2.3 Example $x^2y'' + bxy' + (x^2 - v^2)\,y = 0$

$$x^2y'' + bxy' + \left(x^2 - v^2\right)y = 0 \tag{1}$$

Comparing (1) to the generalized form (C) $x^2y''+(1-2\alpha)\,xy'+(\beta^2\gamma^2x^{2\gamma} - (n^2\gamma^2 - \alpha^2))\,y = 0$ shows that

$$\begin{aligned}(1-2\alpha) &= b\\ 2\gamma &= 2\\ \beta^2\gamma^2 &= 1\\ \left(n^2\gamma^2 - \alpha^2\right) &= v^2\end{aligned}$$

Hence $\gamma = 1, \beta = 1$. From first equation $\alpha = \frac{1}{2}(1-b)$. Using this in the last equation gives

$$\begin{aligned}n^2 - \frac{1}{4}(1-b)^2 &= v^2\\ n &= \sqrt{v^2 + \frac{1}{4}(1-b)^2}\end{aligned}$$

Therefore the solution (C1) is

$$\begin{aligned}y(x) &= x^{\alpha}(c_1J_n(\beta x^{\gamma}) + c_2Y_n(\beta x^{\gamma}))\\ &= x^{\frac{1}{2}(1-b)}(c_1J_n(x) + c_2Y_n(x))\end{aligned}$$

For example, if $b = 4$, then the ode is $x^2y'' + 4xy' + (x^2 - v^2)\,y = 0$ and the solution is

$$y(x) = x^{-\frac{3}{2}}(c_1J_n(x) + Y_n(x))$$

Where $n = \frac{1}{2}\sqrt{\frac{4v^2+9}{2}}$.

2.3.2.13.2.4 Example $xy'' + y' + Ay = 0$

$$xy'' + y' + Ay = 0 \tag{1}$$

Where A is constant. Multiplying by x gives

$$x^2y'' + xy' + Axy = 0$$

Comparing the above to (C) $x^2y''+(1-2\alpha)\,xy'+(\beta^2\gamma^2x^{2\gamma} - (n^2\gamma^2 - \alpha^2))\,y = 0$ shows that

$$\begin{aligned}(1-2\alpha) &= 1\\ Ax &= \beta^2\gamma^2x^{2\gamma}\\ \left(n^2\gamma^2 - \alpha^2\right) &= 0\end{aligned}$$

Which implies $\alpha=0, 2\gamma=1$ or $\gamma=\frac{1}{2}$. Therefore $\beta^2\gamma^2=A$ gives $\beta^2=4A$ or $\beta=2\sqrt{A}$. And $n=0$. Hence the solution (C1) is

$$y(x)=c_1 J_0\left(2\sqrt{A}\sqrt{x}\right)+c_2 Y_0\left(2\sqrt{A}\sqrt{x}\right)$$

Alternative and longer method is the following (this is kept just for illustration, as the above method is more direct).

Using the transformation

$$x=v^2$$

Hence

$$v=\sqrt{x} \tag{2}$$

and $\frac{dv}{dx}=\frac{1}{2\sqrt{x}}$. Therefore

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{dv}\frac{dv}{dx}\\&=\frac{dy}{dv}\frac{1}{2\sqrt{x}}\\&=\frac{dy}{dv}\frac{1}{2v}\end{aligned} \tag{3}$$

And

$$\begin{aligned}\frac{d^2y}{dx^2}&=\frac{d}{dx}\left(\frac{dy}{dx}\right)\\&=\frac{d}{dx}\left(\frac{dy}{dv}\frac{1}{2v}\right)\end{aligned}$$

But $\frac{d}{dx}=\frac{d}{dv}\frac{dv}{dx}$. The above becomes

$$\begin{aligned}\frac{d^2y}{dx^2}&=\frac{d}{dv}\frac{dv}{dx}\left(\frac{dy}{dv}\frac{1}{2v}\right)\\&=\frac{dv}{dx}\frac{d}{dv}\left(\frac{dy}{dv}\frac{1}{2v}\right)\end{aligned}$$

But $\frac{dv}{dx}=\frac{1}{2\sqrt{x}}=\frac{1}{2v}$. Hence the above becomes

$$\frac{d^2y}{dx^2}=\frac{1}{2v}\frac{d}{dv}\left(\frac{dy}{dv}\frac{1}{2v}\right) \tag{4}$$

But

$$\frac{d}{dv}\left(\frac{dy}{dv}\frac{1}{2v}\right)=\frac{1}{2}\left(\frac{d^2y}{dv^2}\frac{1}{v}-\frac{dy}{dv}\frac{1}{v^2}\right)$$

Hence (4) becomes

$$\frac{d^2y}{dx^2} = \frac{1}{4v}\left(\frac{d^2y}{dv^2}\frac{1}{v} - \frac{dy}{dv}\frac{1}{v^2}\right) \tag{5}$$

Substituting (3,5) into (1) gives

$$x\frac{1}{4v}\left(\frac{d^2y}{dv^2}\frac{1}{v} - \frac{dy}{dv}\frac{1}{v^2}\right) + \frac{dy}{dv}\frac{1}{2v} + Ay = 0$$

But $x = v^2$. The above becomes

$$\begin{aligned}
\frac{v}{4}\left(y''\frac{1}{v} - \frac{dy}{dv}\frac{1}{v^2}\right) + y'\frac{1}{2v} + Ay &= 0\\
\frac{1}{4}y'' - \frac{1}{4}y'\frac{1}{v} + y'\frac{1}{2v} + Ay &= 0\\
\frac{1}{4}y'' + \frac{1}{4}y'\frac{1}{v} + Ay &= 0\\
y'' + y'\frac{1}{v} + 4Ay &= 0
\end{aligned}$$

Multiplying through by v^2

$$v^2y'' + vy' + 4Av^2y = 0$$

The above of the form

$$v^2y'' + vy' + \left(a^2v^2 - n^2\right)y = 0$$

Where $n = 0$ and $a^2 = 4A$ which has the standard solution

$$y(v) = c_1 J_n(av) + c_2 Y_n(av)$$

Where $J_n(v)$ is the Bessel function of first kind and $Y_n(v)$ is Bessel function of second kind. Since $v = \sqrt{x}$ and $a = 2\sqrt{A}$ then the solution for (1) becomes (using $n = 0$)

$$y(x) = c_1 J_0\left(2\sqrt{A}\sqrt{x}\right) + c_2 Y_0\left(2\sqrt{A}\sqrt{x}\right)$$

For example, if $A = \frac{1}{4}$. Then the ode $xy'' + y' + \frac{1}{4}y = 0$ and the solution above becomes

$$y(x) = c_1 J_0\left(\sqrt{x}\right) + c_2 Y_0\left(\sqrt{x}\right)$$

2.3.2.13.2.5 Example $y''-\frac{1}{x}y=0$

$$y''-\frac{1}{x}y=0 \tag{1}$$

Multiplying both sides by x^2 gives

$$x^2y''-xy=0$$

Comparing to (C) $x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-(n^2\gamma^2-\alpha^2)\right)y=0$ shows that

$$\begin{aligned}(1-2\alpha)&=0\\ \beta^2\gamma^2x^{2\gamma}&=-x\\ \left(n^2\gamma^2-\alpha^2\right)&=0\end{aligned}$$

First equation gives $\alpha=\frac{1}{2}$. Second equation gives $\gamma=\frac{1}{2}$ and $\beta^2\gamma^2=-1$. Therefore $\beta^2=-4$ or $\beta=\pm 2i$. Last equation gives $n^2\gamma^2=\frac{1}{4}$ or $n=1$ since $\gamma^2=\frac{1}{4}$. Hence the solution (C1) is

$$\begin{aligned}y(x)&=x^{\alpha}(c_1J_n(\beta x^{\gamma})+c_2Y_n(\beta x^{\gamma}))\\ &=\sqrt{x}\left(c_1J_1\left(2i\sqrt{x}\right)+c_2Y_1\left(2i\sqrt{x}\right)\right)\end{aligned}$$

By properties of Bessel functions, where $J_n\left(ai\sqrt{x}\right)=i^nI_n\left(a\sqrt{x}\right)$, then the above becomes

$$y(x)=\sqrt{x}\left(ic_1I_1\left(2\sqrt{x}\right)+c_2Y_1\left(2i\sqrt{x}\right)\right)$$

Alternative longer method is the following:

Trying standard transformation $y=\sqrt{x}Y$. The ode becomes

$$x^2Y''+xY'-\left(x+\frac{1}{4}\right)Y=0$$

Using the transformation $x=t^2$ the above becomes

$$t^2Y''+tY'-\left(4t^2+1\right)Y=0$$

Finally applying the standard transformation $t=\frac{1}{2}z$ to fix the term $(4t^2+1)$ to standard form the above becomes

$$z^2Y''+zY'-\left(t^2+1\right)Y=0$$

This is modified Bessel ODE whose solution is known to be

$$Y(z)=c_1I_1(z)+c_2K_1(z)$$

Where I_1 is modified Bessel function of first kind and K_1 is modified Bessel function of second kind. But $z=2t$. Hence the above becomes

$$Y(t)=c_1 I_1(2t)+c_2 K_1(2t)$$

But $t=\sqrt{x}$. The above becomes

$$Y(x)=c_1 I_1\left(2\sqrt{x}\right)+c_2 K_1\left(2\sqrt{x}\right)$$

But $y(x)=\sqrt{x}Y(z)$ hence

$$y(x)=c_1\sqrt{x} I_1\left(2\sqrt{x}\right)+c_2\sqrt{x} K_1\left(2\sqrt{x}\right)$$

2.3.2.13.2.6 Example $4x^2y''+4xy'+(x-4)\,y=0$

Dividing by 4

$$x^2y''+xy'+\left(\frac{1}{4}x-1\right)y=0$$

Comparing the above to (C) $x^2y''+(1-2\alpha)\,xy'+(\beta^2\gamma^2x^{2\gamma}-(n^2\gamma^2-\alpha^2))\,y=0$ shows that

$$\begin{aligned}(1-2\alpha)&=1\\ \beta^2\gamma^2x^{2\gamma}&=\frac{1}{4}x\\ \left(n^2\gamma^2-\alpha^2\right)&=1\end{aligned}$$

Which implies $\alpha=0, 2\gamma=1, \beta^2\gamma^2=\frac{1}{4}$. Hence $\gamma=\frac{1}{2}$ and $\beta=1$. Last equation now says $n^2\gamma^2=1$ or $n=2$. Hence the solution (C1) is

$$\begin{aligned}y(x)&=x^\alpha(c_1J_n(\beta x^\gamma)+c_2Y_n(\beta x^\gamma))\\ &=c_1J_2\left(\sqrt{x}\right)+c_2Y_2\left(\sqrt{x}\right)\end{aligned}$$

2.3.2.13.2.7 Example $y''-\frac{1}{x^{\frac{3}{2}}}y=0$

Multiplying by $x^{\frac{3}{2}}$

$$x^{\frac{3}{2}}y''-y=0$$

Multiplying by $x^{\frac{1}{2}}$

$$x^2y''-x^{\frac{1}{2}}y=0$$

Comparing the above to (C) $x^2y''+(1-2\alpha)\,xy'+(\beta^2\gamma^2x^{2\gamma}-(n^2\gamma^2-\alpha^2))\,y=0$ shows that

$$\begin{aligned}(1-2\alpha)&=0\\ \beta^2\gamma^2x^{2\gamma}&=-x^{\frac{1}{2}}\\ \left(n^2\gamma^2-\alpha^2\right)&=0\end{aligned}$$

Which implies $\alpha=\frac{1}{2}, 2\gamma=\frac{1}{2}, \beta^2\gamma^2=-1$. Hence $\gamma=\frac{1}{4}$ and $\beta^2=-16$ or $\beta=\pm 4i$. Last equation now says $\left(n^2\frac{1}{16}-\frac{1}{4}\right)=0$ or $n=2$. Hence the solution (C1) is

$$\begin{aligned}
y(x) &= x^{\alpha}(c_1 J_n(\beta x^{\gamma}) + c_2 Y_n(\beta x^{\gamma})) \\
&= \sqrt{x}\left(c_1 J_2\left(4ix^{\frac{1}{4}}\right) + c_2 Y_2\left(4ix^{\frac{1}{4}}\right)\right)
\end{aligned}$$

By properties of Bessel functions, where $J_n(ai\sqrt{x}) = i^n I_n(a\sqrt{x})$, then the above becomes

$$y(x) = \sqrt{x}\left(-c_1 I_2\left(4x^{\frac{1}{4}}\right) + c_2 Y_2\left(4ix^{\frac{1}{4}}\right)\right)$$

2.3.2.13.2.8 Example $x^2y'' - xy + (x^2+1)\,y = 0$

$$x^2y'' - xy + \left(x^2+1\right)y = 0$$

Comparing the above to (C) $x^2y'' + (1-2\alpha)\,xy' + (\beta^2\gamma^2x^{2\gamma} - (n^2\gamma^2-\alpha^2))\,y = 0$ shows that

$$\begin{aligned}
(1-2\alpha) &= -1 \\
\beta^2\gamma^2x^{2\gamma} &= x^2 \\
-\left(n^2\gamma^2-\alpha^2\right) &= 1
\end{aligned}$$

Which implies $\alpha=1$ and $\gamma=1$ and $\beta^2\gamma^2=1$ or $\beta=1$. Last equation now becomes $-(n^2-1)=1$ or $n^2=0$ or $n=0$. Hence the solution (C1) becomes

$$\begin{aligned}
y(x) &= x^{\alpha}(c_1 J_n(\beta x^{\gamma}) + c_2 Y_n(\beta x^{\gamma})) \\
&= x(c_1 J_0(x) + c_2 Y_0(x))
\end{aligned}$$

2.3.2.13.2.9 Example $y'' - x^{-\frac{1}{4}}y = 0$

Multiplying by $x^{\frac{1}{4}}$

$$x^{\frac{1}{4}}y'' - y = 0$$

Multiplying by $x^{\frac{7}{4}}$

$$x^2y'' - x^{\frac{7}{4}}y = 0$$

Comparing the above to (C) $x^2y'' + (1-2\alpha)\,xy' + (\beta^2\gamma^2x^{2\gamma} - (n^2\gamma^2-\alpha^2))\,y = 0$ shows that

$$\begin{aligned}
(1-2\alpha) &= 0 \\
\beta^2\gamma^2x^{2\gamma} &= -x^{\frac{7}{4}} \\
\left(n^2\gamma^2-\alpha^2\right) &= 0
\end{aligned}$$

Which implies $\alpha=\frac{1}{2}$ and $2\gamma=\frac{7}{4}$ or $\gamma=\frac{7}{8}$ and $\beta^2\gamma^2=-1$ or $\beta^2=-\frac{1}{\left(\frac{7}{8}\right)^2}=-\frac{64}{49}$. Hence $\beta=i\frac{8}{7}$. Last equation now becomes $\left(n^2\left(\frac{49}{64}\right)-\frac{1}{4}\right)=0$, or $n=\frac{4}{7}$. Hence the solution (C2) for non integer n becomes

$$
\begin{aligned}
y(x) &= x^{\alpha}(c_1 J_n(\beta x^{\gamma})+c_2 J_{-n}(\beta x^{\gamma}))\\
&= \sqrt{x}\left(c_1 J_{\frac{4}{7}}\left(i\frac{8}{7}x^{\frac{7}{8}}\right)+c_2 J_{-\frac{4}{7}}\left(i\frac{8}{7}x^{\frac{7}{8}}\right)\right)
\end{aligned}
$$

2.3.2.13.2.10 Example $f''+\frac{\lambda}{x}f'-\mu f=0$

Multiplying by x^2

$$x^2f''+\lambda xf'+\left(-\mu x^2\right)f=0 \tag{1}$$

Using the generalized form of Bessel ode

$$x^2f''+xf'+\left(x^2-n^2\right)f=0 \tag{A}$$

Which is given by (Bowman 1958)

$$x^2f''+(1-2\alpha)xf'+\left(\beta^2\gamma^2x^{2\gamma}-\left(n^2\gamma^2-\alpha^2\right)\right)f=0 \tag{C}$$

Comparing (1) and (C) shows that

$$
\begin{aligned}
(1-2\alpha) &= \lambda && (2)\\
\beta^2\gamma^2x^{2\gamma} &= -\mu x^2 && (3)\\
\left(n^2\gamma^2-\alpha^2\right) &= 0 && (4)
\end{aligned}
$$

(2) gives $\alpha=\frac{1}{2}-\frac{1}{2}\lambda$. (3) gives $2\gamma=2$ or $\gamma=1$. And (3) also shows that $\beta^2\gamma^2=-\mu$ or $\beta=\sqrt{-\mu}$. Now (4) gives $\left(n^2-\left(\frac{1}{2}-\frac{1}{2}\lambda\right)^2\right)=0$ or $n=\left(\frac{1}{2}-\frac{1}{2}\lambda\right)$. (taking positive root). But the solution to (C) is gives by

$$y(x)=x^{\alpha}(c_1J_n(\beta x^{\gamma})+c_2Y_n(\beta x^{\gamma}))$$

Therefore the solution to (1) is

$$y(x)=x^{\left(\frac{1}{2}-\frac{1}{2}\lambda\right)}\left(c_1J_{\left(\frac{1}{2}-\frac{1}{2}\lambda\right)}\left(\sqrt{-\mu}x\right)+c_2Y_{\left(\frac{1}{2}-\frac{1}{2}\lambda\right)}\left(\sqrt{-\mu}x\right)\right)$$

Where J is the Bessel function of first kind and Y is the Bessel function of the second kind.

2.3.2.13.2.11 Example $x^2y''+xy'+(x^2-5)y=0$

$$x^2y''+xy'+(x^2-5)y=0 \tag{1}$$

Using the generalized form of Bessel ode

$$x^2y''+xy'+\left(x^2-n^2\right)y=0 \tag{A}$$

Which is given by (Bowman 1958)

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-\left(n^2\gamma^2-\alpha^2\right)\right)y=0 \tag{C}$$

Comparing (1) and (C) shows that

$$(1-2\alpha)=1 \tag{2}$$
$$\beta^2\gamma^2x^{2\gamma}=x^2 \tag{3}$$
$$\left(n^2\gamma^2-\alpha^2\right)=5 \tag{4}$$

(2) gives $\alpha=0$. (3) gives $\gamma=1$ and $\beta^2\gamma^2=1$ or $\beta=1$. Now (4) gives $n^2\gamma^2=5$ or $n=\sqrt{5}$.But the solution to (C) is given by

$$y(x)=x^{\alpha}(c_1J_n(\beta x^{\gamma})+c_2Y_n(\beta x^{\gamma}))$$

Therefore the solution to (1) is

$$y(x)=c_1J_{\sqrt{5}}(x)+c_2Y_{\sqrt{5}}(x)$$

Where J is the Bessel function of first kind and Y is the Bessel function of the second kind.

2.3.2.13.3 References

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2.3.2.14 Bessel form A type ode $ay''+by'+(ce^{rx}-m)y=f(x)$

ode internal name "second_order_bessel_ode_form_A"

These are ode of the above form which can be converted to Bessel using transformation $x=\ln(t)$.

2.3.2.14.1 Example $ay''+by'+(ce^{rx}-m)y=0$

An ode of the form

$$ay''+by'+(ce^{rx}+m)y=0 \tag{1}$$

can be transformed to Bessel ode using the transformation

$$\begin{aligned} x &= \ln(t) \\ e^x &= t \end{aligned}$$

Where a,b,c,m are not functions of x and where b and m are allowed to be be zero. Using this transformation gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{dy}{dt}e^x \\ &= t\frac{dy}{dt} \end{aligned} \tag{2}$$

And

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(t\frac{dy}{dt}\right) \\ &= \frac{d}{dt}\frac{dt}{dx}\left(t\frac{dy}{dt}\right) \\ &= \frac{dt}{dx}\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\ &= t\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\ &= t\left(\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right) \end{aligned} \tag{3}$$

Substituting (2,3) into (1) gives

$$\begin{aligned} at\left(\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right)+bt\frac{dy}{dt}+(ce^{rx}+m)y&=0\\ (aty'+at^2y'')+bty'+(ct^r+m)y&=0\\ at^2y''+(b+a)\,ty'+(ct^r+m)y&=0\\ t^2y''+\frac{b+a}{a}ty'+\left(\frac{c}{a}t^r+\frac{m}{a}\right)y&=0 \qquad (4)\end{aligned}$$

Which is Bessel ODE. Comparing the above to the general known Bowman form of Bessel ode which is

$$t^2y''+(1-2\alpha)\,ty'+\left(\beta^2\gamma^2t^{2\gamma}-\left(n^2\gamma^2-\alpha^2\right)\right)y=0 \qquad (C)$$

And now comparing (4) and (C) shows that

$$\begin{aligned} (1-2\alpha)&=\frac{b+a}{a} &(5)\\ \beta^2\gamma^2&=\frac{c}{a} &(6)\\ 2\gamma&=r &(7)\\ \left(n^2\gamma^2-\alpha^2\right)&=-\frac{m}{a} &(8)\end{aligned}$$

(5) gives $\alpha=\frac{1}{2}-\frac{b+a}{2a}$. (7) gives $\gamma=\frac{r}{2}$. (8) now becomes $\left(n^2\left(\frac{r}{2}\right)^2-\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2\right)=-\frac{m}{a}$ or $n^2=\frac{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}{\left(\frac{r}{2}\right)^2}$. Hence $n=\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}$ by taking the positive root. And finally (6) gives $\beta^2=\frac{c}{a\gamma^2}$ or $\beta=\sqrt{\frac{c}{a}\frac{1}{\gamma}}=\sqrt{\frac{c}{a}\frac{2}{r}}$ (also taking the positive root). Hence

$$\begin{aligned} \alpha&=\frac{1}{2}-\frac{b+a}{2a}\\ n&=\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}\\ \beta&=\sqrt{\frac{c}{a}\frac{2}{r}}\\ \gamma&=\frac{r}{2}\end{aligned}$$

But the solution to (C) which is general form of Bessel ode is known and given by

$$y(t)=t^{\alpha}(c_1J_n(\beta t^{\gamma})+c_2Y_n(\beta t^{\gamma}))$$

Substituting the above values found into this solution gives

$$y(t)=t^{\frac{1}{2}-\frac{b+a}{2a}}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}t^{\frac{r}{2}}\right)+c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}t^{\frac{r}{2}}\right)\right)$$

Since $e^x=t$ then the above becomes

$$\begin{aligned}
y(x)&=e^{x\left(\frac{1}{2}-\frac{b+a}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right)\\
&=e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right)\\
&=e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right)\\
&=e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right)\\
&=e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right) \qquad (9)
\end{aligned}$$

Equation (9) above is the solution to $ay''+by'+(ce^{rx}+m)y=0$. Therefore we just need now to compare this form to the ode given and use (9) to obtain the final solution.

Let us now apply this to an example for illustration. Given the ode

$$y''+(e^{2x}-4)y=0$$

Comparing the above to $ay''+by'+(ce^{rx}+m)y=0$ shows that $a=1,b=0,c=1,r=2,m=-4$. Hence the solution (9) becomes

$$\begin{aligned}
y(x)&=e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right)\\
&=c_1 J_{\frac{1}{2}\sqrt{16}}(e^x)+c_2 Y_{\frac{1}{2}\sqrt{16}}(e^x)\\
&=c_1 J_2(e^x)+c_2 Y_2(e^x)\\
&=c_1 \operatorname{BesselJ}(2,e^x)+c_2 \operatorname{BesselY}(2,e^x)
\end{aligned}$$

Another example for illustration. Given the ode

$$y''+y'+(e^x-4)y=0$$

Comparing the above to $ay''+by'+(ce^{rx}+m)y=0$ shows that $a=1,b=1,c=1,r=1,m=-4$. Hence the solution (9) becomes

$$\begin{aligned}
y(x)&=e^{x\left(\frac{-1}{2}\right)}\left(c_1 J_{\sqrt{16}}\left(2e^{x\frac{1}{2}}\right)+c_2 Y_{\sqrt{16+1}}\left(2e^{x\frac{1}{2}}\right)\right)\\
&=e^{\frac{-x}{2}}\left(c_1 J_{\sqrt{17}}(2e^{\frac{x}{2}})+c_2 Y_{\sqrt{17}}(2e^{\frac{x}{2}})\right)
\end{aligned}$$

Another example for illustration. Given the ode

$$y'' + (e^{2x} - n^2)y = 0$$

Comparing the above to $ay'' + by' + (ce^{rx} + m)y = 0$ shows that $a = 1, b = 0, c = 1, r = 2, m = -n^2$. Hence the solution (9) becomes

$$\begin{aligned} y(x) &= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right) \\ &= c_1 J_{\frac{1}{2}\sqrt{-4(-n^2)}}(e^x) + c_2 Y_{\frac{1}{2}\sqrt{-4(-n^2)}}(e^x) \\ &= c_1 J_n(e^x) + c_2 Y_n(e^x) \\ &= c_1 \operatorname{BesselJ}(n, e^x) + c_2 \operatorname{BesselY}(n, e^x) \end{aligned}$$

2.4 Nonlinear second order ode

2.4.1 Exact nonlinear second order ode $F(x,y,y',y'')=0$

ode internal name "exact_nonlinear_second_order_ode"

(**not implemented yet**)

2.4.1.1 Introduction and terminology used

An ode $F(x,y,y',y'')=0$ is called exact if there exists a function $R(x,y,y')$ with order one less that of the ode, such that

$$F(x,y,y',y'')=\frac{d}{dx}R(x,y,y')$$

Which also implies that $R=c$ some constant, because $F=0$. In the above $R(x,y,y')$ is called the first integral of the ode F (also called the reduced ode), because

$$R=\int F dx+c \tag{1A}$$

An important property of first integral is the following. If we write the ode $F(x,y,y',y'')=0$ as $y''=\Phi(x,y,y')$ which we can always do, then

$$R_x+y'R_y+\Phi R_{y'}=0 \tag{1B}$$

Lets see how this works. Given the ode $y''+xy'+y=0$ which is exact as is from the exactness test $py''+qy'+r=0$ which is $p''-q'+r=0$, hence $p=1,q=x,r=1$, therefore $-1+1=0$ which is true. Therefore we can write because we can write $y''+xy'+y=0=(y'+B(x)\,y)'$ and find that $B=x$, Hence

$$y''+xy'+y=(y'+xy)'$$

Where $y'+xy=0$ is the reduced ode.

$$R=y'+xy$$

For the original ode $y''+xy'+y=0$, it can be written as $y''=-(xy'+y)$, therefore $\Phi=-(xy'+y)$. Eq (1B) now becomes

$$\begin{aligned}
R_x+y'R_y+\Phi R_{y'}&=0\\
y+y'x-(xy'+y)\,(1)&=0\\
y+y'x-xy'-y&=0\\
0&=0
\end{aligned}$$

Verified. Here is another example. Given the ode $(x-1)^2 y''+4y'x+2y-2x=0$, this is exact because we can write $(x-1)^2 y''+4y'x+2y-2x = \frac{d}{dx}((2x+2)\,y+(x^2-2x+1)\,y'-x^2)$, hence the first integral (or the reduced ode) is $R=(2x+2)\,y+(x^2-2x+1)\,y'-x^2$. The original ode can be written as $y''=-\frac{(4y'x+2y-2x)}{(x-1)^2}$, therefore $\Phi=-\frac{(4y'x+2y-2x)}{(x-1)^2}$. Eq (1B) becomes

$$
\begin{aligned}
R_x+y'R_y+\Phi R_{y'} &= 0\\
(2y+2xy'-2y'-2x)+y'(2x+2)-\left(\frac{4y'x+2y-2x}{(x-1)^2}\right)\left(x^2-2x+1\right) &= 0\\
0 &= 0
\end{aligned}
$$

Verified. Equations (1A) and (1B) are important as they will be used to determined an integrating factor when the ode is not exact.

2.4.1.2 Test for exactness

The following shows how to determine if $F(x,y,y',y'')=0$ is exact or not (without having to find the first integral R). This is based on page 164 in Murphy book. The second order ode must be of degree one. If it is, it can not be exact. The ode is exact iff

$$
\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=0 \tag{1}
$$

This turns out to be the same thing as using $p''-q'+r=0$ on the ode $py''-q'+r=0$. Let us apply the above test on second order ode which is known to be exact to see how it works. The ode is

$$
\begin{aligned}
F(x,y,y',y'') &= 0\\
xy''+(y-1)\,y' &= 0
\end{aligned}
$$

Hence the above test gives

$$
\begin{aligned}
y'-\frac{d}{dx}(y-1)+\frac{d^2}{dx^2}(x) &= 0\\
y'-y' &= 0\\
0 &= 0
\end{aligned}
$$

Confirmed. Since the ode is linear, we could also apply $p''-q'+r=0$ to check, which is simpler. Here $p=x, q=(y-1), r=0$. Therefore

$$
\begin{aligned}
p''-q'+r &= 0\\
0-0+0 &= 0
\end{aligned}
$$

The form (1) is given in Murphy book which is more general since it works on nonlinear and linear odes while $p''-q'+r=0$ is meant to be used for linear second order odes.

In implementation of the solver this is the same type of ode as "second order integrable as is" ode which is described below. I should merge these together. if a second order ode is exact, then it is also integrable ode as is. This is by definition of exactness above.

2.4.1.3 Examples showing how to check for exactness

2.4.1.3.1 Example 1

$$
\begin{aligned}
y''+\frac{x}{y^2}y'-\frac{1}{y}&=0\\
F(x,y,y',y'')&=y''+\frac{x}{y^2}y'-\frac{1}{y}
\end{aligned}
$$

Applying the test

$$
\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=0 \tag{1}
$$

Therefore

$$
\begin{aligned}
\frac{\partial F}{\partial y}&=-\frac{2}{y^3}xy'+\frac{1}{y^2}\\
\frac{\partial F}{\partial y'}&=\frac{x}{y^2}\\
\frac{\partial F}{\partial y''}&=1
\end{aligned}
$$

Hence (1) becomes

$$
\begin{aligned}
\left(-\frac{2}{y^3}xy'+\frac{1}{y^2}\right)-\frac{d}{dx}\left(\frac{x}{y^2}\right)+\frac{d^2}{dx^2}(1)&=0\\
\left(-\frac{2}{y^3}xy'+\frac{1}{y^2}\right)-\left(\frac{1}{y^2}-\frac{2xy'}{y^3}\right)&=0\\
0&=0
\end{aligned}
$$

Therefore this exact. We see that $\left(y'-\frac{x}{y}\right)'=y''-\left(\frac{1}{y}+\frac{xy^2}{y'}\right)$. Which implies the ode is integrable as is. Which means

$$
\begin{aligned}
\int\left(y'-\frac{x}{y}\right)'dx&=0\\
y'-\frac{x}{y}&=c
\end{aligned} \tag{2}
$$

Which can now be solved. In the above $R(x,y,y')=\left(y'-\frac{x}{y}\right)$. In other words $F=\frac{d}{dx}R$. Hence

$$\frac{d}{dx}R=0$$

Integrating gives

$$\begin{aligned}\int\frac{d}{dx}Rdx&=c\\ \int dR&=c\\ R&=c\\ y'-\frac{x}{y}&=c\end{aligned}$$

Which is the same as (2) above but shows how it came about more clearly.

2.4.1.3.2 Example 2

$$\begin{aligned}3\beta y''+yy'&=0\\ F(x,y,y',y'')&=3\beta y''+yy'\end{aligned}$$

Applying the test

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=0 \tag{1}$$

Therefore

$$\begin{aligned}\frac{\partial F}{\partial y}&=y'\\ \frac{\partial F}{\partial y'}&=y\\ \frac{\partial F}{\partial y''}&=3\beta\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}(y')-\frac{d}{dx}(y)+\frac{d^2}{dx^2}(3\beta)&=0\\ y'-y'&=0\\ 0&=0\end{aligned}$$

Therefore this exact. Therefore we see that $\left(\frac{y^2}{2} + 3\beta y'\right)' = 3\beta y'' + yy' = 0$. Which implies the ode can be written as

$$\int \left(\frac{y^2}{2} + 3\beta y'\right)' dx = 0$$
$$\frac{y^2}{2} + 3\beta y' = c$$

Solving this first order ode gives the solution

$$y = \tanh\left(\frac{1}{6r}\sqrt{c_1}(c_2 + x)\sqrt{2}\right)\sqrt{2}\sqrt{c_1}$$

2.4.1.4 How to solve the ode once it is determined it is exact

In the examples above we did not show how to obtain or find the first integral $R(x, y, y')$. Given an ode $F(x, y, y', y'') = 0$ which is determined to be exact as above, then how to solve it? This is done by first finding the first integral R. We need to find $R(x, y, y')$ such that

$$F(x, y, y, y'') = \frac{d}{dx}R(x, y, y') = 0$$

Once R is found, then we need to solve the first order ode $R(x, y, y') = c$ where R is now one order less that F so it should be simpler to solve. This ode might require another integration factor to solve depending on what it type turns out to be.

This reduces the order of the ode from second to first order (since R is first order). To find $R(x, y, y')$ the first step is to write the given ode in this form

$$F(x, y, y', y'') = f(x, y, y')\, y'' + g(x, y, y') \tag{1}$$

We know what f, g are in the above by reading them from the given ode. But

$$\begin{aligned} F &= \frac{d}{dx}R(x, y, y') \\ &= \frac{\partial R}{\partial x}\frac{dx}{dx} + \frac{\partial R}{\partial y}\frac{dy}{dx} + \frac{\partial R}{\partial y'}\frac{dy'}{dx} \\ &= R_x + R_y y' + R_{y'} y'' \end{aligned} \tag{1A}$$

And since $y'' = \Phi(x, y, y')$ then the above can also be written as

$$F = R_x + R_y y' + \Phi R_{y'}$$

The above is same as Eq (1B) in the introduction above. Comparing (1,1A) shows that

$$f = R_{y'} \tag{2}$$
$$g = R_x + R_y y' \tag{3}$$

At this point it is easier to replace y' by p. The above becomes

$$f = R_p \tag{2}$$
$$g = R_x + R_y p \tag{3}$$

Using (2,3) we are able to determine R. Note that R must exist since we checked the ode is exact and hence must have a first integral. This method similar to how we find R for an exact first order ode.

Starting with (2) and integrating it w.r.t. p gives

$$R = \int f dp + \psi(x,y) \tag{4}$$

Where $\psi(x,y)$ acts like an integration constant but since R depends on more than one variable, it is now an arbitrary function of the other variables x, y. If we can find $\psi(x,y)$, then R is found, since f is known. To find ψ , we differentiate one time w.r.t. x and another time w.r.t. y and substitute the result in (3). This gives

$$g = \left(\frac{\partial}{\partial x}\left(\int f dp\right) + \psi_x(x,y)\right) + \left(\frac{\partial}{\partial y}\left(\int f dp\right) + \psi_y(x,y)\right) p \tag{5}$$

In the above the terms $\frac{\partial}{\partial x}\left(\int f dp\right), \frac{\partial}{\partial y}\left(\int f dp\right)$ are known, since everything is known. The only unknowns are $\psi_x(x,y), \psi_y(x,y)$. Comparing terms in (5) we can generate two equations for ψ_x, ψ_y and by integrating them we find ψ. Examples below show how to do this as this is easier explained using examples.

2.4.1.4.1 Examples finding first integral $R(x,y,y')$ for an exact second order ode

2.4.1.4.1.1 Example 1

$$yy'' + (y')^2 + 2axyy' + ay^2 = 0$$

Comparing this to $F(x,y,y',y'') = f(x,y,y')\,y'' + g(x,y,y')$ shows that

$$\begin{aligned} f &= y \\ g &= (y')^2 + 2axyy' + ay^2 \\ &= p^2 + 2axyp + ay^2 \end{aligned}$$

Therefore (4) becomes

$$\begin{aligned} R &= \int f dp + \psi(x,y) \\ &= yp + \psi(x,y) \end{aligned} \tag{1A}$$

Hence (5) becomes

$$
\begin{aligned}
g &= \left(\frac{\partial}{\partial x}\left(\int f dp\right) + \psi_x\right) + \left(\frac{\partial}{\partial y}\left(\int f dp\right) + \psi_y\right) p \\
p^2 + 2axyp + ay^2 &= \left(\frac{\partial}{\partial x}(yp) + \psi_x\right) + \left(\frac{\partial}{\partial y}(yp) + \psi_y\right) p
\end{aligned}
$$

But $\frac{\partial}{\partial x}(yp) = 0$ since y,p are held constant. It is important to watch for this here. Given $f(x,y) = 3x + y(x)$ where y is function of x, then when we do $\frac{\partial f}{\partial x}$ the result is 3 and not $3 + y'$ because with partial derivatives the y is held constant. Similarly $\frac{\partial}{\partial y}(yp) = p^2$. The above becomes

$$
\begin{aligned}
p^2 + 2axyp + ay^2 &= \psi_x + (p + \psi_y)\, p \\
&= \psi_x + p^2 + \psi_y p \\
2axyp + ay^2 &= \psi_x + \psi_y p
\end{aligned}
$$

Comparing terms shows that

$$
2axy = \psi_y \tag{2A}
$$

$$
ay^2 = \psi_x \tag{3A}
$$

Integrating (2A) w.r.t y gives

$$
\psi = axy^2 + h(x) \tag{4A}
$$

Differentiating the above w.r.t. x gives $\psi_x = ay^2 + h'(x)$. comparing this to (3A) above gives $ay^2 = ay^2 + h'(x)$, hence $h'(x) = 0$ or $h(x) = c$. Therefore (4A) becomes

$$
\psi = axy^2 + c
$$

Substituting the above in (1A) gives

$$
R = yp + axy^2 + c
$$

Therefore, since $R = c_1$ a constant, then the above becomes (by merging the constants)

$$
\begin{aligned}
yp + axy^2 &= c_2 \\
yy' + axy^2 &= c_2
\end{aligned}
$$

This is the reduced ode which needs to be solved for y. The above says that $R = yy' + axy^2 + c_2$. To verify, let us apply $F = \frac{d}{dx}R$. This gives

$$
\begin{aligned}
yy'' + (y')^2 + 2axyy' + ay^2 &= \frac{d}{dx}\left(yy' + axy^2 + c_2\right) \\
&= y'y' + yy'' + ay^2 + 2axyy' \\
&= yy'' + (y')^2 + 2axyy' + ay^2
\end{aligned}
$$

Verified.

2.4.1.4.1.2 Example 2

$$
\begin{aligned}
y'' + xy' + y &= 0 \\
F(x,y,y',y'') &= 0
\end{aligned}
$$

This ode is not nonlinear, but let us apply this method to it anyway. First we need to determine if it is exact or not. Applying the test

$$
\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) &= 0 \\
1 - \frac{d}{dx}(x) + \frac{d^2}{dx^2}(1) &= 0 \\
1 - 1 &= 0 \\
0 &= 0
\end{aligned}
$$

So it exact. Comparing this ode to $F(x,y,y',y'') = f(x,y,y')\,y'' + g(x,y,y')$ shows that

$$
\begin{aligned}
f &= 1 \\
g &= xy' + y \\
&= xp + y
\end{aligned}
$$

Therefore (4) becomes

$$
\begin{aligned}
R &= \int f dp + \psi(x,y) \\
&= p + \psi(x,y) \qquad \text{(1A)}
\end{aligned}
$$

Hence (5) becomes

$$
\begin{aligned}
g &= \left(\frac{\partial}{\partial x}\left(\int f dp\right) + \psi_x\right) + \left(\frac{\partial}{\partial y}\left(\int f dp\right) + \psi_y\right) p \\
xp + y &= \left(\frac{\partial p}{\partial x} + \psi_x\right) + \left(\frac{\partial p}{\partial y} + \psi_y\right) p
\end{aligned}
$$

But $\frac{\partial p}{\partial x} = 0$ since y is held constant. And $\frac{\partial p}{\partial y} = 0$. The above becomes

$$
xp + y = \psi_x + \psi_y p
$$

Comparing terms shows that

$$
\begin{aligned}
x &= \psi_y \\
y &= \psi_x
\end{aligned}
$$

Integrating the first equation gives $\psi = xy + c$. Hence (1A) becomes

$$R = p + xy + c$$

Therefore, since $R = c_1$ a constant, then the above becomes (by merging the constants)

$$\begin{aligned} p + xy &= c_2 \\ y' + xy &= c_2 \end{aligned}$$

This is the reduced ode which needs to be solved for y. Solving gives

$$y = \operatorname{erf}\left(\frac{i\sqrt{2}x}{2}\right) e^{\frac{-x^2}{2}} c_1 + c_2 e^{\frac{-x^2}{2}}$$

2.4.1.4.1.3 Example 3

$$\begin{aligned} y'' - 2yy' &= 0 \\ F(x,y,y',y'') &= 0 \end{aligned}$$

First we need to determine if it is exact or not. Applying the test

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) &= 0 \\ -2y' - \frac{d}{dx}(-2y) + \frac{d^2}{dx^2}(1) &= 0 \\ -2y' + 2\frac{d}{dx}(y) &= 0 \\ -2y' + 2y' &= 0 \\ 0 &= 0 \end{aligned}$$

So it exact. Comparing this ode to $F(x,y,y',y'') = f(x,y,y')\,y'' + g(x,y,y')$ shows that

$$\begin{aligned} f &= 1 \\ g &= -2yy' \\ &= -2yp \end{aligned}$$

Therefore (4) becomes

$$\begin{aligned} R &= \int f dp + \psi(x,y) \\ &= p + \psi(x,y) \qquad \text{(1A)} \end{aligned}$$

Hence (5) becomes

$$
\begin{aligned}
g &= \left(\frac{\partial}{\partial x}\left(\int f dp\right) + \psi_x\right) + \left(\frac{\partial}{\partial y}\left(\int f dp\right) + \psi_y\right) p \\
-2yp &= \left(\frac{\partial p}{\partial x} + \psi_x\right) + \left(\frac{\partial p}{\partial y} + \psi_y\right) p \\
-2yp &= \psi_x + \psi_y p
\end{aligned}
$$

Comparing terms shows that

$$
\begin{aligned}
-2y &= \psi_y \\
0 &= \psi_x
\end{aligned}
$$

Integrating the first equation gives $\psi = -y^2 + h(x)$. Differentiating this w.r.t. x gives $\psi_x = h'(x)$. comparing this to the second equation above gives $0 = h'(x)$, hence $h(x) = c$. Hence $\psi = -y^2 + c$. Therefore (1A) becomes

$$
R = p - y^2 + c
$$

Therefore, since $R = c_1$ a constant, then the above becomes (by merging the constants)

$$
\begin{aligned}
p - y^2 &= c_2 \\
y' - y^2 &= c_2
\end{aligned}
$$

This is the reduced ode.

2.4.1.4.1.4 Example 4

$$
\begin{aligned}
(x-1)^2 y'' + 4xy' + 2y - 2x &= 0 \\
F(x, y, y', y'') &= 0
\end{aligned}
$$

First we need to determine if it is exact or not. Applying the test

$$
\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) &= 0 \\
2 - \frac{d}{dx}(4x) + \frac{d^2}{dx^2}\left((x-1)^2\right) &= 0 \\
2 - 4 + \frac{d}{dx}(2(x-1)) &= 0 \\
2 - 4 + 2 &= 0 \\
0 &= 0
\end{aligned}
$$

So it exact. Comparing this ode to $F(x,y,y',y'') = f(x,y,y')\,y'' + g(x,y,y')$ shows that

$$
\begin{aligned}
f &= (x-1)^2 \\
g &= 4xy' + 2y - 2x \\
&= 4xp + 2y - 2x
\end{aligned}
$$

Therefore (4) becomes

$$
\begin{aligned}
R &= \int f dp + \psi(x,y) \\
&= (x-1)^2 p + \psi(x,y) \qquad \text{(1A)}
\end{aligned}
$$

Hence (5) becomes

$$
\begin{aligned}
g &= \left(\frac{\partial}{\partial x}\left(\int f dp\right) + \psi_x\right) + \left(\frac{\partial}{\partial y}\left(\int f dp\right) + \psi_y\right) p \\
4xp + 2y - 2x &= \left(\frac{\partial}{\partial x}\left((x-1)^2 p\right) + \psi_x\right) + \left(\frac{\partial}{\partial y}\left((x-1)^2 p\right) + \psi_y\right) p \\
4xp + 2y - 2x &= 2p(x-1) + \psi_x + \psi_y p \\
4xp + 2y - 2x &= p(2(x-1) + \psi_y) + \psi_x
\end{aligned}
$$

Comparing terms shows that

$$
\begin{aligned}
4x &= 2(x-1) + \psi_y \\
2y - 2x &= \psi_x
\end{aligned}
$$

Or

$$
\begin{aligned}
2x + 2 &= \psi_y \\
2y - 2x &= \psi_x
\end{aligned}
$$

Integrating the first equation gives $\psi = 2xy + 2y + h(x)$. Differentiating this w.r.t. x gives $\psi_x = 2y + h'(x)$. comparing this to the second equation above gives $2y - 2x = 2y + h'(x)$, hence $h'(x) = -2x$. Hence $h = -x^2 + c$. Therefore $\psi = 2xy + 2y - x^2 + c$. Eq (1A) becomes

$$
\begin{aligned}
R &= (x-1)^2 p + 2xy + 2y - x^2 + c \\
&= (x-1)^2 y' + 2xy + 2y - x^2 + c
\end{aligned}
$$

Therefore, since $R = c_1$ a constant, then the above becomes (by merging the constants)

$$
(x-1)^2 y' + 2xy + 2y - x^2 = c_2
$$

Which is the reduced ode to solve.

2.4.1.4.1.5 Example 5

$$
\begin{aligned}
y''-y'e^{y}&=0\\
F(x,y,y',y'')&=0
\end{aligned}
$$

First we need to determine if it is exact or not. Applying the test

$$
\begin{aligned}
\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)&=0\\
-y'e^{y}-\frac{d}{dx}(-e^{y})+\frac{d^2}{dx^2}(1)&=0\\
-y'e^{y}+y'e^{y}&=0\\
0&=0
\end{aligned}
$$

So it exact. Comparing this ode to $F(x,y,y',y'')=f(x,y,y')\,y''+g(x,y,y')$ shows that

$$
\begin{aligned}
f&=1\\
g&=-y'e^{y}\\
&=-pe^{y}
\end{aligned}
$$

Therefore (4) becomes

$$
\begin{aligned}
R&=\int f dp+\psi(x,y)\\
&=p+\psi(x,y) \qquad (1A)
\end{aligned}
$$

Hence (5) becomes

$$
\begin{aligned}
g&=\left(\frac{\partial}{\partial x}\left(\int f dp\right)+\psi_x\right)+\left(\frac{\partial}{\partial y}\left(\int f dp\right)+\psi_y\right)p\\
-pe^{y}&=\left(\frac{\partial}{\partial x}p+\psi_x\right)+\left(\frac{\partial}{\partial y}p+\psi_y\right)p\\
-pe^{y}&=\psi_x+\psi_y p
\end{aligned}
$$

Comparing terms shows that

$$
\begin{aligned}
-e^{y}&=\psi_y\\
0&=\psi_x
\end{aligned}
$$

Integrating the first equation gives $\psi=-e^y+h(x)$. Partial differentiating this w.r.t. x gives $\psi_x=h'(x)$. comparing this to the second equation above gives $h'(x)=0$, hence $h(x)=c$. Hence $h=-x^2+c$. Therefore $\psi=-e^y+c$. Eq (1A) becomes

$$\begin{aligned}R &= p-e^y+c\\ &= y'-e^y+c\end{aligned}$$

Therefore, since $\phi=c_1$ a constant, then the above becomes (by merging the constants)

$$y'-e^y=c_2$$

Which is the reduced ode to solve.

2.4.2 nonlinear and not exact second order ode

2.4.2.1 Introduction

There seems in the literature two main approaches for handling this. One is to find an integrating factor μ which makes the ode exact, then it can be solved as shown above. The second approach is to find the first integral directly from the form of the ode itself. There are many methods to do this. I will go over the integrating method first, then the second method after that.

2.4.2.2 Solved by finding an integrating factor μ

ode internal name "exact_nonlinear_second_order_ode_with_integrating_factor"

2.4.2.2.1 Introduction

Not implemented yet. The above section showed how to solve the ode $F(x,y,y',y'')=0$ once it is determined it is exact as is, which is by finding the first integral R. But the real problem is what to do if the ode is not exact as is?. Given the second order nonlinear ode

$$F(x,y,y',y'')=0$$

Which is not exact as is (using the earlier test shown), then we need to either find an integrating factor μ to make it exact (this integrating factor might or might not exist) or try to find the first integral directly without finding an integrating factor first. There are few papers that show how to do this for some types of nonlinear second order odes.

Using an integrating factor approach, If we are able to find μ, then the ode can now be solved as type "second order integrable as is" or as type "exact nonlinear second order ode" as shown in the above section. (need to merge these types).

As mentioned earlier, an ode $F(x,y,y',y'')=0$ is called exact if there exists a function $R(x,y,y')$ (called first integral) with order one less than the order of the ode, such that

$$F(x,y,y',y'')=\frac{d}{dx}R(x,y,y')$$

If the ode is not exact, then we need to find an integrating factor of any of these forms $\mu(x)\,,\mu(y)\,,\mu(y')\,,\mu(x,y)\,,\mu(x,y')\,,\mu(y,y')$ such that $\mu F(x,y,y',y'')$ is now exact and hence

$$\mu F(x,y,y',y'')=\frac{d}{dx}R(x,y,y')$$

The main difficulty is how to find μ. Few papers were written on this (but I found them all not very clear as they give no examples).

Finding μ with first order ODE is easy. But not so easy with second order ode's. Note that in the above, an integrating factor of the form $\mu=\mu(x,y,y')$ will not be considered as finding such an integrating factor requires solving a pde which is harder than solving the original ode. There two relations are important in order to find μ

$$\begin{aligned} R &= G(x,y)+\int \mu dy' \qquad (1)\\ &= G(x,y)+\int \mu dp \end{aligned}$$

Where $p=y'$ and G is some function to be determined. As was derived in the introduction of the earlier section, we also have the relation

$$R_x+y'R_y+\Phi R_{y'}=0 \qquad (2)$$

2.4.2.2.2 Integrating factors by inspection.

These are not yet implemented. Before going through the formal way to find μ for non exact second order nonlinear ode, there is a table given by Murphy which we can utilize before searching for μ as a lookup table. Writing the ode as $y''+g(x,y,y')=0$ the table is

$g(x,y,y')$ form	integrating factor
$g(y)$ (i.e. function of y only)	y'
$g(y')$ (i.e. function of y' only)	$\frac{y'}{g}$
$p(x,y)\,y'+Q(x,y)\,(y')^2$	$\frac{1}{y'}$
$p(x,y)+Q(x,y)\,y'$ such that $\frac{\partial p}{\partial y}=\frac{\partial Q}{\partial x}$	$\frac{1}{y'}$

The above integrating factors are from Murphy book page 165.

2.4.2.2.3 Integrating factor $\mu(x)$ that depends on x only

Not implemented.

2.4.2.2.4 Integrating factor $\mu(y)$ that depends on y only

Not implemented.

2.4.2.2.5 Integrating factor $\mu(y')$ that depends on y' only

Not implemented.

2.4.2.2.6 Integrating factor $\mu(x, y)$

Not implemented.

2.4.2.2.7 Integrating factor $\mu(x, y')$

Not implemented.

2.4.2.2.8 Integrating factor $\mu(y, y')$

Not implemented.

2.4.2.2.9 Checking if an integrating factor exists (but not find it)

An example is

$$xy(2x + y)\, y'' + \left(x^2 + xy\right) y' + \left(3xy + y^2\right) = 0$$

to do.

2.4.2.2.10 References

1. book: Ordinary differential equations and their solutions by George M. Murphy.
2. paper: "Integrating Factors for Second-order ODEs" by E.S. Cheb-Terraba, and A.D. Roche.
3. Handbook of Mathematics for engineers and scientists. By Polyanin and Manzhirov. Page 492.

2.4.2.3 Solved by finding the first integral directly

ode internal name "exact_nonlinear_second_order_ode_using_first_integral"

2.4.2.3.1 Introduction

Not implemented yet. This uses point Lie symmetry.

The above section showed how to solve the nonlinear ode $F(x, y, y', y'') = 0$ once it is determined it is exact as is, which is by finding the first integral R directly without finding an integrating factor first. This below gives few ode forms with the corresponding first integral R to use and how to find R. These are collected from few papers I am studying now.

2.4.2.3.2 ode of the form $y'' + a_2(x,y)\,(y')^2 + a_1(x,y)\,y' + a_0(x,y) = 0$

From paper (On first integrals of second-order ordinary differential equations by Romero et all), this is called class B. The first integral is

$$\frac{d}{dx}R = C(x) + \frac{1}{A\,(x,y)\,y' + B\,(x,y)}$$

where $C_y = 0$. Another class of ode's is called class A with first integral

$$\frac{d}{dx}R = \frac{1}{A\,(x,y)\,y' + B\,(x,y)}$$

This is subset of class B.

2.4.3 ode is Integrable as given

ode internal name "second_order_integrable_as_is"

This is the same as "exact_nonlinear_second_order_ode". Can be linear or nonlinear. But must be of degree one. ODE is integrable as is w.r.t. the independent variable x. Need to merge type names into one.

2.4.3.1 Example 1

$$xyy'' + x(y')^2 - yy' = 0$$

Integrating both sides gives

$$\begin{aligned}\int xyy'' + x(y')^2 - yy'dx &= c_1\\ xyy' - y^2 &= c_1\\ y' &= \frac{c_1}{xy} + \frac{y}{x}\\ &= \frac{c_1 + y^2}{xy}\\ &= \left(\frac{c_1 + y^2}{y}\right)\frac{1}{x}\end{aligned}$$

Which is separable and easily solved.

2.4.3.2 Example 2

$$\begin{aligned}y'' &= -\frac{1}{2\,(y')^2}\\ 2(y')^2\,y'' &= -1\end{aligned}$$

With IC

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Integrating both sides gives

$$\begin{aligned}\int 2(y')^2\,y''dx &= \int -dx\\ \frac{2}{3}(y')^3 &= -x + c\\ (y')^3 &= -\frac{3}{2}x + c_1\end{aligned}$$

Hence

$$y_1' = \left(-\frac{3}{2}x + c_1\right)^{\frac{1}{3}} \tag{1}$$

$$y_2' = -(-1)^{\frac{1}{3}}\left(-\frac{3}{2}x + c_1\right)^{\frac{1}{3}} \tag{2}$$

$$y_3' = (-1)^{\frac{2}{3}}\left(-\frac{3}{2}x + c_1\right)^{\frac{1}{3}} \tag{3}$$

Trying solution (1). Integrating gives

$$\begin{aligned} y_1 &= \int \left(-\frac{3}{2}x+c_1\right)^{\frac{1}{3}} dx + c_2 \\ &= -\frac{1}{2}\left(-\frac{3}{2}x+c_1\right)^{\frac{4}{3}} + c_2 \end{aligned}$$

Applying $y(0)=1$ gives

$$1 = -\frac{1}{2}c_1^{\frac{4}{3}} + c_2 \tag{4}$$

And $y'(x)$ gives

$$y_1' = \left(-\frac{3}{2}x+c_1\right)^{\frac{1}{3}}$$

Hence $y'(0)=-1$ gives

$$-1 = c_1^{\frac{1}{3}}$$

No solution. Trying solution (2). Integrating gives

$$\begin{aligned} y_2 &= -(-1)^{\frac{1}{3}} \int \left(-\frac{3}{2}x+c_1\right)^{\frac{1}{3}} dx + c_2 \\ &= -(-1)^{\frac{1}{3}} \left(-\frac{1}{2}\left(-\frac{3}{2}x+c_1\right)^{\frac{4}{3}}\right) + c_2 \end{aligned} \tag{4A}$$

Applying $y(0)=1$ gives

$$1 = (-1)^{\frac{1}{3}}\left(\frac{1}{2}c_1^{\frac{4}{3}}\right) + c_2 \tag{5}$$

And $y_2'(x)$ gives

$$y_2'(x) = -\left(-\frac{1}{2}\right)^{\frac{1}{3}} (-3x+2c_1)^{\frac{1}{3}}$$

Hence $y'(0)=-1$ gives

$$\begin{aligned} -1 &= -\left(-\frac{1}{2}\right)^{\frac{1}{3}} (2c_1)^{\frac{1}{3}} \\ 1 &= (-1)^{\frac{1}{3}} (c_1)^{\frac{1}{3}} \end{aligned}$$

No solution. Finally we will try y_3. Integrating gives

$$\begin{aligned} y_3 &= (-1)^{\frac{2}{3}} \int \left(-\frac{3}{2}x+c_1\right)^{\frac{1}{3}} + c_2 \\ &= (-1)^{\frac{2}{3}} \left(-\frac{1}{2}\left(-\frac{3}{2}x+c_1\right)^{\frac{4}{3}}\right) + c_2 \end{aligned}$$

Applying $y(0)=1$ gives

$$1=(-1)^{\frac{2}{3}}\left(-\frac{1}{2}c_1^{\frac{4}{3}}\right)+c_2 \tag{6}$$

And $y_3'(x)$ gives

$$y_3'(x)=(-1)^{\frac{2}{3}}\left(-\frac{3}{2}x+c_1\right)^{\frac{1}{3}}$$

Hence $y'(0)=-1$ gives

$$-1=(-1)^{\frac{2}{3}}\,(c_1)^{\frac{1}{3}}$$

Solving gives $c_1=-1$. Substituting into (6) gives

$$\begin{aligned}1&=(-1)^{\frac{2}{3}}\left(-\frac{1}{2}(-1)^{\frac{4}{3}}\right)+c_2\\ c_2&=\frac{3}{2}\end{aligned}$$

Hence solution is

$$\begin{aligned}y_3&=(-1)^{\frac{2}{3}}\left(-\frac{1}{2}\left(-\frac{3}{2}x+c_1\right)^{\frac{4}{3}}\right)+c_2\\ &=(-1)^{\frac{2}{3}}\left(-\frac{1}{2}\left(-\frac{3}{2}x-1\right)^{\frac{4}{3}}\right)+\frac{3}{2}\\ &=\frac{3}{2}-\frac{1}{2}(-1)^{\frac{3}{2}}\left(-\frac{3}{2}x-1\right)^{\frac{4}{3}}\end{aligned}$$

This problem shows that out of the 3 solutions, only one was valid.

2.4.4 ode can be made Integrable $F(x,y,y'')=0$

ode internal name "second_order_ode_can_be_made_integrable"

Can be linear or nonlinear. These are ode's which become integrable if both sides are multiplied by y'. For this method to have chance of working, the original ode must not have y' already in it.

2.4.4.1 Example

$$2y'' - e^y = 0$$

Multiplying both sides by y' gives

$$2y'y'' - y'e^y = 0$$

Integrating

$$\int (2y'y'' - y'e^y)\, dx = c_1$$
$$(y')^2 - e^y = c_1$$

Hence

$$y' = \pm\sqrt{e^y + c_1}$$

Each of the above is separable, which are solved by integration.

2.4.5 Solved using Mainardi Liouville method

ode internal name "second_order_nonlinear_solved_by_mainardi_lioville_method"

2.4.5.1 Introduction

This shows how to solve the nonlinear second order ode of the form

$$y''(x) + p(x)\, y'(x) + q(y)\, (y'(x))^2 = 0 \tag{1}$$

For this method to work, in the above $p(x)$ must be either a function of x or a constant. It can not depend on y. And in the term $q(y)\,[y'(x)]^2$, $q(y)$ must be only a function of y or a constant. It can not depend on x.

For an example this method will work on $y'' + y' + yy^2 = 0$ and on $y'' + \sin(x)\, y'(x) + y(y')^2 = 0$ and on $y'' + \sin(x)\, y' + (1+y)\,(y')^2 = 0$ but not on $y'' + y' + xyy^2 = 0$ and not on $y'' + \sin(y)\, y' + yy^2 = 0$.

This is implemented in my ode solver as type 18. The first step is to divide (1) by $y'(x)$ which gives

$$\frac{y''}{y'} + p(x) + q(y)\, y' = 0 \tag{2}$$
$$\frac{y''}{y'} = -q(y)\, y' - p(x) \tag{3}$$

The LHS is $\frac{d}{dx}(\ln y')$ and the term $q(y)\,y'(x)$ is $\left(\frac{d}{dy}\int q(y)\,dy\right)\frac{dy}{dx}=\frac{d}{dx}\int q(y)\,dy$. This is the reason why q can not depend on x, In order to be able to evaluate the integral. Using this (3) now becomes

$$\begin{aligned}
\frac{y''}{y'} &= -\left(\frac{d}{dx}\int q(y)\,dy\right)-p(x)\\
\frac{d}{dx}(\ln y') &= -\left(\frac{d}{dx}\int q(y)\,dy\right)-p(x)\\
\frac{d}{dx}(\ln y')+\frac{d}{dx}\int q(y)\,dy &= -p(x)\\
\frac{d}{dx}\left(\ln y'+\int q(y)\,dy\right) &= -p(x)
\end{aligned}$$

Integrating gives

$$\ln y'+\int q(y)\,dy=-\int p(x)\,dx \tag{4}$$

And this is the reason why p can not depend on y. In order to able to integrate the RHS above. Once $\int q(y)\,dy$ and $\int p(x)\,dx$ are evaluated, then y' is found and this gives first order ode in y which is easily solved.

2.4.5.2 Example

$$y''+(3+x)\,y'+y[y']^2=0$$

Comparing to

$$y''(x)+p(x)\,y'(x)+q(y)\,[y'(x)]^2=0$$

Show that $p=(3+x)$ and $q(y)=y$. Hence the conditions are satisfied to use this method. Therefore equation (4) becomes

$$\begin{aligned}
\ln y'+\int q(y)\,dy &= -\int p(x)\,dx\\
\ln y'+\int ydy &= -\int(3+x)\,dx\\
\ln y'+\frac{y^2}{2} &= -\frac{(3+x)^2}{2}+c\\
\ln y' &= -\frac{(3+x)^2}{2}-\frac{y^2}{2}+c
\end{aligned}$$

Hence

$$y'=c_1e^{-\frac{(3+x)^2}{2}-\frac{y^2}{2}}$$

This is separable.

$$\frac{dy}{dx} = c_1 e^{-\frac{(3+x)^2}{2}} e^{-\frac{y^2}{2}}$$
$$e^{\frac{y^2}{2}} dy = c_1 e^{-\frac{(3+x)^2}{2}} dx$$

Integrating gives

$$\int e^{\frac{y^2}{2}} dy = \int c_1 e^{-\frac{(3+x)^2}{2}} dx + c_2$$
$$-\frac{i}{2}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{i}{\sqrt{2}}y\right) = -\frac{c_1}{2}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}} + \frac{3}{\sqrt{2}}\right) + c_2$$

And the above is the implicit solution for y.

2.4.6 ode with missing independent variable x or missing dependent variable $y(x)$

A nonlinear ode with missing x has the form

$$y'' + f(y') + g(y) = 0 \tag{1}$$

Where $f(y')$ and $g(y)$ can be nonlinear functions (one or both of them). If both are linear, then there is no need for this method to be used. For missing x the substitution $y'(x) = p(y)$ is used. For nonlinear ode with missing y which has the form

$$y'' + F(x)\,f(y') + g(x) = 0 \tag{2}$$

Where $F(x)\,, g(x)$ are functions of x (or constants) and $f(y')$ is nonlinear in y'. For this case the substitution $y'(x) = p(x)$ is used instead. The following gives examples of each method.

Both methods reduce the order of the ode by one resulting in first order ode where the dependent variable is p which is then easily solved for p. This now results in another first order ode in y which is then easily solved.

2.4.6.1 Missing independent variable x

ode internal name "second_order_ode_missing_x"

Given

$$y'' + f(y') + g(y) = 0 \tag{1}$$

Let $p = y'$ then $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p$ and the ode becomes

$$p\frac{dp}{dy} + f(p) + g(y) = 0 \tag{2}$$

Which is now a first order ode.

2.4.6.1.1 Example 1

$$yy''-(y')^2=1$$

Let $p=y'$ then $y''=p\frac{dp}{dy}$. Hence the ode becomes

$$\begin{aligned}yp\frac{dp}{dy}-p^2&=1\\ \frac{dp}{dy}&=\frac{1+p^2}{p}\frac{1}{y}\end{aligned}$$

This is separable.

$$\begin{aligned}\frac{dp}{dy}\frac{p}{1+p^2}&=\frac{1}{y}\\ \frac{p}{1+p^2}dp&=\frac{1}{y}dy\\ \int\frac{p}{1+p^2}dp&=\int\frac{1}{y}dy\\ \frac{1}{2}\ln(p-1)+\frac{1}{2}\ln(p+1)&=\ln y+c\end{aligned}$$

Or, assuming $p-1>0, p+1>0$

$$\begin{aligned}\ln(p-1)+\ln(p+1)&=2\ln y+2c\\ \ln((p-1)(p+1))&=\ln y^2+c_1\\ (p-1)(p+1)&=c_2y^2\\ p^2-1&=c_2y^2\\ p^2&=c_2y^2+1\end{aligned}$$

Hence

$$p=\pm\sqrt{1+c_2y^2}$$

But $p=y'(x)$. The above becomes

$$y'(x)=\pm\sqrt{1+c_2y^2}$$

This is first order ode which is separable. The first one gives

$$\begin{aligned}y'(x)&=\sqrt{1+c_2y^2}\\ \frac{dy}{\sqrt{1+c_2y^2}}&=dx\\ \int\frac{dy}{\sqrt{1+c_2y^2}}&=\int dx\\ \frac{1}{\sqrt{c_2}}\ln\left(\sqrt{c_2}y+\sqrt{1+c_2y^2}\right)&=x+c_3\\ \ln\left(\sqrt{c_2}y+\sqrt{1+c_2y^2}\right)&=\sqrt{c_2}x+\sqrt{c_2}c_3\end{aligned}$$

Where c_2, c_3 are constants. Similar solution result for the negative ode.

2.4.6.1.2 Example 2

$$y'' + ay(y') + by^3 = 0 \tag{1}$$

Let $p = y'$ then $y'' = p\frac{dp}{dy}$. Hence the ode becomes

$$p\frac{dp}{dy} + ayp + by^3 = 0 \tag{2}$$

Which is now a first order ode.

$$\frac{dp}{dy} = -ay + b\frac{y^3}{p} \tag{3}$$

Solving for p gives

$$\frac{1}{4\sqrt{a^2+8b}}\left(\ln\left(-by^4 + ay^2p + 2p^2\right)\sqrt{a^2+8b} + 2a\operatorname{arctanh}\left(\frac{ax^2+4p}{y^2\sqrt{a^2+8b}}\right)\right) = c_1$$

Then y is found by solving $y' = p$, another first order ode.

$$\frac{1}{4\sqrt{a^2+8b}}\left(\ln\left(-by^4 + ay^2y' + 2(y')^2\right)\sqrt{a^2+8b} + 2a\operatorname{arctanh}\left(\frac{ax^2+4y'}{y^2\sqrt{a^2+8b}}\right)\right) = c_1$$

But this second one could not solve. Actually ode (3) is homogeneous, class G and should use formula given in Kamke's book, p. 19. but I have yet to implement this.

2.4.6.1.3 Example 3

$$2yy'' - y^3 - 2(y')^2 = 0 \tag{1}$$

With IC

$$\begin{aligned} y(0) &= -1 \\ y'(0) &= 0 \end{aligned}$$

Let $p = y'$ then $y'' = p\frac{dp}{dy}$. Hence the ode becomes

$$2yp\frac{dp}{dy} - y^3 - 2p^2 = 0 \tag{2}$$

$$\frac{dp}{dy} = \frac{y^3+2p^2}{2py}$$

Which is first order ode in $p(y)$ of type Bernoulli. There are two solutions

$$p_1 = y\sqrt{y+c_1} \tag{3}$$
$$p_2 = -y\sqrt{y+c_1} \tag{4}$$

But $p = y'$ hence the above becomes

$$y'(x) = y\sqrt{y+c_1} \tag{3}$$
$$y'(x) = -y\sqrt{y+c_1} \tag{4}$$

Solving (3). At $x=0$ we have $y'(0)=0, y(0)=-1$ hence the above becomes

$$\begin{aligned}
0 &= -1\sqrt{-1+c_1}\\
0 &= \sqrt{-1+c_1}\\
c_1 &= 1
\end{aligned}$$

Hence (3) becomes

$$y'(x) = y\sqrt{y+1}$$

This is quadrature. Integrating

$$\begin{aligned}
\frac{dy}{y\sqrt{y+1}} &= dx\\
-2\,\text{arctanh}\left(\sqrt{y+1}\right) &= x + c_2
\end{aligned}$$

At $x=0$ we have $y(0)=-1$ and the above becomes

$$\begin{aligned}
-2\,\text{arctanh}\left(\sqrt{-1+1}\right) &= c_2\\
c_2 &= -2\,\text{arctanh}\left(0\right)\\
c_2 &= 0
\end{aligned}$$

Hence the solution is

$$
\begin{aligned}
-2\operatorname{arctanh}\left(\sqrt{y+1}\right) &= x \\
\operatorname{arctanh}\left(\sqrt{y+1}\right) &= -\frac{x}{2} \\
\sqrt{y+1} &= \tanh\left(-\frac{x}{2}\right) \\
&= -\tanh\left(\frac{x}{2}\right) \\
y+1 &= \tanh^2\left(\frac{x}{2}\right) \\
y &= \tanh^2\left(\frac{x}{2}\right)-1
\end{aligned}
$$

Solving (4)

At $x=0$ we have $y'(0)=0, y(0)=-1$ hence

$$
\begin{aligned}
0 &= 1\sqrt{-1+c_1} \\
0 &= \sqrt{-1+c_1} \\
c_1 &= 1
\end{aligned}
$$

Hence (4) becomes

$$
y'(x) = y\sqrt{y+1}
$$

Which gives same solution as before. $y=\tanh^2\left(\frac{x}{2}\right)-1$

2.4.6.1.4 Example 4

$$
2y''-e^y=0 \tag{1}
$$

With IC

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 1
\end{aligned}
$$

Let $p=y'$ then $y''=p\frac{dp}{dy}$. Hence the ode becomes

$$
\begin{aligned}
2p\frac{dp}{dy}-e^y &= 0 \\
2\frac{dp}{dy}p &= e^y \qquad (2)
\end{aligned}
$$

This is separable.

$$
\begin{aligned}
2\int p dp &= \int e^{y} dy \\
p^2 &= e^y + c_1 \qquad (3)
\end{aligned}
$$

Before solving this, we should apply IC now as it simplifies the solution greatly. This assumes both y, y' are are given at same point x_0. Which is the case here. If only one IC is given (such as $y(0)$ or $y'(0)$ but not both, then we can not apply IC now and have to do it at the end).

We are given that $y'(0) = p = 1, y(0) = 0$, hence the above reduces to

$$
\begin{aligned}
1 &= e^0 + c_1 \\
c_1 &= 0
\end{aligned}
$$

Hence (3) now becomes

$$
p^2 = e^y
$$

but $p = y'$ hence

$$
\begin{aligned}
(y')^2 &= e^y \\
y' &= \pm\sqrt{e^y}
\end{aligned}
$$

This is quadrature. For the positive solution

$$
\begin{aligned}
\frac{dy}{\sqrt{e^y}} &= dx \qquad (4) \\
\frac{2}{\sqrt{e^y}} &= -x + c_2 \qquad (2.3)
\end{aligned}
$$

For $y(0) = 0$ we obtain

$$
2 = c_2
$$

Hence (4) becomes

$$
\begin{aligned}
\frac{2}{\sqrt{e^y}} &= -x + 2 \\
\sqrt{e^y} &= \frac{2}{2-x} \\
e^y &= \left(\frac{2}{2-x}\right)^2 \\
y_1 &= 2\ln\left(\frac{2}{2-x}\right)
\end{aligned}
$$

For the negative solution

$$y' = -\sqrt{e^y}$$

Integrating

$$\frac{2}{\sqrt{e^y}} = x + c_2 \tag{5}$$

At $y(0) = 0$

$$2 = c_2$$

Hence (5) becomes

$$\begin{aligned}\frac{2}{\sqrt{e^y}} &= x + 2\\ \sqrt{e^y} &= \frac{2}{x+2}\\ e^y &= \left(\frac{2}{x+2}\right)^2\\ y_2 &= 2\ln\left(\frac{2}{x+2}\right)\end{aligned}$$

However, this solution do not satisfy $y'(0) = 1$ so it is discarded. Hence the solution is only

$$y_1 = 2\ln\left(\frac{2}{2-x}\right)$$

2.4.6.1.5 Example 5

This is same example as above, but here we delay applying IC to the very end to see the difference. This method is more general, but makes solving for IC harder.

$$2y'' - e^y = 0 \tag{1}$$

With IC

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Let $p = y'$ then $y'' = p\frac{dp}{dy}$. Hence the ode becomes

$$2\frac{dp}{dy}p = e^y$$

This is separable.

$$\begin{aligned}2\int p dp &= \int e^y dy\\ p^2 &= e^y + c_1\end{aligned}$$

but $p=y'$ hence the above becomes

$$
\begin{aligned}
(y')^2 &= e^y + c_1 \\
y' &= \pm\sqrt{e^y + c_1}
\end{aligned}
$$

This is quadrature. For the positive solution

$$
\begin{aligned}
\frac{dy}{\sqrt{e^y+c_1}} &= dx \\
-\frac{2\operatorname{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} &= x + c_2 \\
2\operatorname{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right) &= -x\sqrt{c_1} - c_2\sqrt{c_1} \\
\operatorname{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right) &= -x\frac{\sqrt{c_1}}{2} - \frac{c_2\sqrt{c_1}}{2} \\
\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}} &= \tanh\left(-x\frac{\sqrt{c_1}}{2} - \frac{c_2\sqrt{c_1}}{2}\right) \\
\sqrt{e^y+c_1} &= \sqrt{c_1}\tanh\left(-x\frac{\sqrt{c_1}}{2} - \frac{c_2\sqrt{c_1}}{2}\right) \\
e^y + c_1 &= \left(\sqrt{c_1}\tanh\left(-x\frac{\sqrt{c_1}}{2} - \frac{c_2\sqrt{c_1}}{2}\right)\right)^2 \\
e^y &= \left(\sqrt{c_1}\tanh\left(-x\frac{\sqrt{c_1}}{2} - \frac{c_2\sqrt{c_1}}{2}\right)\right)^2 - c_1 \\
y &= \ln\left(\left(\sqrt{c_1}\tanh\left(-x\frac{\sqrt{c_1}}{2} - \frac{c_2\sqrt{c_1}}{2}\right)\right)^2 - c_1\right) \qquad (2)
\end{aligned}
$$

Now we have to use (2) and take derivative and solve for c_1, c_2. Much harder than if we have applied IC to each solution earlier.

2.4.6.1.6 Example 6

$$
2y'' - \sin(2y) = 0 \qquad (1)
$$

With IC

$$
\begin{aligned}
y(0) &= -\frac{\pi}{2} \\
y'(0) &= 1
\end{aligned}
$$

Let $p=y'$ then $y''=p\frac{dp}{dy}$. Hence the ode becomes

$$\begin{aligned}
2p\frac{dp}{dy} &= \sin(2y) \qquad (2)\\
2pdp &= \sin(2y)\,dy\\
\int 2pdp &= \int \sin(2y)\,dy\\
p^2 &= -\frac{1}{2}\cos(2y)+c_1
\end{aligned}$$

At $x=0$ we have $p=1, y=-\frac{\pi}{2}$. Hence the above becomes

$$\begin{aligned}
1 &= -\frac{1}{2}\cos(-\pi)+c_1\\
&= -\frac{1}{2}\cos(\pi)+c_1\\
1 &= \frac{1}{2}+c_1\\
c_1 &= \frac{1}{2}
\end{aligned}$$

Therefore (2) becomes

$$(y'(x))^2 = -\frac{1}{2}\cos(2y)+\frac{1}{2}$$

Need to solve and apply IC $y(0)=-\frac{\pi}{2}$ to finish.

2.4.6.2 Missing dependent variable $y(x)$

ode internal name "second_order_ode_missing_y"

Given

$$y''+F(x)\,f(y')+g(x)=0$$

Let $p=y'$ then $y''=p'$. Hence the ode becomes

$$p'+F(x)\,f(p)+g(x)=0$$

Which is first order ode.

2.4.6.2.1 Example

$$y'' + (y')^2 + y' = 0 \tag{1}$$

Let $p = y'$ then $y'' = p'$. Hence the ode becomes

$$p' + p^2 + p = 0 \tag{2}$$

Which is now a first order separable ode. Its solution can be easily found to be

$$p = \frac{1}{c_1 e^x - 1}$$

Hence

$$y'(x) = \frac{1}{c_1 e^x - 1}$$

Which is now solved for $y(x)$ as first order, which gives by integration

$$y = \ln\left(c_1 e^x - c_2 + 1\right) - x$$

2.4.7 Higher degree second order ode

ode internal name "second_order_ode_high_degree"

These are ode's with the second derivative raised to power not one. Solved by solving for y'' which generates all roots and now each ode is solved.

CHAPTER 3

Higher order ode $F(x, y, y', y'', y''') = 0$

3.1 Linear higher order ode

3.1.1 Linear ode with constant coefficients $a_3 y''' + a_2 y'' + a_1 y' + a_0 y = f(x)$

3.1.1.1 Solved by finding roots of characteristic equation

ode internal name "Higher order linear constant coefficients ODE"

These are solved finding roots of characteristic equation. This is the standard method. For non-homogeneous ode, The method of Variation of parameters and the method of undetermined coefficients are both used to find the particular solution.

3.1.1.2 Solved by series method

ode internal name "Higher_order_series_method_ordinary_point"

Only ordinary point is supported and for third order ode at this time. See section below.

3.1.1.3 Solved using Laplace transform

ode internal name "higher_order_laplace"

Laplace transform method is used. Currently only linear with constant coefficient ode is supported.

3.1.2 Linear ode with non-constant coefficients

3.1.2.1 Euler type $x^3 y''' + x^2 y'' + x y' + y = f(x)$

ode internal name "higher_order_ODE_non_constant_coefficients_of_type_Euler"

This uses same algorithm as for second order Euler type ode but for higher order.

3.1.2.2 Missing y as in $ay''' + by'' + cy' = f(x)$

ode internal name "higher_order_ODE_missing_y"

Since y is missing, we then assume $y' = u$ and the ode reduces to one order less. Now the lower order ode is solved.

3.1.2.2.1 Example 1

$$x^2y''' + xy'' + y' = 0$$

This is not Euler type as it stands. Let $y' = u$ then the ode becomes

$$x^2u'' + xu' + u = 0$$

This is now Euler type. Solving it gives

$$u = c_2 \cos(\ln x) + c_3 \sin(\ln x)$$

Hence

$$y' = c_2 \cos(\ln x) + c_3 \sin(\ln x)$$

Solving this as first order ode of quadrature type gives

$$\begin{aligned} y &= \frac{c_2}{2} x \cos(\ln x) + \frac{c_2}{2} x \sin(\ln x) - \frac{1}{2} c_3 x \cos(\ln x) + \frac{1}{2} c_3 x \sin(\ln x) + c_1 \\ &= x \cos(\ln x)\left(\frac{c_2}{2} - \frac{1}{2}c_3\right) + x \sin(\ln x)\left(\frac{c_2}{2} + + \frac{1}{2}c_3\right) + c_1 \\ &= C_2 x \cos(\ln x) + C_3 x \sin(\ln x) + c \end{aligned}$$

3.1.2.2.2 Example 2

$$xy'''' + y''' + y'' = 0$$

Let $y' = u$ then the ode becomes

$$xu''' + u'' + u' = 0$$

Since u is missing then let $u' = v$ and the above becomes

$$xv'' + v' + v = 0$$

This is now second order ode. This is Bessel ode whose solution is

$$v = c_3 \operatorname{BesselJ}_0\left(2\sqrt{x}\right) + c_4 \operatorname{BesselY}_0\left(2\sqrt{x}\right)$$

Hence

$$u' = c_3 \operatorname{BesselJ}_0\left(2\sqrt{x}\right) + c_4 \operatorname{BesselY}_0\left(2\sqrt{x}\right)$$

This is solved by quadrature giving

$$u = c_3\sqrt{x}\operatorname{BesselJ}_1\left(2\sqrt{x}\right) + c_4\sqrt{x}\operatorname{BesselY}_1\left(2\sqrt{x}\right) + c_2$$

Hence

$$y' = c_3\sqrt{x}\operatorname{BesselJ}_1\left(2\sqrt{x}\right) + c_4\sqrt{x}\operatorname{BesselY}_1\left(2\sqrt{x}\right) + c_2$$

This is solved by quadrature giving

$$y = c_3 x \operatorname{BesselJ}_2\left(2\sqrt{x}\right) + c_4 x \operatorname{BesselY}_2\left(2\sqrt{x}\right) + c_2 x + c_1$$

3.1.2.2.3 Example 3

$$xy''' - y'' = 0$$

Let $y' = u$ then the ode becomes

$$xu'' - u' = 0$$

Since u is missing then let $u' = v$ and the above becomes

$$xv' - v = 0$$

This is linear first order ode whose solution is $v = c_1 x$. Hence $u' = c_1 x$. Integrating gives $u = c_1 x^2 + c_2$. Hence

$$y' = c_1 x^2 + c_2$$

Integrating gives

$$y = c_1 x^3 + c_2 x + c_3$$

3.1.2.3 Solved by series method

ode internal name "higher_order_taylor_series_method_ordinary_point"

Only ordinary point is supported and for third order ode at this time using Taylor series (not power series) method. Let

$$y''' = f(x, y, y', y'')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y', y'')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$ and $y''(x_0) = y_0''$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \frac{(x-x_0)^4}{4!} y''''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2} y_0'' + \frac{x^3}{3!} f|_{x_0,y_0,y_0',y_0''} + \frac{x^4}{4!} f'|_{x_0,y_0,y_0',y_0''} + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2} y_0'' + \sum_{n=0}^{\infty} \frac{x^{n+3}}{(n+3)!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0',y_0''} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial y''}\frac{dy''}{dx} \qquad (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' + \frac{\partial f}{\partial y'} y''' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' + \frac{\partial f}{\partial y'} f \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) y'' + \frac{\partial}{\partial y''}\left(\frac{df}{dx}\right) y''' \qquad (2) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) y'' + \frac{\partial}{\partial y''}\left(\frac{df}{dx}\right) f \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \frac{\partial}{\partial y}\left(\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) y'' + \frac{\partial}{\partial y''}\left(\frac{d^2 f}{dx^2}\right) f \qquad (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y',y'')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}y'' + \frac{\partial F_0}{\partial y''}y'''\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}y'' + \frac{\partial F_0}{\partial y''}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}y'' + \frac{\partial F_1}{\partial y''}y'''\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}y'' + \frac{\partial F_1}{\partial y''}F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' + \left(\frac{\partial F_{n-1}}{\partial y''}\right)y'''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' + \left(\frac{\partial F_{n-1}}{\partial y''}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \frac{x^2}{2}y_0'' + \sum_{n=0}^{\infty}\frac{x^{n+3}}{(n+3)!}F_n|_{x_0,y_0,y_0',y_0''} \tag{7}
$$

To find $y(x)$ series solution around $x=0$.

3.2 nonlinear higher order ode

Not currently supported.